The geometry of nondegeneracy conditions in completely integrable systems

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Abstract

Nondegeneracy conditions need to be imposed in K.A.M. theorems to insure that the set of diophantine tori has a large measure. Although they are usually expressed in action coordinates, it is possible to give a geometrical formulation using the notion of regular completely integrable systems defined by a fibration of a symplectic manifold by lagrangian tori together with a Hamiltonian function constant on the fibers. In this paper, we give a geometrical definition of different nondegeneracy conditions, we show the implication relations that exist between them, and we show the uniqueness of the fibration for non-degenerate Hamiltonians.

Résumé

Dans les théorèmes de type K.A.M., on doit imposer des conditions de non-dégénérescence pour assurer que l'ensemble des tores diophantiens a une grande mesure. Elles sont habituellement présentées en coordonnées actions, mais il est possible d'en donner une formulation géométrique en considérant des systèmes complètement intégrables définis par la donnée d'une fibration d'une variété symplectique par des tores lagrangiens et d'un Hamiltonien constant sur les fibres. Dans cet article, nous donnons une définition géométrique de différentes conditions de non-dégénérescence, nous montrons les différentes relations d'implication qui existent entre elles, et nous montrons l'unicité de la fibration pour les Hamiltoniens non-dégénérés.

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Introduction

On a symplectic manifold $(\mathcal{M}, \omega)$, the completely integrable systems (CI in short) are the dynamical systems defined by a Hamiltonian $H \in C^{\infty}(\mathcal{M})$ admitting a *momentum map*, i.e. a set $\boldsymbol{A} = (A_1, ..., A_d) : \mathcal{M} \to \mathbb{R}^d$ of smooth functions, d being half of the dimension of $\mathcal{M}$, satisfying $\{A_j, H\} = 0$ and $\{A_j, A_k\} = 0$ for all $j, k : 1...d$, and whose differentials dA_j are linearly independent almost everywhere. Then, the Arnol'd-Mineur-Liouville Theorem [1, 7, 6] insures that in a neighbourhood of any connected component of any compact regular fiber $\boldsymbol{A}^{-1}(a)$, $a \in \mathbb{R}^d$, there exists a fibration in lagrangian tori along which H is constant. These tori are thus invariant by the dynamics generated by the associated Hamiltonian vector field X_H.

Despite the "local" character of the Arnol'd-Mineur-Liouville Theorem, one might be tempted to try to glue together these "local" fibrations in the case of *regular* Hamiltonians, i.e. those for which there exists, near each point of $\mathcal{M}$, a local fibration in invariant lagrangian tori. Nevertheless, we would like to stress the fact that not all regular completely integrable Hamiltonians are constant along the fibers of a fibration in lagrangian tori. For example, a free particle moving on the sphere S^2 can be described by a Hamiltonian system on the symplectic manifold T^*S^2. If we restrict ourself to the symplectic manifold $\mathcal{M} = T^*S^2 \setminus S^2$, we can easily show that $\mathcal{M}$ is diffeomorphic to $SO(3) \times \mathbb{R}$ and that the Hamiltonian H depends only on the second factor. The energy levels $H = cst$ are thus diffeomorphic to $SO(3)$. On the other hand, if there exists a fibration in lagrangian tori such that H is constant along the fibers, then each energy level is itself fibered by tori. But a simple homotopy group argument shows that there exists no fibration of $SO(3)$ by tori.

This example actually belongs to the non-generic (within the class of regular CI Hamiltonians) class of *degenerate* Hamiltonians. Those Hamiltonians might not admit any (global) fibration in lagrangian tori, or they might admit several different ones. But, as we will see, imposing a nondegeneracy condition insures that there exists a global a fibration of $\mathcal{M}$ in lagrangian tori along which H is constant, and moreover that it is unique.

On the other hand, nondegeneracy conditions arise in the K.A.M. theory where one studies the small perturbations $H + \varepsilon K$ of a given CI Hamiltonian H. The K.A.M. Theorem actually deals with the *regular part* of a completely integrable system and is usually expressed in angle-action coordinates. This theorem actually gives two independent statements. The first statement is that the tori on which X_H verifies a certain diophantine relation are only slightly deformed and not destroyed by the perturbation εK, provided ε is sufficiently small. The second statement is that the set of these tori has a large mesure whenever H is non-degenerate.

