THE HOMOLOGY AND HOMOTOPY GROUPS OF THE COMPLEMENT OF CERTAIN FAMILY OF FIBERS IN A CRITICAL POINT PROBLEM

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ABSTRACT Let (E, p, N) be a fibration whose base space N is a manifold having the dimension $n \geq 2$. In this paper we show that the pair $\big(E, E \backslash p^{-1}(A)\big)$, where A is a closed countable subset of N, is $(n-1)$-connected and that $\pi_{n-1}(M \backslash A)$ has free abelian subgroups of arbitrarily large rank. This kind of results will be used in the last section as obstruction to the existence of a function, in a certain class, with a prescribed closed critical set.

1. Introduction

Let M be an n-dimensional differentiable manifold and A be a closed countable subset of M. In the recent paper [6], it is proved that for a given compact connected differntiable manifold P such that $\dim P < \dim M$, in each homotopy class $[f] \in [(P, \partial P), (M, M \backslash A)]$, there exists a mapping whose image avoids the subset A. More precisely we have:

Theorem 1.1. *([6]) Let M be an n-dimensional differentiable manifold ($\partial M = \emptyset$) and A be a closed countable subset of M. If P is a compact differentiable k-dimensional manifold ($k < n$, $\partial P \neq \emptyset$) and $f : P \to M$ is a continuous map such that $f(\partial P) \subseteq M \backslash A$, then there exists a continuous map $g : P \to M$ such that $g(P) \subseteq M \backslash A$, $g|_{\partial P} = f|_{\partial P}$ and $f \simeq g(\mathrm{rel}\, \partial P)$. If M is connected, then one particularly gets, using the particular case $P = [0, 1]$, that $M \backslash A$ is also connected.*

Besides, theorem 1.1 has been firstly proved in [5] for some particular closed countable subsets of M.

Let us finaly observe that any closed countable subset A of a manifold has countably many isolated points. Indeed otherwise the subset $I \subseteq A$ of isolated points would be finite (possible empty) and $A \backslash I$ would be a countable perfect subset of the given manifold. But it is folklore that perfect subsets of complete metric spaces are not countable. Therefore such a subset can be represented as $A = I \cup A' = \{a_1, a_2, \dots\} \cup A'$, where A' is the derived set of A, that is the set of accumulation points.

2. Basic results.

We start this section proving that for a fibration (E, p, N) whose base space N is an n-dimensional manifold and a countable countable closed subset A of N, in each homotopy class $[F] \in [(P, \partial P), (E, E \backslash p^{-1}(A))]$, there exists a mapping whose image avoids the subset $p^{-1}(A)$, where P is a compact connected differntiable manifold such that $\dim P < \dim N$ and $\partial P \neq \emptyset$. Some immediate consequences of this fact will be also pointed out. We will also show that $\pi_{n-1}(M \backslash A)$ has free abelian subgroups of arbitrarily large rank.

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Theorem 2.1. *Let $p : E \to M$ be a fibration whose base space M is n-dimensional differentiable manifold ($\partial M = \emptyset$) and A be a closed countable subset of M. If P is a compact differentiable k-dimensional manifold ($k < n$, $\partial P \neq \emptyset$) and $F : P \to E$ is a continuous map such that $F(\partial P) \subseteq E \backslash p^{-1}(A)$, then there exists a continuous map $G : P \to E$ such that $G(P) \subseteq E \backslash p^{-1}(A)$, $G|_{\partial P} = F|_{\partial P}$ and $F \simeq G(\mathrm{rel}\, \partial P)$. If E is connected, then one particularly gets, using the particular case $P = [0,1]$, that $E \backslash p^{-1}(A)$ is also connected.*

Proof. Applying theorem 1.1, there exists a homotopy $H : P \times [0,1] \to M$ of $p \circ F$ such that $ImH(\cdot, 1) \subseteq M \backslash A$. Because $p : E \to M$ is a fibration, there exists a homotopy H' of F which cover the homotopy H, namely $H = p \circ H'$. It is very easy to check that $ImH'(\cdot, 1) \subseteq E \backslash p^{-1}(A)$

