Realizations of Differential Operators on Conic Manifolds with Boundary

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Abstract. We study the closed extensions (realizations) of differential operators subject to homogeneous boundary conditions on weighted L_p-Sobolev spaces over a manifold with boundary and conical singularities. Under natural ellipticity conditions we determine the domains of the minimal and the maximal extension. We show that both are Fredholm operators and give a formula for the relative index.

Contents

1. Introduction

Operator semigroups are one of the most efficient tools for the analysis of parabolic differential equations. In fact, these problems can often be reformulated as abstract evolution equations of the form $\dot{u}+Au = f$, $u(0) = u_0$, where A is a closed unbounded operator in a Banach space E, induced by the original partial differential equation. Also semihomogeneous boundary value problems can be treated in this way, namely by incorporating the boundary condition into the choice of the domain: Given a boundary condition T, one studies the operator A on a domain contained in the kernel of T. In general, one can think of the operator A to be defined initially on a small space of functions in E. In order to apply the machinery of evolution equations, as a first step one has to determine those closed extensions of A in E which reflect the original problem.

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In the case of a closed manifold, there are at least two distinguished choices: The *minimal* extension is the closure of the operator A acting on all smooth functions, and the *maximal* extension has as domain all those elements of E which are mapped into E by A.

For a manifold with boundary, the latter extension is no longer relevant, since it does not involve any boundary condition. Instead one considers the closed extensions – in this case also called *realizations* – with respect to a given boundary condition T, i.e. one only considers domains $\mathcal{D}$ for which $Tu = 0$ whenever $u \in \mathcal{D}$. Of course, boundary conditions can only be imposed if one has a certain a priori regularity. As the minimal (respectively maximal) extension we therefore choose the closure of the operator acting on all smooth (respectively all sufficiently regular) functions which vanish under the boundary condition.

A classical example is the heat equation in a bounded open set Ω with smooth boundary and Dirichlet boundary condition: In order to solve the problem in $L_p(\Omega)$, for example, one studies $A = -\Delta$, initially defined on all smooth functions on Ω vanishing at the boundary. Then the closure can be shown to coincide with the maximal extension, given by the action of $-\Delta$ on the domain $\{u \in H^2(\Omega) : u|_{\partial\Omega} = 0\}$. Hence there is only one closed realization for the Dirichlet boundary condition, namely the one just described.

For Fuchs type differential operators on manifolds with conical singularities, the situation is less simple. The boundaryless case (for $p = 2$) has been analyzed by Lesch [**12**]. He showed that, under a natural ellipticity assumption (corresponding to $\mathbb{D}$-ellipticity in this paper) both the minimal and the maximal extension are Fredholm operators and the quotient $\mathcal{D}_{\max}/\mathcal{D}_{\min}$ of their domains is finite-dimensional; its dimension can be computed from symbol data (more precisely from the meromorphic structure of the conormal symbol) of A. Gil and Mendoza [**4**], Proposition 3.6, obtained an improved description of $\mathcal{D}_{\min}$, while the structure of the maximal domain was studied in Schrohe and Seiler [**20**], Theorem 2.8.

In the present paper, we extend this work to elliptic boundary value problems on manifolds with boundary and conical singularities. While our final results – given in detail in a)–d) at the end of this section – look similar to those in [**12**], [**4**] and [**20**], the analysis becomes much more difficult. This starts with simple facts: The domains in general are not invariant under multiplication by cut-off functions, since the boundary condition then need not remain fulfilled. This limits localization techniques. Fortunately, there are suitable projections to handle these problems. Also the adjoint of an elliptic boundary value problem is harder to analyze than that of a differential operator, a fact which complicates the theory already in the case of smooth manifolds. We adapt here a construction of Grubb [**5**], Section 1.6, to the conic situation. Moreover, a basic reduction in the analysis of the boundaryless situation consists in switching to an operator whose coefficients are constant near the cone singularity (frozen at the tip of the cone). In the case of boundary value problems, one has to freeze both the operator and the boundary condition, and the corresponding changes of the domain are more difficult to handle. Finally, there is the fact that instead of the pseudodifferential techniques used before for the construction of parametrices we now have to employ Boutet de Monvel's calculus. Indeed, our main tool here will be an L_p-version of the cone calculus for boundary value problems developed by Schrohe and Schulze [**17**], [**18**], which we review in Section 2.

This paper lays the foundations for the study of the resolvents of elliptic boundary value problems on manifolds with conical singularities and questions of maximal regularity or solvability of certain nonlinear equations in the spirit of [**20**], [**2**].

Let us now explain the contents of the present paper in more detail. The intuitive picture of the underlying manifold is given in Figure 1. Technically, we denote by $\operatorname{int}\mathbb{D}$ an n-dimensional riemannian manifold with boundary having finitely many cylindrical ends, i.e. there exists a compact set C such that $\operatorname{int}\mathbb{D} \setminus C$ is the disjoint union $U_1 \cup \ldots \cup U_N$, where each U_i is isometric to the product $]0,1[\, \times X_i$ for a smooth riemannian manifold X_i with boundary; the X_i need not be connected. We fix the coordinate in $]0,1[$ in such a way that every neighborhood of C in $\operatorname{int}\mathbb{D}$ has nonempty intersection with $]\frac{1}{2},1[\, \times X_i$. We complete $\operatorname{int}\mathbb{D}$ with the help of the riemannian metric,

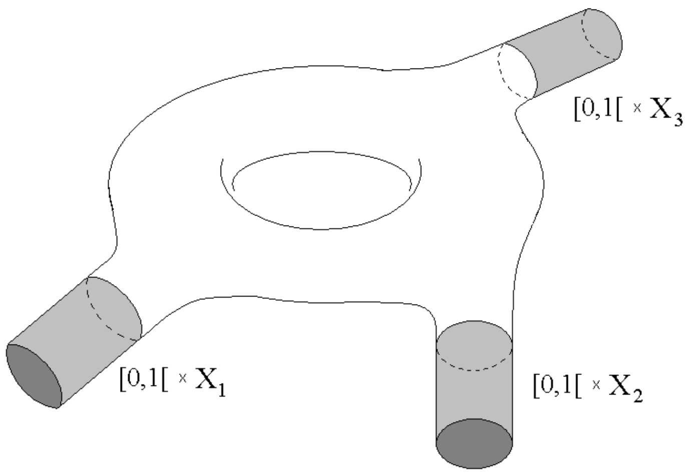

FIGURE 1

so that $\mathbb{D}\setminus C$ can be identified with the disjoint union of all $[0,1[\times X_i$. We next denote by $\operatorname{int}\mathbb{B}$ the boundary of $\operatorname{int}\mathbb{D}$ and by $\mathbb{B}$ its completion in the metric inherited from $\operatorname{int}\mathbb{D}$. Then $\mathbb{B}\setminus C$ can be identified with the disjoint union of all $[0,1[\times\partial X_i$; $\mathbb{B}$ itself is a smooth manifold with boundary.

To make the exposition simpler we shall assume in the following that $\mathbb{D}$ has only one cylindrical end, denoted by $[0,1[\times X$. Here we use the canonical coordinates (t,x), $0\le t<1$, $x\in X$. The subset $\{0\}\times X$ of $\mathbb{D}$, where $t=0$, is occasionally called the *singularity* of $\mathbb{D}$. In the sequel, a vector bundle over $\mathbb{D}$ will be a smooth hermitian vector bundle over $\operatorname{int}\mathbb{D}$ such that $E|_{]0,1[\times X}$ is isometric to the pull-back (under the canonical projection $]0,1[\times X\to X$) of a hermitian bundle E_0 over X.

That we call $\mathbb{D}$ a manifold with *conical* singularity is due to the class of differential operators we consider on it, the so-called *Fuchs type* operators. A μ-th order differential operator A on $\operatorname{int}\mathbb{D}$ with smooth coefficients, acting between sections of vector bundles E and $\widetilde{E}$ over $\mathbb{D}$, is of Fuchs type, if, near the singularity,

$$A=t^{-\mu}\sum_{j=0}^{\mu}a_j(t)(-t\partial_t)^j,\qquad a_j\in\mathcal{C}^\infty([0,1[,\operatorname{Diff}^{\mu-j}(X;E_0,\widetilde{E}_0)),\tag{1.1}$$

where E_0, $\widetilde{E}_0$ are the restrictions of E, $\widetilde{E}$ to $X\cong\{0\}\times X$. The three characteristic properties of a Fuchs type operator are the singular factor $t^{-\mu}$ corresponding to the order of the operator, coefficients which are smooth up to $t=0$, and the totally characteristic differentiation $t\partial_t$ in t-direction. As we are treating elliptic operators, we assume without loss of generality that $E=\widetilde{E}$.

One particular example of such an operator (of order $\mu=2$ and with $E=\operatorname{int}\mathbb{D}\times\mathbb{C}$) is the Laplace-Beltrami operator on $\operatorname{int}\mathbb{D}$ for a so-called *conical* metric, i.e. a metric which is of the form $dt^2+t^2g(t)$ on $]0,1[\times X$ with a smooth (up to $t=0$) family $g(t)$ of metrics on X. In this case

$$\Delta=t^{-2}\Big\{(-t\partial_t)^2-(n-1+a(t))(-t\partial_t)+\Delta_X(t)\Big\}$$

with $a(t)=\frac{1}{2}t\partial_t\log G(t)$ for $G=\det g_{ij}$, and $\Delta_X(t)$ is the Laplacian on X for the metric $g(t)$.

Fuchs type operators naturally act in a scale of weighted cone Sobolev spaces $\mathcal{H}^{s,\gamma}_p(\mathbb{D},E)$ introduced in Definition 2.1, below. Given $s,\gamma\in\mathbb{R}$ and $1<p<\infty$, the operator A defines a continuous map $\mathcal{H}^{s,\gamma}_p(\mathbb{D},E)\to\mathcal{H}^{s-\mu,\gamma-\mu}_p(\mathbb{D},E)$. For $s=0$ the space $\mathcal{H}^{0,\gamma}_p(\mathbb{D},E)$ is isomorphic to the weighted space $t^{\gamma_p+\frac{n}{p}-\gamma}L_p(\mathbb{D},E)$ for $\gamma_p=(n+1)(\frac{1}{2}-\frac{1}{p})$.

The main objective of this paper is the description of the closed realizations of the operator

$$A : \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T \subset \mathcal{H}_p^{0,\gamma}(\mathbb{D},E) \longrightarrow \mathcal{H}_p^{0,\gamma}(\mathbb{D},E), \tag{1.2}$$

where the initial domain consists of all smooth functions that vanish to infinite order in the singularity and that vanish under the *boundary condition* T. We assume that T is of the form $T = (T_0,\ldots,T_{\mu-1})$ with $T_k = \gamma_0 \circ B_k$, where each B_k is a Fuchs type differential operator of order k on $\mathbb{D}$, acting from sections of E to sections of bundles F_k; as usual, γ_0 denotes the operator of restriction to the boundary, i.e. $\gamma_0 : \mathcal{C}^{\infty,\infty}(\mathbb{D},F_k) \to \mathcal{C}^{\infty,\infty}(\mathbb{B},F_k|_{\mathbb{B}})$. We also allow F_k to be zero dimensional; in this case the k-th condition T_k is void. For more details see Section 3.1.

Our main assumption is that A is an elliptic differential operator on $\operatorname{int}\mathbb{D}$, that the boundary condition is *normal* (in the sense of Grubb [**5**], Definition 1.4.3), and that A together with T is an elliptic boundary value problem, i.e. that its boundary symbol satisfies the Shapiro-Lopatinskij condition. From (1.1) it is obvious that the principal symbol of A degenerates in a controlled way as t tends to 0; the same is true for the boundary symbol. After suitable rescaling, we can pass to limit symbols at $t = 0$ which we require to be invertible. We call this $\mathbb{D}$-*ellipticity.*

We shall denote by $A_{T,\min}$ the closure of (1.2) in the Banach space $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$, and by $A_{T,\max}$ the unbounded operator acting as A on the domain $\{u \in \mathcal{H}_p^{\mu,\gamma}(\mathbb{D},E) \mid Tu = 0 \text{ and } Au \in \mathcal{H}_p^{0,\gamma}(\mathbb{D},E)\}$. Assuming $\mathbb{D}$-ellipticity we prove the following:

a) The domain of $A_{T,\min}$ is $\{u \in \bigcap_{\varepsilon>0} \mathcal{H}_p^{\mu,\gamma+\mu-\varepsilon}(\mathbb{D},E) \mid Tu = 0 \text{ and } Au \in \mathcal{H}_p^{0,\gamma}(\mathbb{D},E)\}$; for γ outside a discrete set this coincides with $\{u \in \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E) \mid Tu = 0\}$,
b) $\mathcal{D}(A_{T,\max}) = \mathcal{D}(A_{T,\min}) + \mathcal{E}$ for a finite dimensional space $\mathcal{E}$ of smooth functions that is determined by the (conormal) symbolic structure of A and T,
c) both $A_{T,\min}$ and $A_{T,\max}$ are Fredholm operators; the difference of the indices can be computed,
d) $A^*_{T,\min}$, the adjoint of the closure, can be described explicitly as $(A^t)_{\widetilde{T},\max}$ for the formal adjoint A^t of A and a resulting boundary condition $\widetilde{T}$. Correspondingly, $A^*_{T,\max} = (A^t)_{\widetilde{T},\min}$.

As mentioned above, Lesch [**12**] treated the case of conical manifolds without boundary. Gil and Mendoza [**4**] determined the domain of adjoints of closed extensions of Fuchs type differential operators; in joint work [**3**] with Loya they proved an index theorem. The articles [**20**] and [**2**] focus on the descripton of the resolvent of closed extensions and show existence of bounded imaginary powers. The analysis of boundary value problems on manifolds with conical singularities began with Kondratievs fundamental paper [**11**] and has evolved in many directions. A topic related to this investigation is the work on the asymptotics of the solutions to elliptic equations on singular domains by Mazya, Nazarov, and Plamenevskij [**13**], [**14**].

Notation: Throughout this paper, a *cut-off function* is a non-negative, monotonically decreasing function in $\mathcal{C}^\infty_{\text{comp}}([0,1[)$ that is identically 1 in a neighborhood of 0. In the sequel, $\omega,\omega_0,\omega_1,\ldots$, and $\widetilde{\omega},\widetilde{\omega}_0,\widetilde{\omega}_1,\ldots$ will denote such cut-off functions. We shall write $\omega_0 \prec \omega$ if $\omega \equiv 1$ in a neighborhood of the support of ω_0. The operator of multiplication (in various spaces) with ω shall be denoted by ω, again.

By t we denote a real variable as well as a smooth positive function on $\operatorname{int}\mathbb{D}$ that coincides on the cylindrical part $]0,1[\times X$ with $(t,x) \mapsto t$.

2. Boutet de Monvel's algebra on manifolds with conical singularities

We review Boutet de Monvel's algebra for manifolds with conical singularities. Main references are [**17**] and [**18**], where this algebra is studied in detail. For other expositions we refer to [**10**] and [**19**]. We assume some familiarity with Boutet's calculus on smooth manifolds. However, for completeness, we give an introduction to this calculus in the appendix, cf. Section 8.

In a slight extension of the L_2-based previous work, we shall consider the operators on L_p-spaces with $1 < p < \infty$. This requires two continuity results which we deduce from [**7**].

2.1. Weighted distribution spaces on $\mathbb{D}$ and $\mathbb{B}$. Let X be embedded in Ω, a smooth compact manifold without boundary. Using the product structure of $\mathbb{R} \times \Omega$, it is straightforward to define the Sobolev spaces

$$H_p^s(\mathbb{R} \times \Omega), \qquad s \in \mathbb{R},\ 1 < p < \infty.$$

Extending the map $S_\gamma : \mathcal{C}^\infty_{\text{comp}}(\mathbb{R} \times \Omega) \to \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}_+ \times \Omega)$,

$$(S_\gamma u)(t,x) := t^{-\frac{n+1}{2}+\gamma} u(\log t, x), \qquad \gamma \in \mathbb{R}, \tag{2.1}$$

from functions to distributions, we obtain the weighted Sobolev spaces

$$\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+ \times \Omega), \qquad s, \gamma \in \mathbb{R},\ 1 < p < \infty$$

on the half-cylinder as the image of $H_p^s(\mathbb{R} \times \Omega)$ under S_γ with the canonically induced norm. Restricting to $\mathbb{R}_+ \times X$ we obtain $\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+ \times X)$ with the quotient norm

$$\|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+ \times X)} = \inf \Big\{ \|v\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+ \times \Omega)} \mid v|_{\mathbb{R}_+ \times \operatorname{int} X} = u \Big\}.$$

Gluing together two copies of $\mathbb{D}$ along the singularity $\{0\} \times X$ yields a smooth manifold with boundary $2\mathbb{D}$ and the standard scale of Sobolev spaces $H_p^s(2\mathbb{D})$.

Definition 2.1. *For $s, \gamma \in \mathbb{R}$ and $1 < p < \infty$ let*

$$\mathcal{H}_p^{s,\gamma}(\mathbb{D}) = \Big\{ u \in \mathcal{D}'(\mathring{\mathbb{D}}) \mid \omega u \in \mathcal{H}_p^{s,\gamma}(\mathbb{R}_+ \times X) \text{ and } (1-\omega)u \in H_p^s(2\mathbb{D}) \Big\},$$

where ω is an arbitrary fixed cut-off function and $\mathring{\mathbb{D}}$ denotes the interior of $\operatorname{int}\mathbb{D}$. *The norm is given by*

$$\|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{D})} = \|\omega u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+ \times X)} + \|(1-\omega)u\|_{H_p^s(2\mathbb{D})}.$$

We denote the closure of $\mathcal{C}^\infty_{\text{comp}}(\mathring{\mathbb{D}})$ in $\mathcal{H}_p^{s,\gamma}(\mathbb{D})$ by $\mathring{\mathcal{H}}_p^{s,\gamma}(\mathbb{D})$.

The spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{D})$ and $\mathring{\mathcal{H}}_p^{s,\gamma}(\mathbb{D})$ are Banach spaces and, in case $p = 2$, even Hilbert spaces. While the index s indicates the smoothness of a distribution, the index γ indicates its *flatness* towards the singularity, since

$$u \in \mathcal{H}_p^{s,\gamma}(\mathbb{D}) \iff t^{-\gamma} u \in \mathcal{H}_p^{s,0}(\mathbb{D}). \tag{2.2}$$

The standard embedding and duality properties of Sobolev spaces have corresponding analogues for cone Sobolev spaces:

Remark 2.2. *We have continuous embeddings*

$$\mathcal{H}_p^{s',\gamma'}(\mathbb{D}) \hookrightarrow \mathcal{H}_p^{s,\gamma}(\mathbb{D}), \qquad \mathcal{H}_p^{r,\gamma}(\mathbb{D}) \hookrightarrow \mathcal{H}_q^{t,\gamma}(\mathbb{D}),$$

provided $s' \geq s$, $\gamma' \geq \gamma$, and $q \geq p$, $r - \frac{n+1}{p} \geq t - \frac{n+1}{q}$. The first embedding is compact in case of $s' > s$ and $\gamma' > \gamma$. Via the scalar product of $\mathcal{H}_2^{0,0}(\mathbb{D})$ the spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{D})$ and $\mathring{\mathcal{H}}_{p'}^{-s,-\gamma}(\mathbb{D})$ are dual to each other if $\frac{1}{p} + \frac{1}{p'} = 1$.

Similarly, using the Besov spaces $B_p^s(\mathbb{R} \times \partial X) := B_{p,p}^s(\mathbb{R} \times \partial X)$ on the cylinder and the transformation

$$(S'_\gamma u)(t,x') := t^{-\frac{n}{2}+\gamma} u(\log t, x'), \qquad \gamma \in \mathbb{R}, \tag{2.3}$$

we introduce the Banach spaces (Hilbert spaces in case $p = 2$)

$$\mathcal{B}_p^{s,\gamma}(\mathbb{B}) = \left\{ u \in \mathcal{D}'(\operatorname{int}\mathbb{B}) \mid \omega u \in \mathcal{B}_p^{s,\gamma}(\mathbb{R}_+ \times \partial X) \text{ and } (1-\omega)u \in B_p^s(2\mathbb{B}) \right\}. \tag{2.4}$$

In a neighborhood in $\operatorname{int}\mathbb{D}$ of the boundary $\operatorname{int}\mathbb{B}$ we fix a normal coordinate (using the product structure on the cylindrical part) and a normal derivative ∂_ν. By γ_j we denote the usual boundary operator $\gamma_0 \circ \partial_\nu^j$, where γ_0 denotes restriction to the boundary. From standard theorems on restriction of Sobolev spaces, the following lemma follows immediately.

Lemma 2.3. *For any $1 < p < \infty$ and $s > \frac{1}{p} + j$ the boundary operator γ_j induces continuous maps*

$$\gamma_j : \mathcal{H}_p^{s,\gamma}(\mathbb{D}) \longrightarrow \mathcal{B}_p^{s-j-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B}).$$

Note that the weight γ is shifted to the weight $\gamma - \frac{1}{2}$, since the definition of the weighted spaces involves the dimension of the underlying manifold.

For later purpose we also define subspaces with asymptotics.

Definition 2.4. *For $\gamma \in \mathbb{R}$ and $\theta > 0$ let $\operatorname{As}(X; \gamma, \theta)$ denote the set of all finite sets*

$$P = \Big\{(p, m, M) \mid -\gamma - \theta < \operatorname{Re} p - \frac{n+1}{2} < -\gamma,\ m \in \mathbb{N}_0,\ M \subset \mathcal{C}^\infty(X)\Big\},$$

where M is a finite dimensional vector spaces. Any such P is called an asymptotic type. *We assume that to any $p \in \mathbb{C}$ there exists at most one element (p, m, M) in P. Similarly, $\operatorname{As}(\partial X; \gamma, \theta)$ consists of all finite sets*

$$Q = \Big\{(q, l, L) \mid -\gamma - \theta < \operatorname{Re} q - \frac{n}{2} < -\gamma,\ l \in \mathbb{N}_0,\ L \subset \mathcal{C}^\infty(\partial X)\Big\}$$

with finite dimensional spaces L. We set

$$\operatorname{As}(X, \partial X; \gamma, \theta) = \Big\{(P, Q) \mid P \in \operatorname{As}(X; \gamma, \theta) \text{ and } Q \in \operatorname{As}(\partial X; \gamma - \frac{1}{2}, \theta)\Big\}.$$

With $P \in \operatorname{As}(X; \gamma, \theta)$ we associate a finite dimensional space $\mathcal{E}_P(\mathbb{R}_+ \times X)$ of smooth functions: Its elements are all functions u with

$$u(t, x) = \omega(t) \sum_{(p,m,M)\in P} \sum_{j=0}^{m} c_{pj}(x) t^{-p} \log^j t,$$

where $c_{pj} \in M$ and ω is a fixed cut-off function. We consider u as a function both on $\operatorname{int}\mathbb{D}$ and $\mathbb{R}_+ \times X$. In the same way we obtain the spaces $\mathcal{E}_Q(\mathbb{R}_+ \times \partial X)$ for $Q \in \operatorname{As}(\partial X; \gamma, \theta)$.

Definition 2.5. *For $s, \gamma \in \mathbb{R}$, $1 < p < \infty$, and $\theta > 0$ set*

$$\mathcal{H}_{p,\theta}^{s,\gamma}(\mathbb{D}) = \bigcap_{\varepsilon>0} \mathcal{H}_p^{s,\gamma+\theta-\varepsilon}(\mathbb{D}), \qquad \mathcal{B}_{p,\theta}^{s,\gamma}(\mathbb{B}) = \bigcap_{\varepsilon>0} \mathcal{B}_p^{s,\gamma+\theta-\varepsilon}(\mathbb{B}).$$

For $P \in \operatorname{As}(X; \gamma, \theta)$ and $Q \in \operatorname{As}(\partial X; \gamma, \theta)$ we set

$$\mathcal{H}_{p,P}^{s,\gamma}(\mathbb{D}) = \mathcal{H}_{p,\theta}^{s,\gamma}(\mathbb{D}) \oplus \mathcal{E}_P(\mathbb{R}_+ \times X), \qquad \mathcal{B}_{p,Q}^{s,\gamma}(\mathbb{B}) = \mathcal{B}_{p,\theta}^{s,\gamma}(\mathbb{D}) \oplus \mathcal{E}_Q(\mathbb{R}_+ \times \partial X).$$

All these space carry a natural Fréchet topology as a projective limit of Banach spaces.

Definition 2.6. *For $\gamma \in \mathbb{R}$ set*

$$\mathcal{C}^{\infty,\gamma}(\mathbb{D}) = \Big\{u \in \mathcal{C}^\infty(\operatorname{int}\mathbb{D}) \mid (t\partial_t)^k \log^l t(\omega u) \in \bigcap_{s\in\mathbb{R}} \mathcal{H}_p^{s,\gamma}(\mathbb{D}) \quad \forall\, k, l \in \mathbb{N}_0\Big\},$$

where ω is some cut-off function (the definition is independent of the involved $1 < p < \infty$: if one transports ωu to $\mathbb{R} \times X$ via S_γ from (2.1), the resulting smooth function is rapidly decreasing in t). For $\theta > 0$ and $P \in \operatorname{As}(X; \gamma, \theta)$ we then set

$$\mathcal{C}_\theta^{\infty,\gamma}(\mathbb{D}) := \bigcap_{\varepsilon>0} \mathcal{C}^{\infty,\gamma+\theta-\varepsilon}(\mathbb{D}), \qquad \mathcal{C}_P^{\infty,\gamma}(\mathbb{D}) := \mathcal{C}_\theta^{\infty,\gamma}(\mathbb{D}) \oplus \mathcal{E}_P(\mathbb{R}_+ \times X).$$

Analogously we introduce $\mathcal{C}^{\infty,\gamma}(\mathbb{B})$, $\mathcal{C}_\theta^{\infty,\gamma}(\mathbb{B})$, and $\mathcal{C}_Q^{\infty,\gamma}(\mathbb{B})$ for $Q \in \operatorname{As}(\partial X; \gamma, \theta)$.

By the previous definitions and Lemma 2.3 we obtain:

LEMMA 2.7. *Let $P \in \mathrm{As}(X;\gamma,\theta)$. For any $1<p<\infty$ and $s>\frac{1}{p}+j$ we have*

$$\gamma_j : \mathcal{H}^{s,\gamma}_{p,P}(\mathbb{D}) \longrightarrow \mathcal{B}^{s-j-\frac{1}{p},\gamma-\frac{1}{2}}_{p,Q}(\mathbb{B})$$

with $Q=\{(p,m,\gamma_j M) \mid (p,m,M)\in P\} \in \mathrm{As}(\partial X;\gamma-\frac{1}{2},\theta)$, where $\gamma_j M$ is the trace space of M under $\gamma_j : \mathcal{C}^\infty(X)\to\mathcal{C}^\infty(\partial X)$.