There exist different K.A.M. theorems based on different nondegeneracy conditions, such as the earliest ones of Arnol'd [1] and Kolmogorov [5], or those introduced later by Bryuno [2] and Rüssmann [8]. They are always presented in action-angle coordinates and this hides somehow their geometrical content. But they can be expressed in a geometric way if we consider CI systems defined on a symplectic manifold $\mathcal{M}$ by a fibration in lagrangian tori

$\mathcal{M} \xrightarrow{\pi} \mathcal{B}$, where $\mathcal{B}$ is any manifold, together with a Hamiltonian $H \in C^\infty(\mathcal{M})$ constant along the fibers $\pi^{-1}(b)$, $b \in \mathcal{B}$. Such a Hamiltonian must have the form $H = F \circ \pi$, with $F \in C^\infty(\mathcal{B})$, and all the nondegeneracy conditions express simply in function of F and of a torsion-free and flat connection which naturally exists on the base space $\mathcal{B}$ of the fibration.

In the first section, we review the geometric structures associated with a fibration in lagrangian tori that allow one to define the connection on the base space. In Section 2, we give several nondegeneracy conditions, including those mentionned above, expressed both in a geometric way and in flat coordinates. Then, we show in Section 3 the different implication relations that exist between these different conditions. Finally, in the last section, we give some properties of non-degenerate CI hamiltonians, as for example the uniqueness of the fibration in lagrangian tori.

1 Geometric setting

Let $(\mathcal{M}, \omega)$ be a symplectic manifold of dimension $2d$ and let $\left(H, \mathcal{M} \xrightarrow{\pi} \mathcal{B}\right)$ be a regular CI system composed of a fibration in lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ together with a Hamiltonian $H \in C^\infty(\mathcal{M})$ constant along the fibers $\mathcal{M}_b = \pi^{-1}(b)$, $b \in \mathcal{B}$. Since by definition the fibers are connected, H must be of the form $H = F \circ \pi$, with $F \in C^\infty(\mathcal{B})$. On the other hand, Duistermaat showed in [4] that there exists a natural torsion-free and flat connection ∇ on the base space $\mathcal{B}$ of each fibration in lagrangian tori. It can be seen as follows.

First of all, a theorem due to Weinstein [9, 10] insures that there exists a natural torsion-free and flat connection on each leaf of a lagrangian foliation. Moreover, whenever this foliation defines locally a fibration, then the holonomy of the connection must vanish. Given a fibration in lagrangian tori $\left(H, \mathcal{M} \xrightarrow{\pi} \mathcal{B}\right)$, the space $\mathcal{V}_\nabla(\mathcal{M}_b)$ of parallel vector fields on $\mathcal{M}_b = \pi^{-1}(b)$, for each $b \in \mathcal{B}$, is thus a vector space of dimension d, and the union $\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$ is actually endowed with a structure of a smooth vector bundle over $\mathcal{B}$.

On the other hand, since each fiber $\mathcal{M}_b$ is a standard[1] affine torus, one can define for each $b \in \mathcal{B}$ the space $\Lambda_b \subset \mathcal{V}_\nabla(\mathcal{M}_b)$ of 1-periodic parallel vector fields on $\mathcal{M}_b$, which is easily shown to be a lattice of $\mathcal{V}_\nabla(\mathcal{M}_b)$. Moreover, one can show that the union $\Lambda = \bigcup_{b\in\mathcal{B}} \Lambda_b$ is a smooth lattice subbundle of $\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$, called the *period bundle*. This is in fact the geometrical content of the Arnol'd-Mineur-Liouville Theorem [1, 7, 6]. To prove this, one constructs explicitly smooth sections of $\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$ which are 1-periodic, namely Hamiltonian vector fields $X_{\xi\circ\pi}$ whose Hamiltonian is the pullback of a function $\xi \in C^\infty(\mathcal{M})$ of a special type and called *action*. Now, the symplectic form on $\mathcal{M}$ provides for each b an isomorphism between $\mathcal{V}_\nabla(\mathcal{M}_b)$ and $T_b^*\mathcal{B}$. The image of the bundle Λ by this isomorphism is then a smooth lattice subbundle E^* of $T^*\mathcal{B}$, called the *Action bundle*, and its dual E (called the *Resonance bundle*) is a smooth lattice subbundle of $T\mathcal{B}$. This lattice subbundle E provides a way to associate the tangent spaces T_bB for neighbouring points b. This thus

[1]Here, "standard" means holonomy-free.

implies the existence of a natural integer, torsion-free and flat connection ∇ on the base space $\mathcal{B}$ (as discovered by Duistermaat [4]). Actually, *angle-action* coordinates are semi-global canonical coordinates $(x, \xi) : \pi^{-1}(\mathcal{O}) \to \mathbb{T}^d \times \mathbb{R}^d$, where $\mathcal{O}$ is an open subset of $\mathcal{B}$, with the properties that the x_j's are flat (with respect to Weinstein's connection) coordinates on the tori, and the differentials $d\xi_j$ are smooth sections of the Action bundle E^* (this implies that the coordinates ξ_j are flat with respect to Duistermaat's connection on $\mathcal{B}$).