Consider the following two homotopies $\psi : P \times [0,1] \to P$, and $\varphi : P \times [0,1] \to E$ given by:

$$\psi(x,t) = \begin{cases} x & \text{if } x \in P \backslash Q(\partial P \times [0,2)) \\ Q((\pi_1 \circ Q^{-1})(x), \frac{2}{2-t}(\pi_2 \circ Q^{-1})(x) + \frac{2t}{t-2}) & \text{if } x \in Q(\partial P \times [t,2]) \\ (\pi_1 \circ Q^{-1})(x) & \text{if } x \in Q(\partial P \times [0,t]), \end{cases}$$

$$\varphi(x,t) = \begin{cases} H'(\psi(x,t),t) & \text{if } x \in P \backslash Q(\partial P \times [0,t)) \\ H'(Q^{-1}(x)) & \text{if } x \in Q(\partial P \times [0,t]). \end{cases}$$

where $Q : \partial P \times [0, \infty) \to U \subset P$ is a collar neighbourhood of ∂P and $\pi_1 : \partial P \times [0,\infty) \to \partial P$, $\pi_2 : \partial P \times [0, \infty) \to [0, \infty)$ are obviously the projections.

Denoting $\varphi(\cdot, 1)$ by G and observing that $\varphi(\cdot, 0) = F$ one can be easily seen that $F \simeq_{\varphi} G(\text{ rel } \partial P)$ and also that $G(P) \subseteq E \backslash p^{-1}(A)$, and the theorem is completely proved.□

Corollary 2.2. *If $p : E \to M$ is a fibration whose base space M is an n-dimensional differentiable manifold ($\partial M = \emptyset$) and A is a closed countable subset of M, then the pair $(E, E \backslash p^{-1}(A))$ is (n-1)-connected, that is $\pi_q(E, E \backslash p^{-1}(A)) \simeq 0$ for all $q \in \{1, \dots, n-1\}$. In particular we get that the natural group homomorphism*

$$\chi_{n-1} : \pi_{n-1}(E, E \backslash p^{-1}(A)) \to H_{n-1}(E, E \backslash p^{-1}(A))$$

is an isomorphism. On the other hand the inclusion $i_{E \backslash p^{-1}(A)} : E \backslash p^{-1}(A) \hookrightarrow E$ is $(n-1)$-connected, that is the induced group homomomorrphism

$$\pi_q(i_{E \backslash p^{-1}(A)}) : \pi_q(E \backslash p^{-1}(A)) \to \pi_q(E)$$

is an isomorphism for $q \le n-2$ and it is an epimorphism for $q = n-1$.

Proof. The fact that $\pi_q(E, E \backslash p^{-1}(A)) = 0$ for all $q \in \{1, 2, \dots, n-1\}$ is an imediate consequence of theorem 2.1 and of the fact that $[\alpha] \in \pi_q(E, E \backslash p^{-1}(A))$ is zero if and only if there exists $\beta \in [\alpha]$ such that $\beta(D^q) \subseteq E \backslash p^{-1}(A)$. The fact that $\chi_{n-1} : \pi_{n-1}(E, E \backslash p^{-1}(A)) \to H_{n-1}(E, E \backslash p^{-1}(A))$ is an isomorphism follows from Hurewicz theorem (see for instance [7, pp. 394-398]). Further on, using the exact homotopy sequence

$$\cdots \to \pi_{r+1}(E, E \backslash p^{-1}(A)) \to \pi_r(E \backslash p^{-1}(A)) \to \pi_r(E) \to \pi_r(E, E \backslash p^{-1}(A)) \to \cdots,$$

and $\pi_q(E, E\backslash p^{-1}(A)) = 0$ for all $q \in \{1, 2, \ldots, n-1\}$, it follows that the inclusion $i_{_{E\backslash p^{-1}(A)}} : E\backslash p^{-1}(A) \hookrightarrow E$ is $(n-1)$-connected.□

Proposition 2.3. *If M is an n-dimensional differential manifold $n \geq 2$ and $A = I \cup A'$ is a closed countable subset of M, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of A and A' is its derived set, then $M\backslash A$ is connected and*

$$H_{n-1}(M\backslash A) \cong \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{i \text{ times}} \oplus H_{n-1}(M\backslash A_i),$$

where $A_i = \{a_{i+1}, a_{i+2} \ldots\} \cup A'$. In particular $H_{n-1}(M\backslash A)$ has free abelian subgroups of arbitrarily large rank.