2.2. Integral operators. In the sequel we shall say that $\mathcal{G}=\begin{pmatrix}\mathcal{G}_{11} & \mathcal{G}_{12}\\ \mathcal{G}_{21} & \mathcal{G}_{22}\end{pmatrix}$ is an *integral operator with kernel k with respect to the scalar product in $\mathcal{H}^{0,0}_2(\mathbb{D})\oplus\mathcal{B}^{-\frac{1}{2},-\frac{1}{2}}_2(\mathbb{B})$*, if $\mathcal{G}$ acts on $\mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{D})\oplus\mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B})$ (and then possibly extends by continuity to other spaces) by

$$(\mathcal{G}_{j1}u_1+\mathcal{G}_{j2}u_2)(y_j) = (k_{j1}(y_j,\cdot),\overline{u}_1)_{\mathcal{H}^{0,0}_2(\mathbb{D})} + (k_{j2}(y_j,\cdot),\overline{u}_2)_{\mathcal{B}^{-1/2,-1/2}_2(\mathbb{B})}, \qquad j=1,2. \tag{2.5}$$

In (2.5), y_1 denotes the variable of $\mathbb{D}$ and y_2 that of $\mathbb{B}$. The kernels k_{11}, k_{12}, k_{21}, and k_{22} are locally integrable on $\mathrm{int}\,\mathbb{D}\times\mathrm{int}\,\mathbb{D}$, $\mathrm{int}\,\mathbb{D}\times\mathrm{int}\,\mathbb{B}$, $\mathrm{int}\,\mathbb{B}\times\mathrm{int}\,\mathbb{D}$, and $\mathrm{int}\,\mathbb{B}\times\mathrm{int}\,\mathbb{B}$, respectively. In view of the use of the scalar product of $\mathcal{B}^{-\frac{1}{2},-\frac{1}{2}}_2(\mathbb{B})$ in the second component, this notion differs slightly from the standard notion of integral operators with a kernel k. However, this form makes the exposition easier. We shall use the short-hand notation

$$k=\begin{pmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{pmatrix} \in \begin{pmatrix}V_1\\ \oplus\\ V_2\end{pmatrix} \widehat{\otimes}_\pi (W_1\oplus W_2),$$

if $k_{ij}\in V_i\,\widehat{\otimes}_\pi\,W_j$, $1\le i,j\le 2$, for certain subspaces V_1, W_1 of smooth functions on $\mathrm{int}\,\mathbb{D}$ and subspaces V_2, W_2 of smooth functions on $\mathrm{int}\,\mathbb{B}$. Moreover, $\widehat{\otimes}_\pi$ denotes the completed projective tensor product. If the spaces V_i, W_j are Fréchet spaces, $V_i\,\widehat{\otimes}_\pi\,W_j$ consists of all functions f that have a representation

$$f(y_i,y_j)=\sum_{k=1}^{\infty}\lambda_k\, v_i^k(y_i)\, w_j^k(y_j)$$

with $(\lambda_k)_{k\in\mathbb{N}}$ being an absolutely summable sequence of complex numbers, and $(v_i^k)_{k\in\mathbb{N}}$, $(w_j^k)_{k\in\mathbb{N}}$ being zero sequences in V_i and W_j, respectively.

2.3. The flat cone algebra. A *flat Green operator* $\mathcal{G}$ of type 0 is an integral operator with respect to the scalar product in $\mathcal{H}^{0,0}_2(\mathbb{D})\oplus\mathcal{B}^{-\frac{1}{2},-\frac{1}{2}}_2(\mathbb{B})$ with kernel in

$$\begin{pmatrix}\mathcal{C}^{\infty,\infty}(\mathbb{D})\\ \oplus\\ \mathcal{C}^{\infty,\infty}(\mathbb{B})\end{pmatrix} \widehat{\otimes}_\pi \left(\mathcal{C}^{\infty,\infty}(\mathbb{D})\oplus\mathcal{C}^{\infty,\infty}(\mathbb{B})\right),$$

where

$$\mathcal{C}^{\infty,\infty}(\mathbb{D})=\bigcap_{\gamma\in\mathbb{R}}\mathcal{C}^{\infty,\gamma}(\mathbb{D}), \qquad \mathcal{C}^{\infty,\infty}(\mathbb{B})=\bigcap_{\gamma\in\mathbb{R}}\mathcal{C}^{\infty,\gamma}(\mathbb{B}) \tag{2.6}$$

(these functions vanish to infinite order in the singularity). In particular, $\mathcal{G}$ induces an operator

$$\mathcal{G}: \begin{matrix}\mathcal{C}^{\infty,\infty}(\mathbb{D})\\ \oplus\\ \mathcal{C}^{\infty,\infty}(\mathbb{B})\end{matrix} \longrightarrow \begin{matrix}\mathcal{C}^{\infty,\infty}(\mathbb{D})\\ \oplus\\ \mathcal{C}^{\infty,\infty}(\mathbb{B})\end{matrix}. \tag{2.7}$$

DEFINITION 2.8. *An operator $\mathcal{G}$, acting as in* (2.7)*, is called a* flat Green operator *of type $d\in\mathbb{N}_0$ if it is of the form*

$$\mathcal{G}=\sum_{j=1}^{d}\mathcal{G}_j\begin{pmatrix}D^j & 0\\ 0 & 1\end{pmatrix}+\mathcal{G}_0$$

with flat Green operators $\mathcal{G}_0,\ldots,\mathcal{G}_d$ of type 0, and a differential operator D on $\mathbb{D}$ with coefficients smooth up to $t=0$, that coincides with $-i\partial_\nu$ near the boundary $\mathbb{B}$ of $\mathbb{D}$. The space of all such operators is denoted by $C^d_G(\mathbb{D})_\infty$.

Definition 2.9. *A* holomorphic Mellin symbol *of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}_0$ is a holomorphic function $h : \mathbb{C} \to B^{\mu,d}(X)$ such that*

$$\delta \mapsto h(\delta + i\tau) : \mathbb{R} \longrightarrow B^{\mu,d}(X;\mathbb{R}_\tau)$$

is a continuous function. We denote the space of all such symbols by $M_O^{\mu,d}(X)$ and set

$$M_O^{\mu,d}(\overline{\mathbb{R}}_+ \times X) = \mathcal{C}^\infty(\overline{\mathbb{R}}_+)\,\widehat{\otimes}_\pi\, M_O^{\mu,d}(X).$$

Each $h \in M_O^{\mu,d}(\overline{\mathbb{R}}_+ \times X)$ defines an operator

$$\mathrm{op}_M(h) : \begin{array}{c} \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X) \\ \oplus \\ \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times \partial X) \end{array} \longrightarrow \begin{array}{c} \mathcal{C}^\infty(\mathbb{R}_+ \times X) \\ \oplus \\ \mathcal{C}^\infty(\mathbb{R}_+ \times \partial X) \end{array}$$

by

$$(\mathrm{op}_M(h)u)(t) = \int_\Gamma t^{-z} h(t,z)(\mathcal{M}u)(z)\, đz, \tag{2.8}$$

where $\mathcal{M}$ denotes the Mellin transform, Γ is an arbitrary vertical line in the complex plane, and we identify $\mathcal{C}^\infty(\mathbb{R}_+ \times X) \oplus \mathcal{C}^\infty(\mathbb{R}_+ \times \partial X)$ with $\mathcal{C}^\infty(\mathbb{R}_+, \mathcal{C}^\infty(X) \oplus \mathcal{C}^\infty(\partial X))$. Observe the following useful relation: For any $\sigma \in \mathbb{R}$,

$$\mathrm{op}_M(h)\, t^{-\sigma} = t^{-\sigma}\, \mathrm{op}_M(T^\sigma h), \tag{2.9}$$

where T^σ is the operator of shifting z by σ, i.e.

$$(T^\sigma h)(t,z) = h(t, z+\sigma). \tag{2.10}$$

Definition 2.10. *With $\mu \in \mathbb{Z}$, $d \in \mathbb{N}_0$ let $C_O^{\mu,d}(\mathbb{D})$ denote the space of all operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{C}^{\infty,\infty}(\mathbb{D}) \\ \oplus \\ \mathcal{C}^{\infty,\infty}(\mathbb{B}) \end{array} \longrightarrow \begin{array}{c} \mathcal{C}^{\infty,\infty}(\mathbb{D}) \\ \oplus \\ \mathcal{C}^{\infty,\infty}(\mathbb{B}) \end{array}$$

of the form

$$\mathcal{A} = \omega\, t^{-\mu}\, \mathrm{op}_M(h)\, \omega_0 + (1-\omega)\mathcal{P}(1-\omega_1) + \mathcal{G},$$

where $\omega_1 \prec \omega \prec \omega_0$ are cut-off functions and

i) *$h(t,z) \in M_O^{\mu,d}(\overline{\mathbb{R}}_+ \times X)$ is a holomorphic Mellin symbol,*
ii) *$\mathcal{P} \in B^{\mu,d}(2\mathbb{D})$ is an element of Boutet's algebra on $2\mathbb{D}$,*
iii) *$\mathcal{G} \in C_G^d(\mathbb{D})_\infty$ is a flat Green operator.*

2.4. The full cone algebra. In order to allow the construction of a parametrix to elliptic boundary value problems it is necessary to enlarge the flat cone algebra by a more general class of smoothing remainders.

Flat Green operators of type 0 have, roughly speaking, integral kernels that vanish to infinite order in the conical singularity. General Green operators instead have a specific kind of asymptotic behaviour near the singularity.

A *Green operator* of type 0 associated with asymptotic types $(P,Q) \in \mathrm{As}(X,\partial X; -\gamma, \theta)$ and $(P',Q') \in \mathrm{As}(X,\partial X; \gamma', \theta')$ is an integral operator with respect to the scalar product in $\mathcal{H}_2^{0,0}(\mathbb{D}) \oplus \mathcal{B}_2^{-\frac12,-\frac12}(\mathbb{B})$ with a kernel belonging to the intersection of the two spaces

$$\begin{pmatrix} \mathcal{C}^{\infty,\gamma'}_{P'}(\mathbb{D}) \\ \oplus \\ \mathcal{C}^{\infty,\gamma'-\frac12}_{Q'}(\mathbb{B}) \end{pmatrix} \widehat{\otimes}_\pi \left(\mathcal{C}^{\infty,-\gamma}(\mathbb{D}) \oplus \mathcal{C}^{\infty,-\gamma-\frac12}(\mathbb{B})\right)$$

and

$$\begin{pmatrix} \mathcal{C}^{\infty,\gamma'}(\mathbb{D}) \\ \oplus \\ \mathcal{C}^{\infty,\gamma'-\frac{1}{2}}(\mathbb{B}) \end{pmatrix} \widehat{\otimes}_\pi \left(\mathcal{C}^{\infty,-\gamma}_{\overline{P}}(\mathbb{D}) \oplus \mathcal{C}^{\infty,-\gamma-\frac{1}{2}}_{\overline{Q}}(\mathbb{B})\right),$$

where $\overline{P} = \{(\overline{p}, m, \overline{M}) \mid (p,m,M) \in P\}$ and analogously $\overline{Q}$ is given; $\overline{M}$ denotes the space of all complex conjugates of elements of M.

Using the technique from [**22**], Section 4, it can be seen that Green operators equivalently can be characterized by mapping properties in Sobolev (Besov) spaces:

Theorem 2.11. *For an operator $\mathcal{G}$ to have a kernel as above, it is necessary and sufficient that, for some fixed $1 < p < \infty$,*

$$\mathcal{G} : \begin{matrix} \mathcal{H}^{s,\gamma}_p(\mathbb{D}) \\ \oplus \\ \mathcal{B}^{r,\gamma-\frac{1}{2}}_p(\mathbb{B}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{C}^{\infty,\gamma'}_{P'}(\mathbb{D}) \\ \oplus \\ \mathcal{C}^{\infty,\gamma'-\frac{1}{2}}_{Q'}(\mathbb{B}) \end{matrix}$$

for all $s > -1 + \frac{1}{p}$ and all $r \in \mathbb{R}$, and that

$$\mathcal{G}^t : \begin{matrix} \mathcal{H}^{s,-\gamma'}_{p'}(\mathbb{D}) \\ \oplus \\ \mathcal{B}^{r,-\gamma'-\frac{1}{2}}_{p'}(\mathbb{B}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{C}^{\infty,-\gamma}_{P}(\mathbb{D}) \\ \oplus \\ \mathcal{C}^{\infty,-\gamma-\frac{1}{2}}_{Q}(\mathbb{B}) \end{matrix}$$

for all $s > -1+\frac{1}{p'}$ and all $r \in \mathbb{R}$. Here, the exponent t refers to the formal adjoint of $\mathcal{G}$ with respect to the scalar product in $\mathcal{H}^{0,0}_2(\mathbb{D}) \oplus \mathcal{B}^{-\frac{1}{2},-\frac{1}{2}}_2(\mathbb{B})$ and p' is the dual number to p.

Definition 2.12. *Let $(P,Q) \in \mathrm{As}(X, \partial X; -\gamma, \theta)$, $(P', Q') \in \mathrm{As}(X, \partial X; \gamma', \theta')$, and $d \in \mathbb{N}_0$. Then $C^d_G(\mathbb{D}; \gamma, \theta; \gamma', \theta')$ denotes the space of all operators $\mathcal{G}$ that are of the form*

$$\mathcal{G} = \sum_{j=1}^{d} \mathcal{G}_j \begin{pmatrix} D^j & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{G}_0$$

with operators $\mathcal{G}_0, \ldots, \mathcal{G}_d$ having a kernel as described above (with certain asymptotic types depending on $\mathcal{G}_j$), and D as in Definition 2.8. We write $C^d_G(\mathbb{D}; \gamma, \gamma', \theta) = C^d_G(\mathbb{D}; \gamma, \theta; \gamma', \theta)$.

Besides Green operators we need another kind of smoothing remainders, built on *meromorphic Mellin symbols.*

Definition 2.13. *An* asymptotic type *for Mellin symbols P of type $d \in \mathbb{N}_0$ is a set of triples (p, n, N) with $p \in \mathbb{C}$, $n \in \mathbb{N}_0$, and N a finite dimensional subspace of finite rank operators from $B^{-\infty,d}(X)$. Moreover, we require that, for each $\delta > 0$,*

$$\{p \in \mathbb{C} \mid |\mathrm{Re}\, p| \le \delta \text{ and } (p,n,N) \in P \text{ for some } n,\, N\}$$

is a finite set. A meromorphic Mellin symbol with asymptotic type P of type $d \in \mathbb{N}_0$ is a meromorphic function $f : \mathbb{C} \to B^{-\infty,d}(X)$ with poles at most in points $p \in \mathbb{C}$ with $(p,n,N) \in P$ for some n, N. Moreover f satisfies: If $(p,n,N) \in P$, then the principal part of the Laurent series of f in p is of the form $\sum_{k=0}^{n} R_k (z-p)^{-k-1}$ with $R_k \in N$; if $\chi \in \mathcal{C}^\infty(\mathbb{C})$ is identically zero in a small neighborhood around each pole and identically 1 outside another neighborhood, then $(\chi f)(\delta + i\tau)$ is a continuous function of $\delta \in \mathbb{R}$ with values in $B^{-\infty,d}(X; \mathbb{R}_\tau) = \mathcal{S}(\mathbb{R}_\tau, B^{-\infty,d}(X))$. We shall then write $f \in M^{-\infty,d}_P(X)$.

As in (2.8) we can associate with meromorphic Mellin symbols a pseudodifferential operator. Now, however, the operator will depend on the choice of the line Γ. Denoting the vertical line $\mathrm{Re}\, z = \frac{1}{2} - \delta$ by $\Gamma_{\frac{1}{2}-\delta}$, we define $\mathrm{op}^\delta_M(f)$ by

$$(\mathrm{op}^\delta_M(f)u)(t) = \int_{\Gamma_{\frac{1}{2}-\delta}} t^{-z} f(t,z)(\mathcal{M}u)(z)\, đz.$$

Using this notation, we always require that none of the poles of f lies on the chosen line.

Definition 2.14. *Let $\gamma \in \mathbb{R}$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}_0$, and $k \in \mathbb{N}$. Then $C^{\mu,d}_{M+G}(\mathbb{D};\gamma,\gamma-\mu,k)$ is the space of all operators*

$$\mathcal{A}: \begin{array}{c}\mathcal{C}^{\infty,\gamma}(\mathbb{D})\\ \oplus \\ \mathcal{C}^{\infty,\gamma-\frac12}(\mathbb{B})\end{array} \longrightarrow \begin{array}{c}\mathcal{C}^{\infty,\gamma-\mu}(\mathbb{D})\\ \oplus \\ \mathcal{C}^{\infty,\gamma-\mu-\frac12}(\mathbb{B})\end{array}$$

which are of the form

$$\mathcal{A} = \omega \sum_{j=0}^{k-1} t^{-\mu+j} \operatorname{op}_M^{\gamma_j-\frac{n}{2}}(f_j)\,\omega_0 + \mathcal{G}, \tag{2.11}$$

where ω,ω_0 are some cut-off functions, $\mathcal{G} \in C^d_G(\mathbb{D};\gamma,\gamma-\mu,k)$, the f_j are meromorphic Mellin symbols with asymptotic types P_j of type d, and $\gamma - j \le \gamma_j \le \gamma$.

Now we are in the position to define the full cone algebra.

Definition 2.15. *Let $\gamma \in \mathbb{R}$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}_0$, and $k \in \mathbb{N}$. We set*

$$C^{\mu,d}(\mathbb{D};\gamma,\gamma-\mu,k) = C^{\mu,d}_O(\mathbb{D}) + C^{\mu,d}_{M+G}(\mathbb{D};\gamma,\gamma-\mu,k),$$

acting as in Definition 2.14.

The terminology 'algebra' is due to the following fact:

Theorem 2.16. *Composition of operators, $(\mathcal{A}_0,\mathcal{A}_1) \mapsto \mathcal{A}_0\mathcal{A}_1$, induces a map*

$$C^{\mu_0,d_0}(\mathbb{D};\gamma-\mu_1,\gamma-\mu_0-\mu_1,k) \times C^{\mu_1,d_1}(\mathbb{D};\gamma,\gamma-\mu_1,k) \longrightarrow C^{\mu,d}(\mathbb{D};\gamma,\gamma-\mu,k),$$

with $\mu = \mu_0+\mu_1$ and $d = \max(\mu_1+d_0,d_1)$. If one of the factors $\mathcal{A}_0$ or $\mathcal{A}_1$ belongs to the C_{M+G}- or C_G-classes, the same holds true for the product.

Theorem 2.17. *Each $\mathcal{A} \in C^{\mu,d}(\mathbb{D};\gamma,\gamma-\mu,k)$ extends by continuity to mappings*

$$\mathcal{A}: \begin{array}{c}\mathcal{H}^{s,\gamma}_p(\mathbb{D})\\ \oplus \\ \mathcal{B}^{s-\frac1p,\gamma-\frac12}_p(\mathbb{B})\end{array} \longrightarrow \begin{array}{c}\mathcal{H}^{s-\mu,\gamma-\mu}_p(\mathbb{D})\\ \oplus \\ \mathcal{B}^{s-\mu-\frac1p,\gamma-\mu-\frac12}_p(\mathbb{B})\end{array}, \qquad s > d-1+\frac{1}{p}. \tag{2.12}$$

For each asymptotic type $(P,Q) \in \operatorname{As}(X,\partial X;\gamma,k)$ there exists a $(P',Q') \in \operatorname{As}(X,\partial X;\gamma-\mu,k)$ such that

$$\mathcal{A}: \begin{array}{c}\mathcal{H}^{s,\gamma}_{p,P}(\mathbb{D})\\ \oplus \\ \mathcal{B}^{s-\frac1p,\gamma-\frac12}_{p,Q}(\mathbb{B})\end{array} \longrightarrow \begin{array}{c}\mathcal{H}^{s-\mu,\gamma-\mu}_{p,P'}(\mathbb{D})\\ \oplus \\ \mathcal{B}^{s-\mu-\frac1p,\gamma-\mu-\frac12}_{p,Q'}(\mathbb{B})\end{array}, \qquad s > d-1+\frac{1}{p}. \tag{2.13}$$

Proof. We may assume type $d=0$. Let $\mathcal{A}$ be as described in Definitions 2.10 and 2.15. Then $(1-\omega)\mathcal{P}(1-\omega_1)$ has the required mapping properties by the results of [**7**], since away from the tip cone Sobolev and cone Besov spaces coincide with the usual ones. Also $\mathcal{G}$ behaves correctly due to Theorem 2.11. Thus we may assume that

$$\mathcal{A} = \omega\, t^{-\mu}\Big(\operatorname{op}_M(h) + \sum_{j=0}^{k-1} t^j \operatorname{op}_M^{\gamma_j-\frac{n}{2}}(f_j)\Big)\,\omega_0,$$

cf. (2.11). In order to obtain (2.12) it is then enough to consider

$$\widetilde{\mathcal{A}} = \omega\, t^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\,\omega_0$$

for some $f(t,\gamma+i\tau) \in \mathcal{C}^\infty(\overline{\mathbb{R}}_+, B^{\mu,0}(X;\mathbb{R}_\tau))$. We pull back $t^\mu\widetilde{\mathcal{A}}$ to an operator on $\mathbb{R}\times X$ using the maps S_γ and $S_{\gamma-\frac12}$ in (2.1) and (2.3). Since

$$\begin{pmatrix} S_\gamma^{-1} & 0\\ 0 & S'^{-1}_\gamma\end{pmatrix} \operatorname{op}_M^{\gamma-\frac{n}{2}}(f) \begin{pmatrix} S_\gamma & 0\\ 0 & S'_\gamma\end{pmatrix} = \operatorname{op}(f_\gamma), \qquad f_\gamma(r,\varrho) = f(e^r, \frac{n+1}{2} - \gamma + i\varrho),$$

this yields a 'standard' operator in Boutet's calculus on $\mathbb{R} \times X$ and the assertion follows from [**7**]. Let us turn to the proof of (2.13). From the known case $p = 2$, cf. Theorem 4.1.14 in [**17**] and Lemma 3.1.8 in [**18**], we have that

$$\mathcal{A} : \begin{array}{c} \mathcal{C}_P^{\infty,\gamma}(\mathbb{D}) \\ \oplus \\ \mathcal{C}_Q^{\infty,\gamma-\frac{1}{2}}(\mathbb{B}) \end{array} \longrightarrow \begin{array}{c} \mathcal{C}_{P'}^{\infty,\gamma-\mu}(\mathbb{D}) \\ \oplus \\ \mathcal{C}_{Q'}^{\infty,\gamma-\mu-\frac{1}{2}}(\mathbb{B}) \end{array}.$$

By the definition of the spaces with asymptotics, cf. Definition 2.5, we thus may assume that P and Q are the empty asymptotic types. Since h is holomorphic, $\mathrm{op}_M^\sigma(h)$ on $\mathcal{C}_{\mathrm{comp}}^\infty(\mathrm{int}\,\mathbb{D}) \oplus \mathcal{C}_{\mathrm{comp}}^\infty(\mathrm{int}\,\mathbb{B})$ is independent of the choice of σ. By density and (2.12) this yields that

$$\omega\, t^{-\mu}\, \mathrm{op}_M(h)\, \omega_0 : \begin{array}{c} \mathcal{H}_{p,k}^{s,\gamma}(\mathbb{D}) \\ \oplus \\ \mathcal{B}_{p,k}^{s-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B}) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}_{p,k}^{s-\mu,\gamma-\mu}(\mathbb{D}) \\ \oplus \\ \mathcal{B}_{p,k}^{s-\mu-\frac{1}{p},\gamma-\mu-\frac{1}{2}}(\mathbb{B}) \end{array}$$

is continuous. It remains to consider $\widetilde{\mathcal{A}}$ as above, but now with a meromorphic Mellin symbol $f \in M_R^{-\infty,0}(X)$. For small $\varepsilon > 0$ we write

$$\widetilde{\mathcal{A}} = \omega\, t^{-\mu}\, \mathrm{op}_M^{\gamma+k-\varepsilon-\frac{n}{2}}(f)\, \omega_0 + t^{-\mu}\, \omega \left(\mathrm{op}_M^{\gamma-\frac{n}{2}}(f) - \mathrm{op}_M^{\gamma+k-\varepsilon-\frac{n}{2}}(f)\right) \omega_0.$$

If ε is sufficiently small f will be holomorphic in the strip $-\gamma - k < \mathrm{Re}\, z - \frac{n+1}{2} < -\gamma - k + \varepsilon$, and thus the action of $\mathrm{op}_M^{\gamma+k-\varepsilon-\frac{n}{2}}(f)$ will be independent of $\varepsilon > 0$. Hence the first summand maps $\mathcal{H}_{p,k}^{s,\gamma}(\mathbb{D}) \oplus \mathcal{B}_{p,k}^{s-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B})$ into $\mathcal{C}_k^{\infty,\gamma-\mu}(\mathbb{D}) \oplus \mathcal{C}_k^{\infty,\gamma-\mu-\frac{1}{2}}(\mathbb{B})$ The second summand can be calculated explicitly by means of Cauchy's integral formula, see Proposition 5.1, giving the desired result. □

THEOREM 2.18. *Let $\mathcal{A} \in C^{\mu,0}(\mathbb{D}; \gamma, \gamma-\mu, k)$ with $\mu \le 0$. Then the formal adjoint of $\mathcal{A}$ (with respect to the scalar product of $\mathcal{H}_2^{0,0}(\mathbb{D}) \oplus \mathcal{B}_2^{-\frac{1}{2},-\frac{1}{2}}(\mathbb{B})$) belongs to $C^{\mu,0}(\mathbb{D}; -\gamma+\mu, -\gamma, k)$. If $\mathcal{A}$ belongs to the C_{M+G}- or C_G-classes, the same is true for the formal adjoint.*

2.5. Symbolic structure, ellipticity, and parametrix.

Let $\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} \in C^{\mu,d}(\mathbb{D}; \gamma, \gamma-\mu, k)$ be given. On $\mathrm{int}\,\mathbb{D}$ (i.e. away from the conical singularity) the operator $\mathcal{A}$ is a usual element of Boutet de Monvel's algebra. Hence we can associate with $\mathcal{A}$ the usual (homogeneous) *principal symbol*

$$\sigma_\psi^\mu(\mathcal{A}) = \sigma_\psi^\mu(A) \in \mathcal{C}^\infty(T^*\mathrm{int}\,\mathbb{D} \setminus 0)$$

and the usual (principal) *boundary symbol*

$$\sigma_\partial^\mu(\mathcal{A}) \in \mathcal{C}^\infty(T^*\mathrm{int}\,\mathbb{B} \setminus 0, \mathcal{L}(H_p^s(\mathbb{R}_+) \oplus \mathbb{C}, H_p^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C})). \tag{2.14}$$

Near the conical singularity these symbols are 'cone degenerate', i.e. the limits

$$\widetilde{\sigma}_\psi^\mu(\mathcal{A})(x,\tau,\xi) = \lim_{t\to 0+} t^\mu\, \sigma_\psi^\mu(\mathcal{A})(t,x,t^{-1}\tau,\xi), \qquad \widetilde{\sigma}_\partial^\mu(\mathcal{A})(x',\tau,\xi') = \lim_{t\to 0+} t^\mu\, \sigma_\partial^\mu(\mathcal{A})(t,x',t^{-1}\tau,\xi')$$

exist. We call

$$\widetilde{\sigma}_\psi^\mu(\mathcal{A}) \in \mathcal{C}^\infty((T^*X \times \mathbb{R}) \setminus 0)$$

the *rescaled principal symbol*. The *rescaled boundary symbol* is

$$\widetilde{\sigma}_\partial^\mu(\mathcal{A}) \in \mathcal{C}^\infty((T^*\partial X \times \mathbb{R}) \setminus 0, \mathcal{L}(H_p^s(\mathbb{R}_+) \oplus \mathbb{C}, H_p^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C})). \tag{2.15}$$

According to Definition 2.15 $\mathcal{A}$ is of the form

$$\mathcal{A} = \omega\, t^{-\mu}\, \mathrm{op}_M(h)\, \omega_0 + (1-\omega)\mathcal{P}(1-\omega_1) + \mathcal{M} + \mathcal{G},$$

where h and $\mathcal{P}$ are as in Definition 2.10.i,ii), and $\mathcal{M}+\mathcal{G}$ are as in definition 2.14. Then, by definition, the *conormal symbol* of $\mathcal{A}$ is

$$\sigma_M^\mu(\mathcal{A})(z) = h(0,z) + f_0(z).$$

It is a meromorphic function with values in $B^{\mu,d}(X)$ or, alternatively,

$$\sigma_M^\mu(\mathcal{A})(z) \in \mathcal{L}(H_p^s(X) \oplus B_p^{s-\frac{1}{p}}(\partial X), H_p^{s-\mu}(X) \oplus B_p^{s-\mu-\frac{1}{p}}(\partial X)) \tag{2.16}$$

for any $s > d - 1 + \frac{1}{p}$. Recall also that $\sigma_M^\mu(\mathcal{A})(\delta + i\tau) \in B^{\mu,d}(X; \mathbb{R}_\tau)$ whenever f_0 has no pole on the line $\operatorname{Re} z = \delta$. We then have the following compatibility relations:

$$\sigma_\psi^\mu(\sigma_M^\mu(\mathcal{A}))(x,\xi,\tau) = \widetilde{\sigma}_\psi^\mu(\mathcal{A})(x,-\tau,\xi), \qquad \sigma_\partial^\mu(\sigma_M^\mu(\mathcal{A}))(x',\xi',\tau) = \widetilde{\sigma}_\partial^\mu(\mathcal{A})(x',-\tau,\xi')$$

independently of $\delta \in \mathbb{R}$ (note that f_0 is of order $-\infty$ and thus does not contribute to the principal symbolic levels).