In the sequel, the space of parallel vector fields will be denoted by $\mathcal{V}_\nabla(\mathcal{B})$ and the space of parallel 1-forms by $\Omega^1_\nabla(\mathcal{B})$. We mention that in general the holonomy of Duistermaat's connection does not vanish. As a consequence, the spaces $\mathcal{V}_\nabla(\mathcal{B})$ and $\Omega^1_\nabla(\mathcal{B})$ might be empty. Nevertheless, when one works locally in a simply connected subset $\mathcal{O} \subset \mathcal{B}$, the spaces of local parallel sections $\mathcal{V}_\nabla(\mathcal{O})$ and $\Omega^1_\nabla(\mathcal{O})$ are d-dimensional vector spaces.

2 Different nondegeneracy conditions

Let $\left(H, \mathcal{M} \xrightarrow{\pi} \mathcal{B}\right)$ be a regular CI system composed of a fibration in lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ together with a Hamiltonian $H \in C^\infty(\mathcal{M})$ constant along the fibers. As mentionned before, H is of the form $H = F \circ \pi$, with $F \in C^\infty(\mathcal{B})$. It turns out that all the nondegeneracy conditions are expressed in terms of the function F and Duistermaat's connection ∇ that naturally exists on $\mathcal{B}$. On the other hand, these conditions are local : F is said to be *non-degenerate* if is non-degenerate at each $b \in \mathcal{B}$. Moreover, some of these conditions are expressed in terms of the space of parallel vector fields $\mathcal{V}_\nabla(\mathcal{B})$, but the local character of the nondegeneracy conditions means that one needs actually only the spaces $\mathcal{V}_\nabla(\mathcal{O})$ of local parallel vector fields in a neighbourhood $\mathcal{O} \subset \mathcal{B}$ of each point $b \in \mathcal{B}$. We will use a slight misuse of language and say "for each $X \in \mathcal{V}_\nabla(\mathcal{B})$" instead of "for each $b \in \mathcal{B}$, each neighbourhood $\mathcal{O} \subset \mathcal{B}$ of b and each $X \in \mathcal{V}_\nabla(\mathcal{O})$".

For our purposes, let's define for each $X \in \mathcal{V}_\nabla(\mathcal{B})$ the function $\Omega_X \in C^\infty(\mathcal{B})$ by $\Omega_X = dF(X)$ and the associated *resonance set*

$$\Sigma_X = \{b \in \mathcal{B} \mid \Omega_X(b) = 0\}.$$

We will also denote by $\mathcal{K} = \bigcup_b \mathcal{K}_b$ the integrable distibution of hyperplanes $\mathcal{K}_b \subset T_b\mathcal{B}$ tangent to the hypersurfaces $F = cst$, i.e. $\mathcal{K}_b = \ker dF|_b$. The *Hessian* $\nabla\nabla F$, which is a $(0,2)$-tensor field on $\mathcal{B}$, will be denoted by F''. It is symmetric since ∇ is torsion-free. The connection ∇ also yields an identification of the cotangent spaces $T_b^*\mathcal{B}$ at neighbouring points b and allows us to define the *frequency map* $\varphi : \mathcal{B} \to \Omega^1_\nabla(\mathcal{B})$ by $\varphi(b) = dF_b^\nabla$, where $dF_b^\nabla \in \Omega^1_\nabla(\mathcal{B})$ is the parallel 1-form which coincides with dF at the point b. In the sequel, the expression $A \propto B$ means that the vectors A and B are linearly dependent.

We now review different nondegeneracy conditions, including those used in the literature. We give both a geometrical formulation and the corresponding (usual) formulation in flat coordinates.

Condition "Kolmogorov" : For each $b \in \mathcal{B}$, the Hessian, seen as a linear map $F''_b : T_b\mathcal{B} \to T^*_b\mathcal{B}$, is invertible, i.e

$$X \in T_b\mathcal{B}, \nabla_X \nabla F = 0 \Longrightarrow X = 0.$$

In flat coordinates, this condition reads :

$$\det\left(\frac{\partial^2 F}{\partial \xi_j \partial \xi_k}\right) \neq 0.$$

Condition "Locally diffeomorphic frequency map" : The frequency map is a local diffeomorphism. In flat coordinates, this condition means that the map $\varphi : \xi_j \in \mathbb{R}^d \to \frac{\partial F}{\partial \xi_k} \in \mathbb{R}^d$ is a local diffeomorphism.

