ON THE TANGENT ALGEBRA OF ALGEBRAIC COMMUTATIVE MOUFANG LOOPS

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Introduction

In the theory of loops, in view of the deepness of the methods and result, the most developed parts are certainly the theories of abstract, topological and differentiable loops. The standard references on the theory are [1] and [2], while recent contributions to the continuous theory are the monographs [8] of P.T. Nagy and K. Strambach and [9] by L.V. Sabinin.

The richness of the analytic theory of loops enables us even to speak of the *Lie theory* of some special loop classes (Moufang, Bruck, local Bol). On the other hand, the theory of algebraic groups, built up along the same lines as the theory of Lie groups in some sense, is one of the main branches of group theory. This makes the fact that *algebraic loops and quasigroups* were so far only sporadically investigated ([8, Section 12]) surprising.

The main object of the author's PhD thesis [6] was the category of *algebraic commutative Moufang loops*. The adjective *algebraic* is used in the sense it is used in the theory of algebraic groups: The underlying set of the loop under consideration is an (affine) algebraic variety over an algebraically closed field K and the loop operations are morphisms of algebraic varieties.

The results of [6] concern the abstract and algebraic properties of algebraic commutative Moufang loops ([5]) and their tangential structure ([7]). The latter is constructed via the localization process; the method shows that any algebraic Bruck loop has a tangential Lie triple system. (The abstractly defined class of Bruck loop contains the class of abstract commutative Moufang loops.)

In this paper, we consider the special properties of the tangent Lie triple systems of algebraic commutative Moufang loops (Theorem 3.1) and relate them with the tangent Lie algebras of the multiplication group of the loops (Proposition 2.3). Using this relation, we obtain an algebraic homomorphism of the group of inner mappings of the loop in the (linear) group of automorphisms of the tangent L.t.s. (Theorem 3.7). This homomorphism implies well expected results on tangent spaces of closed normal and associator subloops (Propositions 3.8 and 3.9).

1. Preliminaries from the theory of algebraic loops

Definition. *The set L endowed with the binary operation "$\cdot$" is a* loop *if there is a unit element $1 \in L$ such that for all $x \in L$ holds $x = 1 \cdot x = x \cdot 1$*

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and, furthermore, for any $a, b, c, d \in L$, the equations $x \cdot a = b$, $c \cdot y = d$ have unique solutions in x and y. Bruck loops. Commutative Moufang loop.

We denote the solutions by $x = a\backslash b$ and $y = d/c$. For a loop L, the *left* and *right translation maps*

$$\lambda_x(y) = xy, \quad \rho_x(y) = yx$$

are also defined. These translations generate the *multiplication group*

$$\mathcal{M}(L) = \langle \lambda_x, \rho_x \mid x \in L \rangle$$

of L. The stabilizer $\mathcal{M}(L)_1$ of the unit element in $\mathcal{M}(L)$ is generated by the maps $\lambda_{xy}^{-1}\lambda_x\lambda_y$, $\rho_{yx}^{-1}\rho_x\rho_y$, $\lambda_x^{-1}\rho_x$ $(x, y \in L)$ ([1, Theorem ...]). The elements of $\mathcal{M}(L)_1$ are called *inner maps* of L.

In order to consider special loop classes, we need the following identities.

$$x \cdot (y \cdot xz) = (x \cdot yx) \cdot z \quad \text{(Bol identity)} \tag{1}$$

$$(xy)^{-1} = x^{-1}y^{-1} \quad \text{(Automorph inverse identity)} \tag{2}$$

$$(xy)(xz) = x^2(yz) \quad \text{(Commutative Moufang identity).} \tag{3}$$

Loops, which satisfy (1) and (2) are called *Bruck loops*. A proper subclass of such loops is the class of *commutative Moufang loops*, this can be defined by identity (3), as well. The basic result of the theory of commutative Moufang loops is the following theorem.

Theorem 1.1 (Theorem of Bruck and Slaby [1])**.** *Let L be a commutative Moufang loop which is generated by n elements. Then L is nilpotent with nilpotency class at most $n - 1$.*

Next, we define the category of algebraic loops.

Definition. *Let L be an algebraic variety over the algebraically closed field K and an abstract loop $L = (L, \cdot, /, \backslash)$. We say that L is an* algebraic loop *if the $L \times L \to L$ maps*

$$(x, y) \mapsto x \cdot y, x/y, x\backslash y$$

are morphisms of algebraic varieties.

The loop L is an *algebraic Bruck* or *algebraic commutative Moufang loop*, if L is an algebraic loop and, at the same time, a Bruck or commutative Moufang loop, respectively. We have the following important result.

Theorem 1.2 ([5, Theorem 1.6])**.** *Algebraic commutative Moufang loops are nilpotent.*

The main consequences of the nilpotence of algebraic commutative Moufang loops are:

Corollary 1.3 ([5, Section 1])**.** *(i) Let L be a proper connected algebraic commutative Moufang loop over K. Then* $\mathrm{char}(\mathsf{K}) = 3$.

(ii) For any closed normal subloop $N \lhd L$ of L, the factor loop L/N is an algebraic loop.

(iii) L has closed normal subloops $U, S \lhd L$, where S is a semisimple commutative group, U has exponent 3^k for some $k \in \mathbb{N}$ and $L = S \times U$ holds. Moreover, the component U is isomorphic to K^n $(n = \dim(U))$ as a variety.

(iv) We can introduce coordinates on U such that $U = \mathsf{K}^n$ and $\boldsymbol{x}^{-1} = -\boldsymbol{x}$ holds for all $\boldsymbol{x} \in \mathsf{K}^n$.

Coordinates satisfying this last property is called *normal coordinates.* In a normal coordinate system, the loop multiplication is given by

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{\mu}(\boldsymbol{x}, \boldsymbol{y}),$$

where $\boldsymbol{\mu} = (\mu^i)$ and $\mu^i(X^1, \dots, X^n, Y^1, \dots, Y^n)$ are polynomials in $2n$ variables over K.