Remark 2.19. *Let $\mathcal{A} \in C^{\mu,d}(\mathbb{D}; \gamma, \gamma-\mu, k)$ with $d = \max(0,\mu)$ and assume that both $\widetilde{\sigma}_\psi^\mu(\mathcal{A})$ and $\widetilde{\sigma}_\partial^\mu(\mathcal{A})$ are pointwise everywhere invertible. Then the before mentioned compatibility relations ensure that $h(0, \delta + i\tau) \in B^{\mu,d}(X; \mathbb{R}_\tau)$ is (parameter-dependent) elliptic for each δ, hence $\sigma_M^\mu(\mathcal{A})(\delta + i\tau)$ is invertible for large $|\tau|$. By a classical theorem on the invertibility of Fredholm families one then can deduce that $\sigma_M^\mu(\mathcal{A})$ is meromorphically invertible with*

$$\sigma_M^\mu(\mathcal{A})^{-1}(z) = h'(z) + f_0'(z),$$

where $h' \in M_O^{\mu,d'}(X)$ and $f_0' \in M_{P'}^{-\infty,d'}(X)$ with $d' = \max(0,-\mu)$. For details see [**18**].

Note that the invertibilty of the (rescaled) boundary symbol (2.14), (2.15), and the conormal symbol (2.16) is independent of s and p.

Definition 2.20. *We call $\mathcal{A} \in C^{\mu,d}(\mathbb{D}; \gamma, \gamma-\mu, k)$* elliptic *if the following three conditions are satisfied:*

i) *Both $\sigma_\psi^\mu(\mathcal{A})$ and $\widetilde{\sigma}_\psi^\mu(\mathcal{A})$ are pointwise everywhere invertible,*
ii) *both $\sigma_\partial^\mu(\mathcal{A})$ and $\widetilde{\sigma}_\partial^\mu(\mathcal{A})$ are pointwise everywhere invertible,*
iii) *$\sigma_M^\mu(\mathcal{A})^{-1}$ has no pole on the line $\operatorname{Re} z = \frac{n+1}{2} - \gamma$.*

Theorem 2.21. *Let $\mathcal{A} \in C^{\mu,d}(\mathbb{D}; \gamma, \gamma-\mu, k)$, $d = \max(0,\mu)$, be elliptic. Then there exists a $\mathcal{B} \in C^{-\mu,d'}(\mathbb{D}; \gamma-\mu, \gamma, k)$, $d' = \max(0,-\mu)$, such that*

$$\mathcal{B}\mathcal{A} - 1 \in C_G^d(\mathbb{D}; \gamma, \gamma, k), \qquad \mathcal{A}\mathcal{B} - 1 \in C_G^{d'}(\mathbb{D}; \gamma-\mu, \gamma-\mu, k).$$

Any such $\mathcal{B}$ with these properties we call a *parametrix* of $\mathcal{A}$. Using the compactness of Green operators and the mapping properties of elements of the cone algebra, one can easily deduce the following results on Fredholm property and elliptic regularity:

Theorem 2.22. *Let $\mathcal{A} \in C^{\mu,d}(\mathbb{D}; \gamma, \gamma-\mu, k)$, $d = \max(0,\mu)$, be elliptic. Then*

$$\mathcal{A} : \begin{array}{c} \mathcal{H}_p^{s,\gamma}(\mathbb{D}) \\ \oplus \\ \mathcal{B}_p^{s-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B}) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{D}) \\ \oplus \\ \mathcal{B}_p^{s-\mu-\frac{1}{p},\gamma-\mu-\frac{1}{2}}(\mathbb{B}) \end{array}$$

is a Fredholm operator for any $1 < p < \infty$ and $s > d - 1 + \frac{1}{p}$.

Theorem 2.23. *Let $\mathcal{A} \in C^{\mu,d}(\mathbb{D}; \gamma, \gamma-\mu, k)$, $d = \max(0,\mu)$, be elliptic and consider the equation $Au = f$ Then:*

a) *If $f \in \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{D}) \oplus \mathcal{B}_p^{s-\mu-\frac{1}{p},\gamma-\mu-\frac{1}{2}}(\mathbb{B})$, $s > d - 1 + \frac{1}{p}$, and $u \in \mathcal{H}_p^{t,\gamma}(\mathbb{D}) \oplus \mathcal{B}_p^{t-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B})$ for some $t > \max(0,-\mu) - 1 + \frac{1}{p}$, then $u \in \mathcal{H}_p^{s,\gamma}(\mathbb{D}) \oplus \mathcal{B}_p^{s-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B})$.*
b) *If $f \in \mathcal{H}_{p,P'}^{s-\mu,\gamma-\mu}(\mathbb{D}) \oplus \mathcal{B}_{p,Q'}^{s-\mu-\frac{1}{p},\gamma-\mu-\frac{1}{2}}(\mathbb{B})$, $s > d-1+\frac{1}{p}$, for some asymptotic type $(P',Q') \in \operatorname{As}(X, \partial X; \gamma-\mu, k)$ and $u \in \mathcal{H}_p^{t,\gamma}(\mathbb{D}) \oplus \mathcal{B}_p^{t-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B})$ for some $t > \max(0,-\mu) - 1 + \frac{1}{p}$, then there exists a $(P,Q) \in \operatorname{As}(X, \partial X; \gamma, k)$ such that $u \in \mathcal{H}_{p,P}^{s,\gamma}(\mathbb{D}) \oplus \mathcal{B}_{p,Q}^{s-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B})$.*

2.6. Operators acting on sections into vector bundles. For simplicity of the presentation, we so far restricted ourselves to operators acting on scalar valued functions. However, the results in Sections 2.1 to 2.5 readily extend to operators acting on sections of vector bundles.

If $E, \widetilde{E}$ and $F, \widetilde{F}$ are vector bundles over $\mathbb{D}$ and $\mathbb{B}$ (see the introduction), respectively, we can speak of the cone algebra $C^{\mu,d}(\mathbb{D};\gamma,\gamma-\mu,k;E,F;\widetilde{E},\widetilde{F})$. Elements $\mathcal{A}$ of this space then induce operators

$$\mathcal{A}:\begin{array}{c}\mathcal{H}_p^{s,\gamma}(\mathbb{D},E)\\ \oplus\\ \mathcal{B}_p^{s-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B},F)\end{array}\longrightarrow\begin{array}{c}\mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{D},\widetilde{E})\\ \oplus\\ \mathcal{B}_p^{s-\mu-\frac{1}{p},\gamma-\mu-\frac{1}{2}}(\mathbb{B},\widetilde{F})\end{array}$$

for any $1<p<\infty$ and $s>d-1+\frac{1}{p}$. It is straightforward to adapt the previous definitions to the vector bundle situation. The holomorphic Mellin symbol, for example, takes values in $B^{\mu,d}(X;E_0,F_0;\widetilde{E}_0,\widetilde{F}_0)$, where subscript 0 indicates restriction of the bundles to $t=0$.

The principal and the rescaled principal symbols are then homomorphisms acting between the pull-backs of E and $\widetilde{E}$ to the cotangent bundle. Similar statements for the (rescaled) boundary symbol. More details will be given in Section 3.2.

If the bundles F or $\widetilde{F}$ are zero dimensional, we shall omit them from the notation, e.g. we write $C^{\mu,d}(\mathbb{D};\gamma,\gamma-\mu,k;E;\widetilde{E})$ if $F=\widetilde{F}=0$.

3. Differential boundary value problems on manifolds with conical singularities

In the following let A be a Fuchs type differential operator of order $\mu\in\mathbb{N}$ on $\mathbb{D}$, acting between sections of a vector bundle E over $\mathbb{D}$. Using the notation from (1.1), the principal symbol of A near $t=0$ is

$$\sigma_\psi^\mu(A)(t,x,\tau,\xi)=t^{-\mu}\sum_{j=0}^{\mu}\sigma_\psi^{\mu-j}(a_j)(t,x,\xi)(-it\tau)^j,$$

and thus the rescaled principal symbol is

$$\widetilde{\sigma}_\psi^\mu(A)(x,\tau,\xi)=\sum_{j=0}^{\mu}\sigma_\psi^{\mu-j}(a_j)(0,x,\xi)(-i\tau)^j.$$

We shall say that A has *t-independent coefficients* (near the singularity $t=0$), if all the a_j are constant in t near $t=0$.

3.1. Differential boundary conditions. A typical boundary condition for A is of the form $\gamma_0 B$, where γ_0 denotes the operator of restriction to $\mathbb{B}$, and B is a Fuchs type differential operator on $\mathbb{D}$, acting from sections of E to sections of a bundle $\widetilde{F}$ over $\mathbb{D}$. Near $t=0$ let

$$B=t^{-\nu}\sum_{j=0}^{\nu}b_j(t)(-t\partial_t)^j,\qquad b_j\in\mathcal{C}^\infty([0,1[,\mathrm{Diff}^{\nu-j}(X;E_0,\widetilde{F}_0)).$$

Using the splitting $x=(x',x_n)$ near the boundary ∂X, we write

$$b_j(t)=\sum_{k=0}^{\nu-j}b_{jk}(t)D_{x_n}^k,\qquad b_{jk}\in\mathcal{C}^\infty([0,1[,\mathrm{Diff}^{\nu-j-k}(\partial X;E_0',\widetilde{F}_0')),$$

where $'$ indicates restriction of the bundles to ∂X. Then

$$\gamma_0 B=\sum_{k=0}^{\nu}t^{-k}\Big(t^{-\nu+k}\sum_{j=0}^{\nu-k}b_{jk}(t)(-t\partial_t)^j\Big)\gamma_k=\sum_{k=0}^{\nu}\underline{S}_k\,t^{-k}\,\gamma_k,\tag{3.1}$$

with $\gamma_k=\gamma_0 D_n^k$ and Fuchs type differential operators $\underline{S}_k$ of order $\nu-k$ on $\mathbb{B}$ (acting from $E_{\mathbb{B}}$ to $\widetilde{F}_{\mathbb{B}}$, the restrictions of the bundles to $\mathbb{B}$).

Given μ such boundary conditions

$$T_k=\gamma_0 B_k,\qquad k=0,\dots,\mu-1,$$

with Fuchs type differential operators B_k of order k acting from sections of E to sections of $\widetilde{F}_k$ and if we set

$$T = \begin{pmatrix} T_0 & T_1 & \ldots & T_{\mu-1}\end{pmatrix}^t, \qquad \varrho = \begin{pmatrix}\gamma_0 & \gamma_1 & \ldots & \gamma_{\mu-1}\end{pmatrix}^t, \tag{3.2}$$

we obtain $T = S\varrho$ with a left-lower triangular matrix

$$S = (\underline{S}_{jk})_{0\le j,k<\mu}\,\mathrm{diag}(1, t^{-1}, \ldots, t^{-\mu+1}). \tag{3.3}$$

Here the $\underline{S}_{jk}$ are Fuchs type operators of order $j-k$ on $\mathbb{B}$, and $\underline{S}_{jk} = 0$ if $k > j$. Each $\underline{S}_{jk}$ acts from sections of $E_{\mathbb{B}}$ to sections of $F_k := \widetilde{F}_{k\mathbb{B}}$.

For the remaining part of the paper $T = S\varrho$ will be the standard form of a boundary condition for A. Note that we allow F_k to be zero dimensional. In this case the k-th boundary condition T_k is void. Nevertheless, for systematic reasons, it is convenient to work with the full matrix S.

We shall say that T has *t-independent coefficients* (near the singularity $t = 0$), if all the $\underline{S}_{jk}$ have t-independent coefficients.

3.2. Relation with the cone algebra. Let $T = S\varrho$ be as above and consider the boundary value problem

$$\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix} : \mathcal{H}_p^{s,\gamma}(\mathbb{D}, E) \longrightarrow \begin{matrix} \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{D}, E) \\ \oplus \\ \bigoplus\limits_{k=0}^{\mu-1} \mathcal{B}^{s-k-\frac{1}{p},\gamma-k-\frac{1}{2}}(\mathbb{B}, F_k) \end{matrix} \tag{3.4}$$

for $s > \mu - 1 + \frac{1}{p}$. We choose order reductions on the boundary

$$\Lambda_k \in C^{\mu-k}(\mathbb{B}; \gamma - k, \gamma - \mu, \theta; F_k, F_k)$$

where $\theta \in \mathbb{N}$ can be chosen arbitrarily large, and let $\Lambda := \mathrm{diag}(\Lambda_0, \ldots, \Lambda_{\mu-1})$. In particular,

$$\Lambda : \bigoplus_{k=0}^{\mu-1} \mathcal{B}^{s-k-\frac{1}{p},\gamma-k-\frac{1}{2}}(\mathbb{B}, F_k) \longrightarrow \bigoplus_{k=0}^{\mu-1} \mathcal{B}^{s-\mu-\frac{1}{p},\gamma-\mu-\frac{1}{2}}(\mathbb{B}, F_k) = \mathcal{B}^{s-\mu-\frac{1}{p},\gamma-\mu-\frac{1}{2}}(\mathbb{B}, F)$$

for $F := F_0 \oplus \ldots \oplus F_{\mu-1}$ is an isomorphism. Then

$$\mathcal{A}_\Lambda := \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} A \\ T \end{pmatrix} = \begin{pmatrix} A \\ \Lambda T \end{pmatrix} : \mathcal{H}_p^{s,\gamma}(\mathbb{D}, E) \longrightarrow \begin{matrix} \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{D}, E) \\ \oplus \\ \mathcal{B}^{s-\mu-\frac{1}{p},\gamma-\mu-\frac{1}{2}}(\mathbb{B}, F) \end{matrix}$$

is an element of the cone algebra as it has been described in Section 2, i.e. $\mathcal{A}_\Lambda \in C^{\mu,\mu}(\mathbb{D}; \gamma, \gamma - \mu, \theta; E; E, F)$.

Definition 3.1. *The boundary value problem $\mathcal{A}$ is called* elliptic with respect to γ *if the resulting $\mathcal{A}_\Lambda$ is elliptic in the sense of Definition* 2.20.

For later application it is convenient to describe ellipticity of $\mathcal{A}$ intrinsically and not refering to $\mathcal{A}_\Lambda$. To this end we introduce a symbolic structure for $\mathcal{A}$, using the representation of $T = S\varrho$ as in (3.3).

The principal and rescaled principal symbol of $\mathcal{A}$ are

$$\sigma_\psi^\mu(\mathcal{A}) := \sigma_\psi^\mu(A), \qquad \widetilde{\sigma}_\psi^\mu(\mathcal{A}) := \widetilde{\sigma}_\psi^\mu(A).$$

The boundary symbol

$$\sigma_\partial^\mu(\mathcal{A}) : \pi_{\mathbb{B}}^* E_{\mathbb{B}} \otimes H_p^s(\mathbb{R}_+) \longrightarrow \pi_{\mathbb{B}}^* \begin{pmatrix} E_{\mathbb{B}} \otimes H_p^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ F \end{pmatrix}$$

is given by

$$\sigma_\partial^\mu(\mathcal{A})(y,\eta) = \begin{pmatrix} \sigma_\psi^\mu(A)(y, 0, \eta, D_s) \\ \left(\sigma_\psi^{j-k}(\underline{S}_{jk}(y,\eta)\right)_{0\le j,k<\mu} \varrho \end{pmatrix}.$$

Here, $\pi_{\mathbb{B}} : T^*\mathrm{int}\,\mathbb{B} \setminus 0 \to \mathrm{int}\,\mathbb{B}$ is the canonical projection, (y, s) refers to a splitting of coordinates in a collar neighborhood of $\mathbb{B}$, and $\varrho = (\gamma_0, \ldots, \gamma_{\mu-1})_s$ acts on the half-axis $\mathbb{R}_+$. Similarly, with the projection $\pi_{\partial X} : T^*\partial X \setminus 0 \to \partial X$, we define

$$\widetilde{\sigma}^{\mu}_{\partial}(\mathcal{A}) : \pi^*_{\partial X} E_0' \otimes H^s_p(\mathbb{R}_+) \longrightarrow \pi^*_{\partial X} \begin{pmatrix} E_0' \otimes H^{s-\mu}_p(\mathbb{R}_+) \\ \oplus \\ F_0 \end{pmatrix}$$

by

$$\widetilde{\sigma}^{\mu}_{\partial}(\mathcal{A})(x', \xi', \tau) = \begin{pmatrix} \widetilde{\sigma}^{\mu}_{\psi}(A)(x', 0, \xi', \tau, D_{x_n}) \\ \left(\widetilde{\sigma}^{j-k}_{\psi}(\underline{S}_{jk}(x', \xi', \tau)\right)_{0 \le j,k < \mu} \varrho \end{pmatrix}.$$

Finally, the conormal symbol of $\mathcal{A}$,

$$\sigma^{\mu}_M(\mathcal{A})(z) : H^s_p(X, E_0) \longrightarrow \begin{matrix} H^{s-\mu}_p(X, E_0) \\ \oplus \\ \bigoplus_{k=0}^{\mu-1} B^{s-k-\frac{1}{p}}_p(\partial X, F_{k0}) \end{matrix}$$

for $s > \mu - 1 + \frac{1}{p}$ is defined by

$$\sigma^{\mu}_M(\mathcal{A})(z) = \begin{pmatrix} \sigma^{\mu}_M(A)(z) \\ \left(T^k \sigma^{j-k}_M(\underline{S}_{jk})(z)\right)_{0 \le j,k < \mu} \varrho_{x_n} \end{pmatrix}.$$

Here, T^k acts on functions as operator of translation by k, i.e. $(T^k f)(z) = f(z + k)$. Note that this shift appears here, since we commuted in (3.1) the factor t^{-k} to the right.
A straightforward calculation then shows that

$$\sigma^{\mu}(\mathcal{A}_\Lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma(\Lambda) \end{pmatrix} \sigma^{\mu}(\mathcal{A}),$$

where σ is any of the symbols σ_ψ, σ_∂, $\widetilde{\sigma}_\psi$, $\widetilde{\sigma}_\partial$, or σ_M and

$$\sigma(\Lambda) := \mathrm{diag}(\sigma^{\mu}(\Lambda_0), \sigma^{\mu-1}(\Lambda_1), \ldots, \sigma^1(\Lambda_{\mu-1}))$$

in all of the first four cases, while for $\sigma = \sigma_M$ we have

$$\sigma_M(\Lambda) := \mathrm{diag}(T^0 \sigma^{\mu}_M(\Lambda_0), T^1 \sigma^{\mu-1}_M(\Lambda_1), \ldots, T^{\mu-1} \sigma^1_M(\Lambda_{\mu-1})).$$

From these relations the following statement is immediately clear.

Proposition 3.2. *$\mathcal{A}$ of (3.4) is elliptic with respect to γ if and only if*

i) *Both $\sigma^{\mu}_\psi(\mathcal{A})$ and $\widetilde{\sigma}^{\mu}_\psi(\mathcal{A})$ are invertible,*
ii) *both $\sigma^{\mu}_\partial(\mathcal{A})$ and $\widetilde{\sigma}^{\mu}_\partial(\mathcal{A})$ are invertible,*
iii) *$\sigma^{\mu}_M(\mathcal{A})(z)$ is invertible for each z with $\mathrm{Re}\, z = \frac{n+1}{2} - \gamma$.*

3.3. A parametrix construction. Let $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ be elliptic with respect to γ. Then there exists a parametrix $\mathcal{B}_\Lambda$ to $\mathcal{A}_\Lambda$ in the sense of Theorem 2.21 for any arbitrarily large prescribed $k \in \mathbb{N}$. We call

$$\begin{pmatrix} R & K \end{pmatrix} := \mathcal{B}_\Lambda \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}$$

a parametrix of the original problem $\mathcal{A}$. We then have that

$$R \in C^{-\mu,0}(\mathbb{D}; \gamma - \mu, \gamma, k; E; E) \tag{3.5}$$

and, modulo Green remainders,

$$RA + KT \equiv 1, \qquad \begin{pmatrix} AR & AK \\ TR & TK \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This construction can be improved. For notational convenience let us introduce the following abbreviation: Whenever $\mathcal{F}$ is a function or distribution space on $\mathbb{D}$ on which the boundary operator T acts continuously, we set

$$\mathcal{F}_T = \{u \in \mathcal{F} \mid Tu = 0\}.$$

This is then a closed subspace of $\mathcal{F}$.

Proposition 3.3. *Let* $\begin{pmatrix} A \\ T \end{pmatrix}$ *be elliptic with respect to the weight* γ. *Then there exists a parametrix* $\begin{pmatrix} R_0 & K_0 \end{pmatrix}$ *such that*

a) $TR_0 = 0$, *i.e.* R_0 *maps* $\mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{D}, E)$, $s > -1 + \frac{1}{p}$, *into the kernel of* T *in* $\mathcal{H}_p^{s,\gamma}(\mathbb{D}, E)$. *We denote this space by* $\mathcal{H}_p^{s,\gamma}(\mathbb{D}, E)_T$.
b) $AR_0 - 1 \in C_G^{\mu}(\mathbb{D}; \gamma - \mu, \gamma - \mu, k; E; E)$ *is a finite rank operator on* $\mathcal{H}_p^{s,\gamma}(\mathbb{D}, E)$ *with image in* $\mathcal{C}_P^{\infty,\gamma-\mu}(\mathbb{D}, E)$ *for a suitable asymptotic type* P.
c) *On* $\mathcal{H}_p^{s,\gamma}(\mathbb{D}, E)_T$, *the operator* $R_0 A - 1$ *has finite rank; it coincides with an element in* $C_G^0(\mathbb{D}; \gamma, \gamma, k; E; E)$ *and has its image in* $\mathcal{C}_{P'}^{\infty,\gamma}(\mathbb{D}, E)$ *for a suitable asymptotic type* P'.

Proof. Our construction is based on an argument by Grubb [**5**], Proposition 1.4.2. We first reduce order and weight to zero: Let Λ be the operator constructed in Section 3.2, and denote by $\Lambda_-^{-\mu}$ an invertible cone differential operator of order $-\mu$, cf. Theorem 2.10 in [**8**]. For $s > -1 + 1/p$ we consider the operator

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A \\ \Lambda T \end{pmatrix} \Lambda_-^{-\mu} : \mathcal{H}_p^{s,\gamma-\mu}(\mathbb{D}, E) \longrightarrow \begin{matrix} \mathcal{H}_p^{s,\gamma-\mu}(\mathbb{D}, E) \\ \oplus \\ \mathcal{B}_p^{s-1/p,\gamma-\mu-1/2}(\mathbb{B}, F) \end{matrix}. \tag{3.6}$$

By conjugating with $t^{\gamma-\mu}$ we may also assume that $\gamma = \mu$.
As a composition of an elliptic operator with an invertible operator, $\widetilde{\mathcal{A}} \in C^{0,0}(\mathbb{D}, 0, 0, k; E; E, F)$ is elliptic of order and type zero. Hence there exists a parametrix $\mathcal{C}$ of order and type zero such that

$$\widetilde{\mathcal{A}}\mathcal{C} = 1 + \mathcal{S} \quad \text{and} \quad \mathcal{C}\widetilde{\mathcal{A}} = 1 + \mathcal{S}' \tag{3.7}$$

with Green operators $\mathcal{S} \in C_G^0(\mathbb{D}; 0, 0, k; E, F; E, F)$ and $\mathcal{S}' \in C_G^0(\mathbb{D}; 0, 0, k; E; E)$, respectively.
We denote by $\widetilde{\mathcal{A}}^*$ the formal adjoint of $\widetilde{\mathcal{A}}$ with respect to the scalar products in $\mathcal{H}_2^{0,0}(\mathbb{D}, E)$ and $\mathcal{H}_2^{0,0}(\mathbb{D}, E) \oplus \mathcal{B}_2^{-1/2,-1/2}(\mathbb{B}, F)$, respectively. $\widetilde{\mathcal{A}}^*$ also is elliptic of order and type zero; for $1 < q < \infty$ and $s' > -1 + 1/q$ it extends to a Fredholm operator

$$\widetilde{\mathcal{A}}^* : \begin{matrix} \mathcal{H}_q^{s',0}(\mathbb{D}, E) \\ \oplus \\ \mathcal{B}_q^{s'-1/q,-1/2}(\mathbb{B}, F) \end{matrix} \longrightarrow \mathcal{H}_q^{s',0}(\mathbb{D}, E).$$

By elliptic regularity, the kernel is finite dimensional and independent of q and s'; it consists of functions in $\mathcal{C}_P^{\infty,0}(\mathbb{D}, E) \oplus \mathcal{C}_Q^{\infty,-1/2}(\mathbb{B}, F)$ for a suitable asymptotic type (P, Q). We choose a basis $\{v_1, \ldots, v_m\}$ which is orthonormal with respect to the scalar product in $\mathcal{H}_2^{0,0}(\mathbb{D}, E) \oplus \mathcal{B}_2^{-1/2,-1/2}(\mathbb{B}, F)$.
Let us show that $\ker \widetilde{\mathcal{A}}^* = \operatorname{span}\{v_1, \ldots, v_m\}$ is a complement of the image of $\widetilde{\mathcal{A}}$ in $\mathcal{H}_p^{s,0}(\mathbb{D}, E) \oplus \mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B}, F)$. In fact, for $u \in \mathcal{H}_p^{s,0}(\mathbb{D}, E)$,

$$\langle \widetilde{\mathcal{A}}u, v_j \rangle_{\mathcal{H}_p^{s,0} \oplus \mathcal{B}_p^{s-1/p,-1/2}, \mathring{\mathcal{H}}_{p'}^{-s,0} \oplus \mathcal{B}_{p'}^{-s-1/p',-1/2}} = \langle u, \widetilde{\mathcal{A}}^* v_j \rangle_{\mathcal{H}_p^{s,0}, \mathring{\mathcal{H}}_{p'}^{-s,0}} = 0, \qquad j = 1, \ldots, m. \tag{3.8}$$

For v in $\ker \widetilde{\mathcal{A}}^* \cap \operatorname{im} \widetilde{\mathcal{A}}$, (3.8) implies that

$$0 = \langle v, v \rangle_{\mathcal{H}_p^{s,0} \oplus \mathcal{B}_p^{s-1/p,-1/2}, \mathring{\mathcal{H}}_{p'}^{-s,0} \oplus \mathcal{B}_{p'}^{-s-1/p',-1/2}} = \|v\|^2_{\mathcal{H}_2^{0,0} \oplus \mathcal{B}_2^{-1/2,-1/2}},$$

so that $v=0$. On the other hand, for an element v in $\mathring{\mathcal{H}}_{p'}^{-s,0}(\mathbb{D},E)\oplus\mathcal{B}_{p'}^{-s-1/p',-1/2}(\mathbb{B},F)$ we have

$$\langle v,\widetilde{\mathcal{A}}u\rangle_{\mathring{\mathcal{H}}_{p'}^{-s,0}\oplus\mathcal{B}_{p'}^{-s-1/p',-1/2},\mathcal{H}_p^{s,0}\oplus\mathcal{B}_p^{s-1/p,-1/2}}=0\qquad u\in\mathcal{H}_p^{s,0}(\mathbb{D},E)$$

if and only if v is in the kernel of $\widetilde{\mathcal{A}}^*$ on $\mathring{\mathcal{H}}_{p'}^{-s,0}(\mathbb{D},E)\oplus\mathcal{B}_{p'}^{-s-1/p',-1/2}$. Since the image of $\widetilde{\mathcal{A}}$ is closed, the quotient $\mathcal{H}_p^{s,0}(\mathbb{D},E)\oplus\mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B},F)/\operatorname{im}\widetilde{\mathcal{A}}$ is isomorphic to the space of all functionals in the dual of $\mathcal{H}_p^{s,0}(\mathbb{D},E)\oplus\mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B},F)$ which vanish on $\operatorname{im}\widetilde{\mathcal{A}}$, thus to $\ker\widetilde{\mathcal{A}}^*$. Hence $\ker\widetilde{\mathcal{A}}^*$ is a complement to $\operatorname{im}\widetilde{\mathcal{A}}$, for it has the right dimension.