Condition "Iso-energetic" or "Arnol'd" : For each $b \in \mathcal{B}$ the restriction to $\mathcal{K}_b$ (for the two slots) of the Hessian $F''|_{\mathcal{K}_b} : \mathcal{K}_b \to \mathcal{K}^*_b$ is invertible, i.e.

$$X \in \mathcal{K}_b, \nabla_X \nabla F \propto dF \Longrightarrow X = 0$$

In flat coordinates, this condition reads :

$$\det\begin{pmatrix} \begin{bmatrix} \ddots & & \\ & \frac{\partial^2 F}{\partial \xi_j \partial \xi_k} & \\ & & \ddots \end{bmatrix} & \begin{bmatrix} \vdots \\ \frac{\partial F}{\partial \xi_j} \\ \vdots \end{bmatrix} \\ \begin{bmatrix} \cdots & \frac{\partial F}{\partial \xi_k} & \cdots \end{bmatrix} & 0 \end{pmatrix} \neq 0.$$

Condition "Bryuno" : For each $b \in \mathcal{B}$, the set of vectors $X \in T_b\mathcal{B}$ satisfying $\nabla_X \nabla F \propto dF$ is 1-dimensionnal, i.e. :

$$X, Y \in T_b\mathcal{B}, \nabla_X \nabla F \propto dF \text{ and } \nabla_Y \nabla F \propto dF \Longrightarrow X \propto Y.$$

This amounts to requiring that for each b, the linear map $\mathbf{U} : T_b\mathcal{B} \oplus \mathbb{R} \to T^*_b\mathcal{B}$ defined by

$$\mathbf{U}(X, \alpha) = \nabla_X \nabla F + \alpha dF$$

has a rank equal to d. In flat coordinates, this condition reads :

$$\operatorname{rank}\begin{pmatrix} \begin{bmatrix} \ddots & & \\ & \frac{\partial^2 F}{\partial \xi_j \partial \xi_k} & \\ & & \ddots \end{bmatrix} & \begin{bmatrix} \vdots \\ \frac{\partial F}{\partial \xi_j} \\ \vdots \end{bmatrix} \end{pmatrix} = d.$$

Condition "N" : For each $b \in \mathcal{B}$, the restriction to $\mathcal{K}_b$ (for the first slot) of the Hessian $F''|_{\mathcal{K}_b} : \mathcal{K}_b \to T^*_b\mathcal{B}$ is injective, i.e. :

$$X \in \mathcal{K}_b, \nabla_X \nabla F = 0 \Longrightarrow X = 0.$$

This is equivalent to requiring that for each b, the linear map $\mathbf{V} : T_b\mathcal{B} \to T_b^*\mathcal{B} \oplus \mathbb{R}$ defined by

$$\mathbf{V}(X) = (\nabla_X \nabla F, dF(X))$$

has a rank equal to d. In flat coordinates, this condition reads :

$$\text{rang}\left(\begin{array}{c}\begin{bmatrix}\ddots & & \\ & \frac{\partial^2 F}{\partial \xi_j \partial \xi_k} & \\ & & \ddots\end{bmatrix} \\ \begin{bmatrix}\cdots & \frac{\partial F}{\partial \xi_k} & \cdots\end{bmatrix}\end{array}\right) = d.$$

Condition "Turning frequencies" : Let $P\left(\Omega^1_\nabla(\mathcal{B})\right)$ denote the projective space of $\Omega^1_\nabla(\mathcal{B})$ and $\pi : \Omega^1_\nabla(\mathcal{B}) \to P\left(\Omega^1_\nabla(\mathcal{B})\right)$ the associated projection. We require the map $\pi \circ \varphi : \mathcal{B} \to P\left(\Omega^1_\nabla(\mathcal{B})\right)$ to be a submersion. In flat coordinates, this condition amounts to requiring that the map $\pi \circ \varphi : \mathbb{R}^d \to P\left(\mathbb{R}^d\right)$ defined by $\xi \to \left[\frac{\partial F}{\partial \xi^j}(\xi)\right]$ is a submersion.

Condition "Iso-energetic turning frequencies" : The restriction of the frequency map $\varphi : \mathcal{B} \to \Omega^1_\nabla(\mathcal{B})$ to each energy level $S_E = \{b \mid F(b) = E\}$ is a local diffeomorphism $\pi \circ \varphi$ between S_E and $P\left(\Omega^1_\nabla(\mathcal{B})\right)$.

Condition "Regular resonant set" : For each point $b \in \mathcal{B}$ and each non-vanishing parallel vector field $X \in \mathcal{V}_\nabla(\mathcal{B})$, one has

$$d(\Omega_X)_b \neq 0.$$

This implies that for each $X \in \mathcal{V}_\nabla(\mathcal{B})$ the resonant set Σ_X is a 1-codimensionnal submanifold of $\mathcal{B}$.

Condition "Resonant set with empty interior " : For each non-vanishing parallel vector field $X \in \mathcal{V}_\nabla(\mathcal{B})$, the resonant set Σ_X has an empty interior.

Condition "Rüssmann" : For each non-vanishing parallel vector field $X \in \mathcal{V}_\nabla(\mathcal{B})$, the image of the frequency map does not annihilate X on an open set. In flat coordinates, this condition means that the image of $\varphi : \xi_j \in \mathbb{R}^d \to \frac{\partial F}{\partial \xi_k} \in \mathbb{R}^d$ does not lie in any hyperplane passing through the origin.