The main result concerning algebraic commutative Moufang loops is the following.

Theorem 1.4 ([5, Theorem 4.2])**.** *Let L be an algebraic commutative Moufang loop. Then the multiplication group $\mathcal{M}(L)$ of L has the structure of an algebraic transformation group, acting morphically on L.*

In the theory of algebraic loops, an other important aspect is the definition of the tangent algebra. The next concept occured first in connection with the tangential structure of totally geodesic submanifolds and symmetric spaces.

Definition. *A finite dimensional vector space $\mathfrak{b}$ over a field K equipped with a trilinear operation $(.,.,.)$ is called a* Lie triple system *(abbrev. L.t.s.), if for all $x, y, z, u, v \in \mathfrak{b}$,*

$$(x, x, y) = 0, \tag{4}$$

$$(x, y, z) + (y, z, x) + (z, x, y) = 0, \tag{5}$$

$$(u, v, (x, y, z)) = ((u, v, x), y, z) + (x, (u, v, y), z) + (x, y, (u, v, z)). \tag{6}$$

In the same way than for Lie algebras, the theory of Lie triple systems gets more complex in characteristic $p > 0$.

Definition. *A* restricted Lie triple system *over a field K of prime characteristic $p > 2$ is a L.t.s. $\mathfrak{b}$ together with a mapping $z \mapsto z^{[p]}$ of $\mathfrak{b}$ into $\mathfrak{b}$ satisfying the identities:*

$$(x, y^{[p]}, z) = (((x, \underbrace{y, y), \dots y, y), y}_{p}, z), \tag{7}$$

$$(\alpha z)^{[p]} = \alpha^p z^{[p]}, \tag{8}$$

$$(y + z)^{[p]} = y^{[p]} + z^{[p]} + \Lambda_p(x, y), \tag{9}$$

where $\Lambda_p(x, y)$ is a universally defined $(.,.,.)$-polynomial over the prime field of K.

Remark. The $(.,.,.)$-polynomial $\Lambda_p(x, y)$ is uniquely defined by the following property. Let A be any associative K-algebra, let us define the Lie bracket $[x, y] = xy - yx$ and the ternary operation $(x, y, z) = [[x, y], z]$. Then the equation

$$\Lambda_p(x, y) = (x + y)^p - x^p - y^p$$

holds identically in A.

It is well known that for a given point $x \in L$, the set of point derivations can be identified with the tangent space $T_x(L)$ of L in x (see [3, p. 38]). Let us denote by $\mathfrak{l}$ the tangent space $T_e(L)$ at the unit element. One can associate any tangent vector $\alpha \in \mathfrak{l}$ to a derivation D_α in the following way. For any $f \in \mathsf{K}[L]$, we define $D_\alpha(f)$ by

$$D_\alpha(f)(x) = \alpha(\tau_x f),$$

where the $\mathsf{K}(L) \to \mathsf{K}(L)$ mapping τ_x is defined by $(\tau_x f)(y) = f(yx)$. Indeed, one can use the calculations on [3, p. 66 and 68] to show that $D : \alpha \mapsto D_\alpha$ is an linear embedding $\mathrm{PDer}(\mathsf{K}(L)) \hookrightarrow \mathrm{Der}(\mathsf{K}(L))$ and $D_\alpha = (\alpha \otimes 1) \circ \mu^*$ where μ^* is the loop coproduct $\mathsf{K}[L] \to \mathsf{K}[L] \otimes \mathsf{K}[L]$.

Definition. *Let L be an algebraic Bruck loops. The* tangent algebra $\mathfrak{l}$ *of L is the linear K-space of the derivations $\{D_\alpha \,|\, \alpha \in T_1(L)\}$.*

Theorem 1.5 ([7, Theorem 3.7])**.** *The tangent algebra of an algebraic Bruck loop forms a Lie triple system with respect to the operation $(A, B, C) = [[A, B], C]$. If $\mathrm{char}(\mathsf{K}) = p > 0$, then the tangent L.t.s. is restricted with $[p]$-map $A \mapsto A^p$.*

Let now L be an algebraic commutative Moufang loop. By Corollary 1.3, we don't loose much if we suppose that L has exponent 3^k for some $k \in \mathbb{N}$. Moreover, we can assume that normal coordinates on $L = \mathsf{K}^n$ are given. Then, by the automorph inverse property (2) the polynomials μ^i have the form

$$\mu^i(\boldsymbol{X}, \boldsymbol{Y}) = X^i + Y^i + \mu_3^i(\boldsymbol{X}, \boldsymbol{Y}) + o(5), \tag{10}$$

where $\mu_3^i(\boldsymbol{X}, \boldsymbol{Y})$ is a homogenous polynomial of degree 3 in $X^1, \dots, Y^n$. Moreover, $\mu_3^i(\boldsymbol{X}, -\boldsymbol{X}) = 0$ and $\mu_3^i(-\boldsymbol{X}, -\boldsymbol{Y}) = -\mu_3^i(\boldsymbol{X}, \boldsymbol{Y})$ hold. We define the *associator series*

$$\alpha^i(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = \mu^i(\boldsymbol{\mu}(\boldsymbol{\mu}(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z}), -\boldsymbol{\mu}(\boldsymbol{X}, \boldsymbol{\mu}(\boldsymbol{Y}, \boldsymbol{Z}))).$$

On the one hand,

$$\alpha^i(\boldsymbol{0}, \boldsymbol{Y}, \boldsymbol{Z}) = \alpha^i(\boldsymbol{X}, \boldsymbol{0}, \boldsymbol{Z}) = \alpha^i(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{0}) = 0$$

implies $\alpha^i(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ to have the form

$$\alpha^i(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = \alpha_3^i(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) + o(5),$$

where α_3^i is a homogenous polynomial

$$\alpha_3^i(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = \sum_{k,\ell,m=1}^{n} \omega_{k\ell m}^i \, X^k Y^\ell Z^m$$

of degree 3. Putting $\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle = \alpha_3^i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ for the elements $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathsf{K}^n$, we get a trilinear map $(\mathsf{K}^n)^3 \to \mathsf{K}^n$. We call $\langle ., ., . \rangle$ *the infinitesimal formal associator* of the loop L. It is clear that this concept is the analogue of the local analytic construction of the infinitesimal associator.