As a consequence, we can write the projection $\pi_{\ker\widetilde{\mathcal{A}}^*}$ onto $\ker\widetilde{\mathcal{A}}^*$ along the image of $\widetilde{\mathcal{A}}$,

$$\pi_{\ker\widetilde{\mathcal{A}}^*}:\begin{matrix}\mathcal{H}_p^{s,0}(\mathbb{D},E)\\ \oplus\\ \mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B},F)\end{matrix}\longrightarrow\ker\widetilde{\mathcal{A}}^*\subset\begin{matrix}\mathcal{H}_p^{s,0}(\mathbb{D},E)\\ \oplus\\ \mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B},F)\end{matrix},$$

in the form

$$\pi_{\ker\widetilde{\mathcal{A}}^*}(v)=\sum_{j=1}^m\langle v,v_j\rangle_{\mathcal{H}_p^{s,0}\oplus\mathcal{B}_p^{s-1/p,-1/2},\mathcal{H}_{p'}^{-s,0}\oplus\mathcal{B}_{p'}^{-s-1/p',-1/2}}\ v_j.\tag{3.9}$$

This shows that it is an element of $C_G^0(\mathbb{D};0,0,k;E,F;E,F)$ with integral kernel $\sum_{j=1}^m v_j\otimes\overline{v}_j$, cf. Section 2.4.

In an analogous way we can choose a basis $\{u_1,\ldots,u_k\}$ of the kernel of $\widetilde{\mathcal{A}}$ in $\mathcal{H}_p^{s,0}(\mathbb{D},E)$. It is a subset of $\mathcal{C}_{P'}^{\infty,0}(\mathbb{D},E)$ for some asymptotic type P', thus independent of s and p; we choose it orthonormal with respect to the scalar product of $\mathcal{H}_2^{0,0}(\mathbb{D},E)$. The projection $\pi_{\ker\widetilde{\mathcal{A}}}$ onto $\ker\widetilde{\mathcal{A}}$ along the range of $\widetilde{\mathcal{A}}^*:\mathcal{H}_p^{s,0}(\mathbb{D},E)\oplus\ \mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B},F)\longrightarrow\mathcal{H}_p^{s,0}(\mathbb{D},E)$ is given by $\pi_{\ker\widetilde{\mathcal{A}}}(u)=\sum_{j=1}^k\langle u,u_j\rangle_{\mathcal{H}_p^{s,0},\mathring{\mathcal{H}}_{p'}^{-s,0}}u_j$ and therefore an element of $C_G^0(\mathbb{D};0,0,k;E;E)$.

The kernel of $\pi_{\ker\widetilde{\mathcal{A}}}$ is the range of $\widetilde{\mathcal{A}}^*$ and thus a complement of the kernel of $\widetilde{\mathcal{A}}$. Let $\mathcal{C}'$ denote the operator which acts like an inverse of

$$\widetilde{\mathcal{A}}:\ker\pi_{\ker\widetilde{\mathcal{A}}}\longrightarrow\operatorname{im}\widetilde{\mathcal{A}}=\ker\pi_{\ker\widetilde{\mathcal{A}}^*}$$

and is zero on $\ker\widetilde{\mathcal{A}}^*$. In fact, the operator $\mathcal{C}'$ is defined for all $1<p<\infty$ and $s>1/p-1$; its action is independent of s and p; since this is the case for $\widetilde{\mathcal{A}}$. We have

$$\mathcal{C}'\widetilde{\mathcal{A}}=1-\pi_{\ker\widetilde{\mathcal{A}}}\quad\text{on }\mathcal{H}_p^{s,0}(\mathbb{D},E)\tag{3.10}$$

$$\widetilde{\mathcal{A}}\mathcal{C}'=1-\pi_{\ker\widetilde{\mathcal{A}}^*}\quad\text{on }\mathcal{H}_p^{s,0}(\mathbb{D},E)\oplus\mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B},F).\tag{3.11}$$

For the difference of $\mathcal{C}'$ and the above parametrix $\mathcal{C}$ we have by (3.7):

$$\mathcal{C}'-\mathcal{C}=(\mathcal{C}\widetilde{\mathcal{A}}-\mathcal{S}')(\mathcal{C}'-\mathcal{C})=\mathcal{C}(1-\pi_{\ker\widetilde{\mathcal{A}}^*})-\mathcal{C}(1+\mathcal{S})-\mathcal{S}'(\mathcal{C}'-\mathcal{C})=-\mathcal{C}(\pi_{\ker\widetilde{\mathcal{A}}^*}+\mathcal{S})-\mathcal{S}'(\mathcal{C}'-\mathcal{C}).$$

This operator maps $\mathcal{H}_p^{s,0}(\mathbb{D},E)\oplus\mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B},F)$ to $\mathcal{C}_{P_1}^{\infty,0}(\mathbb{D},E)\oplus\mathcal{C}_{Q_1}^{\infty,-1/2}(\mathbb{B},F)$ with suitable asymptotic types. For the adjoint we have

$$\begin{aligned}(\mathcal{C}'-\mathcal{C})^*&=(\mathcal{C}^*\widetilde{\mathcal{A}}^*-\mathcal{S}^*)(\mathcal{C}'^*-\mathcal{C}^*)=\mathcal{C}^*(1-\pi^*_{\ker\widetilde{\mathcal{A}}})-\mathcal{C}^*(1+\mathcal{S}'^*)-\mathcal{S}^*(\mathcal{C}'-\mathcal{C})^*\\&=-\mathcal{C}^*(\pi^*_{\ker\widetilde{\mathcal{A}}}+\mathcal{S}'^*)-\mathcal{S}^*(\mathcal{C}'-\mathcal{C})^*.\end{aligned}$$

Again, this operator maps $\mathcal{H}_p^{s,0}(\mathbb{D},E)\oplus\mathcal{B}_p^{s-1/p,-1/2}(\mathbb{B},F)$ to $\mathcal{C}_{P_2}^{\infty,0}(\mathbb{D},E)\oplus\mathcal{C}_{Q_2}^{\infty,-1/2}(\mathbb{B},F)$ with suitable asymptotic types and thus is a Green operator of type zero. We write $\mathcal{C}=\begin{pmatrix}R & K\end{pmatrix}$, $\mathcal{C}'=\begin{pmatrix}R' & K'\end{pmatrix}$, and $\widetilde{\mathcal{A}}=\begin{pmatrix}\widetilde{A}\\ \widetilde{T}\end{pmatrix}$ with $\widetilde{A}=A\Lambda_-^{-\mu}$ and $\widetilde{T}=\Lambda T\Lambda_-^{-\mu}$. Then there are Green operators S_1 and S_2 in $C_G^0(\mathbb{D};0,0,k;E;E)$ such that

$$R'\widetilde{A}=1+S_2\quad\text{on }\mathcal{H}_p^{s,0}(\mathbb{D},E)_{\widetilde{T}},\qquad\widetilde{A}R'=1+S_1\quad\text{on }\mathcal{H}_p^{s,0}(\mathbb{D},E).$$

Indeed, the first identity follows from (3.10), since, on $\mathcal{H}^{s,0}_p(\mathbb{D},E)_{\widetilde{T}}$, we have

$$R'\widetilde{A} = R'\widetilde{A} + K'\widetilde{T} = \mathcal{C}'\widetilde{\mathcal{A}} = 1 - \pi_{\ker \widetilde{\mathcal{A}}},$$

while the second follows from (3.11). Next we write

$$\pi_{\ker \widetilde{\mathcal{A}}^*} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} : \begin{array}{c} \mathcal{H}^{s,0}_p(\mathbb{D},E) \\ \oplus \\ \mathcal{B}^{s-1/p,-1/2}_p(\mathbb{B},F) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{s,0}_p(\mathbb{D},E) \\ \oplus \\ \mathcal{B}^{s-1/p,-1/2}_p(\mathbb{B},F) \end{array}.$$

We know that S_{21} is of finite rank. The adjoint, S_{21}^*, also is of finite rank. By (3.9), its range lies in $\mathcal{C}^{\infty,-1/2}_{Q'}(\mathbb{D},E)$, for a suitable asymptotic type and is independent of p. We denote this range by Z and note that the projection π_Z onto Z along the kernel of S_{21} also is an element of $C^0_G(\mathbb{D};0,0,k;E,E)$. In fact, choosing a basis $\{e_1,\ldots e_r\}$ of Z, which is orthonormal with respect to the $\mathcal{H}^{0,0}_2(\mathbb{D},E)$ scalar product, we can write π_Z as the integral operator with the kernel $\sum_{j=1}^r e_j\otimes\overline{e}_j$ with respect to the pairing between $\mathcal{H}^{s,0}_p(\mathbb{D},E)$ and $\mathring{\mathcal{H}}^{-s,0}_{p'}(\mathbb{D},E)$. We now let

$$R'' = R'(1-\pi_Z).$$

We infer from (3.11) that

$$\widetilde{\mathcal{A}}R'' = \begin{pmatrix} \widetilde{A}R'' \\ \widetilde{T}R'' \end{pmatrix} = \begin{pmatrix} 1-S_{11} \\ -S_{21} \end{pmatrix}(1-\pi_Z) = \begin{pmatrix} (1-S_{11})(1-\pi_Z) \\ 0 \end{pmatrix},$$

so that R'' maps $\mathcal{H}^{0,0}_p(\mathbb{D},E)$ to $\mathcal{H}^{0,0}_p(\mathbb{D},E)_{\widetilde{T}}$, while $\widetilde{A}R''$ differs from the identity by a finite rank operator, say S'', in $C^0_G(\mathbb{D};0,0,k;E;E)$.

We can now conclude the proof: We let $R_0 = \Lambda_-^{-\mu}R'(1-\pi_Z) = \Lambda_-^{-\mu}R''$. Since $TR_0 = \Lambda^{-1}\widetilde{T}R'' = 0$, this operator maps $\mathcal{H}^{s,0}_p(\mathbb{D},E)$ into $\mathcal{H}^{s+\mu,0}_p(\mathbb{D},E)_T$, and we obtain assertion a). Moreover, $AR_0-1 = \widetilde{A}R''-1 = S'' \in C^0_G(\mathbb{D};0,0,k;E;E)$, showing b). Finally, on $\mathcal{H}^{s,0}(\mathbb{D},E)_T$,

$$\begin{aligned} R_0A-1 &= \Lambda_-^{-\mu}R'(1-\pi_Z)\widetilde{A}\Lambda_-^{\mu} - 1 = \Lambda_-^{-\mu}(R'\widetilde{A}+K'\widetilde{T})\Lambda_-^{\mu} - 1 - \Lambda_-^{-\mu}R'\pi_Z\Lambda_-^{\mu} \\ &= \Lambda_-^{-\mu}\mathcal{C}'\widetilde{\mathcal{A}}\Lambda_-^{\mu} - 1 - \Lambda_-^{-\mu}R'\pi_Z\Lambda_-^{\mu} = -\Lambda_-^{-\mu}(\pi_{\ker\widetilde{\mathcal{A}}} + R'\pi_Z)\Lambda_-^{\mu}. \end{aligned}$$

Hence $R_0A-1 \in C_G(\mathbb{D};\mu,\mu,k;E,E)$, which implies c). ☐

3.4. Normal boundary conditions.

LEMMA 3.4. *There exists a map $K : \bigoplus_{k=0}^{\mu-1}\mathcal{C}^{\infty,\infty}(\mathbb{B},F_k) \to \mathcal{C}^{\infty,\infty}(\mathbb{D},E)$ that extends continuously to*

$$K : \bigoplus_{k=0}^{\mu-1}\mathcal{B}^{s-k-\frac{1}{p},\gamma-\frac{1}{2}}_p(\mathbb{B},F_k) \longrightarrow \mathcal{H}^{s,\gamma}_p(\mathbb{D},E)$$

for any $\gamma\in\mathbb{R}$, $1<p<\infty$, and $s>\mu-1+\frac{1}{p}$, such that $\varrho K = 1$ and both $\varphi K\omega$ and $\omega K\varphi$ map into $\mathcal{C}^{\infty,\infty}(\mathbb{D},E)$ whenever the cut-off function ω and $\varphi\in\mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{D})$ have disjoint support.

PROOF. We shall construct K in the form $K = \omega K'\omega_0 + (1-\omega)K''(1-\omega_1)$ with cut-off functions $\omega_1 \prec \omega \prec \omega_0$. For K'' take any right-inverse to ϱ on the smooth manifold with boundary $2\mathbb{D}$, as for example constructed in Lemma 1.6.4 of [**5**]. Then $(1-\omega)K''(1-\omega_1)$ has the required mapping property. It remains to construct K' on $\mathbb{R}_+\times X$. The construction is of local nature, so we may assume all bundles to be trivial and, for notational convenience, one dimensional, and we may work on $\mathbb{R}_+\times\mathbb{R}^n_+$. We then define $K' = \begin{pmatrix} K_0 & K_1 & \ldots & K_{\mu-1}\end{pmatrix}$ by

$$(K_ju)(t,x',x_n) = \int\int_\Gamma e^{ix'\xi'}t^{-z}k_j(x_n,z,\xi')(\mathcal{M}_{s\to z}\mathcal{F}_{y'\to\xi'}u)(z,\xi')\,đz đ\xi'$$

with the symbol

$$\begin{aligned}k_j(x_n,z,\xi') &= \frac{x_n^j}{j!}\iint_0^\infty s^{i\sigma-z}\varphi(s)\zeta(\langle\xi',\sigma\rangle\,x_n))\,\frac{ds}{s}đ\sigma\\ &= \frac{x_n^j}{j!}\int \mathcal{M}(s^{-\delta}\varphi)(i\sigma)\zeta(\langle\xi',\tau+\sigma\rangle\,x_n)\,đ\sigma, \qquad z=\delta+i\tau.\end{aligned}$$

Here, $\varphi\in\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+)$ with $\varphi\equiv 1$ near 1 and $\zeta\in\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R})$ with $\zeta\equiv 1$ near 0 (the symbols k_j arise by applying a so-called kernel cut-off procedure, cf. [**19**], to the symbols used in in the proof of [**5**], Lemma 1.6.4). Moreover, Γ is any vertical line in the complex plane; due to the holomorphy of k_j in z and Cauchy's theorem, its choice does not effect the action of K_j on $\mathcal{C}^{\infty,\infty}(\mathbb{R}_+)\,\widehat{\otimes}_\pi\,\mathcal{S}(\overline{\mathbb{R}}^n_+)$. Since $\frac{d^l}{dx_n^l}k_j(x_n,z,\xi')|_{x_n=0}=\delta_{jl}$ (Kronecker symbol) for all $0\le l\le j$, K' defines a right-inverse to ϱ. Also K' has the desired continuous extensions (where one takes Γ as the line $\mathrm{Re}\,z=\frac{n+1}{2}-\gamma$), since (up to an order reduction of order j) K_j is the typical local model for a Poisson operator in the flat cone algebra. This then also induces the stated smoothing behaviour of $\varphi_0K\varphi_1$. $\square$

Remark 3.5. *The proof of Lemma* 3.4 *yields the following: Given* $0<\varepsilon<1$*, we can construct* K *in such a way that* $K(\omega u_0,\ldots,\omega u_{\mu-1})$ *is supported in* $[0,\varepsilon[\times X$*, whenever* ω *is supported sufficiently close to zero.*

The previous lemma shows that ϱ is surjective. This is also true for general boundary conditions, provided they are normal in a sense we now specify. To this end we identify zero order Fuchs type differential operators on $\mathbb{B}$ with vector bundle morphisms.

Definition 3.6. *Let* $T=S\varrho$ *as in* (3.3)*. We call* T *a* normal *boundary condition if* $\underline{S}_{kk}:E_{\mathbb{B}}\to F_k$ *is surjective for all* $0\le k<\mu$*.*

Proposition 3.7. *Let* $T=S\varrho$ *be a normal boundary condition. For* $0\le k<\mu$ *let* $\underline{S}'_{kk}:E_{\mathbb{B}}\to E_{\mathbb{B}}$ *be projections onto* $Z_k:=\ker\underline{S}_{kk}$ *(these are subbundles of* $E_{\mathbb{B}}$*, since the* $\underline{S}_{kk}$ *are surjective). Then*

$$\begin{pmatrix}\underline{S}\\ \underline{S}'\end{pmatrix}:\bigoplus_{j=0}^{\mu-1}\mathcal{B}_p^{s-j-\frac1p,\gamma-j-\frac12}(\mathbb{B},E_{\mathbb{B}})\longrightarrow\begin{array}{c}\bigoplus_{k=0}^{\mu-1}\mathcal{B}_p^{s-k-\frac1p,\gamma-k-\frac12}(\mathbb{B},F_k)\\ \oplus\\ \bigoplus_{k=0}^{\mu-1}\mathcal{B}_p^{s-k-\frac1p,\gamma-k-\frac12}(\mathbb{B},Z_k)\end{array}\tag{3.12}$$

with $\underline{S}'=\mathrm{diag}(\underline{S}'_{00},\ldots,\underline{S}'_{\mu-1,\mu-1})$ *is an isomorphism for any* $\gamma,s\in\mathbb{R}$*, and* $1<p<\infty$*. If*

$$\begin{pmatrix}\underline{C} & \underline{C}'\end{pmatrix}:=\begin{pmatrix}\underline{S}\\ \underline{S}'\end{pmatrix}^{-1},$$

then both $\underline{C}$ *and* $\underline{C}'$ *have the same structure as* $\underline{S}$*, i.e. they are left lower triangular matrices whose entry in the* j*-th row and* k*-th column are Fuchs type differential operators of order* $j-k$*.*

Proof. The proof is parallel to Lemma 1.6.1 of [**5**]. The map $\widetilde{\underline{S}}=\begin{pmatrix}\mathrm{diag}(\underline{S}_{00},\ldots,\underline{S}_{\mu-1,\mu-1})\\ \underline{S}'\end{pmatrix}$, acting as in (3.12), is invertible by construction of $\underline{S}'$. Then $\underline{U}=\widetilde{\underline{S}}^{-1}\begin{pmatrix}\underline{S}\\ \underline{S}'\end{pmatrix}$ is a left lower triangular matrix of Fuchs type operators acting on sections of $E_{\mathbb{B}}$, with identities in the diagonal. Hence $\underline{U}$ is invertible, since $\underline{U}=1-\underline{V}$ with a nilpotent matrix $\underline{V}$. By Neumann series,

$$\begin{pmatrix}\underline{C} & \underline{C}'\end{pmatrix}=\underline{U}^{-1}\widetilde{\underline{S}}^{-1}=(1+\underline{V}+\ldots+\underline{V}^{\mu-1})\widetilde{\underline{S}}^{-1}$$

is of the described structure. $\square$

As a corollary, any normal boundary condition $T=S\varrho$ is surjective. In fact, if

$$C:=\mathrm{diag}(1,t,\ldots,t^{\mu-1})\,\underline{C}\tag{3.13}$$

with $\underline{C}$ from the previous proposition, and K is as in Lemma 3.4, then

$$TKC = S\varrho KC = \underline{S}\,\mathrm{diag}(1,t^{-1},\ldots,t^{-\mu+1})\,\mathrm{diag}(1,t,\ldots,t^{\mu-1})\,\underline{C} = 1.$$

Thus KC is a right-inverse to T. This together with the mapping properties of K and C immediately yields the following lemma.

LEMMA 3.8. *Let $T = S\varrho$ be normal, K as in Lemma 3.6, and C from (3.13). Then*

$$P := 1 - KCT : \mathcal{H}_p^{s,\gamma}(\mathbb{D},E) \longrightarrow \mathcal{H}_p^{s,\gamma}(\mathbb{D},E) \tag{3.14}$$

continuously for any $\gamma \in \mathbb{R}$, $1 < p < \infty$, and $s > \mu - 1 + \frac{1}{p}$. Moreover, P is a projection, i.e. $P^2 = P$, and $\mathrm{im}\,P = \mathcal{H}_p^{s,\gamma}(\mathbb{D},E)_T$.

For later purpose it is useful to know further properties of P, stated in the next lemma.

LEMMA 3.9. *Let $T = S\varrho$ be a normal boundary condition and P as in (3.14). Let $\omega, \omega_0, \omega_1$ be cut-off functions with $\omega_1 \prec \omega \prec \omega_0$. Then*

i) *$\omega u = 0$ implies $\omega_1 Pu \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)$, and $(1-\omega)u = 0$ implies $(1-\omega_0)Pu \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)$,*
ii) *$\omega Tu = 0$ implies $\omega_1 Pu - \omega_1 u \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)$.*
iii) *$\omega Tu \in \bigoplus_{k=0}^{\mu-1} \mathcal{B}_p^{s-k-\frac{1}{p},\gamma+r-k-\frac{1}{2}}(\mathbb{B},F_k)$ for some $r \geq 0$ implies $\omega_1 Pu - \omega_1 u \in \mathcal{H}_p^{s,\gamma+r}(\mathbb{D},E)$.*

Here, $u \in \mathcal{H}_p^{s,\gamma}(\mathbb{D},E)$ with $\gamma \in \mathbb{R}$, $1 < p < \infty$, and $s > \mu - 1 + \frac{1}{p}$.

PROOF. Let us show the first part of i). We write

$$\omega_1 Pu = \omega_1 u - \omega_1 KCT\omega u + \omega_1 KCT(1-\omega)u.$$

The second summand vanishes by assumption, the third belongs to $\mathcal{C}^{\infty,\infty}(\mathbb{D},E)$ due to the locality of C and T, and due to Lemma 3.4. The other claims are proved analogously. □

COROLLARY 3.10. *Let T be a normal boundary condition. Then $\mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T$ is a dense subset of the closed subspace $\mathcal{H}_p^{s,\gamma}(\mathbb{D},E)_T$ of $\mathcal{H}_p^{s,\gamma}(\mathbb{D},E)$ for any $\gamma \in \mathbb{R}$, $1 < p < \infty$, and $s > \mu - 1 + \frac{1}{p}$.*

PROOF. Let $u \in \mathcal{H}_p^{s,\gamma}(\mathbb{D},E)$ with $Tu = 0$ be given. There exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{D},E)$ converging to u in $\mathcal{H}_p^{s,\gamma}(\mathbb{D},E)$. Now $u_n := Pv_n$ vanishes under T and belongs to $\mathcal{C}^{\infty,\infty}(\mathbb{D},E)$ by Lemma 3.9.i). By Lemma 3.8, $u_n \to Pu = u$ in $\mathcal{H}_p^{s,\gamma}(\mathbb{D},E)$. □

3.5. Green's formula and the adjoint problem. We denote by A^t the formal adjoint of A. In coordinates (t,x',x_n) near $]0,1] \times \partial X$ let us write

$$A = t^{-\mu} \sum_{l=0}^{\mu} \sum_{j=0}^{\mu-l} s_{jl}(t,x',x_n,D')(-t\partial_t)^j D_n^l \tag{3.15}$$

with differential operators $s_{jl}(t,x',x_n,D')$ of order $\mu - j - l$ on ∂X (depending on the parameter (t,x_n)).

THEOREM 3.11. *For all $u \in \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)$ and $v \in \mathcal{H}_{p'}^{\mu,-\gamma}(\mathbb{D},E)$ we have*

$$(Au,v)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)} - (u,A^t v)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)} = (\mathfrak{A}\varrho u, \varrho v)_{\mathcal{B}_2^{0,0}(\mathbb{B},E_{\mathbb{B}}^\mu)}, \tag{3.16}$$

where $E_{\mathbb{B}}^\mu$ is the μ-fold direct sum of $E_{\mathbb{B}}$ and $\mathfrak{A} = (\mathfrak{A}_{jk})_{0\leq j,k<\mu}$ is a left upper triangular matrix with

$$\mathfrak{A}_{jk} = t^{-j}\,\underline{\mathfrak{A}}_{jk}\,t^{-k}$$

and cone differential operators $\underline{\mathfrak{A}}_{jk}$ of order $\mu - 1 - j - k$ acting on sections of $E_{\mathbb{B}}$ and $\underline{\mathfrak{A}}_{jk} = 0$ if $j + k \geq \mu$. Setting $\underline{\mathfrak{A}} = (\underline{\mathfrak{A}}_{jk})_{0\leq j,k<\mu}$ we can also write

$$\mathfrak{A} = \mathrm{diag}(1,t^{-1},\ldots,t^{-\mu+1})\,\underline{\mathfrak{A}}\,\mathrm{diag}(1,t^{-1},\ldots,t^{-\mu+1}).$$

Explicitly, near the singularity,

$$\underline{\mathfrak{A}}_{jk} = t^{-\mu+j+k+1} \sum_{m=0}^{\mu-j-k-1} \sum_{l=j+k+1}^{\mu-m} i((-D_n)^{l-j-k-1} s_{ml})(t,x',0,D')(-t\partial_t - k)^m,$$

where we use the notation from (3.15). *In particular, for* $0 \le j < \mu$,

$$\underline{\mathfrak{A}}_{j,\mu-j-1} = \underline{\mathfrak{A}}_{0,\mu-1} = s_{0\mu}(t,x',0,D').$$

The proof is a standard but lengthy induction based on the relation

$$(D_n u, v)_{\mathcal{H}_2^{0,0}(\mathbb{R}_+\times\overline{\mathbb{R}}^n_+)} - (u, D_n v)_{\mathcal{H}_2^{0,0}(\mathbb{R}_+\times\overline{\mathbb{R}}^n_+)} = i(\gamma_0 u, \gamma_0 v)_{\mathcal{B}_2^{0,-1/2}(\mathbb{R}_+\times\mathbb{R}^{n-1})},$$

for functions $u, v \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}_+ \times \overline{\mathbb{R}}^n_+)$ and the fact that $(f,g)_{\mathcal{B}_2^{0,-1/2}(\mathbb{R}_+\times\mathbb{R}^{n-1})} = (tf,g)_{\mathcal{B}_2^{0,0}(\mathbb{R}_+\times\mathbb{R}^{n-1})}$. To this end note that $\mathcal{H}_2^{0,0}(\mathbb{R}_+ \times \overline{\mathbb{R}}^n_+) = L_2(\mathbb{R}_+ \times \overline{\mathbb{R}}^n_+, t^{n+1}dtdx)$ and $\mathcal{B}_2^{0,0}(\mathbb{R}_+ \times \mathbb{R}^{n-1}) = L_2(\mathbb{R}_+ \times \mathbb{R}^{n-1}, t^n dtdx')$. We shall omit the details.