3 Hierarchy of conditions

First of all, we will show that

$$\boxed{\text{Turning frequencies}} \Leftrightarrow \boxed{\text{Bryuno}} \Leftrightarrow \boxed{\text{N}} \Leftrightarrow \boxed{\text{Regular resonant set}}$$

Consequently, those four equivalent conditions will be denoted by "*Weak nondegeneracy*". Then, we will show the following equivalences.

$$\boxed{\text{Iso-energetic}} \iff \boxed{\text{Iso-energetic turning frequencies}}$$

$$\boxed{\text{Kolmogorov}} \iff \boxed{\text{Locally diffeomorphic frequency map}}$$

$$\boxed{\text{Rüssmann}} \iff \boxed{\text{Resonant set with empty interior}}$$

In the second subsection, we will show that we have the following implications :

$$\boxed{\text{Rüssmann}} \Longleftarrow \boxed{\text{Weak}} \iff \boxed{\begin{array}{c}\text{Kolmogorov}\\ \text{or}\\ \text{Iso-energetic}\end{array}}$$

3.1 Equivalent conditions

Proposition 1. *Condition "Bryuno" is equivalent to Condition "N".*

Proof. Consider the linear maps $\mathbf{U} : T_b\mathcal{B} \oplus \mathbb{R} \to T_b^*\mathcal{B}$ and $\mathbf{V} : T_b\mathcal{B} \to T_b^*\mathcal{B} \oplus \mathbb{R}$ of Conditions "Bryuno" and "N". We will show that $\mathbf{U}^t = \mathbf{V}$. Indeed, for each $X \in T_b\mathcal{B}$ and each $(Y, \alpha) \in T_b\mathcal{B} \oplus \mathbb{R}$, the transposed $\mathbf{U}^t : T_b\mathcal{B} \to T_b^*\mathcal{B} \oplus \mathbb{R}$ verifies :

$$\begin{aligned}
\mathbf{U}^t (X) (Y, \alpha) &= \mathbf{U} (Y, \alpha) (X) \\
&= F^{''} (Y, X) + \alpha dF (X) \\
&= F^{''} (X, Y) + \alpha dF (X) ,
\end{aligned}$$

where the symmetry of $F^{''}$ has been used. Therefore, one has

$$\mathbf{U}^t (X) (Y, \alpha) = \mathbf{V} (X) (Y, \alpha) .$$

This implies that rank $(\mathbf{V})$ = rank $\left(\mathbf{U}^t\right)$ = rank $(\mathbf{U})$ and thus that Conditions "N" and "Bryuno" are equivalent. $\square$

Proposition 2. *Condition "Bryuno" is equivalent to Condition "Turning frequencies".*

Proof. Let $dF_b^{\nabla} = \varphi (b)$ be the parallel 1-form which coincides with dF at the point b. Condition "TF" ("Turning frequencies") means that the derivative $(\pi \circ \varphi)_*$ of the map $\pi \circ \varphi : \mathcal{B} \to P\left(\Omega^1_{\nabla} (\mathcal{B})\right)$ is surjective. Since $P\left(\Omega^1_{\nabla} (\mathcal{B})\right)$ is $(d-1)$-dimensionnal, Condition "TF" means that the kernel of $(\pi \circ \varphi)_*$ is 1-dimensionnal. Now, the kernel of $(\pi \circ \varphi)_*$ is precisely the space of X such that $\varphi_* (X)$ is in the kernel of π_*, i.e. tangent to the fibers π^{-1}. Using the natural isomorphism between $\Omega^1_{\nabla} (\mathcal{B})$ and its tangent space $T\left(\Omega^1_{\nabla} (\mathcal{B})\right)$, one easily sees that $\ker (\pi \circ \varphi)_*$ is composed of the vectors X such that $\varphi_* X_b \propto \varphi (b)$. Condition "TF" is thus equivalent to requiring that if X_b and Y_b satisfy $\varphi_* X_b \propto \varphi (b)$ and $\varphi_* Y_b \propto \varphi (b)$, then $X_b \propto Y_b$.