Proof. The connectedness follows easily from theorem 1.1. To prove the stated isomorphism, we will use the Mayer-Vietoris sequence for the following two spaces

$$X = M\backslash\{a_1, a_2, \ldots, a_i\},\ Y = M\backslash A_i.$$

Taking into account the fact that $X \cap Y = M\backslash A$ and $X \cup Y = M$ we get the following sequence

$$\cdots \to H_n(M) \to H_{n-1}(M\backslash A) \to H_{n-1}(X) \oplus H_{n-1}(Y) \to H_{n-1}(M) \to \cdots .$$

But since M is n-connected it follows, by Hurewicz Theorem, that $H_q(M) = 0\ \forall\, q \in \{1, 2, \ldots, n\}$ and we therefore obtain that

$$H_{n-1}(M\backslash A) \cong H_{n-1}(X) \oplus H_{n-1}(Y).$$

Using the homology sequence of the pair (M, X) and excising a suitable open subset of X, one can get that

$$H_{n-1}(X) \cong \underbrace{\mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{i \text{ times}},$$

which finishes the proof of our theorem.□

Corollary 2.4. *Let M be an n-dimensional differential manifold $n \geq 2$ and $A = I \cup A'$ be a closed countable subset of M, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of A a,d A' is its derived set. If M is n-connected, then $M\backslash A$ is $(n-2)$-connected and*

$$\pi_{n-1}(M\backslash A) \cong \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{i \text{ times}} \oplus \pi_{n-1}(M\backslash A_i),$$

where $A_i = \{a_{i+1}, a_{i+2} \ldots\} \cup A'$. In particular $\pi_{n-1}(M\backslash A)$ has free abelian subgroups of arbitrarily large rank.

Proof. The $(n-2)$-connectedness follows easily from corollary **??**. The isomorphism

$$\pi_{n-1}(M\backslash A) \cong \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{i \text{ times}} \oplus \pi_{n-1}(M\backslash A_i)$$

follows from proposition 2.3 using the Hurewicz Theorem.□

Corollary 2.5. *Let M be an n-dimensional differential manifold $n \geq 2$ and $A = I \cup A'$ be a closed countable subset of M, where $I = \{a_1, a_2, \dots\}$ is the set of isolated points of A a,d A' is its derived set.*

(i) If G is a finitely generated abelian group, then there is no any surjective group homomorphism $\varphi : G \to H_{n-1}(M \backslash A)$

(ii) If M is n-connected and G is a finitely generated abelian group, then there is no any surjective group homomorphism $\psi : G \to \pi_{n-1}(M \backslash A)$.

Proof. (i) Assume that such a surjective group homomorphism φ exists. This implies, taking into account the inclusion $\varphi(t(G)) \subseteq t(H_{n-1}(M \backslash A))$, that the mapping

$$\frac{\varphi}{t(\varphi)} : \frac{G}{t(G)} \to \frac{H_{n-1}(M \backslash A)}{t(H_{n-1}(M \backslash A))}$$

given by $\frac{\varphi}{t(\varphi)}(x + t(G)) := \varphi(x) + t(H_{n-1}(M \backslash A))$ is well defined and it is a surjective group homomorphism. But since $\mathcal{F}$ is a compact manifold its homology group G is finitely generated, that is there exists a unique natural number k such that $G \simeq \mathbf{Z}^k \oplus t(G)$ which implies that $\frac{G}{t(G)} \simeq \mathbf{Z}^k$. On the other hand $H_{n-1}(M \backslash A) \simeq \mathbf{Z}^{k+1} \oplus H_{n-1}(A_{k+1})$ and $t(H_{n-1}(M \backslash A)) \subseteq H_{n-1}(A_{k+1})$ which implies that

$$\frac{H_{n-1}(M \backslash A)}{t(H_{n-1}(M \backslash A))} \simeq \mathbf{Z}^{k+1} \oplus \frac{H_{n-1}(M \backslash A_{k+1})}{t(H_{n-1}(M \backslash A))}.$$

Therefore, up to some isomorphisms, the group homomorphism $\frac{\varphi}{t(\varphi)}$ acts surjectively from $\mathbf{Z}^k$ to $\mathbf{Z}^{k+1} \oplus \frac{H_{n-1}(M \backslash A_{k+1})}{t(H_{n-1}(M \backslash A))}$. Because the projection

$$q : \mathbf{Z}^{k+1} \oplus \frac{H_{n-1}(M \backslash A_{k+1})}{t(H_{n-1}(M \backslash A))} \to \mathbf{Z}^{k+1}$$

is also surjective, it follows that $q \circ \frac{\varphi}{t(\varphi)} : \mathbf{Z}^k \to \mathbf{Z}^{k+1}$ is a surjective group homomorphism which means that

$$0 \to \ker(q \circ \frac{\varphi}{t(\varphi)}) \to \mathbf{Z}^k \to \mathbf{Z}^{k+1} \to 0$$

is an exact sequence which obviuosly splits, that is $\mathbf{Z}^k \simeq \ker(q \circ \frac{\varphi}{t(\varphi)}) \oplus \mathbf{Z}^{k+1}$. But this is impossible because all the subgroups of $\mathbf{Z}^k$ have smaler rank than k. The proof of statement (ii) is the same with that of (i).□

3. Application

Let $f : M \to N$ be a differentiable mapping acting between differentiable manifolds. Recall that $C(f) \big/ R(f)$ denotes the critical$\big/$regular set of f and $B(f) = f(C(f))$ the set of its critical values.