Definition 3.12. *The adjoint problem of* $\begin{pmatrix} A \\ T \end{pmatrix}$ *is* $\begin{pmatrix} A^t \\ \widetilde{T} \end{pmatrix}$ *with*

$$\widetilde{T} = \widetilde{S}\varrho = \underline{\widetilde{S}}\,\mathrm{diag}(1, t^{-1}, \ldots, t^{-\mu+1})\,\varrho, \qquad \underline{\widetilde{S}} := I^\times(\underline{\mathfrak{A}}\,\underline{C}')^t.$$

Here, $\underline{\mathfrak{A}}$ *is from* (3.16), $\underline{C}'$ *from Proposition* 3.7, *and* $I^\times = (\delta_{j+k,\mu-1})_{0\le j,k<\mu}$ *is the skew unit matrix. The superscript t refers to taking formal adjoints in each entry of the matrix and then passing to the transposed matrix.*

We note that

$$\underline{\widetilde{S}}_{jk} = \sum_{l=0}^{\mu-1} \underline{C}'^t_{l,\mu-j-1}\underline{\mathfrak{A}}^t_{kl} \tag{3.17}$$

is a cone differential operator of order $j-k$ and equal to zero if $k > j$. Letting $Z_k = \ker \underline{S}_{kk}$ for $0 \le k < \mu$ the fact that $\underline{C}'^t_{jk}$ maps to sections of Z_k implies that

$$\widetilde{T} : \mathcal{H}_p^{s,\gamma}(\mathbb{D}, E) \longrightarrow \bigoplus_{k=0}^{\mu-1} \mathcal{B}_p^{s-k-\frac{1}{p},\gamma-k-\frac{1}{2}}(\mathbb{B}, Z_{\mu-1-k}). \tag{3.18}$$

In general, $\widetilde{T}$ is not uniquely determined, since in Proposition 3.7 we may have some freedom in constructing $\underline{C}'$. However, any of the possible choices for $\widetilde{T}$ is equally good.

Proposition 3.13. *If* $\underline{\mathfrak{A}}_{0,\mu-1}$ *is a homeomorphism then* $\widetilde{T}$ *is normal.*

Proof. By (3.17), $\underline{\widetilde{S}}_{kk} = \underline{C}'^t_{\mu-k-1,\mu-k-1}\underline{\mathfrak{A}}^t_{k,\mu-k-1} = \underline{C}'^t_{\mu-k-1,\mu-k-1}\underline{\mathfrak{A}}^t_{0,\mu-1}$, since the only nonzero summand is that for $l = \mu-k-1$. However, by construction the $\underline{C}'_{kk}$ are injective morphisms, hence the $\underline{C}'^t_{kk}$ are surjective. The assumption yields the surjectivity of all $\underline{\widetilde{S}}_{kk}$, i.e. normality of $\widetilde{T}$. □

We shall say that $\mathbb{B}$ *non-characteristic for A* if the assumption on $\underline{\mathfrak{A}}_{0,\mu-1}$ in Proposition 3.13 is satisfied.

4. The minimal extension and its adjoint

Let us now consider A as the unbounded operator

$$A : \mathcal{C}^{\infty,\infty}(\mathbb{D}, E)_T \subset \mathcal{H}_p^{0,\gamma}(\mathbb{D}, E) \longrightarrow \mathcal{H}_p^{0,\gamma}(\mathbb{D}, E) \tag{4.1}$$

with a *normal* boundary condition $T = S\varrho$. This operator is densely defined and closable, since A acts continuously in the scale of Sobolev spaces. We shall assume that $\begin{pmatrix} A \\ T \end{pmatrix}$ satisfies the ellipticity conditions i) and ii) of Proposition 3.2. This we shall refer to as $\mathbb{D}$-*ellipticity.* Note that then the conormal symbol is meromorphically invertible, cf. Remark 2.19, and that $\mathbb{B}$ is non-characteristic for A.

Definition 4.1. *We denote the closure (respectively minimal extension) of A from* (4.1) *by $A_{T,\min}$.*

Proposition 4.2. *We have the inclusions*

$$\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)_T \subset \mathcal{D}(A_{T,\min}) \subset \Big\{u \in \bigcap_{\varepsilon>0} \mathcal{H}_p^{\mu,\gamma+\mu-\varepsilon}(\mathbb{D},E)_T \mid Au \in \mathcal{H}_p^{0,\gamma}(\mathbb{D},E)\Big\}. \tag{4.2}$$

If, additionally, $\begin{pmatrix} A \\ T \end{pmatrix}$ is elliptic with respect to $\gamma+\mu$ then

$$\mathcal{D}(A_{T,\min}) = \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)_T.$$

Proof. Let $u \in \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)_T$. By Corollary 3.10 there exist $u_n \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T$ such that $u_n \to u$ in $\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)$. Then $Au_n \to Au$ in $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$ and the first inclusion of (4.2) is verified. Next assume ellipticity with respect to $\gamma+\mu$ and let $u \in \mathcal{D}(A_{T,\min})$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T$ such that $u_n \to u$ and $Au_n \to Au$ in $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$. Let $\begin{pmatrix} R & K \end{pmatrix}$ be the parametrix as constructed in Proposition 3.3 (with γ replaced by $\gamma+\mu$). Then, in particular,

$$G := RA - 1 : \mathcal{H}_p^{0,\gamma+\mu}(\mathbb{D},E)_T \longrightarrow \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)$$

is a finite rank operator. The sequence $Gu_n = R(Au_n) - u_n$ converges in $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$, hence also in $\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)$ due to $\dim \operatorname{im} G < \infty$. Since $TR = 0$, it even converges in $\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)_T$. Consequently, $u_n = R(Au_n) - Gu_n$ converges in $\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)_T$. Thus we obtain the above stated identity. For the remaining inclusion in (4.2) note that in any case $\begin{pmatrix} A \\ T \end{pmatrix}$ is elliptic with respect to $\gamma+\mu-\varepsilon$ for all sufficiently small $\varepsilon>0$, cf. Remark 2.19. Then we argue as before. $\square$

We shall improve the second inclusion in (4.2) to an equality, cf. Theorem 4.12.

4.1. The adjoint of the closure. Our next aim is to study the adjoint of $A_{T,\min}$. A first simple result is the following lemma.

Lemma 4.3. *$A_{T,\min}^*$ is given by the action of A^t on $\mathcal{D}(A_{T,\min}^*)$.*

Proof. Since $A_{T,\min}$ is the closure of the operator in (4.1) their adjoints coincide and

$$\begin{aligned} \mathcal{D}(A_{T,\min}^*) = \Big\{u \in \mathcal{H}_{p'}^{0,-\gamma}(\mathbb{D},E) \mid \exists\, v \in \mathcal{H}_{p'}^{0,-\gamma}(\mathbb{D},E) \quad \forall\, w \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T : \\ (w,v)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)} = (Aw,u)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)}\Big\} \end{aligned} \tag{4.3}$$

and then $A_{T,\min}^* u = v$. To given $u \in \mathcal{D}(A_{T,\min}^*)$ there exists a sequence $u_n \in \mathcal{C}_{\mathrm{comp}}^\infty(\mathring{\mathbb{D}},E)$ with $u_n \to u$ in $\mathcal{H}_{p'}^{0,-\gamma}(\mathbb{D},E)$. Therefore, with $(\cdot,\cdot)$ denoting the scalar-product of $\mathcal{H}_2^{0,0}(\mathbb{D},E)$,

$$(w,v) = (Aw,u) \longleftarrow (Aw,u_n) = (w,A^t u_n) \longrightarrow (w,A^t u)$$

due to the duality of $\mathcal{H}_p^{s,\gamma}(\mathbb{D},E)$ and $\mathring{\mathcal{H}}_{p'}^{-s,-\gamma}(\mathbb{D},E)$. In particular, this is true for all $w \in \mathcal{C}_{\mathrm{comp}}^\infty(\mathring{\mathbb{D}},E)$ and thus, by density, $v = A^t u$. $\square$

Definition 4.4. *The operator $A_{T,\max}$ is defined by the action of A on the domain*

$$\mathcal{D}(A_{T,\max}) = \{u \in \mathcal{H}_p^{\mu,\gamma}(\mathbb{D},E)_T \mid Au \in \mathcal{H}_p^{0,\gamma}(\mathbb{D},E)\}.$$

Observe that $A_{T,\max}$ is not the maximal extension of A from (4.1) in the usual sense (which would have the domain $\{u \in \mathcal{H}^{0,\gamma}(\mathbb{D},E) \mid Au \in \mathcal{H}^{0,\gamma}(\mathbb{D},E)\}$). Instead, it is the largest extension of A in $\mathcal{H}_p^{\mu,\gamma}(\mathbb{D},E)_T$. We shall call it the maximal extension of A. This is also justified by the following result.

Theorem 4.5. *As unbounded operators in $\mathcal{H}_{p'}^{0,-\gamma}(\mathbb{D},E)$ and $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$, respectively, we have*

$$A_{T,\min}^* = (A^t)_{\widetilde{T},\max}, \qquad A_{T,\min} = (A^t)_{\widetilde{T},\max}^*, \tag{4.4}$$

where $\widetilde{T}$ is the adjoint boundary condition of Definition 3.12.

PROOF. It is enough to prove the first identity, for if it is valid then

$$(A^t)^*_{\widetilde{T},\max} = A^{**}_{T,\min} = \overline{A_{T,\min}} = A_{T,\min}.$$

To do so, we first shall show that

$$\{u \in \mathcal{H}^{\mu,-\gamma}_{p'}(\mathbb{D},E)_{\widetilde{T}} \mid A^t u \in \mathcal{H}^{0,-\gamma}_{p'}(\mathbb{D},E)\} = \mathcal{D}(A^*_{T,\min}) \cap \mathcal{H}^{\mu,-\gamma}_{p'}(\mathbb{D},E). \tag{4.5}$$

Let $u \in \mathcal{H}^{\mu,-\gamma}_{p'}(\mathbb{D},E)$, $w \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T$, and $\begin{pmatrix} \underline{S} \\ \underline{S}' \end{pmatrix}$ as in Proposition 3.7 with corresponding inverse $\begin{pmatrix} \underline{C} & \underline{C}' \end{pmatrix}$. If we set $D = \mathrm{diag}(1, t^{-1}, \ldots, t^{-\mu+1})$, and write $\mathfrak{A} = D\underline{\mathfrak{A}}(\underline{C}\,\underline{S} + \underline{C}'\underline{S}')D$ we obtain

$$(\mathfrak{A}\varrho w, \varrho u) = (Tw, (\underline{\mathfrak{A}}\,\underline{C})^t D\varrho u) + (\underline{S}' D\varrho w, I^\times \widetilde{T}u) = (\underline{S}' D\varrho w, I^\times \widetilde{T}u),$$

where $(\cdot,\cdot)$ denotes the scalar product of $\mathcal{B}^{0,0}_2(\mathbb{B}, E^\mu_{\mathbb{B}})$. Hence, if $\widetilde{T}u = 0$, Greens formula (3.15) yields $(Aw,u)_{0,0} = (w, A^t u)_{0,0}$, with $(\cdot,\cdot)_{0,0}$ denoting the scalar product of $\mathcal{H}^{0,0}_2(\mathbb{D},E)$. By (4.3), this shows that the right-hand side of (4.5) contains the left-hand side. Vice versa, if u is an element from the right-hand side,

$$(Aw,u)_{0,0} = (w, A^*_{T,\min}u)_{0,0} = (w, A^t u)_{0,0}$$

since $u \in \mathcal{D}(A^*_{T,\min})$. On the other hand, Greens formula for $u \in \mathcal{H}^{\mu,-\gamma}_{p'}(\mathbb{D},E)$ shows that

$$(Aw,u)_{0,0} = (w, A^t u)_{0,0} + (\underline{S}' D\varrho w, I^\times \widetilde{T}u).$$

Hence $(\underline{S}' D\varrho w, I^\times \widetilde{T}u) = 0$ for all $w \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T$. But now, by construction, $\underline{S}' D\varrho : \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T \to \bigoplus_{k=0}^{\mu-1} \mathcal{C}^{\infty,\infty}(\mathbb{B}, Z_k)$ with $Z_k = \ker \underline{S}_{kk}$ is surjective. We infer from (3.18) that $\widetilde{T}u$ must vanish.

We have verified (4.5). Next we shall show that $\mathcal{D}(A^*_{T,\min}) \subset \mathcal{H}^{\mu,-\gamma}_{p'}(\mathbb{D},E)$.

First let us assume that, additionally, $\begin{pmatrix} A \\ T \end{pmatrix}$ is elliptic with respect to $\gamma+\mu$. Then, by Proposition 3.3, there exists an $R \in C^{-\mu,0}(\mathbb{D};\gamma,\gamma+\mu,k;E;E)$ such that

$$R : \mathcal{H}^{0,\gamma}_p(\mathbb{D},E) \longrightarrow \mathcal{H}^{\mu,\gamma+\mu}_p(\mathbb{D},E)_T = \mathcal{D}(A_{T,\min})$$

and that $G := AR - 1 \in C^0_G(\mathbb{D};\gamma,\gamma,k;E;E)$ is a Green operator. By general facts on the adjoint of compositions, $R^* A^*_{T,\min} \subset (AR)^* = 1 + G^*$. Then $u = R^* A^*_{T,\min}u - G^*u$ for $u \in \mathcal{D}(A^*_{T,\min})$ yields $u \in \mathcal{H}^{\mu,-\gamma}_{p'}(\mathbb{D},E)$ by Theorem 2.18.

In general, we have ellipticity with respect to $\gamma+\mu+\varepsilon$ for any sufficiently small $\varepsilon > 0$. If we now denote by $\widetilde{A}$ the unbounded operator in $\mathcal{H}^{0,\gamma+\varepsilon}_p(\mathbb{D},E)$ acting as A on the domain $\mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T$, the previous step shows that

$$\{u \in \mathcal{H}^{\mu,-\gamma-\varepsilon}_{p'}(\mathbb{D},E)_{\widetilde{T}} \mid A^t u \in \mathcal{H}^{0,-\gamma-\varepsilon}_{p'}(\mathbb{D},E)\} = \mathcal{D}(\widetilde{A}^*_{T,\min}).$$

By the definition of the domain of the adjoint, it is clear that $\mathcal{D}(A^*_{T,\min}) \subset \mathcal{D}(\widetilde{A}^*_{T,\min})$. Passing to the intersection over all ε, we obtain

$$\mathcal{D}(A^*_{T,\min}) \subset \bigcap_{\varepsilon>0} \{u \in \mathcal{H}^{\mu,-\gamma-\varepsilon}_{p'}(\mathbb{D},E)_{\widetilde{T}} \mid A^t u \in \mathcal{H}^{0,-\gamma-\varepsilon}_{p'}(\mathbb{D},E)\} \cap \mathcal{H}^{0,-\gamma}_{p'}(\mathbb{D},E).$$

Now let u be an element of the right-hand side. As shown in Corollary 6.3, the adjoint problem $\begin{pmatrix} A^t \\ \widetilde{T} \end{pmatrix}$ is elliptic with respect to $-\gamma-\varepsilon$ for small ε. From the inclusion $\mathcal{H}^{0,-\gamma-\varepsilon}_{p'}(\mathbb{D},E) \subset \mathcal{H}^{0,-\gamma-\mu-\varepsilon}_{p',\mu}(\mathbb{D},E)$, cf. Definition 2.5, and elliptic regularity, cf. Theorem 2.23.b), we obtain

$$u \in \mathcal{H}^{\mu,-\gamma-\varepsilon}_{p',P}(\mathbb{D},E) = \bigcap_{\delta>0} \mathcal{H}^{\mu,-\gamma-\varepsilon+\mu-\delta}_{p'}(\mathbb{D},E) \oplus \mathcal{E}_P(\mathbb{R}_+ \times X) \subset \mathcal{H}^{\mu,-\gamma}_{p'}(\mathbb{D},E) + \mathcal{E}_P(\mathbb{R}_+ \times X) \tag{4.6}$$

for some asymptotic type $P = \{(p,m,M)\} \in \mathrm{As}(X; -\gamma-\varepsilon, \mu)$. Since $u \in \mathcal{H}^{0,-\gamma}_{p'}(\mathbb{D},E)$ we must have $\mathrm{Re}\, p < \frac{n+1}{2} + \gamma$ for all $(p,m,M) \in P$, and therefore $u \in \mathcal{H}^{\mu,-\gamma}_{p'}(\mathbb{D},E)$. □

From (4.2) and (4.4) we conclude:

COROLLARY 4.6. $(Au,v)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)} = (u,A^t v)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)}$ *for all* $u \in \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)_T$, $v \in \mathcal{H}_{p'}^{\mu,-\gamma}(\mathbb{D},E)_{\widetilde{T}}$.

Identity (4.6) applied to A shows the following:

COROLLARY 4.7. *There exists an* $\varepsilon > 0$ *such that* $\mathcal{D}(A_{T,\max}) \subset \mathcal{H}_p^{\mu,\gamma+\varepsilon}(\mathbb{D},E)_T$.

THEOREM 4.8. *As unbounded operators in* $\mathcal{H}_{p'}^{0,-\gamma}(\mathbb{D},E)$ *and* $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$*, respectively, we have*

$$A_{T,\max}^* = (A^t)_{\widetilde{T},\min}, \qquad A_{T,\max} = (A^t)_{\widetilde{T},\min}^*,$$

where $\widetilde{T}$ *is the adjoint boundary condition of Definition* 3.12.

PROOF. It is enough to prove the second identity, since the first follows from passing to the adjoints. To do so, note that $\mathcal{D}((A^t)^*_{\widetilde{T},\min})$ is given by

$$\Big\{u \in \mathcal{H}_p^{0,\gamma}(\mathbb{D},E) \mid \exists\, v \in \mathcal{H}_p^{0,\gamma}(\mathbb{D},E) \quad \forall\, w \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_{\widetilde{T}}: \quad (w,v)_{0,0} = (A^t w,u)_{0,0}\Big\},$$

and that Green's formula for the formal adjoint A^t reads as

$$(A^t w,u)_{0,0} - (w,Au)_{0,0} = (-\mathfrak{A}^t \varrho w, \varrho u),$$

with $(\cdot,\cdot)_{0,0}$ and $(\cdot,\cdot)$ denoting the scalar product of $\mathcal{H}_2^{0,0}(\mathbb{D},E)$ and $\mathcal{B}_2^{0,0}(\mathbb{D},E^\mu_{\mathbb{B}})$, respectively.
Now let $u \in \mathcal{H}_p^{\mu,\gamma}(\mathbb{D},E)$ be given and $w \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_{\widetilde{T}}$ be arbitrary.
Using $\underline{\mathfrak{A}} = \underline{\mathfrak{A}}(\underline{C}\,\underline{S} + \underline{C}'\underline{S}')$ and setting $D = \mathrm{diag}(1,t^{-1},\ldots,t^{-\mu+1})$, we can write

$$\mathfrak{A}^t\varrho = D\underline{S}^t\underline{C}^t\underline{\mathfrak{A}}^t D\varrho + D\underline{S}'^t\underline{C}'^t\underline{\mathfrak{A}}^t D\varrho = D\underline{S}^t\underline{C}^t\underline{\mathfrak{A}}^t D\varrho + D\underline{S}'^t I^\times \widetilde{T}.$$

This yields

$$(-\mathfrak{A}^t\varrho w,\varrho u) = -(D\underline{S}^t\underline{C}^t\underline{\mathfrak{A}}^t D\varrho w,\varrho u) = -(\underline{C}^t\underline{\mathfrak{A}}^t D\varrho w, Tu). \tag{4.7}$$

Thus if $Tu = 0$, i.e. $u \in \mathcal{D}(A_{T,\max})$, the right-hand side equals zero and together with the above Green's formula we obtain $u \in \mathcal{D}((A^t)^*_{\widetilde{T},\min})$ and $A_{T,\max} \subset (A^t)^*_{\widetilde{T},\min}$.
By Theorem 4.5 we know that $(A^t)^*_{\widetilde{T},\min} = A_{\widetilde{\widetilde{T}},\max}$, where $\widetilde{\widetilde{T}} = \underline{\widetilde{\widetilde{S}}}D\varrho$ with $\underline{\widetilde{\widetilde{S}}} = -I^\times\underline{\widetilde{C}}'^t\underline{\mathfrak{A}}$ is the adjoint boundary condition to $\widetilde{T}$, obtained via Proposition 3.7. Thus, for $u \in \mathcal{D}((A^t)^*_{\widetilde{T},\min}) \subset \mathcal{H}_p^{\mu,\gamma}(\mathbb{D},E)$, it follows from Green's formula and (4.7) that

$$(D\varrho w, \underline{\mathfrak{A}}\underline{C}Tu) = (\underline{C}^t\underline{\mathfrak{A}}^t D\varrho w, Tu) = 0$$

for all $w \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)_{\widetilde{T}}$. Using $\underline{\widetilde{C}}\underline{\widetilde{S}} + \underline{\widetilde{C}}'\underline{\widetilde{S}}' = 1$, we write $D\varrho w = \underline{\widetilde{C}}\widetilde{T}w + \underline{\widetilde{C}}'\underline{\widetilde{S}}'D\varrho w$. Since $\widetilde{T}w = 0$ and by surjectivity of $\underline{\widetilde{S}}'D\varrho$, we conclude that $\underline{\widetilde{C}}'^t\underline{\mathfrak{A}}\underline{C}Tu = 0$. Multiplying this equation with $\underline{\widetilde{S}}'^t$ and using $\underline{\widetilde{C}}'\underline{\widetilde{S}}' = 1 - \underline{\widetilde{C}}\underline{\widetilde{S}}$ and $\underline{\widetilde{S}}^t = \underline{\mathfrak{A}}\underline{C}'I^\times$ yields

$$0 = \underline{\mathfrak{A}}\begin{pmatrix}\underline{C} & \underline{C}'\end{pmatrix}\begin{pmatrix} Tu \\ -I^\times\underline{\widetilde{C}}^t\underline{\mathfrak{A}}\underline{C}Tu\end{pmatrix}.$$

However, $\begin{pmatrix}\underline{C} & \underline{C}'\end{pmatrix}$ is bijective by construction and $\underline{\mathfrak{A}}$ is, since $\partial\mathbb{B}$ is non-characteristic. In particular, it follows that $Tu = 0$, and therefore $(A^t)^*_{\widetilde{T},\min} \subset A_{T,\max}$. $\square$

4.2. Fredholm property and description of the minimal domain. Let $\begin{pmatrix} r_0 & k_0 \end{pmatrix}$ denote the inverted conormal symbol of $\begin{pmatrix} A \\ T \end{pmatrix}$. If $\begin{pmatrix} R & K \end{pmatrix}$ is the parametrix from Proposition 3.3 (with γ replaced by $\gamma+\mu-\varepsilon$) and P denotes the projection from (3.14), we define

$$B = P\,\omega\,\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^{\mu}\,\omega_0 + P\,(1-\omega)\,R\,(1-\omega_1). \tag{4.8}$$

Here, $\omega, \omega_0, \omega_1$ are cut-off functions with $\omega_1 \prec \omega \prec \omega_0$, and $0<\varepsilon<1$ is chosen so small that r_0 has no pole in the strip $\frac{n+1}{2}-\gamma-\mu < \operatorname{Re} z \le \frac{n+1}{2}-\gamma-\mu+\varepsilon$. This choice guarantees that B as an operator on $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$ does not depend on the choice of ε.

Lemma 4.9. *For given $T = S\varrho$ define*

$$T_0 = \omega\,\mathrm{diag}(1, t^{-1}, \ldots, t^{-\mu+1})\,\mathrm{op}_M(\sigma_M^{\mu}(T)) + (1-\omega)T.$$

Then the cut-off function ω can be chosen in such a way that T_0 is a normal boundary condition, $T - T_0 = tT_1$ for a resulting boundary condition T_1, and $\begin{pmatrix} A \\ T_0 \end{pmatrix}$ is $\mathbb{D}$-elliptic.

Proof. Since the $\underline{S}_{jk}$ are Fuchs type differential operators of order $j-k$, we can write

$$\omega\,\underline{S}_{jk} = \omega\,t^{k-j}\,\mathrm{op}_M(\sigma_M^{j-k}(\underline{S}_{jk})) + \omega\,t\,\underline{S}_{jk}^1$$

with Fuchs type operators $\underline{S}_{jk}^1$ of order $j-k$. Therefore

$$\omega\,\underline{S}_{jk}\,t^{-k} = \omega\,t^{-j}\,\mathrm{op}_M(T^k\sigma_M^{j-k}(\underline{S}_{jk})) + \omega\,t\,\underline{S}_{jk}^1\,t^{-k}.$$

By definition of the conormal symbol of T it follows that $T - T_0 = tS_1\varrho$, where $\underline{S}_1 = (\omega\,\underline{S}_{jk}^1)_{j,k}$. If we write $T_0 = S_0\varrho$ with $\underline{S}_0 = (\underline{S}_{jk}^0)_{j,k}$ then, by construction,

$$\underline{S}_{kk} - \underline{S}_{kk}^0 = \omega\,t\,\underline{S}_{kk}^1, \qquad 0 \le k < \mu.$$

If ω is supported sufficiently close to $t=0$ we see that with $\underline{S}_{kk}$ also $\underline{S}_{kk}^0$ is surjective.
By construction the rescaled symbols of $\begin{pmatrix} A \\ T \end{pmatrix}$ and $\begin{pmatrix} A \\ T_0 \end{pmatrix}$ coincide, and are invertible in case of ellipticity. Then also the (not rescaled) symbols are invertible for small t, say $t \le \varepsilon$ (recall that one defines the rescaled symbols by a limit procedure). Replacing ω by an $\widetilde{\omega} \prec \omega$ with support $[0,\varepsilon]$, we obtain invertibility of the symbols of $\begin{pmatrix} A \\ T_0 \end{pmatrix}$ for $t<\varepsilon$. For $t \ge \varepsilon$ they coincide with the symbols of $\begin{pmatrix} A \\ T \end{pmatrix}$, hence are invertible there. □

Lemma 4.10. *Let T and T_0 be as in the previous Lemma* 4.9 *and let P be any projection associated to T by* (3.14). *Then we can choose a projection P_0 associated with T_0 such that*

$$P - P_0 : \mathcal{H}_p^{s,\gamma}(\mathbb{D},E) \longrightarrow \mathcal{H}_p^{s,\gamma+1}(\mathbb{D},E) \tag{4.9}$$

for any $\gamma \in \mathbb{R}$, $1<p<\infty$ and $s > \mu - 1 + \frac{1}{p}$.