On the other hand, we show that $\varphi_* X_b = \nabla_{X_b} \nabla F$. Indeed, let $t \to b (t)$ be a geodesic $t \to b (t)$, passing through b at $t = 0$, and let X_b the tangent vector of $b (t)$ at b. In a neigbourhood of b, one can extend X_b in a unique way to a parallel vector field X. We thus have $\phi_X^t(b) = b(t)$. We want to calculate $\varphi_* X_b = \frac{d}{dt}(\varphi(b(t)))_{t=0}$. By definition, for each t, $\varphi(b(t)) = dF^{\nabla}_{b(t)}$ is the parallel 1-form which coincides with $dF_{b(t)}$ at the point $b(t)$. It is invariant by the flow of any parallel vector field, and thus by the one of X, i.e.

$$(\varphi(b(t)))_b = \left(\phi_X^t\right)^* \left(dF^{\nabla}_{b(t)}\right)_{b(t)} = \left(\left(\phi_X^t\right)^* dF\right)_b .$$

Using the definition of the Lie derivative, one obtains $\varphi_* X_b = (\mathcal{L}_X(dF))_b$. The Cartan's magic formula then gives $\varphi_* X_b = d(dF(X))_b$. Finally, since X is parallel, we find that $\varphi_* X_b = (\nabla_X \nabla F)_b$.

Together with the previous result, we have thus proved that is X_b and X_b satisfy $(\nabla_{X_b}\nabla F)_b \propto dF_b$ et $(\nabla_{Y_b}\nabla F)_b \propto dF_b$, then they must be linearly dependent. This is precisely Condition "Bryuno". ☐

Proposition 3. *Condition "N" is equivalent to Condition "Regular resonant set".*

Proof. Indeed, for each $X \in \mathcal{V}_\nabla(\mathcal{B})$, one has $\Omega_X = \nabla_X F$ and thus $d(\Omega_X) = \nabla\nabla_X F = \nabla_X \nabla F$. Conditions ""Regular resonant set" thus reads

$$\forall X \in \mathcal{V}_\nabla(\mathcal{B}), \forall b \text{ with } X \in \mathcal{K}_b \Longrightarrow (\nabla_X \nabla F)_b \neq 0.$$

This is equivalent to

$$\forall b, \forall X \in \mathcal{K}_b \Longrightarrow (\nabla_X \nabla F)_b \neq 0,$$

i.e. precisely Condition "N". ☐

Proposition 4. *Condition "Iso-energetic" is equivalent to Condition "Iso-energetic turning frequencies".*

Proof. Condition "Iso-energetic turning frequencies" amounts to requiring that at each point b the map $(\pi \circ \varphi)_*$ restricted to an energy level $S_E = F^{-1}(E)$ is a isomorphism between $T_b S_E$ and $T_{\pi(\varphi(b))} P\left(\Omega^1_\nabla(\mathcal{B})\right)$, i.e. the kernel of $(\pi \circ \varphi)_*$ is transverse to S_E. Arguing as is the proof of Proposition 2, this is equivalent to requiring that if X_b verifies $(\nabla_{X_b}\nabla H)_b \propto dH_b$, then X_b must be transverse to S_E, i.e. $dF(X_b) \neq 0$, which is precisely Condition "Iso-energetic". ☐

Proposition 5. *Condition "Russmann" is equivalent to Condition "Resonant set with empy interior".*

Proof. By definition of the frequency map φ, for each $X \in \mathcal{V}_\nabla(\mathcal{B})$, one has $\varphi(b)(X) = dF^{\nabla}_b(X)$. Since both X and dF^{∇}_b are parallel, the contraction $dF^{\nabla}_b(X)$ is a constant function on $\mathcal{B}$ and thus equals to its value, e.g. at the point b which is nothing but $dF_b(X) = \Omega_X(b)$. We thus have $\varphi(.)(X) = \Omega_X(.)$. Now, $\varphi(b)$ annihilates X on an open subset iff the resonant set $\Sigma_X = \Omega_X^{-1}(0)$ contains this open subset, and therefore its interior is not empty. ☐

3.2 Stronger and weaker conditions

Proposition 6. *Condition "Iso-energetic" implies Condition "Weak".*

Proof. Condition "Iso-energetic" means that for each $X \in \mathcal{K}_b$, the 1-form $\nabla_X \nabla F|_{\mathcal{K}_b}$ is non-vanishing. This property remains true whithout the restriction to $\mathcal{K}_b$, i.e.

$$X \in \mathcal{K}_b \Longrightarrow \nabla_X \nabla F \neq 0,$$

which is Condition "N". $\square$

Proposition 7. *Condition "Kolmogorov" implies Condition "Weak".*

Proof. Condition "Kolmogorov" means that for each $X \in T_b\mathcal{B}$, one has $\nabla_X \nabla F \neq 0$. By restriction, this remains true for each $X \in \mathcal{K}_b$, which is Condition "N". $\square$