The subject of this section arises from the well known critical point problem of the existence of a function with a prescribed critical set and it was intensively studied in [3], [4].

Definition 3.1. We say that a differentiable mapping $f : M \to N$ *is separating the critical values by the regular ones* if $B(f) \cap f(R(f)) = \emptyset$. Denote by $CS^\infty(M, N)$ the set of all of these mappings.

Proposition 3.2. (i) $f \in CS^\infty(M,N)$ *iff* $C(f) = f^{-1}(B(f))$.

(ii) If M is a connected differentiable manifold and $f \in CS^\infty(M,\mathbf{R})$ is such that $R(f) = M\backslash C(f)$ is also connected, then $f(R(f)) = (m_f, M_f)$, where $m_f = \inf_{x\in M} f(x), M_f = \sup_{x\in M} f(x)$ and $B(f) \subseteq \{m_f, M_f\} \cap \mathbf{R}$. Moreover, If M is compact, then $m_f, M_f \in \mathbf{R}$ and $B(f) = \{m_f, M_f\}$.

Proof. (i) Indeed if $f \in CS^\infty(M,N)$ we only have to show that $f^{-1}(B(f)) \subseteq C(f)$ because the other inclusion is always true. Hence if $p \in f^{-1}(B(f))$ it follows that $f(p) \in B(f) = f(C(f)) \subseteq N\backslash f(R(f))$, that is $f(p) \notin f(R(f))$ meaning that $p \notin R(f)$.

Conversely, if $C(f) = f^{-1}(B(f))$ asume that $B(f) \cap f(R(f)) \neq \emptyset$ and cosnsider $q \in f(C((f)) \cap f(R(f))$. This means that there exists $p_1 \in C(f), p_2 \in R(f)$ such that $f(p_1) = f(p_2) = q$, that is $p_2 \in f^{-1}(q) \subseteq f^{-1}(B(f)) = C(f)$, which is of course a contradiction with the obvious fact $R(f) \cap C(f) = \emptyset$.

(ii) If f is non constant, then oviously $f(R(f)) \subseteq (m_f, M_f)$. On the other hand, we can choose two regular values q_1, q_2 in $Imf \subseteq [m_f, M_f]$ arbitrarily close to m_f and M_f respectively. We can also take two regular points $p_1 \in f^{-1}(q_1), p_2 \in f^{-1}(q_2)$ and connect p_1, p_2 by a differential path in $R(f)$. Because f is separating critical points by the regular ones, this path is applied by f on a segment of regular values in Imf connecting q_1 with q_2. Hence the interval $[q_1, q_2]$ is completely contained in $Imf\backslash B(f)$. Consequently we have shown in this way that $(m_f, M_f) \subseteq Imf\backslash B(f) = f(R(f))$ and that $B(f) \subseteq \{m_f, M_f\} \cap \mathbf{R}$.

Theorem 3.3. *Let $p : E \to M^n$, $n \geq 2$ be a differential fibration with fiber F and A be a closed countable subset of M.*

(i) If $H_i(M) \simeq 0$ for $0 < i < s$, $H_j(F) \simeq 0$ for $0 < j < t$ $(s + t \geq n)$ and $H_{n-2}(F) \simeq 0$, then there is no any proper mapping $f \in CS^\infty(E,\mathbf{R})$ such that $C(f) = p^{-1}(A)$.

(ii) If $\pi_{n-2}(F) \simeq 0$, and M is n-connected, then there is no any proper mapping $f \in CS^\infty(E,\mathbf{R})$ such that $C(f) = p^{-1}(A)$.