Proof. To prove this statement we need to have a closer look at the construction of P and P_0, which relies on Proposition 3.7. We write $P - P_0 = \widetilde{\omega}(P-P_0) + (1-\widetilde{\omega})(P-P_0)$ for a cut-off function $\widetilde{\omega}$. The second summand trivially has the desired mapping property. We rewrite the first as

$$\widetilde{\omega}(P-P_0) = \widetilde{\omega}(KCT - KC_0T_0) = \widetilde{\omega}K(C(T-T_0) + (C-C_0)T_0) = \widetilde{\omega}\,Kt\widetilde{C}T_1 + \widetilde{\omega}\,KtC_1T_0$$

with $\widetilde{C} = t^{-1}Ct$, T_1 as in Lemma 4.9, and $C_1 = t^{-1}(C_0 - C)$. We conclude from Lemma 3.4 and (2.9) that the first summand has the mapping property in (4.9) and focus on the second. Write $T_0 = S_0\varrho$ and $T_1 = S_1\varrho$ as in the proof of Lemma 4.9. Let $\begin{pmatrix} \underline{C} & \underline{C}' \end{pmatrix} = \begin{pmatrix} \underline{S} \\ \underline{S}' \end{pmatrix}^{-1}$ and

$\begin{pmatrix}\underline{C}_0 & \underline{C}_0'\end{pmatrix} = \begin{pmatrix}\underline{S}_0\\ \underline{S}_0'\end{pmatrix}^{-1}$ be as in Proposition 3.7. Now assume the existence of a bundle morphism $\underline{S}_1' = \mathrm{diag}(\underline{S}_{00}^{1\prime},\ldots,\underline{S}_{\mu-1,\mu-1}^{1\prime})$ such that

$$\begin{pmatrix}\underline{S}\\ \underline{S}'\end{pmatrix} - \begin{pmatrix}\underline{S}_0\\ \underline{S}_0'\end{pmatrix} = t\begin{pmatrix}\underline{S}_1\\ \underline{S}_1'\end{pmatrix}. \tag{4.10}$$

Multiplying this equation from the left with $\begin{pmatrix}\underline{C} & \underline{C}'\end{pmatrix}$ and with $\begin{pmatrix}\underline{C}_0 & \underline{C}_0'\end{pmatrix}$ from the right yields

$$\begin{pmatrix}\underline{C}_0-\underline{C} & \underline{C}_0'-\underline{C}'\end{pmatrix} = \begin{pmatrix}\underline{C}t\underline{S}_1\underline{C}_0 + \underline{C}'t\underline{S}_1'\underline{C}_0 & \underline{C}t\underline{S}_1\underline{C}_0' + \underline{C}'t\underline{S}_1'\underline{C}_0'\end{pmatrix}$$

so that $\underline{C}_1 = t^{-1}(\underline{C}_0 - \underline{C})$ is a left lower triangular matrix of Fuchs type differential operators of order $j-k$ at position (j,k). Multiplying from the left with $\mathrm{diag}(1,\ldots,t^{-\mu-1})$ we obtain C_1 and conclude that $\widetilde{\omega}\,KtC_1T_0$ has the mapping property in (4.9).

Thus it remains to verify (4.10). To this end fix a $0\le k<\mu$ and let $\underline{s}'$ be a projector onto $E' := \ker \underline{S}_{kk}$. By construction of $\underline{S}_0$, the bundle $E_0' := \ker \underline{S}_{0,kk}$ is constant in t near the singularity (i.e. the fibre at (t,x) only depends on x), and E_0' coincides with E' in $t=0$. Thus, if $\underline{s}_{0,\perp}'$ is the orthogonal projection onto E_0', also

$$\underline{s}_0' := \omega\underline{s}'(0,\cdot) + (1-\omega)\underline{s}_{0,\perp}'$$

is a projection onto E_0' (note that convex combinations of projections having the same image again are projections). Obviously, $\underline{s}'-\underline{s}_0' = t\underline{s}_1'$ for a resulting $\underline{s}_1'$. $\square$

Theorem 4.11. *If B is as in (4.8), then $B:\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)\to\mathcal{D}(A_{T,\min})$.*

Proof. Since $P(1-\omega)R(1-\omega_1)$ maps to $\mathcal{H}_p^{\mu,\infty}(\mathbb{D},E)_T$ by Lemma 3.9.i), it remains to analyze the first summand in (4.8), which we now denote by B_1. By construction, B_1 maps into $\bigcap_{\delta>0}\mathcal{H}_p^{\mu,\gamma+\mu-\delta}(\mathbb{D},E)_T$. Next we shall show that B_1 maps into the domain of $A_{T,\max}$, i.e. $ABu\in\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$ whenever $u\in\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$.

Since, near $t=0$, $A - t^{-\mu}\,\mathrm{op}_M(\sigma_M^\mu(A)) = tA_1$ for a Fuchs type differential operator A_1 of order μ, it suffices to show that

$$t^{-\mu}\,\mathrm{op}_M(\sigma_M^\mu(A))\,\widetilde{\omega}\,B_1\,u\in\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$$

for some cut-off function $\widetilde{\omega}$. Since $T=\mathrm{diag}(1,t^{-1},\ldots,t^{-\mu+1})\mathrm{op}_M(\sigma_M^\mu(T))+tT_1$ near $t=0$ for another boundary condition T_1 and since $\sigma_M^\mu(T)r_0=0$, we obtain

$$\widetilde{\omega}\,T\Big(\omega\,\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0u\Big)\in\bigoplus_{k=0}^{\mu-1}\mathcal{B}_p^{\mu-k-\frac1p,\gamma+\mu+1-\varepsilon-k-\frac12}(\mathbb{B},F_k)\qquad\forall\,\varepsilon>0$$

for any $u\in\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$ and any cut-off function $\widetilde{\omega}$ with $\widetilde{\omega}\prec\omega$. Thus, by Lemma 3.9.iii),

$$\widetilde{\omega}\,P\omega\,\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0u\equiv\widetilde{\omega}\,\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0u$$

modulo $\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)$. Therefore, if $\widetilde{\omega},\omega_2$ are cut-off functions supported sufficiently close to zero with $\omega_2\prec\widetilde{\omega}$,

$$\begin{aligned}t^{-\mu}\,\mathrm{op}_M(\sigma_M^\mu(A))\,\widetilde{\omega}\,B_1\,u&\equiv t^{-\mu}\,\mathrm{op}_M(\sigma_M^\mu(A))\,\widetilde{\omega}\,\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0u\\&\equiv\omega_2\omega_0u+(1-\omega_2)t^{-\mu}\,\mathrm{op}_M(\sigma_M^\mu(A))\,\widetilde{\omega}\,\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0\,u\equiv0\end{aligned}$$

modulo $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$. We used that $\sigma_M^\mu(A)r_0=1$. This shows, as claimed, that $\mathrm{im}\,B_1\subset\mathcal{D}(A_{T,\max})$.

Now let $u\in\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$ and $f\in\mathcal{D}((A^t)_{\widetilde{T},\max})$. Since then $f\in\mathcal{H}_{p'}^{\mu,-\gamma+\delta}(\mathbb{D},E)_{\widetilde{T}}$ for some $\delta>0$ by Corollary 4.7, we have

$$(A^tf,B_1u)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)}=(f,AB_1u)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)}=(f,v)_{\mathcal{H}_2^{0,0}(\mathbb{D},E)}$$

with $v:=AB_1u\in\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$. Here, we used Corollary 4.6 (with γ replaced by $\gamma-\delta$). Thus $\mathrm{im}\,B_1\subset\mathcal{D}((A^t)_{\widetilde{T},\max}^*)=\mathcal{D}(A_{T,\min})$ by Theorem 4.5. $\square$

THEOREM 4.12. (4.2) *can be improved to*

$$\begin{aligned}\mathcal{D}(A_{T,\min}) &= \Big\{u \in \bigcap_{\varepsilon>0} \mathcal{H}_p^{\mu,\gamma+\mu-\varepsilon}(\mathbb{D},E)_T \mid Au \in \mathcal{H}_p^{0,\gamma}(\mathbb{D},E)\Big\}\\ &= \Big\{u \in \bigcap_{\varepsilon>0} \mathcal{H}_p^{\mu,\gamma+\mu-\varepsilon}(\mathbb{D},E)_T \mid t^{-\mu}\operatorname{op}_M(\sigma_M^\mu(A))\,\widetilde{\omega}\,u \in \mathcal{H}_p^{0,\gamma}(\mathbb{D},E)\Big\},\end{aligned}$$

where $\widetilde{\omega}$ *is an arbitrary cut-off function.*

PROOF. That the second identity holds is already contained in the previous proof. The independence of the choice of $\widetilde{\omega}$ is clear. Now let u belong to the right-hand side. If $\omega_2 \prec \widetilde{\omega}$, then $\omega_2 T\widetilde{\omega}u = 0$ implies $\omega_2 P\widetilde{\omega}u - \omega_2 u \in \mathcal{C}^{\infty,\infty}(\mathbb{D},E)$ by Lemma 3.9.ii). If P_0 denotes the projection from Lemma 4.10, then $(P-P_0)\widetilde{\omega}u \in \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)$. By Remark 3.5 we may assume that $P_0\widetilde{\omega}u$ is supported arbitrarily near to the tip. Therefore

$$t^{-\mu}\operatorname{op}_M(\sigma_M^\mu(A))\,P_0\,\widetilde{\omega}\,u \equiv t^{-\mu}\operatorname{op}_M(\sigma_M^\mu(A))(\omega_2 u + (1-\omega_2)P\widetilde{\omega}u) \equiv 0$$

modulo $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$ and, if B is as in (4.8), then

$$B\,t^{-\mu}\operatorname{op}_M(\sigma_M^\mu(A))\,P_0\,\widetilde{\omega}\,u = P\,\omega\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0\sigma_M^\mu(A))\,P_0\,\widetilde{\omega}\,u = P\,\widetilde{\omega}\,u + P(P_0-P)\,\widetilde{\omega}\,u.$$

The last equality holds, since $r_0\sigma_M^\mu(A) = 1 - k_0\sigma_M^\mu(T)$ and $\operatorname{op}_M(\sigma_M^\mu(T))P_0\widetilde{\omega} = T_0P_0\widetilde{\omega} = 0$. Since P vanishes under T and by Lemma 3.8, $P(P_0-P)\,\widetilde{\omega}\,u$ belongs to $\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)_T \subset \mathcal{D}(A_{T,\min})$. Then

$$u = Pu = P\,\widetilde{\omega}\,u + P(1-\widetilde{\omega})\,u \in \mathcal{D}(A_{T,\min}),$$

since $P(1-\widetilde{\omega})\,u \in \mathcal{H}_p^{\mu,\infty}(\mathbb{D},E)_T$. □

THEOREM 4.13. *Both minimal and maximal extension are Fredholm operators.*

PROOF. Let B as in (4.8). Since B maps into the domain of $A_{T,\min}$, we have $A_{T,\min/\max}B = AB$ on $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$. Write

$$AB = A\,P\,\omega\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0 + A\,P\,(1-\omega)\,R\,(1-\omega_1). \tag{4.11}$$

Let $\omega_2 \prec \omega_3 \prec \omega_4 \prec \omega$. Then, if $[\cdot,\cdot]$ denotes the commutator,

$$[AP, 1-\omega] = \omega_3[\omega,AP]\omega_4 + (1-\omega_3)[\omega,AP](1-\omega_2) + \omega_3[\omega,AP](1-\omega_4) + (1-\omega_3)[\omega,AP]\omega_2.$$

The first summand vanishes, the second is an operator of order $\mu-1$ located away from the singularity, the third and fourth map into $\mathcal{C}^{\infty,\infty}(\mathbb{D},E)$ by Lemma 3.9.i) and the locality of A. In particular, $[AP,\omega] \in \mathcal{K}$, the compact operators in $\mathcal{H}_p^{0,\gamma}(\mathbb{D},E)$. Using that $PR = R$ and that R inverts A up to a Green remainder, we obtain

$$A\,P\,(1-\omega)\,R\,(1-\omega_1) \equiv 1-\omega \quad \text{modulo } \mathcal{K}.$$

To treat the first summand in (4.11), note that, if ω is chosen to be supported sufficiently close to zero, then each of the operators $P\omega$, $P_0\omega$, ωP, and ωP_0 can be assumed to be located arbitrarily close to the singularity, cf. Remark 3.5. In view of the compactness of the commutator above, we get

$$A\,P\,\omega\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0 \equiv \omega\,A\,P_0\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0 + \omega\,A\,(P-P_0)\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0$$

modulo $\mathcal{K}$. In the first term, P_0 can be dropped, since $\operatorname{op}_M(\sigma_M^\mu(T))\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0) = 0$. Writing $A = t^{-\mu}\operatorname{op}_M(\sigma_M^\mu(A)) + tA_1$ near $t=0$ for a μ-th order Fuchs type operator A_1,

$$AB \equiv 1 + \omega\,t\,A_1\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0 + \omega\,A\,(P-P_0)\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\,t^\mu\,\omega_0 =: 1 + C$$

modulo $\mathcal{K}$. Due to the presence of the t-factor and Lemma 4.10, $\|C\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{D},E))} < \frac{1}{2}$ if ω is located sufficiently close to 0 (since the sup-norm of $t\widetilde{\omega}$ tends to zero if the support of $\widetilde{\omega}$ does), and $1+C$ is invertible. Altogether, we found a B_R such that $A_{T,\min/\max}B_R = 1 + C_R$ for a $C_R \in \mathcal{K}$. This yields that $\operatorname{im} A_{T,\min/\max}$ is finite codimensional, since the image of the Fredholm operator $1 + C_R$ is. Analogously one can construct a right-inverse modulo compact operators for $A^t_{\widetilde{T},\min/\max}$. Passing to the adjoint yields a B_L such that $B_L A_{T,\min/\max} = 1 + C_L$ on $\mathcal{D}(A_{T,\min/\max})$ with a $C_L \in \mathcal{K}$. This shows that $\ker A_{T,\min/\max}$ is finite dimensional, since the kernel of $1 + C_L$ is. $\square$

We conclude this section with an observation that will have some importance in describing the maximal domain.

LEMMA 4.14. *Let T and T_0 be as in Lemma 4.9 and P, P_0 be as in Lemma 4.10. Let $0 < \delta < 1$. Then P_0 induces an isomorphism*

$$\mathcal{D}(A_{T,\max}) \cap \mathcal{H}_p^{\mu,\gamma+\mu-\delta}(\mathbb{D}, E)\Big/\mathcal{D}(A_{T,\min}) \longrightarrow \mathcal{D}(A_{T_0,\max}) \cap \mathcal{H}_p^{\mu,\gamma+\mu-\delta}(\mathbb{D}, E)\Big/\mathcal{D}(A_{T_0,\min}).$$

Its inverse is induced by P.

PROOF. We use the description of minimal domains established in Theorem 4.12. If $u \in \mathcal{D}(A_{T,\min})$, then $P_0 u \in \bigcap_{\varepsilon>0} \mathcal{H}_p^{\mu,\gamma+\mu-\varepsilon}(\mathbb{D}, E)_{T_0}$ by (3.14). Moreover, $AP_0u = Au + A(P_0 - P)u \in \mathcal{H}_p^{0,\gamma}(\mathbb{D}, E)$, since $Pu = u$ and by Lemma 4.10. Thus $P_0 : \mathcal{D}(A_{T,\min}) \to \mathcal{D}(A_{T_0,\min})$ and, analogously $P : \mathcal{D}(A_{T_0,\min}) \to \mathcal{D}(A_{T,\min})$. If $u \in \mathcal{D}(A_{T,\max}) \cap \mathcal{H}_p^{\mu,\gamma+\mu-\delta}(\mathbb{D}, E)$ the same argument shows that $P_0 u \in \mathcal{D}(A_{T_0,\max}) \cap \mathcal{H}_p^{\mu,\gamma+\mu-\delta}(\mathbb{D}, E)$. Similarly for P. Hence the maps $[u] \mapsto [P_0 u]$ and $[v] \mapsto [Pv]$ are well-defined. Moreover,

$$[PP_0u] = [P^2u + P(P_0 - P)u] = [u],$$

since $P^2u = Pu = u$ and $P(P_0 - P)u \in \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D}, E)_T$ by Lemma 4.10. Analogously, $[P_0Pv] = [v]$. $\square$

5. The maximal extension

In this section we shall discuss the maximal extension of the operator A in (4.1). As before we shall assume normality and $\mathbb{D}$-ellipticity. Let T_0 be as in Lemma 4.9 and define

$$A_0 = \omega\, t^{-\mu} \operatorname{op}_M(\sigma_M^\mu(A)) + (1 - \omega)\, A. \tag{5.1}$$

Then $A - A_0 = tA_1$ with a Fuchs type differential operator A_1 of order μ. Arguing as in the proof of Lemma 4.9, we see that the function ω can be chosen in such a way that both $\begin{pmatrix} A \\ T_0 \end{pmatrix}$ and $\begin{pmatrix} A_0 \\ T_0 \end{pmatrix}$ are $\mathbb{D}$-elliptic. Observe that both A_0 and T_0 have t-independent coefficients near $t = 0$. We choose projections P and P_0 for T and T_0 as in Lemma 4.10.

5.1. A class of smoothing operators.

Let us recall basic properties of operators of the form

$$G = \omega\left(\operatorname{op}_M^{\gamma_1-\frac{n}{2}}(r) - \operatorname{op}_M^{\gamma_2-\frac{n}{2}}(r)\right) : \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X, E_0) \longrightarrow \mathcal{C}^\infty(\operatorname{int}\mathbb{D}, E),$$

where $\gamma_1 < \gamma_2$ and $r \in M_O^{\mu,0}(X; E_0; E_0) + M_P^{-\infty,0}(X; E_0; E_0)$ is a meromorphic Mellin symbol of type 0 with asymptotic type P as described in Section 2. An analysis of such operators for closed X can be found in [**12**]; the results in the present case with boundary are completely analogous. Let

$$\sum_{k=0}^{n_p} R_{pk}(z - p)^{-(k+1)}, \qquad R_{pk} \in N_p \subset \mathcal{B}^{-\infty,0}(X; E_0; E_0),$$

denote the principal part of r around p for $(p, n_p, N_p) \in P$. The R_{pk} have finite rank by definition. Thus, also each of the mappings

$$C_p : \bigoplus_{j=0}^{n_p} \mathcal{C}^\infty(X, E_0) \longrightarrow \bigoplus_{k=0}^{n_p} \mathcal{C}^\infty(X, E_0)$$

defined by the left-upper triangular block-matrix $(c^p_{jk})_{0\le j,k\le n_p}$ with $c^p_{jk} = R_{p,j+k}$ if $j + k \le n_p$ and $c^p_{jk} = 0$ otherwise, is of finite rank. Let us set $M(r, p) := \operatorname{rank} C_p$ and $M(r, p) = 0$ if r is holomorphic in p.

Proposition 5.1. *If G is as above, then G is of finite rank and, for each $u \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X, E_0)$,*

$$(Gu)(t,x) = \omega(t) \sum_{\substack{(p,n_p,N_p)\in P\\ -\gamma_2 < \mathrm{Re}\, p - \frac{n+1}{2} < -\gamma_1}} \sum_{l=0}^{n_p} \zeta_{pl}(u)(x)\, t^{-p} (\log t)^l$$

with the linear maps $\zeta_{pl} : \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X, E_0) \to \mathrm{im}\, R_{pl} + \ldots + \mathrm{im}\, R_{pn_p} \subset \mathcal{C}^\infty(X, E_0)$ given by

$$\zeta_{pl}(u)(x) = \sum_{k=l}^{n_p} \frac{(-1)^l}{l!(k-l)!} R_{pk} \frac{\partial^{k-l}}{\partial z^{k-l}} (\mathcal{M}u)(p,x),$$

where $\mathcal{M} = \mathcal{M}_{t\to z}$ denotes the Mellin transform. The rank is

$$\mathrm{rank}\, G = \sum_{-\gamma_2 < \mathrm{Re}\, z - \frac{n+1}{2} < -\gamma_1} M(r,z).$$

Note that G only depends on the meromorphic structure of r; a change by a holomorphic symbol leaves G invariant. Changing the domain of G to be $\mathcal{H}^{0,\gamma_1}_p(\mathbb{R}_+ \times X, E_0) \cap \mathcal{H}^{0,\gamma_2}_p(\mathbb{R}_+ \times X, E_0)$ does not alter the image.

5.2. The maximal domain in case of t-independent coefficients. Let $\begin{pmatrix} r_0 & k_0 \end{pmatrix}$ be the inverted conormal symbol of $\begin{pmatrix} A \\ T \end{pmatrix}$ and thus of $\begin{pmatrix} A_0 \\ T_0 \end{pmatrix}$.

Proposition 5.2. *Let $\varepsilon > 0$ be so small that r_0 has no pole in the strips $-\gamma - \mu < \mathrm{Re}\, z - \frac{n+1}{2} \le -\gamma - \mu + \varepsilon$ and $-\gamma - \varepsilon \le \mathrm{Re}\, z - \frac{n+1}{2} < -\gamma$. Define*

$$G_0 = \omega \left\{ \mathrm{op}_M^{\gamma+\varepsilon-\frac{n}{2}}(r_0) - \mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0) \right\}$$

for an arbitrary fixed cut-off function ω. Then

a) $\mathrm{Im}\, P_0\, G_0 \cap \mathcal{H}^{\mu,\gamma+\mu-\delta}_p(\mathbb{D}, E) = \{0\}$, *if $\delta > 0$ is sufficiently small. Moreover,* $\mathrm{im}\, P_0\, G_0 \subset \mathcal{D}((A_0)_{T_0,\max})$ *and*

$$\mathrm{rank}\, P_0\, G_0 = \mathrm{rank}\, G_0 = \sum_{-\gamma-\mu < \mathrm{Re}\, z - \frac{n+1}{2} < -\gamma} M(r_0, z).$$

b) $\mathcal{D}((A_0)_{T_0,\max}) = \mathcal{D}((A_0)_{T_0,\min}) \oplus \mathrm{im}\, P_0\, G_0$.

Proof. a) Since $\sigma^\mu_M(T_0) r_0 = 0$ and T_0 is a differential operator with t-independent coefficients, $T_0 G_0 u = 0$ near $t = 0$ for $u \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X, E_0)$. By Lemma 3.9.i), $P_0 G_0 u \equiv G_0 u$ modulo $\mathcal{C}^{\infty,\infty}(\mathbb{D}, E)$. Hence also $A_0 P_0 G_0 u \in \mathcal{C}^{\infty,\infty}(\mathbb{D}, E)$, since $A_0 G_0 u = 0$ near $t = 0$. Moreover, since the image of G_0 has trivial intersection with $\mathcal{H}^{\mu,\gamma+\mu-\delta}_p(\mathbb{D}, E)$ for small $\delta > 0$ by Proposition 5.1, we conclude that also $\mathrm{im} P_0 G_0$ has trivial intersection and that the two ranks coincide.
b) The directness of the sum and the inclusion '$\supset$' follow from a). So let $u \in \mathcal{D}((A_0)_{T_0,\max})$ and let ω_0 be a cut-off function with $\omega_0 \prec \omega$. Then $P_0(1-\omega_0)u \in \mathcal{H}^{\mu,\infty}_p(\mathbb{D}, E)$ by Corollary 4.7 and Lemma 3.9.i). Hence $P_0 \omega_0 u = u - P_0(1-\omega_0)u \in \mathcal{D}((A_0)_{T_0,\max})$, and thus $A_0 P_0 \omega_0 u \in \mathcal{H}^{0,\gamma}_p(\mathbb{D}, E)$. As A_0 has t-independent coefficients and $P_0 \omega_0 u$ is supported close to the singularity, we can write

$$\begin{aligned} P_0\, \omega_0\, u &= P_0\, \omega\, \mathrm{op}_M^{\gamma+\varepsilon-\frac{n}{2}}(r_0)\, t^\mu A_0\, P_0\, \omega_0\, u \\ &= P_0\, \omega\, \mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)(t^\mu A_0\, P_0\, \omega_0\, u) + P_0\, G_0(t^\mu A_0\, P_0\, \omega_0\, u), \end{aligned}$$

which is in $\mathcal{D}((A_0)_{T_0,\min}) \oplus \mathrm{im}\, P_0\, G_0$ by Theorem 4.11. $\square$

5.3. On the index of the minimal and the maximal extension.

LEMMA 5.3. *For sufficiently small $\varepsilon > 0$,*

$$\mathcal{D}(A_{T,\max}) \cap \mathcal{H}_p^{\mu,\gamma+\mu-\varepsilon}(\mathbb{D},E) = \mathcal{D}(A_{T,\min}).$$

PROOF. The identity $A - A_0 = tA_1$ implies that

$$\mathcal{D}(A_{T_0,\max}) \cap \mathcal{H}_p^{\mu,\gamma+\mu-\varepsilon}(\mathbb{D},E) = \mathcal{D}((A_0)_{T_0,\max}) \cap \mathcal{H}_p^{\mu,\gamma+\mu-\varepsilon}(\mathbb{D},E), \quad 0<\varepsilon<1.$$

For small ε, Proposition 5.2 implies that the right hand side equals $\mathcal{D}((A_0)_{T_0,\min})$ which in turn equals $\mathcal{D}(A_{T_0,\min})$ by Theorem 4.12. Now the assertion follows from Lemma 4.14. $\square$

The next proposition says that for considerations of the index one may assume that $\begin{pmatrix} A \\ T \end{pmatrix}$ is elliptic with respect to $\gamma+\mu$, and thus the domain of $A_{T,\min}$ coincides with the space $\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{D},E)_T$. This observation was made in [**3**] for operators on conic manifolds without boundary.

PROPOSITION 5.4. *Let $\delta > 0$ and let $\widetilde{A}$ denote the unbounded operator*

$$\mathcal{C}^{\infty,\infty}(\mathbb{D},E)_T \subset \mathcal{H}_p^{0,\gamma-\delta}(\mathbb{D},E) \longrightarrow \mathcal{H}_p^{0,\gamma-\delta}(\mathbb{D},E),$$

given by the action of A. Both $A_{T,\min}$ and $\widetilde{A}_{T,\min}$ are Fredholm operators according to Theorem 4.13*. Moreover, their indices coincide, provided δ is small enough.*

PROOF. By Theorem 4.12, $\ker A_{T,\min} \subset \ker \widetilde{A}_{T,\min}$. If $u \in \ker \widetilde{A}_{T,\min}$, then

$$u \in \mathcal{D}(A_{T,\max}) \cap \mathcal{H}_p^{\mu,\gamma+\mu-\delta}(\mathbb{D},E) = \mathcal{D}(A_{T,\min})$$

for small δ according to Lemma 5.3. Thus $\ker A_{T,\min} = \ker \widetilde{A}_{T,\min}$. Taking adjoints of the minimal extensions leads to $A^t_{\widetilde{T},\max}$ and $\widetilde{A}^t_{\widetilde{T},\max}$ acting as unbounded operators in $\mathcal{H}_{p'}^{0,-\gamma}(\mathbb{D},E)$ and $\mathcal{H}_{p'}^{0,-\gamma+\delta}(\mathbb{D},E)$, respectively, by Theorem 4.5. Obviously $\ker \widetilde{A}^t_{\widetilde{T},\max} \subset \ker A^t_{\widetilde{T},\max}$. By Corollary 4.7 the reverse inclusion also holds for small δ. Hence $\operatorname{ind} \widetilde{A}_{T,\min} = \dim\ker \widetilde{A}_{T,\min} - \dim\ker \widetilde{A}^*_{T,\min} = \operatorname{ind} A_{T,\min}$. $\square$

THEOREM 5.5. $\operatorname{ind}(A_0)_{T_0,\min} = \operatorname{ind} A_{T,\min}$.