Proposition 1.6 ([7, Proposition 3.8])**.** *The tangent Lie triple system of an algebraic commutative Moufang loop is isomorphic to the infinitesimal associator operation.*

2. Loops as sections in algebraic groups

The method of considering loops as sections in their translation groups is originally due to A.A. Albert, for a very recent and excessive reference see [8]. This method is extremely useful in the investigation of Bruck and in particular commutative Moufang loops.

It is not the aim of this section to explain the method in details, but we use it in order to *recover the tangent algebra of an algebraic commutative Moufang loop in the Lie algebra of its multiplication group.*

From now on, we consider the connected unipotent algebraic commutative Moufang loop L. We denote by H the group generated by the inner maps $\lambda_{x,y}$ $(x, y \in L)$ and by G the multiplication group of L. From the proof of Theorem 1.4 we know that G is isomorphic to $L \times H$ as a variety. Hence, we have the morphisms

$$\psi : G \to L \quad \text{and} \quad i : L \to G$$

such that $i(x) = \lambda_x$ and $\psi(\lambda_x h) = x$ for all $x \in L$ and $h \in H$. Moreover, for any $x \in L$ and $g \in G$, the K-homomorphisms $\tau_x : \mathsf{K}[L] \to \mathsf{K}[L]$ and $\bar{\tau}_g : \mathsf{K}[G] \to \mathsf{K}[G]$ are defined by

$$(\tau_x f)(x') = f(x'x) \quad \text{and} \quad (\bar{\tau}_g p)(g') = p(g'g)$$

for all $x' \in L$, $f \in \mathsf{K}[L]$ and $g' \in G$ and $p \in \mathsf{K}[G]$.

Lemma 2.1. *For all $x \in L$ and $h \in H$, the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{K}[L] \arrow[r, "\tau_x"] \arrow[d, "\psi^*"'] & \mathsf{K}[L] \\
\mathsf{K}[G] \arrow[r, "\bar{\tau}_{\lambda_x h}"'] & \mathsf{K}[G] \arrow[u, "i^*"']
\end{tikzcd}$$

Proof. With $h' = \lambda_{yx}\lambda_y\lambda_x h \in H$, we have

$$\begin{aligned}
(i^*\bar{\tau}_{\lambda_x h}\psi^* f)(y) &= (\bar{\tau}_{\lambda_x h}\psi^* f)(\lambda_y) \\
&= (\psi^* f)(\lambda_y\lambda_x h) = (\psi^* f)(\lambda_{yx} h') \\
&= f(yx) = (\tau_x f)(y)
\end{aligned}$$

for all $y \in L$ and $f \in \mathsf{K}[L]$. □

Let us denote by $\mathfrak{g}$ the Lie algebra of G. Recall that the elements of $\mathfrak{g}$ are the (right) invariant derivations of $\mathsf{K}[G]$; these are precisely the derivations of the form D_α with point derivation $\alpha \in T_eG$ (cf. [3, Chapter III]); by definition:

$$(D_\alpha f)(g) = \alpha(\bar{\tau}_g f), \qquad g \in G.$$

Since $i(e) = e$, the morphism i defines a K-linear map $di : T_eL \to T_eG$.

Lemma 2.2. *For any $\alpha \in T_eL$, the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{K}[L] \arrow[r, "D_\alpha"] \arrow[d, "\psi^*"'] & \mathsf{K}[L] \arrow[d, "\psi^*"] \\
\mathsf{K}[G] \arrow[r, "D_{di(\alpha)}"'] & \mathsf{K}[G]
\end{tikzcd}$$

Proof. On the one hand, $(\psi^* D_\alpha f)(\lambda_x h) = (D_\alpha f)(x) = \alpha(\tau_x f)$ holds for all $f \in \mathsf{K}[L]$, $x \in L$ and $h \in H$. On the other hand, we have

$$\begin{aligned} (D_{di(\alpha)}\psi^* f)(\lambda_x h) &= di(\alpha)(\bar\tau_{\lambda_x h}\psi^* f) \\ &= \alpha(i^*\bar\tau_{\lambda_x h}\psi^* f) \\ &= \alpha(\tau_x f) \end{aligned}$$

by Lemma 2.1. □

We define the automorphism σ of G by $\sigma(\lambda_x h) = \lambda_x^{-1} h$. It is clear that σ is an involutive morphism of G (cf. [1]), in the terminology of [8, Section 12], $G\langle\sigma\rangle$ is an algebraic reflection group with respect to the conjugacy class of σ.

Since $\sigma(e) = e$, the map $d\sigma : \alpha \mapsto \alpha \circ \sigma^*$ is an involutive automorphism of T_eG. Using the definitions given so far, we obtain

$$\begin{aligned} ((\sigma^* D_{d\sigma(\alpha)})f)(g) &= (D_{d\sigma(\alpha)}f)(\sigma(g)) = d\sigma(\alpha)(\bar\tau_{\sigma(g)}f) \\ &= \alpha(\sigma^*\bar\tau_{\sigma(g)}f) = \alpha(\bar\tau_g\sigma^* f) \\ &= ((D_\alpha\sigma^*)f)(g) \end{aligned}$$

for all $f \in \mathsf{K}[G]$ and $g \in G$, whence $D_{d\sigma(\alpha)} = \sigma^* D_\alpha \sigma^*$. With other words, the action of $d\sigma$ on T_eG is compatible the action $\delta \mapsto \sigma^*\delta\sigma^*$ of σ^* on $\mathfrak{g}$. We denote by $(T_eG)^-$ and $\mathfrak{g}^-$ the nontrivial eigenspace of $d\sigma$ and σ^*, respectively. Clearly, $\mathfrak{g}^-$ is closed under $[[.,.],.]$.