Proof. Assume that there exists a proper mapping $f \in CS^\infty(E,\mathbf{R})$ such that $C(f) = p^{-1}(A)$. This means that $B(f) = \{m_f, M_f\}$ and on the other hand that its restriction

$$E\backslash C(f) \to Imf\backslash B(f) = (m_f, M_f),\ p \mapsto f(p)$$

is a proper submersion, that is a localy trivial fibration, via Ehresmann's theorem, whose compact fiber we are denoting by $\mathcal{F}$. Its base space (m_f, M_f) being contractible, it follows that the inclusion $i_{\mathcal{F}} : \mathcal{F} \hookrightarrow E\backslash C(f)$ is a weak homotopy equivalence, namely the induced group homomorphisms

$$\pi_q(i_{\mathcal{F}}) : \pi_q(\mathcal{F}) \to \pi_q(E\backslash C(f))$$

are all isomorphisms. Consequentely, using the Whitehead theorem (see for instance [1, pp. 167] or [7, pp. 399]), it follows that the induced group homomorphisms

$$H_q(i_{\mathcal{F}}) : H_q(\mathcal{F}) \to H_q(E\backslash C(f)) = H_q(E\backslash p^{-1}(A))$$

are also all isomorphisms.

(i) The Serre homology sequence of the fibration $F \hookrightarrow E\backslash p^{-1}(A) \xrightarrow{p} M\backslash A$ is:

$$H_{s+t-1}(F) \to \cdots \to H_{n-1}(F) \to H_{n-1}\big(E\backslash p^{-1}(A)\big) \xrightarrow{H_{n-1}(p)} H_{n-1}(M\backslash A) \to H_{n-2}(F) \to \cdots .$$

Because $H_{n-2}(F) \simeq 0$, it follows that $H_{n-1}(p) : H_{n-1}\big(E\backslash p^{-1}(A)\big) \to H_{n-1}(M\backslash A)$ is surjective. Therefore

$$H_{n-1}(p) \circ H_{n-1}(i_{\mathcal{F}}) : H_{n-1}(\mathcal{F}) \to H_{n-1}(M\backslash A)$$

is a surjective group homomorphism from the finitely generated abelian group $H_{n-1}(\mathcal{F})$ to $H_{n-1}(M\backslash A)$, which is a contradiction with corollary 2.5 (i).

(ii) The homotopy sequence of the fibration $F \hookrightarrow E\backslash p^{-1}(A) \xrightarrow{p} M\backslash A$ is:

$$\cdots \to \pi_{n-1}(F) \to \pi_{n-1}\big(E\backslash p^{-1}(A)\big) \xrightarrow{\pi_{n-1}(p)} \pi_{n-1}(M\backslash A) \to \pi_{n-2}(F) \to \cdots .$$

Because $\pi_{n-2}(F) \simeq 0$, it follows that $\pi_{n-1}(p) : \pi_{n-1}\big(E\backslash p^{-1}(A)\big) \to \pi_{n-1}(M\backslash A)$ is surjective. On the other hand, according to corollary 2.2, the natural group homorphism

$$\chi_{n-1} : \pi_{n-1}(E, E\backslash p^{-1}(A)) \to H_{n-1}(E, E\backslash p^{-1}(A))$$

is an isomorphism. Therefore

$$\pi_{n-1}(p) \circ \chi_{n-1}^{-1} \circ H_{n-1}(i_{\mathcal{F}}) : H_{n-1}(\mathcal{F}) \to \pi_{n-1}(M\backslash A)$$

is surjective group homomorphism from the finitely generated abelian group $H_{n-1}(\mathcal{F})$ to $\pi_{n-1}(M\backslash A)$, which is a contradiction with corollary 2.5 (ii).□

Corollary 3.4. *Let $p : E \to M$ be a differential fibration with fiber F, having the total space E compact and A be a closed countable subset of M. If $H_i(M) \simeq 0$ for $0 < i < s$, $H_j(F) \simeq 0$ for $0 < j < t$ $(s + t \geq n)$ and $H_{n-2}(F) \simeq 0$, where $2 \leq n := \dim M$, then there is no any mapping $f \in CS^\infty(E, \mathbf{R})$ such that $C(f) = p^{-1}(A)$.*

Example 3.5. *(i)* Let $S^3 \hookrightarrow S^7 \to S^4$ be the quaternionic Hopf fibration and A be a closed countable subset of S^4. There is no any mapping $f \in CS^\infty(S^7, \mathbf{R})$ such that $C(f) = p^{-1}(A)$.

(ii) If A is a closed countable subset of $\mathbf{R}^n$, $n \geq 4$ then thre is no any proper mapping f in $CS^\infty(\mathbf{R}^n \times T^{n-2}, \mathbf{R})$ such that $C(f) = p^{-1}(A) = A \times T^{n-2}$, where $p : \mathbf{R}^n \times T^{n-2} \to \mathbf{R}^n$ is the projection.

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