PROOF. In view of Proposition 5.4 we may assume ellipticity with respect to $\gamma+\mu$. By Theorem 7.3 of the Appendix, the statement is equivalent to $\operatorname{ind}\begin{pmatrix} A \\ T \end{pmatrix} = \operatorname{ind}\begin{pmatrix} A_0 \\ T_0 \end{pmatrix}$. However, $\begin{pmatrix} A \\ T \end{pmatrix}$ and $\begin{pmatrix} A_0 \\ T_0 \end{pmatrix}$ can be connected by a homotopy through elliptic elements, hence have the same index: In fact, if $A = t^{-\mu}\operatorname{op}_M(h)$ near $t=0$, let $h_\varepsilon(t,z) = h(\varepsilon t, z)$ and set

$$A_\varepsilon = \omega\, t^{-\mu}\operatorname{op}_M(h_\varepsilon) + (1-\omega)A, \qquad 0 \le \varepsilon \le 1,$$

with ω supported sufficiently close to 0. Similarly we define T_ε. $\square$

COROLLARY 5.6. *The domain of the maximal extension differs from the domain of the minimal extension by a finite-dimensional space. More precisely,*

$$\dim \mathcal{D}(A_{T,\max}) \Big/ \mathcal{D}(A_{T,\min}) = \dim \mathcal{D}((A_0)_{T_0,\max}) \Big/ \mathcal{D}((A_0)_{T_0,\min}) = \sum_{-\gamma-\mu<\operatorname{Re} z-\frac{n+1}{2}<-\gamma} M(r_0,z).$$

PROOF. Let $\iota : \mathcal{D}(A_{T,\min}) \hookrightarrow \mathcal{D}(A_{T,\max})$ denote the inclusion, so that $A_{T,\min} = A_{T,\max}\iota$. By Theorem 4.13, ι is a Fredholm operator, and

$$\dim \mathcal{D}(A_{T,\max}) \Big/ \mathcal{D}(A_{T,\min}) = -\operatorname{ind}\iota = \operatorname{ind} A_{T,\max} - \operatorname{ind} A_{T,\min} = -\operatorname{ind}(A^t)_{\widetilde{T},\min} - \operatorname{ind} A_{T,\min},$$

where the last identity follows from Theorem 4.8. Next we observe that the adjoint boundary condition $(T_0)^{\sim}$ can be constructed in such a way that it coincides with $(\widetilde{T})_0$ near $t=0$. From Theorem 5.5 we obtain

$$\begin{aligned}\dim \mathcal{D}(A_{T,\max})\Big/\mathcal{D}(A_{T,\min}) &= -\mathrm{ind}\,(A_0^t)_{\widetilde{T}_0,\min} - \mathrm{ind}\,(A_0)_{T_0,\min}\\ &= \dim \mathcal{D}((A_0)_{T_0,\max})\Big/\mathcal{D}((A_0)_{T_0,\min}),\end{aligned}$$

and the assertion follows from Proposition 5.2. $\square$

5.4. Lower order conormal symbols. In order to describe the maximal domain for operators with t-dependent coefficients we have to take into account the first μ Taylor coefficients of the Mellin symbols of A and T at $t=0$. Writing A as in (1.1), we let

$$f_j = \frac{1}{j!}\sum_{k=0}^{\mu}\frac{d^j a_k(0)}{dt^j} z^k \in M_O^{\mu,0}(X,E_0), \quad j=0,\dots,\mu-1,$$

so that, near $t=0$,

$$A = t^{-\mu}\sum_{j=0}^{\mu-1} t^j\,\mathrm{op}_M(f_j) + t^\mu\,A_\mu \tag{5.2}$$

with a Fuchs type differential operator A_μ of order μ. Similarly, writing the boundary condition $T=S\varrho$ near $t=0$ in the form $T=\mathrm{diag}(1,t^{-1},\dots,t^{-\mu+1})\,\mathrm{op}_M(s)\varrho$, we let

$$s_j = \frac{1}{j!}\,\frac{d^j s(0)}{dt^j}\varrho, \quad j=0,\dots,\mu-1,$$

so that, near $t=0$,

$$T = \mathrm{diag}(1,t^{-1},\dots,t^{-\mu+1})\sum_{j=0}^{\mu-1} t^j\,\mathrm{op}_M(s_j) + t^\mu\,T_\mu. \tag{5.3}$$

with a resulting boundary condition T_μ. We then call

$$\sigma_M^{\mu-j}\left(\begin{pmatrix}A\\T\end{pmatrix}\right) = \begin{pmatrix}f_j\\s_j\end{pmatrix}, \quad j=0,\dots,\mu-1,$$

the conormal symbol of order $\mu-j$.

5.5. Description of the maximal domain. Following the method developed in [**20**] for the boundaryless case, we define recursively $\begin{pmatrix}r_j & k_j\end{pmatrix}$, $0\le j<\mu$, by

$$\begin{aligned}\begin{pmatrix}r_0 & k_0\end{pmatrix} &= \begin{pmatrix}f_0\\s_0\end{pmatrix}^{-1}\\ \begin{pmatrix}r_j & k_j\end{pmatrix} &= -\begin{pmatrix}T^{-j}r_0 & T^{-j}k_0\end{pmatrix}\sum_{l=0}^{j-1}\begin{pmatrix}T^{-l}f_{j-l}\\T^{-l}s_{j-l}\end{pmatrix}\begin{pmatrix}r_l & k_l\end{pmatrix}, \qquad j=1,\dots,\mu-1.\end{aligned}$$

Recall that T^k denotes the operator of shifting the argument by k, i.e. $(T^k g)(z)=g(z+k)$. These equations are equivalent to

$$\sum_{j=0}^{l}\begin{pmatrix}T^{-j}f_{l-j}\\T^{-j}s_{l-j}\end{pmatrix}\begin{pmatrix}r_j & k_j\end{pmatrix} = \delta_{0l}\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \forall\, 0\le l<\mu,$$

with the Kronecker symbol δ_{jk}. Written componentwise, we particularly obtain

$$\sum_{j=0}^{l} T^{-j}f_{l-j}\,r_j = \delta_{0l}, \qquad \sum_{j=0}^{l} T^{-j}s_{l-j}\,r_j = 0 \qquad \forall\, 0\le l<\mu. \tag{5.4}$$

Definition 5.7. *We define the operators* $G_k = \sum_{l=0}^{k} G_{kl} : \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X, E_0) \longrightarrow \mathcal{C}^{\infty,\gamma}(\mathbb{D}, E)$, $0 \le k < \mu$, *by*

$$G_{00} = \omega \left(\mathrm{op}_M^{\gamma+\mu+\varepsilon-1-\frac{n}{2}}(r_0) - \mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\right),$$

and, if $1 \le k \le \mu - 1$ *and* $0 \le l \le k$,

$$G_{kl} = \omega\, t^l \left(\mathrm{op}_M^{\gamma+\mu+\varepsilon-k-1-\frac{n}{2}}(r_l) - \mathrm{op}_M^{\gamma+\mu+\varepsilon-k-\frac{n}{2}}(r_l)\right).$$

Here, ω *is an arbitrary fixed cut-off function. Moreover,* $\varepsilon > 0$ *is fixed and so small that for each pole* p *of one of the symbols* $r_0, \ldots, r_\mu - 1$ *all the distances* $|\frac{n+1}{2} - \gamma - \mu + j - \mathrm{Re}\, p|$, $j = 0, \ldots, \mu - 1$, *are either zero or* $> \varepsilon$.

Note that, by construction, G_{kl} maps into $\mathcal{C}^{\infty,\gamma+\mu-k+l-1+\varepsilon}(\mathbb{D}, E) \hookrightarrow \mathcal{C}^{\infty,\gamma+\varepsilon}(\mathbb{D}, E)$.

Lemma 5.8. *For* $k = 0 \ldots, \mu - 1$,

$$A : \mathrm{im}\, G_k \longrightarrow \mathcal{C}^{\infty,\gamma+\varepsilon}(\mathbb{D}, E), \qquad T : \mathrm{im}\, G_k \longrightarrow \bigoplus_{j=0}^{\mu-1} \mathcal{C}^{\infty,\gamma+\mu+\varepsilon-j-\frac{1}{2}}(\mathbb{B}, F_j).$$

Proof. Let $u \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X, E_0)$. Writing A as in (5.2) and using the mapping properties of the G_{kl}, we see that $AG_k u \in \mathcal{C}^{\infty,\gamma+\varepsilon}(\mathbb{D}, E)$ if and only if

$$\widetilde{\omega} \sum_{j=0}^{k} \sum_{l=0}^{k-j} t^j\, \mathrm{op}_M(f_j)\, G_{kl}\, u \ \in\ \mathcal{C}^{\infty,\gamma+\mu+\varepsilon}(\mathbb{D}, E)$$

for every cut-off function $\widetilde{\omega}$. Choosing $\widetilde{\omega}$ with $\widetilde{\omega} \prec \omega$, inserting the definition of G_{kl} and rearranging the order of summation, this is equivalent to

$$\widetilde{\omega} \sum_{j=0}^{k} t^j \sum_{l=0}^{j} \left(\mathrm{op}_M^{\gamma+\mu+\varepsilon-k-1-\frac{n}{2}}((T^{-l} f_{j-l}) r_l) - \mathrm{op}_M^{\gamma+\mu+\varepsilon-k-\frac{n}{2}}((T^{-l} f_{j-l}) r_l)\right) u \in \mathcal{C}^{\infty,\gamma+\mu+\varepsilon}(\mathbb{D}, E).$$

However, this expression actually equals zero by (5.4). For T we argue in the same way, using the representation (5.3). □

Corollary 5.9. $\mathrm{Im}\, PG_k \subset \mathcal{D}(A_{T,\max})$ *for* $k = 0, \ldots, \mu - 1$.

Proof. The mapping property of T in Lemma 5.8 in connection with Lemma 3.9.iii) implies that $PG_k u \equiv G_k u$ modulo $\mathcal{C}^{\infty,\gamma+\mu+\varepsilon}(\mathbb{D}, E)$ for $u \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X, E_0)$. Therefore $APG_k u \in \mathcal{C}^{\infty,\gamma+\varepsilon}(\mathbb{D}, E)$ according to Lemma 5.8. □

Definition 5.10. *We set* $\mathcal{E} := \mathrm{im}\, PG_0 + \ldots + \mathrm{im}\, PG_{\mu-1} \subset \mathcal{D}(A_{T,\max})$.

$\mathcal{E}$ is a finite-dimensional subspace of $\mathcal{C}^{\infty,\gamma+\varepsilon}(\mathbb{D}, E)_T$. As in the proof of Corollary 5.9 we see that

$$\mathcal{E} + \mathcal{C}^{\infty,\gamma+\mu+\varepsilon}(\mathbb{D}, E) = \mathrm{im}\, G_0 + \ldots + \mathrm{im}\, G_{\mu-1} + \mathcal{C}^{\infty,\gamma+\mu+\varepsilon}(\mathbb{D}, E). \tag{5.5}$$

In particular, there exist complex numbers q_j, determined from the meromorphic structure of the symbols r_l, with

$$\frac{n+1}{2} - \gamma - \mu \le \mathrm{Re}\, q_j < \frac{n+1}{2} - \gamma \tag{5.6}$$

such that any element u of $\mathcal{E}$ is of the form

$$u(t, x) \equiv \omega(t) \sum_{j=0}^{N} \sum_{k=0}^{l_j} u_{jk}(x)\, t^{-q_j}\, \log^k t \qquad \mathrm{mod}\ \mathcal{C}^{\infty,\gamma+\mu+\varepsilon}(\mathbb{D}, E) \tag{5.7}$$

with smooth functions $u_{jk} \in \mathcal{C}^\infty(X, E_0)$.

In case $A = A_0$ and $T = T_0$ have t-independent coefficients near $t = 0$, all G_{kl}, $l \ge 1$, vanish and

$$\mathcal{E} = \mathcal{E}_0 = \mathrm{im}\, P_0\, \omega \left(\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0) - \mathrm{op}_M^{\gamma+\varepsilon-\frac{n}{2}}(r_0)\right),$$

cf. Proposition 5.2. In particular, we have twice strict inequality '$<$' in (5.6). In general, equality in (5.6) is possible.

Proposition 5.11. *The dimension of $\mathcal{E}$ can be estimated by*

$$\dim \mathcal{E} \geq \dim \mathcal{E}_0 = \dim \operatorname{im} \omega \left(\operatorname{op}_M^{\gamma+\varepsilon-\frac{n}{2}}(r_0) - \operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0) \right).$$

Moreover, there exists a subspace of $\mathcal{E}$ which has the same dimension as $\mathcal{E}_0$ and has trivial intersection with $\mathcal{D}(A_{T,\min})$.

Proof. Proposition 5.2 shows the identity for $\dim \mathcal{E}_0$. For $0 \leq k < \mu$ pick $u_{k1}, \ldots, u_{kn_k} \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}_+ \times X, E_0)$ in such a way that $\{G_{k0} u_{kj} \mid 1 \leq j \leq n_k\}$ is a basis of $\operatorname{im} G_{k0}$. Then $n_1 + \ldots + n_{\mu-1} = \dim \mathcal{E}_0$. We shall show that

$$\{PG_k u_{kj} \mid 0 \leq k < \mu,\, 1 \leq j \leq n_k\} \subset \mathcal{E}$$

is a set of linearly independent functions with

$$\operatorname{span}\{PG_k u_{kj} \mid 0 \leq k < \mu,\, 1 \leq j \leq n_k\} \cap \mathcal{D}(A_{T,\min}) = \{0\}:$$

Let $\alpha_{jk} \in \mathbb{C}$ and $\sum\limits_{k=0}^{\mu-1} \sum\limits_{j=1}^{n_k} \alpha_{jk} PG_k u_{kj} \in \mathcal{D}(A_{T,\min})$. Since $PG_k u_{kj} \equiv G_k u_{kj}$ modulo $\mathcal{C}^{\infty,\gamma+\mu}(\mathbb{D}, E)$, cf. the proof of Corollary 5.9, we obtain that

$$\sum_{k=0}^{\mu-1} \sum_{j=1}^{n_k} \alpha_{jk} G_k u_{kj} = u \;\in\; \mathcal{H}_p^{0,\gamma+\mu-\varepsilon}(\mathbb{D}, E).$$

Setting $l = \mu - 1$, we conclude that

$$\sum_{j=1}^{n_l} \alpha_{jl} G_{l0} u_{lj} = u - \sum_{k=0}^{l-1} \sum_{j=1}^{n_k} \alpha_{jk} G_k u_{kj} - \sum_{j=1}^{n_l} \alpha_{kl} (G_l - G_{l0}) u_{lj}.$$

Now the right-hand side belongs to $\mathcal{H}_p^{0,\gamma+\mu-\varepsilon}(\mathbb{D}, E) + \mathcal{H}_p^{0,\gamma+1+\varepsilon}(\mathbb{D}, E)$ which has trivial intersection with $\operatorname{im} G_{l0}$. Hence the left hand side is zero, and $\alpha_{jl} = 0$ for all $1 \leq j \leq n_l$, since the $G_{l0} u_{lj}$ are linearly independent by assumption. Taking $l = \mu - 2, \ldots, 0$, we see that all α_{jk} must equal zero. □

Theorem 5.12. *With $\mathcal{E}$ from Definition 5.10, the domain of the maximal extension is*

$$\mathcal{D}(A_{T,\max}) = \mathcal{D}(A_{T,\min}) + \mathcal{E}.$$

The sum is direct at least in the cases where A has t-independent coefficients near $t = 0$ or r_0 has no pole on the line $\operatorname{Re} z = \frac{n+1}{2} - \gamma - \mu$. In any case,

$$\mathcal{D}(A_{T,\min}) \cap \mathcal{E} \subset \operatorname{im} P\, \omega \left(\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0) - \operatorname{op}_M^{\gamma+\mu+\varepsilon-\frac{n}{2}}(r_0) \right).$$

Proof. By Corollary 5.6, $\dim \mathcal{D}(A_{T,\max}) / \mathcal{D}(A_{T,\min}) = \dim \mathcal{E}_0$. Thus $\mathcal{D}(A_{T,\max}) = \mathcal{D}(A_{T,\min}) + \mathcal{E}$ by Proposition 5.11. If A has t-independent coefficients near $t = 0$, the intersection of $\mathcal{D}(A_{\min})$ and $\mathcal{E} = \mathcal{E}_0$ is zero by Proposition 5.2a).

In order to see that the sum is also direct if r_0 has no pole on the line in question, suppose that $u \in \mathcal{D}(A_{T,\min})$ and $u = Pv$ with $v \in \operatorname{im} G_0 + \ldots + \operatorname{im} G_{\mu-1}$, cf. Definition 5.10. We can write $v = \omega(t) \sum\limits_{j=0}^{N} \sum\limits_{k=0}^{l_j} u_{jk}(x)\, t^{-q_j} \log^k t$ with q_j satisfying (5.6). Since $u - v \in \mathcal{C}^{\infty,\gamma+\mu+\varepsilon}(\mathbb{D}, E)$, cf. the proof of Corollary 5.9, we have $v \in \mathcal{H}_p^{\mu,\gamma+\mu-\delta}(\mathbb{D}, E)$ for all $\delta > 0$ and $Av \in \mathcal{H}_p^{0,\gamma}(\mathbb{D}, E)$. The first property of v implies that $\operatorname{Re} q_j = (n+1)/2 - \gamma - \mu$ for all q_j. Now A maps v to a function with a similar structure, where t has exponents $-q_j + l$, $l \in \mathbb{N}_0$. Noting that $\omega\, t^{-p_j} \log^k t \in \mathcal{C}^{\infty,\gamma}$ if and only if $\operatorname{Re} p_j < (n+1)/2 - \gamma$, we conclude from the second property that $Av \in \mathcal{C}^{\infty,\gamma+\delta}(\mathbb{D}, E)$ for all $\delta < 1$. Moreover, $P_0 v \equiv P_0 u = u + (P_0 - P)u \equiv v$ modulo $\mathcal{H}_p^{\mu,\gamma+\mu+\varepsilon}(\mathbb{D}, E)$ by (4.9). We next use the facts that we may assume P_0v to be supported near the singularity by Remark 3.5 and that $r_0f_0 + k_0s_0 = 1$ and $\mathrm{op}_M(s_0)P_0v = T_0P_0v = 0$. This implies that

$$v \equiv P_0v = \widetilde{\omega}\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0f_0 + k_0s_0)P_0v = \widetilde{\omega}\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0)\mathrm{op}_M(f_0)P_0v \tag{5.8}$$

modulo $\mathcal{H}_p^{\mu,\gamma+\mu+\varepsilon}(\mathbb{D},E)$ with a suitable cut-off function $\widetilde{\omega}$. Now $\widetilde{u} := \mathrm{op}_M(f_0)P_0v \equiv \mathrm{op}_M(f_0)v = t^\mu A_0v$ modulo $\mathcal{H}_p^{0,\gamma+\mu+\varepsilon}(\mathbb{D},E)$. Moreover, $A_0v \in \mathcal{C}^{\infty,\gamma+\delta}(\mathbb{D},E)$, since $Av \in \mathcal{C}^{\infty,\gamma+\delta}(\mathbb{D},E)$, $\delta < 1$, and $A - A_0$ gains a factor t. Thus $\widetilde{u} \in \mathcal{H}_p^{0,\gamma+\mu+\varepsilon}(\mathbb{D},E)$ and with (5.8)

$$v - \omega\left(\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(r_0) - \mathrm{op}_M^{\gamma+\mu+\varepsilon-\frac{n}{2}}(r_0)\right)\widetilde{u} \equiv \widetilde{\omega}\mathrm{op}_M^{\gamma+\mu+\varepsilon-\frac{n}{2}}(r_0)\widetilde{u} \equiv 0$$

modulo $\mathcal{H}_p^{\mu,\gamma+\mu+\varepsilon}(\mathbb{D},E)$ (note that changing ω to $\widetilde{\omega}$ induces only a remainder in $\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{D},E)$). By the structure of the left-hand side, this is only possible if the left-hand side equals 0. This finishes the proof. ☐

Remark 5.13. *If* $\widetilde{\mathcal{E}} = \mathrm{span}\{PG_ku_{kj} \mid 0 \le k < \mu,\ 1 \le j \le n_k\}$, *as constructed in the proof of Proposition* 5.11, *then* $\mathcal{D}(A_{T,\max}) = \mathcal{D}(A_{T,\min}) \oplus \widetilde{\mathcal{E}}$.

6. Ellipticity of the adjoint problem

Let $\widetilde{\mathcal{A}} = \begin{pmatrix} A^t \\ \widetilde{T}\end{pmatrix}$ denote the adjoint problem to $\mathcal{A} = \begin{pmatrix} A \\ T\end{pmatrix}$, cf. Definition 3.12.

Proposition 6.1. *If* $\mathcal{A}$ *is* $\mathbb{D}$*-elliptic then so is* $\widetilde{\mathcal{A}}$.

Proof. Clearly with $\sigma_\psi^\mu(\mathcal{A})$ and $\widetilde{\sigma}_\psi^\mu(\mathcal{A})$ also $\sigma_\psi^\mu(\widetilde{\mathcal{A}})$ and $\widetilde{\sigma}_\psi^\mu(\widetilde{\mathcal{A}})$ are invertible (note that these symbols only involve the differential operators A and A^t, and not the boundary conditions). The fact that invertibility of $\sigma_\partial^\mu(\mathcal{A})$ implies that of $\sigma_\partial^\mu(\widetilde{\mathcal{A}})$ is proven in [**5**], Theorem 1.6.9. This proof passes over to the rescaled boundary symbols. ☐

Our next goal is to show that ellipticity of $\mathcal{A}$ with respect to $\gamma \in \mathbb{R}$ in the sense of Definition 3.1 implies ellipticity of the adjoint problem with respect to $-\gamma + \mu$. According to Proposition 6.1 and Proposition 3.2 reduces to a question on the invertibility of the conormal symbol. We shall show:

Proposition 6.2. *The conormal symbol of the adjoint problem satisfies*

$$\sigma_M(\widetilde{\mathcal{A}})(z) = \begin{pmatrix}\sigma_M^\mu(A^t) \\ \sigma_M^\mu(\widetilde{T})\end{pmatrix}(z) = \begin{pmatrix}\sigma_M^\mu(A)^t \\ \sigma_M^\mu(T)\widetilde{\ }\end{pmatrix}(n+1-\mu-\overline{z}), \qquad z \in \mathbb{C}.$$

The statement contains some notation we shall now explain: For each fixed z, $\sigma_M^\mu(A)(z)$ is a differential operator on X; $\sigma_M^\mu(A)^t(z)$ denotes its formal adjoint. Moreover $\begin{pmatrix}\sigma_M^\mu(A)(z) \\ \sigma_M^\mu(T)(z)\end{pmatrix}$ is a boundary value problem on the smooth manifold X; $\sigma_M^\mu(T)\widetilde{\ }(z)$ denotes the adjoint boundary condition in the sense of [**5**]. Note that $\sigma_M^\mu(T)(z)$ is a normal boundary condition: The entries $\underline{S}_{kk}$ in (3.3) are Fuchs type operators of order zero so that the Mellin symbol – just like the operator – is simply multiplication by the surjective morphism $\underline{S}_{kk}$.

If one goes through the construction (Proposition 1.3.2, Lemma 1.6.1, (1.6.26) in [**5**] and Proposition 3.8, Theorem 3.11, Definition 3.12 in this paper) one finds that

$$\sigma_M^\mu(T)\widetilde{\ } = I^\times(\sigma_M(\mathfrak{A})\sigma_M(\underline{C}'))^t,$$

where

$$\sigma_M(\underline{C}') = (T^k\sigma_M^{j-k}(\underline{C}'_{jk}))_{0\le j,k<\mu}, \qquad \sigma_M(\mathfrak{A}) = (T^k\sigma_M^{\mu-j-k-1}(\mathfrak{A}_{jk}))_{0\le j,k<\mu}.$$

Now the proof is a purely algebraic calculation, using the rules for computing conormal symbols of formally adjoint operators, cf. [**19**]. Here are the details:

For abbreviation let us write $f^{[t]}(z) := f(n-\overline{z})^t$ and $f^{(t)}(z) := f(n+1-\overline{z})^t$. We have

$$\widetilde{T} = I^\times(\mathfrak{A}C')^t = (\underline{\widetilde{S}}_{jk})_{j,k}\,\mathrm{diag}(1, t^{-1}, \ldots, t^{-\mu+1}), \qquad \underline{\widetilde{S}}_{jk} = \sum_l \underline{C}'^t_{l,\mu-j-1}\underline{\mathfrak{A}}^t_{kl}.$$

Therefore, $\sigma_M(\widetilde{T}) = \big(T^k\sigma_M^{j-k}(\underline{\widetilde{S}}_{jk})\big)_{jk}$ with

$$\begin{aligned} T^k\sigma_M^{j-k}(\underline{\widetilde{S}}_{jk}) &= \sum_l T^{\mu-l-1}\sigma_M^{l-\mu+j+1}(\underline{C}'^t_{l,\mu-j-1})\,T^k\sigma_M^{\mu-k-l-1}(\underline{\mathfrak{A}}^t_{kl}) \\ &= \sum_l T^j\sigma_M^{l-\mu+j+1}(\underline{C}'_{l,\mu-j-1})^{[t]}\,T^{\mu-l-1}\sigma_M^{\mu-k-l-1}(\underline{\mathfrak{A}}_{kl})^{[t]}. \end{aligned}$$

On the other hand, using $f^{(t)} = T^{-1}f^{[t]}$, we obtain

$$\begin{aligned} I^\times(\sigma_M(\mathfrak{A})\sigma_M(\underline{C}'))^{(t)} &= \Big(\sum_l T^{j-\mu+1}\sigma_M^{l-\mu+j+1}(\underline{C}'_{l,\mu-j-1})^{(t)}\,T^{-l}\sigma_M^{\mu-k-l-1}(\underline{\mathfrak{A}}_{kl})^{(t)}\Big)_{j,k} \\ &= T^{-\mu}\Big(\sum_l T^j\sigma_M^{l-\mu+j+1}(\underline{C}'_{l,\mu-j-1})^{[t]}\,T^{\mu-l-1}\sigma_M^{\mu-k-l-1}(\underline{\mathfrak{A}}_{kl})^{[t]}\Big)_{j,k} \\ &= T^{-\mu}(T^k\sigma_M^{j-k}(\underline{\widetilde{S}}_{jk}))_{j,k}. \end{aligned}$$

Hence $\sigma_M(T)^\sim(n+1-\overline{z}) = \sigma_M(\widetilde{T})(z-\mu)$, and this proves Proposition 6.2.