Proposition 2.3. *Let L be an algebraic commutative Moufang loop with tangent algebra $\mathfrak{l}$. Let G be the multiplication group of L with Lie algebra $\mathfrak{g}$. Then, the map $D_\alpha \mapsto D_{di(\alpha)}$ ($\alpha \in T_eL$) defines an isomorphism between the Lie triple systems $(\mathfrak{l}, [[.,.],.])$ and $(\mathfrak{g}^-, [[.,.],.])$.*

Proof. Clearly, the map $\Theta : D_\alpha \mapsto D_{di(\alpha)}$ ($\alpha \in T_eL$) is injective. Moreover, since $\mathrm{char}(\mathsf{K}) \neq 2$, σ is a semisimple element, and by [3, Proposition 18.1], $\mathfrak{g}^-$ is the tangent space of

$$\{g^{-1}\sigma(g) | g \in G\} = \{\lambda_x | x \in L\},$$

which is isomorphic to L. Thus, $\dim \mathfrak{l} = \dim \mathfrak{g}^-$. We still have to show $di(\alpha) \in (T_eG)^-$ in order to see that Θ is a linear isomorphism between $\mathfrak{l}$ and $\mathfrak{g}^-$. And indeed, the composition $L \xrightarrow{(i,\sigma i)} G \times G \xrightarrow{\mu} G$ maps x to e, its differential $d\mu \circ (di, d\sigma di)$ is therefore 0. This means $di(\alpha) + d\sigma(di(\alpha)) = 0$ for all $\alpha \in T_eL$ (cf. [3, Proposition 10.1]), and implies $\sigma^* D_{di(\alpha)}\sigma^* = -D_{di(\alpha)}$ by the calculation above.

Finally, Lemma 2.2 implies $\psi^* D_\alpha D_\beta D_\gamma = D_{di(\alpha)} D_{di(\beta)} D_{di(\gamma)} \psi^*$ for all $\alpha, \beta, \gamma \in T_eL$, whence $\psi^*[[D_\alpha, D_\beta], D_\gamma] = [[D_{di(\alpha)}, D_{di(\beta)}], D_{di(\gamma)}]\psi^*$.

Let us now choose a basis $\alpha_1, \dots, \alpha_n$ for T_eL and consider the structure constants $w^j_{k\ell m}$, $\bar w^j_{k\ell m}$ of $\mathfrak{l}$ and $\mathfrak{g}^-$, with respect to the basis $D_{\alpha_1}, \dots, D_{\alpha_n}$ and $D_{di(\alpha_1)}, \dots, D_{di(\alpha_n)}$, respectively. As we just have seen,

$$\psi^*(w^j_{k\ell m} D_{\alpha_j}) = (\bar w^j_{k\ell m} D_{di(\alpha_j)})\psi^* = \psi^*(\bar w^j_{k\ell m} D_{\alpha_j}).$$

Substituting $\beta = (w^j_{k\ell m} - \bar w^j_{k\ell m})\alpha_j \in T_eL$, we obtain $\psi^* D_\beta = 0$. This implies

$$(D_\beta f)(x) = (\psi^* D_\beta f)(\lambda_x) = 0$$

for all $f \in \mathsf{K}[L]$ and $x \in L$, thus $\beta = 0$ and $w^j_{k\ell m} = \bar{w}^j_{k\ell m}$ holds for all $k, \ell, m, j = 1, \dots, n$. $\square$

3. Tangent algebras of algebraic commutative Moufang loops

Theorem 1.5 shows that the tangent space of an algebraic Bol loop can be endowed with a structure of a Lie triple system. In this section, we show that for an algebraic commutative Moufang loop, the tangent algebra defined in this way plays exactly the same role that Lie algebras do in the case of algebraic groups.

In this section, we denote by $\alpha(x, y, z)$ the associator map $(x \cdot yz)^{-1} \cdot (xy \cdot z)$ of a commutative Moufang loop. For any non-negative integer i, Bruck [1] defines the functions f_i on the commutative Moufang loop L by

$$f_0(x, y, z) = \alpha(x, y, z),$$
$$f_{i+1}(x, y, z; a_1, \dots, a_i, u) = \alpha(f_i(x, y, u; a_1, \dots, a_i), u, z).$$

Let $(\mathfrak{b}, (.,.,.))$ be a Lie triple system over a field K. For each non-negative integer i, we define a function F_i on $\mathfrak{b}$ by

$$F_0(x, y, z) = (x, y, z),$$
$$F_{i+1}(x, y, z; a_1, \dots, a_i, u) = (F_i(x, y, u; a_1, \dots, a_i), u, z).$$

Theorem 3.1. *Let L be an algebraic commutative Moufang loop with tangent algebra $\mathfrak{l}$. Then, $\mathfrak{l}$ is a restricted Lie triple system w.r.t. the operations $(D_\alpha, D_\beta, D_\gamma) = [[D_\alpha, D_\beta], D_\gamma]$ and $D^{[3]}_\alpha = D^3_\alpha$ $(\alpha, \beta, \gamma \in T_eL)$. Moreover, for each non-negative integer i, the functions $F_i = F_i(x, y, z; a_1, \dots, a_i)$ are symmetric in $a_1, \dots, a_n$ and skew-symmetric in x, y, z.*

Proof. We assume L to be connected w.l.o.g. and proper in order to avoid trivialities. By Corollary 1.3, the base field has to have characteristic 3, thus, by Theorem 1.5, $(\mathfrak{l}, (.,.,.))$ is a restricted L.t.s. By Corollary 1.3, L has a Jordan decomposition $L = U \times S$ with $S \leq Z(L)$. It is very easy to see that this induces a decomposition $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{s}$, where $\mathfrak{u}, \mathfrak{s} \lhd \mathfrak{l}$ are the tangent subalgebras of U and S, respectively, and the L.t.s. $\mathfrak{s}$ is trivial. This means that we may suppose $L = U$ when considering non-trivial properties of the tangent L.t.s. $\mathfrak{l}$.