Proposition 8. *Condition "Weak" implies "Kolmogorov or Iso-energetic".*

Proof. We will actually show the following equivalent logical statement : if Condition "Weak" is fulfilled but not Condition "Kolmogorov", then Condition "Iso-energetic" is fulfilled. Suppose that there is a vector $X \in T_b\mathcal{B}$ such that $\nabla_X \nabla F = 0$ at b. We can then extend X around b to a parallel vector field, and thanks to the symmetry of F'', one has $\nabla \nabla_X F = 0$ at b, i.e. $(d\Omega_X)_b = 0$. On the other hand, if Condition "Regular resonant set" is fulfilled, this implies that b cannot belong to the resonant set Σ_X, and thus $X \notin \mathcal{K}_b$. Moreover, Condition "Bryuno" insures that each Y satisfying $\nabla_X \nabla F \propto dF$ at b must be linearly dependent on X, i.e. $Y \propto X$. We thus have showed that for each $Y \in \mathcal{K}_b$ satisfying $\nabla_Y \nabla F \propto dF$, one has $Y \propto X$ and therefore $Y \notin \mathcal{K}_b$. This implies that $Y = 0$. It is precisely Condition "Iso-energetic". $\square$

Proposition 9. *Condition "Weak" implies Condition "Rüssmann".*

Proof. Indeed, Condition "Regular resonant set" implies that for each non-vanishing parallel vector field $X \in \mathcal{V}_\nabla(\mathcal{B})$, the resonant set Σ_X is a 1-codimensional submanifold, and thus has an empty interior. $\square$

3.3 Examples

Example. "Kolmogorov" and "Iso-energetic". On $\mathcal{B} = \mathbb{R}^d \setminus 0$, consider the function $F(\xi) = \frac{1}{2}|\xi|^2$, where $|\xi|^2 = \sum_{j=1}^d (\xi_j)^2$. The differential is $dF = \sum_{j=1}^d \xi_j d\xi_j$ and the Hessian $F_{ij}(\xi) = \delta_{ij}$ is the identity matrix at each point ξ. . Therefore, one has $\det(F_{ij}) = 1$, which means that F actually satisfies Condition "Kolmogorov" on $\mathbb{R}^d$. On the other hand, a straightforward calculation yields

$$\det \begin{pmatrix} \begin{bmatrix} \ddots & & \\ & \frac{\partial^2 F}{\partial \xi_j \partial \xi_k} & \\ & & \ddots \end{bmatrix} & \begin{bmatrix} \vdots \\ \frac{\partial F}{\partial \xi_j} \\ \vdots \end{bmatrix} \\ \begin{bmatrix} \cdots & \frac{\partial F}{\partial \xi_k} & \cdots \end{bmatrix} & 0 \end{pmatrix} = -|\xi|^2 .$$

This is non-zero whenever $\xi \neq 0$, and thus F satisfies both Conditions "Kolmogorov" and "Iso-energetic" on $\mathcal{B} = \mathbb{R}^d \setminus 0$.

Example. "Iso-energetic" but not "Kolmogorov". On $\mathcal{B} = \mathbb{R}^d \setminus 0$, let us consider the function $F(\xi) = |\xi|$. The differential is $dF = \frac{\sum_{j=1}^{d} \xi_j d\xi_j}{|\xi|}$ and the Hessian is $F_{ij}(\xi) = \frac{\delta_{ij}}{|\xi|} - \frac{\xi_i \xi_j}{|\xi|^3}$. We can see that Condition "Kolmogorov" is nowhere satisfied since for each ξ, the vector $X_j = \xi_j$ verifies $\nabla_X \nabla F = 0$. Indeed, for each i one has

$$\sum_j F_{ij} X_j = \frac{\xi_i}{|\xi|} - \frac{\xi_i |\xi|^2}{|\xi|^3} = 0.$$

Nevertheless, Condition "Iso-Energetic" is satisfied since whenever a vector X verifies $\nabla_X \nabla F \propto \nabla F$ and $\nabla_X F = 0$, this means that

$$\begin{cases} X_i - \sum_j \frac{\xi_i \xi_j X_j}{|\xi|^2} = \lambda \xi_i \\ \sum_j X_j \xi_j = 0. \end{cases}$$

By inserting the second equation into the first one, one must have

$$\begin{cases} X_i = \lambda \xi_i \\ \sum_j X_j \xi_j = 0, \end{cases}$$

and this is possible only for $\xi = 0$, i.e. Condition "Iso-Energetic" is satisfied on $\mathcal{B} = \mathbb{R}^d \setminus 0$.

Example. "Kolmogorov" but not "Iso-energetic". On $\mathcal{B} = \mathbb{R}^+ \times \mathbb{R}$, consider the function $F(\xi) = \frac{\xi_1^3}{3} + \frac{\xi_2^2}{2}$. The differential is $dF = \xi_1^2 d\xi_1 + \xi_2 d\xi_2$ and the Hessian is $F''(\xi) = \begin{pmatrix} 2\xi_1 & 0 \\ 0 & 1 \end{pmatrix}$. The determinant of F'' is thus simply $\det\left(F''\right) = 2\xi_1$ which is non-zero on $\mathcal{B}$. Condition "Kolmogorov" is thus satisfied. Nevertheless, one easily verifies that

$$\det \begin{pmatrix} 2\xi_1 & 0 & \xi_1^2 \\ 0 & 1 & \xi_2 \\ \xi_1^2 & \xi_2 & 0 \end{pmatrix} = -\xi_1^4 - 2\xi_1 \xi_2^2 = -\xi_1 \left(\xi_1^3 + 2\xi_2^2\right).$$

Outside from $\xi_1 = 0$, this determinant vanishes on the curve give by the equation $\xi_1^3 + 2\xi_2^2 = 0$. This means that Condition "Iso-energetic" is not satisfied on this curve.