Corollary 6.3. *If $\mathcal{A}$ is elliptic with respect to γ then the adjoint problem is elliptic with respect to $-\gamma+\mu$.*

Proof. $\mathbb{D}$-ellipticity of $\widetilde{\mathcal{A}}$ is shown in Proposition 6.1. By assumption, $\sigma_M(\mathcal{A})(z)$ is invertible whenever $\mathrm{Re}\,z = \frac{n+1}{2}-\gamma$. This is equivalent, by Corollary 7.2, to the invertibility of

$$\sigma_M(A)(z) : H^\mu_p(X,E_0)_{\sigma_M(T)(z)} \subset L_p(X,E_0) \to L_p(X,E_0).$$

Thus also the adjoint of this operator is invertible. But this adjoint is just given by

$$\sigma_M(A)(z)^t : H^\mu_{p'}(X,E_0)_{\sigma_M(T)(z)^\sim} \subset L_{p'}(X,E_0) \to L_{p'}(X,E_0).$$

Equivalently, $\begin{pmatrix}\sigma_M(A)^t\\ \sigma_M(T)^\sim\end{pmatrix}(z)$ is invertible whenever $\mathrm{Re}\,z = \frac{n+1}{2}-\gamma$. Then, due to Proposition 6.2, also $\sigma_M(\widetilde{\mathcal{A}})(n+1-\mu-\overline{z})$ is invertible. This gives the assertion, since $\mathrm{Re}\,(n+1-\mu-\overline{z}) = \frac{n+1}{2}+\gamma-\mu$. □

7. Appendix: A theorem on Fredholm operators

Let E, F, H be vector spaces and $\mathcal{A} = \begin{pmatrix}A\\T\end{pmatrix} : H \longrightarrow \begin{matrix}E\\ \oplus\\ F\end{matrix}$ be a linear map. We shall recall here a few known results relating the invertibility and Fredholm property of $\mathcal{A}$ to that of the induced operator $A_T : \ker T \to E$ defined by the action of A. Being a Fredholm operator in this context just means that the dimension of the kernel and the codimension of the image are finite. Clearly,

$$\ker\mathcal{A} = \ker A \cap \ker T = \ker A_T. \tag{7.1}$$

Lemma 7.1. *$\mathcal{A}$ is surjective if and only if A_T is surjective and $T : H \to F$ is surjective.*

Proof. Surjectivity of $\mathcal{A}$ implies that of A_T and T, since $E\oplus 0$ and $0\oplus F$ are contained in the image of $\mathcal{A}$. Vice versa, if $e\oplus f$ is given, there exists a u_1 with $Tu_1 = f$, and a u_2 with $A_Tu_2 = e - Au_1$. Then $\mathcal{A}u = e\oplus f$ for $u := u_1+u_2$. □

Together with (7.1) we at once obtain the following corollary:

Corollary 7.2. *$\mathcal{A}$ is invertible if and only if A_T is invertible and $T : H \to F$ is surjective.*

THEOREM 7.3. *$\mathcal{A}$ is a Fredholm operator if and only if A_T is a Fredholm operator and the image of T has finite codimension in F. In this case,*

$$\operatorname{ind}\mathcal{A} = \operatorname{ind}A_T - \operatorname{codim}\operatorname{im}T.$$

PROOF. i) Without loss of generality, we may assume that T is surjective. In fact, if $F = \operatorname{im}T \oplus F_0$ we pass to $\mathcal{A}_0$ given by the action of $\mathcal{A}$ on H but mapping to $E \oplus \operatorname{im}T$. Then obviously $\mathcal{A}$ is a Fredholm operator if and only if F_0 is finite dimensional and $\mathcal{A}_0$ is Fredholm. Then $\operatorname{ind}\mathcal{A}_0 = \operatorname{ind}\mathcal{A} + \dim F_0$, and it suffices to show that $\operatorname{ind}\mathcal{A}_0 = \operatorname{ind}A_T$.
ii) Let us assume that $\mathcal{A}$ is a Fredholm operator. Let $e_1 \oplus f_1, \ldots, e_k \oplus f_k$ be a basis of a complement to $\operatorname{im}\mathcal{A}$. If we define

$$K : \mathbb{C}^k \longrightarrow E,\ c \mapsto \sum_{j=1}^{k} c_j\, e_j, \qquad Q : \mathbb{C}^k \longrightarrow E,\ c \mapsto \sum_{j=1}^{k} c_j\, f_j,$$

and set $\widetilde{H} = H \oplus \mathbb{C}^k$, $\widetilde{A} = \begin{pmatrix} A & K \end{pmatrix}$, and $\widetilde{T} = \begin{pmatrix} T & Q \end{pmatrix}$, then $\widetilde{\mathcal{A}} := \begin{pmatrix} \widetilde{A} \\ \widetilde{T} \end{pmatrix} : \widetilde{H} \longrightarrow \begin{matrix} E \\ \oplus \\ F \end{matrix}$ is surjective.
According to Lemma 7.1, also $\widetilde{A}_{\widetilde{T}} : \ker \widetilde{T} \to E$ is surjective. If $\ker \widetilde{T} = \ker \widetilde{A}_{\widetilde{T}} \oplus V$ then $\widetilde{A}_V : V \to E$, defined by the action of $\widetilde{A}$, is bijective. Let us define the map S by

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} : E \xrightarrow{\widetilde{A}_V^{-1}} V \hookrightarrow \widetilde{H} = \begin{matrix} H \\ \oplus \\ \mathbb{C}^k \end{matrix}.$$

For an $e \in \ker S_2$ we then obtain

$$\begin{pmatrix} A \\ T \end{pmatrix} S_1 e = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} \begin{pmatrix} S_1 e \\ S_2 e \end{pmatrix} = \begin{pmatrix} \widetilde{A} \\ \widetilde{T} \end{pmatrix} S e = \begin{pmatrix} e \\ 0 \end{pmatrix},$$

since S maps into the kernel of $\widetilde{T}$. Thus $\ker S_2 \subset \operatorname{im} A_T$ and therefore

$$\text{codim im}\, A_T \le \text{codim ker}\, S_2 = \dim \operatorname{im} S_2 \le k = \operatorname{codim}\operatorname{im}\mathcal{A}.$$

Together with (7.1) this shows that A_T is a Fredholm operator with $\operatorname{ind}A_T \ge \operatorname{ind}\mathcal{A}$.
iii) Now let A_T be a Fredholm operator (and T surjective, cf. i)). Let us write $E = \operatorname{im}A_T \oplus E_0$ with a finite dimensional E_0, and $H = \ker T \oplus V$. Then $T_V : V \to F$, defined by the action of T, is bijective. We define the map S by $S : F \xrightarrow{T_V^{-1}} V \hookrightarrow H$ and set

$$W = \Big\{ \begin{pmatrix} e + ASf \\ f \end{pmatrix} \mid e \in \operatorname{im} A_T,\ f \in F \Big\}.$$

Then $W \subset \operatorname{im}\mathcal{A}$, since for $e = A_T u \in \operatorname{im}A_T$ and $f \in F$

$$\mathcal{A}(u + Sf) = \begin{pmatrix} A \\ T \end{pmatrix}(u + Sf) = \begin{pmatrix} e + ASf \\ TSf \end{pmatrix} = \begin{pmatrix} e + ASf \\ f \end{pmatrix}.$$

The proof if finished if we can show that the codimension of W in $E \oplus F$ equals $\dim E_0$, for then

$$\text{codim im}\,\mathcal{A} \le \operatorname{codim} W = \dim E_0 = \text{codim im}\, A_T;$$

this, together with (7.1), yields that $\mathcal{A}$ is a Fredholm operator with $\operatorname{ind}\mathcal{A} \ge \operatorname{ind}A_T$. We define

$$R = \begin{pmatrix} 1 & \pi AS & 0 \\ 0 & 1 & 0 \\ 0 & (1-\pi)AS & 1 \end{pmatrix} : \begin{matrix} \operatorname{im} A_T \\ \oplus \\ F \\ \oplus \\ E_0 \end{matrix} \longrightarrow \begin{matrix} \operatorname{im} A_T \\ \oplus \\ F \\ \oplus \\ E_0 \end{matrix},$$

where π denotes the projection in E onto $\operatorname{im}A_T$ along E_0. Then R is Fredholm with index 0, since it differs from an invertible operator by a finite rank operator (the latter is the 3×3-matix with $(1-\pi)AS$ as the only non-zero entry). Even more is true: R is invertible, since its kernel is trivial.

If $\alpha : E \oplus F = \operatorname{im} A_T \oplus E_0 \oplus F \to \operatorname{im} A_T \oplus F \oplus E_0$ is the isomorphism given by exchanging the last two components, then $W = (\alpha^{-1} R \alpha)(\operatorname{im} A_T \oplus 0 \oplus F)$. Hence $\operatorname{codim} W = \dim E_0$ as desired. $\square$

8. Appendix: The parameter-dependent Boutet de Monvel algebra on smooth manifolds

We recall some basic facts about Boutet de Monvel's calculus on smooth manifolds [**1**]. For details we refer to [**5**], also to [**15**], [**16**], [**17**] for an approach based on operator-valued symbols. In the sequel, X is a smooth compact n-dimensional manifold with boundary, s is a real number, and p is a real number with $1 < p < \infty$.

8.1. Function spaces. The Sobolev space

$$H_p^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle D\rangle^s u \in L_p(\mathbb{R}^n)\},$$

carries the norm $\|u\|_{H_p^s(\mathbb{R}^n)} = \| \langle D\rangle^s u\|_{L_p(\mathbb{R}^n)}$. A closed subspace is

$$\mathring{H}_p^s(\overline{\mathbb{R}}_+^n) = \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\mathbb{R}}_+^n\}.$$

Restriction of $H_p^s(\mathbb{R}^n)$ to the half-space $\mathbb{R}_+^n$ leads to

$$H_p^s(\overline{\mathbb{R}}_+^n) = r^+ H_p^s(\mathbb{R}^n),$$

endowed with the quotient norm $\|u\|_{H_p^s(\overline{\mathbb{R}}_+^n)} = \inf\{\|v\|_{H_p^s(\mathbb{R}^n)} \mid r^+ v = u\}$.
We write $B_p^s(\mathbb{R}^n)$ for the Besov space $B_{p,p}^s(\mathbb{R}^n)$; $B_{p,q}^s(\mathbb{R}^n)$ consists of all tempered distributions u with finite norm

$$\|u\|_{B_{p,q}^s(\mathbb{R}^n)} = \Big(\|\varphi_0(D)u\|_{L_p(\mathbb{R}^n)}^q + \sum_{k=1}^{\infty} 2^{skq}\|\varphi(2^{-k}D)u\|_{L_p(\mathbb{R}^n)}^q\Big)^{1/q}.$$

Here, $\varphi \in \mathcal{C}_{\mathrm{comp}}^\infty(\mathbb{R}^n)$ is chosen such that (i) $\varphi(\xi) > 0$ for $2^{-1} < |\xi| < 2$ and $\varphi(\xi) = 0$ otherwise, (ii) $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1$ when $\xi \neq 0$, and φ_0 is given by $\varphi_0(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi)$. Note that $B_2^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n)$.

Remark 8.1. *The L_2-scalar product allows an identification of $\mathring{H}_{p'}^{-s}(\overline{\mathbb{R}}_+^n)$ with the dual of $H_p^s(\overline{\mathbb{R}}_+^n)$, and of $B_{p'}^{-s}(\mathbb{R}^n)$ with the dual of $B_p^s(\mathbb{R}^n)$. Here, p' is the dual number to p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.*

Remark 8.2. *The trace operator γ_j, first defined on $\mathcal{C}_{\mathrm{comp}}^\infty(\overline{\mathbb{R}}_+^n)$ by $\gamma_j u(x') = (D_{x_n}^j u)(x', 0)$, extends by continuity to*

$$\gamma_j : H_p^s(\overline{\mathbb{R}}_+^n) \longrightarrow B_p^{s-j-\frac{1}{p}}(\mathbb{R}^{n-1}), \qquad s > j + \frac{1}{p}.$$

Using a partition of unity we then define $H_p^s(X)$, $\mathring{H}_p^s(X)$, and $B_p^s(X)$ for a manifold X (with or without boundary), as well as $H_p^s(X, E)$, $\mathring{H}_p^s(X, E)$, and $B_p^s(X, E)$ for a vector bundle E over X.

8.2. The transmission condition. Let us denote by $2X$ the double of X (or any smooth manifold without boundary which contains X as a submanifold). With a classical parameter-dependent pseudodifferential operator $P(\tau)$ on $2X$ we can form $P_+(\tau) = r^+ P(\tau) e^+$, where r^+ and e^+ are the operators of restriction to X and extension-by-zero to $2X$, respectively. In general, $P_+(\tau)$ does not map the space $\mathcal{C}^\infty(X)$ to itself: when this is true, $P(\tau)$ is said to satisfy the transmission property (cf. [**9**]). We will assume the so-called (parameter-dependent) *transmission condition*, described in Definition 8.3 below in terms of properties of the local symbols of P.

DEFINITION 8.3. *A classical parameter-dependent symbol* $p(x,\xi,\tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n_\xi; \mathbb{R}^l_\tau)$ *with integer order* $\mu \in \mathbb{Z}$*, such that* $p \sim \sum_{j \in \mathbb{N}_0} p_{(\mu-j)}$ *with* $p_{(\mu-j)}$ *being positively homogeneous of degree* $\mu - j$ *in* (ξ,τ) *for* $(\xi,\tau) \neq 0$*, has the transmission condition at* $x_n = 0$ *if,*

$$D^k_{x_n} D^\alpha_{(\xi,\tau)} p_{(\mu-j)}(x',0,0,0,1,0) = (-1)^{\mu-j-|\alpha|} D^k_{x_n} D^\alpha_{(\xi,\tau)} p_{(\mu-j)}(x',0,0,-1,0) \qquad \forall\, k,\alpha. \tag{8.1}$$

Here, we used the splittings $x = (x', x_n)$ *and* $\xi = (\xi', \xi_n)$*.*

The difference between the transmission condition and the transmission property is that the first ensures the transmission property for P on both sides of ∂X. For a detailed description of the transmission condition see, e.g., [**5**], [**6**], [**16**].

8.3. Parameter-dependent Boutet de Monvel's algebra. The parameter-dependent version of Boutet de Monvel's calculus deals with families of block-matrix operators $\mathcal{A} = \mathcal{A}(\tau)$, $\tau \in \mathbb{R}^l$, of the form

$$\mathcal{A} = \begin{pmatrix} P_+(\tau) + G(\tau) & K(\tau) \\ T(\tau) & Q(\tau) \end{pmatrix} : \begin{matrix} \mathcal{C}^\infty(X,E_0) \\ \oplus \\ \mathcal{C}^\infty(\partial X, F_0) \end{matrix} \longrightarrow \begin{matrix} \mathcal{C}^\infty(X,E_1) \\ \oplus \\ \mathcal{C}^\infty(\partial X, F_1) \end{matrix}, \tag{8.2}$$

where E_0, E_1 are vector bundles over X and F_0, F_1 are vector bundles over ∂X (possibly zero-dimensional). As described above, $P_+(\tau) = r^+ P(\tau) e^+$, satisfies the transmission condition. $G(\tau)$ is an operator family on X called (*parameter-dependent*) *singular Green operator*, arising, in particular, in the composition of two block-matrix operators by $(PP')_+(\tau) - P_+(\tau)P'_+(\tau)$. Moreover, $T(\tau)$ is a (*parameter-dependent*) *trace operator*, $K(\tau)$ a (*parameter-dependent*) *Poisson operator*, and $Q(\tau)$ is a parameter-dependent pseudodifferential operator on ∂X.

We now describe the various entries. For simplicity, we switch to the half-space case $X = \overline{\mathbb{R}}^n_+$ (local situation) and take all bundles trivial one dimensional.

Trace operators. A parameter-dependent trace operator of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}$ is a family of operators $T(\tau) : \mathcal{C}^\infty_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to \mathcal{C}^\infty(\mathbb{R}^{n-1})$, $\tau \in \mathbb{R}^l$, defined as

$$T(\tau) = \sum_{j=0}^{d-1} S_j(\tau)\gamma_j + \widetilde{T}(\tau),$$

where γ_j is as in Remark 8.2, $S_j(\tau)$ is a parameter-dependent pseudodifferential operator on $\mathbb{R}^{n-1}$ of order $\mu - j$ and $\widetilde{T}(\tau)$ is an operator family of the form

$$(\widetilde{T}(\tau)u)(x') = \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{ix'\xi'}\, \tilde{t}(x', x_n, \xi', \tau)\, (\mathcal{F}_{x'\to\xi'} u)(\xi', x_n)\, dx_n đ\xi',$$

where $\tilde{t}$ is in $\mathcal{S}(\overline{\mathbb{R}}_+)$ as a function of x_n and satisfies

$$\tilde{t}(x', x_n, \xi', \tau) = [\xi', \tau]^{\frac12} t(x', \xi', \tau; [\xi', \tau] x_n)$$

with

$$t(x', \xi', \tau; x_n) \in S^{\mu+\frac12}_{\mathrm{cl}}(\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^{n-1}_{\xi'}; \mathbb{R}^l_\tau)\, \widehat{\otimes}_\pi\, \mathcal{S}(\mathbb{R}_+).$$

Here, $[\,\cdot\,]$ denotes a smooth positive function with $[\,\cdot\,] = |\cdot|$ outside a neighborhood of the origin. Note that this structure implies the estimates

$$\|x_n^l D^{l'}_{x_n} D^\beta_{x'} D^\alpha_{(\xi',\tau)}\, \tilde{t}(x', x_n, \xi', \tau)\|_{L_2(\mathbb{R}_{+,x_n})} \le c\, \langle \xi', \tau \rangle^{\mu + \frac12 - l + l' - |\alpha|}$$

for every choice of the indices l, l', α, β with a resulting constant c. The *principal boundary symbol* of T,

$$\sigma^\mu_\partial(T)(x', \xi', \tau) : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathbb{C},$$

defined for $(\xi', \tau) \neq 0$, is given by

$$u \mapsto \sum_{j=0}^{d} \sigma^{\mu-j}_\psi(S_j)(x', \xi', \tau)\gamma_j u + |(\xi', \tau)|^{\frac12} \int_0^\infty \sigma^{\mu+\frac12}_\psi(t)(x', \xi', \tau; |(\xi', \tau)| x_n) u(x_n)\, dx_n.$$

Here, σ_ψ^m denotes the usual homogeneous principal symbol of order m and γ_j acts on functions on $\overline{\mathbb{R}}_+$. With these choices, it turns out that any operator of the form $T(\tau) = \gamma_0 P_+(\tau)$ is a parameter-dependent trace operator if P satisfies the transmission condition.

Poisson operators. A parameter-dependent Poisson operator of order $\mu \in \mathbb{R}$ is of the form

$$(K(\tau)v)(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} \tilde{k}(x', x_n, \xi', \tau)\hat{v}(\xi') đ\xi',$$

where $\tilde{k}$ has the same structure as $\tilde{t}$ above, but with $\mu - 1$ in place of μ. The *principal boundary symbol*, defined for $(\xi', \tau) \neq 0$, is

$$\sigma_\partial^\mu(K)(x', \xi', \tau) : \mathbb{C} \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+), \quad a \mapsto |(\xi', \tau)|^{\frac{1}{2}} \sigma_\psi^{\mu - \frac{1}{2}}(k)(x', \xi', \tau, |(\xi', \tau)| \cdot)\, a.$$

Singular Green operators. A parameter-dependent singular Green operator of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ is a family

$$G(\tau) = \sum_{j=0}^{d-1} K_j(\tau)\gamma_j + \widetilde{G}(\tau)$$

with parameter-dependent Poisson operators $K_j(\tau)$ of order $\mu - j$, the standard trace operators γ_j and an operator $\widetilde{G}(\tau)$ of the form

$$(\widetilde{G}(\tau)u)(x) = \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{ix'\xi'} \tilde{g}(x', x_n, y_n, \xi', \tau)(\mathcal{F}_{x' \to \xi'} u)(\xi', y_n) dy_n đ\xi'.$$

The symbol-kernel $\tilde{g}$ is of the form

$$\tilde{g}(x', x_n, y_n, \xi', \tau) = [\xi', \tau]\, g(x', \xi', \tau, [\xi', \tau]x_n, [\xi', \tau]y_n)$$

with

$$g(x', \xi', \tau, x_n, y_n) \in S_{\mathrm{cl}}^\mu(\mathbb{R}_{x'}^{n-1} \times \mathbb{R}_{\xi'}^{n-1}; \mathbb{R}_\tau^l) \,\widehat{\otimes}_\pi\, \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+).$$

In particular, $\tilde{g}$ satisfies, for all indices, the estimates

$$\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_{x'}^\beta D_{(\xi', \tau)}^\alpha \tilde{g}(x', x_n, y_n, \xi', \tau)\|_{L_2(\mathbb{R}_{+,x_n} \times \mathbb{R}_{+,y_n})} \leq c \langle \xi', \tau \rangle^{\mu - k + k' - m + m' - |\alpha|}.$$

The *principal boundary symbol* of G,

$$\sigma_\partial^\mu(G)(x', \xi', \tau) : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+),$$

defined for $(\xi', \tau) \neq 0$, is given by

$$u \mapsto \sum_{j=0}^{d-1} \sigma_\partial^{\mu - j}(K_j)(x', \xi', \tau)\gamma_j u \,+\, |(\xi', \tau)| \int_0^\infty \sigma_\psi^\mu(g)(x', \xi', \tau, |(\xi', \tau)| \cdot, |(\xi', \tau)| y_n) u(y_n)\, dy_n.$$

The *principal boundary symbols* of the remaining elements are

$$\sigma_\partial^\mu(P_+)(x', \xi', \tau) = r^+ \sigma_\psi^\mu(P)(x', D_{x_n}, \xi', \tau)\, e^+ : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+),$$

and $\sigma_\partial^\mu(Q)(x', \xi', \tau) = \sigma_\psi^\mu(Q)(x', \xi', \tau)$ is the usual homogeneous principal symbol.
The definitions made above are all invariant under smooth change of coordinates which preserve the boundary, so operators can be defined on the manifold X via a partition of unity.

Definition 8.4. *$B^{\mu,d}(X; \mathbb{R}^l; E_0, F_0; E_1, F_1)$, $\mu \in \mathbb{Z}$ and $d \in \mathbb{N}_0$, consists of all operator-families $\mathcal{A} = \mathcal{A}(\tau)$ of the form* (8.2) *with each entry being of order μ and type d.*

8.4. Algebra and mapping properties, ellipticity. The next result is due to [**7**].

THEOREM 8.5. *Let $\mathcal{A} \in B^{\mu,d}(X;\mathbb{R}^l;E_0,F_0;E_1,F_1)$. Then, for each τ, $\mathcal{A}(\tau)$ extends to continuous operators*

$$\mathcal{A}(\tau): \begin{array}{c} H_p^s(X,E_0) \\ \oplus \\ B_p^{s-\frac{1}{p}}(\partial X,F_0) \end{array} \longrightarrow \begin{array}{c} H_p^{s-\mu}(X,E_1) \\ \oplus \\ B_p^{s-\mu-\frac{1}{p}}(\partial X,F_1) \end{array}, \qquad s > d-1+\frac{1}{p}. \tag{8.3}$$

The notion algebra refers to the following behaviour under composition:

THEOREM 8.6. *Let (E_0,F_0), (E_1,F_1) and (E_2,F_2) be pairs of vector bundles (on X and ∂X, respectively) such that composition of operators*

$$\mathcal{A}_0: \begin{array}{c} \mathcal{C}^\infty(X,E_1) \\ \oplus \\ \mathcal{C}^\infty(\partial X,F_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{C}^\infty(X,E_2) \\ \oplus \\ \mathcal{C}^\infty(\partial X,F_2) \end{array}, \qquad \mathcal{A}_1: \begin{array}{c} \mathcal{C}^\infty(X,E_0) \\ \oplus \\ \mathcal{C}^\infty(\partial X,F_0) \end{array} \longrightarrow \begin{array}{c} \mathcal{C}^\infty(X,E_1) \\ \oplus \\ \mathcal{C}^\infty(\partial X,F_1) \end{array}$$

makes sense. The pointwise composition $(\mathcal{A}_0(\tau),\mathcal{A}_1(\tau)) \mapsto \mathcal{A}_0(\tau)\,\mathcal{A}_1(\tau)$ then induces a map

$$B^{\mu_0,d_0}(X;\mathbb{R}^l;E_1,F_1;E_2,F_2) \times B^{\mu_1,d_1}(X;\mathbb{R}^l;E_0,F_0;E_1,F_1) \longrightarrow B^{\mu,d}(X;\mathbb{R}^l;E_0,F_0;E_2,F_2),$$

with $\mu = \mu_0+\mu_1$ and $d = \max\{\mu_1+d_0,d_1\}$.

Assuming all bundles to be equipped with a hermitian metric (compatible with the product structure near the boundary), we have scalar products in $L_2(X,E_j)$ and $H_2^{-\frac{1}{2}}(\partial X,F_j)$. If $\mathcal{A}$ is an operator as in (8.3) of order and type 0, we can define its adjoint $\mathcal{A}^*$ by

$$(\mathcal{A}u_1,u_2)_{L_2(X,E_1)\oplus H_2^{-1/2}(\partial X,F_1)} = (u_1,\mathcal{A}^*u_2)_{L_2(X,E_0)\oplus H_2^{-1/2}(\partial X,F_0)},$$

where $u_j \in \mathcal{C}^\infty(X,E_j) \oplus \mathcal{C}^\infty(\partial X,F_j)$ are arbitrary. Note that we take the scalar product of $H_2^{-\frac{1}{2}}(\partial X,F_j) = B_2^{-\frac{1}{2}}(\partial X,F_j)$, since by this the duality of $B_p^{s-\frac{1}{p}}(\partial X,F_j)$ and $B_{p'}^{-s-\frac{1}{p'}}(\partial X,F_j)$ is induced. As the following theorem states, the subalgebra of such operators is closed under taking adjoints.

THEOREM 8.7. *Let $\mu \le 0$. Taking pointwise the formal adjoint induces a map*

$$\mathcal{A} \mapsto \mathcal{A}^*: B^{\mu,0}(X;\mathbb{R}^l;E_0,F_0;E_1,F_1) \longrightarrow B^{\mu,0}(X;\mathbb{R}^l;E_1,F_1;E_0,F_0).$$

Finally, we give a brief account to ellipticity. It is determined by the invertibility of two principal symbols. If $\mathcal{A} \in B^{\mu,d}(X;\mathbb{R}^l;E_0,F_0;E_1,F_1)$, its *homogeneous principal symbol* is

$$\sigma_\psi(\mathcal{A}) = \sigma_\psi^\mu(P): \pi_X^*E_0 \longrightarrow \pi_X^*E_1,$$

where π_X^* denotes the bundle pull-back under the canonical projection $\pi_X: (T^*X \times \mathbb{R}^l)\setminus 0 \to X$. The *principal boundary symbol*

$$\sigma_\partial^\mu(\mathcal{A}): \pi_{\partial X}^* \begin{pmatrix} E_0|_{\partial X} \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ F_0 \end{pmatrix} \longrightarrow \pi_{\partial X}^* \begin{pmatrix} E_1|_{\partial X} \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ F_1 \end{pmatrix},$$

$\pi_{\partial X}: (T^*\partial X \times \mathbb{R}^l)\setminus 0 \to \partial X$ denoting the canonical projection, is defined by

$$\sigma_\partial^\mu(\mathcal{A})(x',\xi',\tau) = \begin{pmatrix} \sigma_\partial^\mu(P_+ + G) & \sigma_\partial^\mu(K) \\ \sigma_\partial^\mu(T) & \sigma_\partial^\mu(Q) \end{pmatrix}(x',\xi',\tau)$$

with the single components as introduced in Section 8.3.

DEFINITION 8.8. *$\mathcal{A} \in B^{\mu,d}(X;\mathbb{R}^l;E_0,F_0;E_1,F_1)$ is parameter-dependent elliptic if both $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial^\mu(\mathcal{A})$ are invertible.*

As for the usual pseudodifferential calculus, ellipticity implies (in fact, is equivalent to) the existence of a parametrix within the calculus.

THEOREM 8.9. *Let $\mathcal{A} \in B^{\mu,d}(X;\mathbb{R}^l;E_0,F_0;E_1,F_1)$ with $d = \max(\mu,0)$ be elliptic. Then there is an operator $\mathcal{B} \in B^{-\mu,d'}(X;\mathbb{R}^l;E_1,F_1;E_0,F_0)$, $d' = \max(-\mu,0)$, such that*

$$\mathcal{A}\mathcal{B} - I \in B^{-\infty,d'}(X;\mathbb{R}^l;E_1,F_1;E_1,F_1), \qquad \mathcal{B}\mathcal{A} - I \in B^{-\infty,d}(X;\mathbb{R}^l;E_0,F_0;E_0,F_0).$$

In particular, $\mathcal{A}(\tau)$ acting as in (8.3) is a Fredholm operator of index 0 for each τ and is invertible for $|\tau|$ sufficiently large.

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