By Corollary 1.3, L can be supposed to be identical with K^n $(n = \dim L)$ such that $X^{-1} = -X$ holds. Then, $\mathsf{K}[L] = \mathsf{K}[\boldsymbol{T}]$ and we can identify T_eL with K^n by

$$(a_1, \dots, a_n) \longleftrightarrow a^i \left. \frac{\partial}{\partial T^i} \right|_{\boldsymbol{T}=\boldsymbol{0}}.$$

Let us define the L.t.s. operation $(.,.,.)$ on K^n by

$$D_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})} = [[D_{\boldsymbol{x}}, D_{\boldsymbol{y}}], D_{\boldsymbol{z}}],$$

clearly $(\mathsf{K}^n, (.,.,.)) \cong \mathfrak{l}$ holds. Furthermore, by Proposition 1.6, we have

$$\alpha(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + o(5).$$

This implies

$$f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_i) = F_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_i) + o(2i + 5)$$

immediately. The properties of the functions f_i given in [1, Lemma VIII.7.2.] imply the second part of the theorem. $\square$

Remark. One of the most important property of the tangent algebra of an algebraic commutative Moufang loop is that the L.t.s. operation *alternates*, i.e.,

$$(x,y,z) = -(y,x,z) \quad \text{and} \quad (x,y,z) = (y,z,x)$$

hold. Much to our regrets, this property is far not sufficient: If K is a field of characteristic 3, then we can construct an L.t.s. on K^3 with structure constants

$$w^3_{123} = -w^3_{132} = w^3_{231} = -w^3_{213} = w^3_{312} = -w^3_{321} = 1$$

and $w^i_{k\ell m} = 0$ for all other k, ℓ, m, i. We define the [3]-map to be the trivial map $x \mapsto 0$. The resulting ternary algebra is an "alternating" restricted L.t.s., which does not happen to be the tangent algebra of a commutative Moufang loop since $F_1(x,y,y;a) = 0$ is not satisfied.

Proposition 3.2. *Let L and M be algebraic commutative Moufang loops over the algebraically closed field K and let $u : L \to M$ be a morphism of algebraic loops over K. Then, the map $\Psi : D_\alpha \to D_{du(\alpha)}$ $(\alpha \in T_eL)$ defines a homomorphism between the Lie triple systems $\mathfrak{l}$ and $\mathfrak{m}$ of L and M, respectively.*

Proof. First we show that the diagram

$$\begin{array}{ccc} \mathsf{K}[M] & \xrightarrow{D_{du(\alpha)}} & \mathsf{K}[M] \\ {\scriptstyle u^*}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathsf{K}[L] & \xrightarrow[D_\alpha]{} & \mathsf{K}[L] \end{array} \tag{11}$$

commutes. Indeed, for any $f \in \mathsf{K}[M]$ and $x, y \in L$, we have $(\tau_x u^* f)(y) = f(u(yx))$ and $(u^*\tau_{u(x)} f)(y) = f(u(y)u(x))$, hence $\tau_x u^* f = u^*\tau_{u(x)} f$. Therefore,

$$(D_\alpha u^* f)(x) = \alpha(\tau_x u^* f) = \alpha(u^*\tau_{u(x)} f) = du(\alpha)(\tau_{u(x)} f) = (u^* D_{du(\alpha)} f)(x).$$

The diagram implies $D_\alpha D_\beta D_\gamma u^* = u^* D_{du(\alpha)} D_{du(\beta)} D_{du(\gamma)}$ as well, hence

$$[[D_\alpha, D_\beta], D_\gamma] u^* = u^*[[D_{du(\alpha)}, D_{du(\beta)}], D_{du(\gamma)}]$$

holds for all $\alpha, \beta, \gamma \in T_eL$. Let us put

$$D_\delta = [[D_\alpha, D_\beta], D_\gamma] \quad \text{and} \quad D_\epsilon = [[D_{du(\alpha)}, D_{du(\beta)}], D_{du(\gamma)}]$$

with $\delta \in T_eL$ and $\epsilon \in T_eM$. All we have to show is $\epsilon = du(\delta)$. However, we have $D_\delta u^* = u^* D_\epsilon$, thus $u^* D_\varphi = 0$ with $\varphi = du(\delta) - \epsilon$. This means that

$$0 = (u^* D_\varphi f)(e) = (D_\varphi f)(u(e)) = \varphi(f)$$

holds for all $f \in \mathsf{K}[L]$, whence $\varphi = 0$ and $du(\delta) = \epsilon$. $\square$

The following property of $\mathfrak{l}$ is analog to the ones of Lie algebras given in [3, Lemma 8.5. and Lemma 9.4.].

Let H be a closed subloop of the algebraic commutative Moufang loop L. The inclusion $u : H \to L$ is an isomorphism onto a closed subloop, u^* maps $\mathsf{K}[L]$ onto $\mathsf{K}[H] = \mathsf{K}[L]/I$ (I the ideal vanishing on H). Therefore,

du identifies T_eH with a subspace of T_eG consisting of those α for which $\alpha(I) = 0$. But u is also a morphism of algebraic loops, so $du : \mathfrak{h} \to \mathfrak{l}$ is a L.t.s. homomorphism, which allows us to view $\mathfrak{h}$ as a Lie triples subsystem of $\mathfrak{l}$ (see Proposition 3.2).

Proposition 3.3. *Let H be a closed subloop of the algebraic commutative Moufang loop L, I the ideal of $\mathsf{K}[L]$ vanishing on H. Then $H = \{x \in L | \tau_x(I) \subseteq I\}$ and $\mathfrak{h} = \{D_\alpha | \alpha \in T_eL,\, D_\alpha(I) \subseteq I\}$.*

Proof. See the proofs of [3, Lemma 8.5. and Lemma 9.4.]. □

We have the following elementary formulas (cf. [3, Proposition 10.1]).

Proposition 3.4. *Let L be an algebraic commutative Moufang loop with multiplication $\mu(x, y) = xy$ and inverse mapping $\iota : x \mapsto x^{-1}$. Then, for all $\alpha, \beta \in T_eL$:*

(i) $d\mu(\alpha, \beta) = \alpha + \beta$.
(ii) $d\iota(\alpha) = -\alpha$.
(iii) $(dg_{x,y})(\alpha) = (1 - d\lambda_{x,y})(\alpha)$, *where* $g_{x,y}(z) = (x, y, z)$, $(x, y, z \in L)$.

Proof. For (i) and (ii), see the proof of [3, Proposition 10.1]. Since by [1, Lemma VIII.5.4.] we have $g_{x,y}(z) = (x, y, z) = z\lambda_{x,y}(z^{-1})$, hence $g_{x,y}$ is the composition $\mu \circ (id, \lambda_{x,y}) \circ (id, \iota)$. If $\alpha \in T_eL$, then by (i) and (ii), $dg_{x,y}(\alpha) = d\mu(\alpha, d\lambda_{x,y}(-\alpha)) = (1 - d\lambda_{x,y})(\alpha)$. □

Let us take an algebraic automorphism h of L. By diagram (11), we have the action

$$D_{dh(\alpha)} = (h^{-1})^* D_\alpha h^* \tag{12}$$

of dh on $\mathfrak{l}$. In particular, for any $x, y \in L$, we have the algebraic automorphism $\lambda_{x,y}$ of L; we denote by $\mathrm{Ad}_{x,y}$ the action (12) on $\mathfrak{l}$. For any $x \in L$, we have thus a map $u_x : L \to \hom(\mathfrak{l}, \mathfrak{l})$, mapping y to $\mathrm{Ad}_{x,y}$. $\hom(\mathfrak{l}, \mathfrak{l})$ has a natural structure of a variety. If u_x is a morphism, then the map $u : L \to \hom(\mathfrak{l}, \hom(\mathfrak{l}, \mathfrak{l}))$, $x \mapsto du_x$ is well defined. We will show that for all $x \in L$, the maps u_x and v are morphisms, and $(dv)(\alpha)(\beta)(\gamma) = \frac{1}{2}(\alpha, \beta, \gamma)$ holds. This fact is in analogy with [3, Theorem 10.4].

Lemma 3.5. *Let G be a 2-divisible algebraic group, that is, we assume that the map $x \to x^2$ is a automorphism of the variety G. Let us denote by $\nu : x \mapsto x^{\frac{1}{2}}$ the inverse of $x \mapsto x^2$. For any $x, y \in G$, we define the map $t_{x,y} : G \to G$ by*

$$t_{x,y}(z) = (x^{-\frac{1}{2}}y^{-1}x^{\frac{1}{2}}y)z(x^{-\frac{1}{2}}y^{-1}x^{\frac{1}{2}}y)^{-1}.$$

Then, for all $x, y \in G$, $t_{x,y}$ is a morphism. For any $x \in G$, let us define the map $u_x : G \to \hom(\mathfrak{g}, \mathfrak{g})$ by $u_x(y) = dt_{x,y}$. Then, for all $x \in G$, u_x is a morphism. Finally, we define the map $v : G \mapsto \hom(\mathfrak{g}, \hom(\mathfrak{g}, \mathfrak{g}))$ by $v(x) = du_x$. Then, v is a morphism and

$$(dv)(\alpha)(\beta)(\gamma) = \frac{1}{2}[[\alpha, \beta], \gamma]$$

holds for all $\alpha, \beta, \gamma \in \mathfrak{g}$.

Proof. For any $x \in L$, we define the $G \to G$ morphisms $\mathrm{Int}(x)\,(y) = xyx^{-1}$ and $\psi(x)\,(y) = x^{-1}y^{-1}xy$. Then, $t_{x,y} = \mathrm{Int}(x^{-\frac{1}{2}}y^{-1}x^{\frac{1}{2}}y)$ and

$$dt_{x,y} = \mathrm{Ad}(x^{-\frac{1}{2}}y^{-1}x^{\frac{1}{2}}y) = (\mathrm{Ad}\circ\psi(x^{\frac{1}{2}}))(y)$$

in the notation of [3]. Since for any $x \in L$, $\psi(x^{\frac{1}{2}}) : G \to G$ is a morphism and $\mathrm{Ad} : G \to \hom(\mathfrak{g}, \mathfrak{g})$ is a morphism by [3, Proposition 10.3], v_x is a morphism, as well. Moreover, $d\,\mathrm{Ad} = \mathrm{ad}$ and $d\psi(x) = 1 - \mathrm{Ad}(x^{-1})$ by [3, Proposition 10.1 and Theorem 10.4], thus, $du_x = \mathrm{ad}\circ(1 - \mathrm{Ad}(x^{-\frac{1}{2}}))$. $\mathrm{ad} : \mathfrak{g} \to \hom(\mathfrak{g}, \mathfrak{g})$ is linear map and Ad and $x \mapsto x^{-\frac{1}{2}}$ are morphisms, thus $v : x \mapsto du_x$ is a morphism with differential

$$dv = \frac{1}{2}\,\mathrm{ad}\circ\mathrm{ad} : \mathfrak{g} \to \hom(\mathfrak{g}, \hom(\mathfrak{g}, \mathfrak{g})).$$

This finishes the proof of the lemma. $\square$

Let X_1, X_2, Y_1, Y_2 be (abstract) sets. We say that the maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are (φ, ψ)-*equivalent*, if the maps $\varphi : X_1 \to X_2$ and $\psi : Y_1 \to Y_2$ are bijections and $f_1 \circ \psi = \varphi \circ f_2$ holds. We express this fact with the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ X_2 & \xrightarrow[f_2]{} & Y_2 \end{array} .$$

If $Y_1 = Y_2$ and $\varphi = \psi$, we simply speak of φ-equivalence. Clearly, if X_1, X_2, Y_1, Y_2 are varieties with morphism f_1 and automorphisms φ, ψ of varieties, then f_2 is a morphism, as well.

Let V_1, V_2 be vector spaces, then $\hom(V_1, V_2) \cong V_1^* \otimes V_2$ in a natural way. Now, if $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ are linear isomorphisms, then they induce a $\hom(V_1, V_2) \to \hom(W_1, W_2)$ isomorphism, which we denote by $(f_1^{-1})^* \otimes f_2$.

Lemma 3.6. *The map $L \times L \to \hom(\mathfrak{l}, \mathfrak{l})$, $(x, y) \mapsto d\lambda_{x,y}$ is a morphism of algebraic varieties.*

Proof. An element $h \in H$ is an algebraic automorphism of L, normalizing the set $S = S(L)$. Moreover, h acts on L in the same way as it does on S by conjugation. This is easy to show abstractly, and it follows from the construction in the proof of Theorem 1.4 that the two actions are i-equivalent. Thus, by Proposition 2.3, the linear action of dh on T_eL is di-equivalent to the linear action of Ad_h on $\mathfrak{g}^- \le \mathfrak{g}$. By [3, Proposition 10.3], the map $H \to \hom(\mathfrak{g}^-, \mathfrak{g}^-)$, $h \mapsto \mathrm{Ad}_h$ is a morphism. On the other hand, the map $(x, y) \mapsto \mathrm{Ad}_{x,y}$ is precisely the composition of the morphisms

$$\begin{cases} L \times L \to H \\ (x, y) \mapsto \lambda_{x,y} \end{cases}, \qquad \begin{cases} H \to GL(\mathfrak{g}^-) \\ h \mapsto \mathrm{Ad}_h\,|_{\mathfrak{g}^-} \end{cases}$$

and $(di)^{-1} \otimes (di)^{-1} : \hom(\mathfrak{g}^-, \mathfrak{g}^-) \to \hom(\mathfrak{l}, \mathfrak{l})$. This gives us the diagrams

$$\begin{tikzcd} L \arrow[r, "\lambda_{x,y}"] \arrow[d, "i"'] & L \arrow[d, "i"] \\ S(L) \arrow[r, "t_{i(x),i(y)}"'] & S(L) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{l} \arrow[r, "d\lambda_{x,y}"] \arrow[d, "di"'] & \mathfrak{l} \arrow[d, "di"] \\ \mathfrak{g}^- \arrow[r, "dt_{i(x),i(y)}"'] & \mathfrak{g}^- \end{tikzcd} . \qquad \square$$

We make two remarks for the next theorem. Due to the Jordan decomposition $L = U \times S$, we can assume L to be a connected, unipotent and proper algebraic commutative Moufang loop. Let us denote by G and H the translation group and the group generated by the inner mappings of L. Let $S(L)$ be the set of left translations λ_x of L. Then $G = S(L) \times H \cong L \times H$ as variety. We denote by i the automorphism $x \mapsto \lambda_x$ between the varieties L and $S(L)$.

By [5, Theorem 1.11], L has finite exponent of the form 3^e, hence by [1, Lemma VIII.11.5.], G has finite exponent of the form 3^f, thus, the group G is 2-divisible.

Theorem 3.7. *Let L be an algebraic commutative Moufang loop with tangent L.t.s. $(\mathfrak{l}, (.,.,.))$. The map $\mathrm{Ad} : L \times L \to GL(\mathfrak{l})$, defined by $(x, y) \mapsto \mathrm{Ad}_{x,y}$ is a morphism of algebraic varieties. The map $U_x : L \to \hom(\mathfrak{l}, \mathfrak{l})$ defined by $U_x(y) = \mathrm{Ad}_{x,y}$ is a morphism. Finally, the map $V : L \to \hom(\mathfrak{l}, \hom(\mathfrak{l}, \mathfrak{l}))$ defined by $V(x) = dU_x$ is a morphism with differential*

$$(dV)(\alpha)(\beta)(\gamma) = \frac{1}{2}(\alpha, \beta, \gamma).$$

Proof. Applying the definition of morphically equivalent actions, we get first the diagrams

$$\begin{tikzcd} L \arrow[r, "U_x"] \arrow[d, "i"'] & \hom(\mathfrak{l}, \mathfrak{l}) \arrow[d, "(di^{-1})^* \otimes di"] \\ S(L) \arrow[r, "u_{i(x)}"'] & \hom(\mathfrak{g}^-, \mathfrak{g}^-) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{l} \arrow[r, "dU_x"] \arrow[d, "di"'] & \hom(\mathfrak{l}, \mathfrak{l}) \arrow[d, "(di^{-1})^* \otimes di"] \\ \mathfrak{g}^- \arrow[r, "du_{i(x)}"'] & \hom(\mathfrak{g}^-, \mathfrak{g}^-) \end{tikzcd} ,$$

explaining us the equivalences of the mappings dU_x and $du_{i(x)}$ for all $x \in L$. Using the same thing once more, we get

$$\left.\begin{array}{c} \begin{tikzcd} L \arrow[r, "V"] \arrow[d, "i"'] & \hom(\mathfrak{l}, \hom(\mathfrak{l}, \mathfrak{l})) \arrow[d, "(di^{-1})^* \otimes (di^{-1})^* \otimes di"] \\ S(L) \arrow[r, "v"'] & \hom(\mathfrak{g}^-, \hom(\mathfrak{g}^-, \mathfrak{g}^-)) \end{tikzcd} \\ \begin{tikzcd} \mathfrak{l} \arrow[r, "dV"] \arrow[d, "di"'] & \hom(\mathfrak{l}, \hom(\mathfrak{l}, \mathfrak{l})) \arrow[d, "(di^{-1})^* \otimes (di^{-1})^* \otimes di"] \\ \mathfrak{g}^- \arrow[r, "dv"'] & \hom(\mathfrak{g}^-, \hom(\mathfrak{g}^-, \mathfrak{g}^-)) \end{tikzcd} \end{array}\right\} \tag{13}$$

representing the equivalences of the mappings V and v. Using Lemma 3.5, we obtain that all maps mentioned in the proposition are morphism, and by

Proposition 2.3, the diagrams (13) give precisely

$$(dV)(\alpha)(\beta)(\gamma) = \frac{1}{2}(\alpha,\beta,\gamma). \qquad \square$$

An *ideal* (see [4]) of a L.t.s. $\mathfrak{b}$ is a subspace $\mathfrak{h}$ such that for

$$(x,y,\mathfrak{h}), (x,\mathfrak{h},y), (\mathfrak{h},x,y) \leq \mathfrak{h}$$

holds all $x,y \in \mathfrak{b}$. The *center* $\mathfrak{z}(\mathfrak{b})$ of $\mathfrak{b}$ is the subspace consisting of the elements $z \in \mathfrak{b}$ such that

$$(z,\mathfrak{b},\mathfrak{b}) = (\mathfrak{b},z,\mathfrak{b}) = (\mathfrak{b},\mathfrak{b},z) = 0.$$

Proposition 3.8. *Let K be a closed normal subloop of the algebraic commutative Moufang loop L. Then, the tangent L.t.s. $\mathfrak{k}$ of K is an ideal in $\mathfrak{l}$.*

Proof. To say that K is normal is to say that all $\lambda_{x,y}$ $(x,y \in L)$ stabilize K; hence $\mathrm{Ad}_{x,y}$ $(x,y \in L)$ stabilize $\mathfrak{k}$. Thus, in an appropriate basis, $\mathrm{Ad}_{x,y}$ has the form

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right). \tag{14}$$

Let us denote by $\mathfrak{M}$ the subspace of $\hom(\mathfrak{l},\mathfrak{l})$, consisting of matrices of the form (14). Then, U_x maps L to $\mathfrak{M}$ and $V(x) = dU_x \in \hom(\mathfrak{l},\mathfrak{M})$. Since $\hom(\mathfrak{l},\mathfrak{M})$ is a linear subspace of $\hom(\mathfrak{l},\hom(\mathfrak{l},\mathfrak{l}))$, we have $(dV)(\alpha) \in \hom(\mathfrak{l},\mathfrak{M})$ for all $\alpha \in \mathfrak{l}$. This means that $(dV)(\alpha)(\beta)$ has the form (14) for all $\alpha,\beta \in \mathfrak{l}$, hence

$$(dV)(\alpha)(\beta)(\gamma) = \frac{1}{2}(\alpha,\beta,\gamma) \in \mathfrak{k}$$

for all $\gamma \in \mathfrak{k}$. By the alternation property of $\mathfrak{l}$, this suffices to have $\mathfrak{k} \triangleleft \mathfrak{l}$. $\square$

Proposition 3.9. *Let A, B and C closed subloops of the algebraic commutative Moufang loop L with tangent algebras $\mathfrak{a},\mathfrak{b},\mathfrak{c}$, respectively. Let D be the subloop (A,B,C), generated by the associators. Then, the tangent algebra $\mathfrak{d}$ of D contains all elements*

$$\gamma - \mathrm{Ad}_{x,y}(\gamma), \beta - \mathrm{Ad}_{x,z}(\beta), \alpha - \mathrm{Ad}_{y,z}(\alpha),$$
$$V(x)(\beta)(\gamma), V(y)(\alpha)(\gamma), V(z)(\alpha)(\beta) \ \text{ and } \ (\alpha,\beta,\gamma)$$

($x \in A, y \in B, z \in C$, $\alpha \in \mathfrak{a}, \beta \in \mathfrak{b}, \gamma \in \mathfrak{c}$).

Proof. All concepts of the statement are well defined, since by Proposition [5, Proposition 1.4], D is a closed subloop of L. We use the notation of Proposition 3.4. For $x \in A$, $y \in B$, $g_{x,y}$ maps $C \to D$, so the differential $1 - \mathrm{Ad}_{x,y}$ maps $\mathfrak{c} \to \mathfrak{d}$. This yields the elements of the first type listed, similarly for the second and third type.

For fixed $x \in A$ and $\gamma \in \mathfrak{c}$, consider the morphism $B \to \mathfrak{d}$, $y \mapsto \gamma - \mathrm{Ad}_{x,y}(\gamma) = \gamma - U_x(y)(\gamma)$. Since it maps e to 0, we can compute its derivative $\mathfrak{b} \to \mathfrak{d}$, mapping β to $V(x)(\beta)(\gamma) \in \mathfrak{d}$ (since $V(x) = dU_x$). This works for elements of the 4th, 5th and 6th types.

Finally, for fixed $\beta \in \mathfrak{b}$ and $\gamma \in \mathfrak{c}$, we have the $A \to \mathfrak{d}$ morphism $x \mapsto V(x)(\beta)(\gamma)$, its derivative maps an element $\alpha \in \mathfrak{a}$ to the element

$$(dV)(\alpha)(\beta)(\gamma) = \frac{1}{2}(\alpha,\beta,\gamma)$$

of $\mathfrak{d}$, showing $(\alpha, \beta, \gamma) \in \mathfrak{d}$. $\square$

Corollary 3.10. *Let L be an algebraic commutative Moufang loop with associator subloop $L' = (L, L, L)$. Then, the tangent L.t.s. $\mathfrak{l}'$ of L' includes $(\mathfrak{l}, \mathfrak{l}, \mathfrak{l})$ (= the set of all linear combinations of elements (α, β, γ), $\alpha, \beta, \gamma \in \mathfrak{l}$).* $\square$

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