Example. "Rüssmann" but not "Weak". On $\mathcal{B} = \mathbb{R}^2 \setminus 0$, let us consider the function $F(\xi) = \xi_1^4 + \xi_2^4$. The differential is $dF = 4\xi_1^3 d\xi_1 + 4\xi_2^3 d\xi_2$, implying that for each $X \in \mathbb{R}^2$, one has $\Omega_X = dF(X) = 4\xi_1^3 X_1 + 4\xi_2^3 X_2$. The associated resonant set $\Sigma_X = \left\{(\xi_1, \xi_2) \mid \xi_1^3 X_1 + \xi_2^3 X_2 = 0\right\}$ is simply a line passing through the origine and with slope equal to $-\left(\frac{X_1}{X_2}\right)^{\frac{1}{3}}$, without the origin point. This set has an empty interior and therefore Condition "Rüssmann" is satisfied. On the other hand, the differential of Ω_X is given by $d\Omega_X = 12\left(\xi_1^2 X_1 d\xi_1 + \xi_2^2 X_2 d\xi_2\right)$. We can see that for some vectors X, the differential $d\Omega_X$ vanishes at some points belonging to the Σ_X, and thus Condition "Weak" is not satisfied. For example, for $X = (X_1, 0)$ the resonant set Σ_X is the vertical axis $\{(0,\xi_2)\,;\xi_2 \neq 0\}$ without the origin point. Now, at each point of this set, one has $d\Omega_X = 0$.

4 Somes properties of ND hamiltonians

Let $\left(H, \mathcal{M} \xrightarrow{\pi} \mathcal{B}\right)$ be a regular CI system composed of a fibration in lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ together with a Hamiltonian $H \in C^\infty(\mathcal{M})$ constant along the fibers, i.e. $H = F \circ \pi$ for some $F \in C^\infty(\mathcal{B})$. As explained in Section 1, such a fibration implies the existence of a natural torsion-free and flat connection on the base space $\mathcal{B}$, the *Duistermaat connection*. Moreover, one can easily show that the Hamiltonian vector field X_H associated with $H = F \circ \pi$ is tangent to the fibration and its restriction to each torus $\mathcal{M}_b$ is an element of $\mathcal{V}_\nabla(\mathcal{M}_b)$, i.e. is parallel with respect to Weintein's connection. Therefore, on each torus, X_H generates a linear dynamics that can be periodic, resonant or non-resonant.

The resonance properties are actually well-parametrized by using the *Resonance bundle E*. Indeed, for any given $k \in \Gamma(E)$, the previously defined set $\Sigma_k = \Omega_k^{-1}(0)$, with $\Omega_k = dF(k)$, is actually the set of tori on which the Hamiltonian vector field X_H satisfies at least one resonance relation which reads $\sum_{j=1}^d k_i X^j = 0$ in action-angle coordinates. On such tori, the dynamics is actually confined in $(d-1)$-dimensionnal subtori. If the dynamics of X_H is periodic on a torus $\mathcal{M}_b$, this means that it satisfies $d-1$ resonance relations, i.e. b belongs to the intersection of resonant sets $\Sigma_{k_1}, ..., \Sigma_{k_{d-1}}$, where the k_j are linearly independent sections of $\Gamma(E)$. On the other hand, if X_H is ergodic on some torus $\mathcal{M}_b$, this means that b does not belong to any resonant set.

Lemma 10. *If $F \in C^\infty(\mathcal{B})$ is "Rüssmann" non-degenerate, then the set of tori on which the dynamics is ergodic is dense in $\mathcal{B}$.*

Proof. If $\mathcal{M}_b$ is an ergodic torus, this means that b does not belong to any resonant set Σ_k, i.e. $b \in \mathcal{B} \setminus \bigcup_k \Sigma_k$. We will show that $\mathcal{B} \setminus \bigcup_k \Sigma_k$ is dense in $\mathcal{B}$ by showing that the interior of $\bigcup_k \Sigma_k$ is empty. Indeed, whenever the function F satisfies Condition "Rüssmann", then for each non-vanishing $k \in \Gamma(E)$, the subset Σ_k has empty interior. Moreover, Σ_k is a closed subset since it is the inverse image of the point $0 \in \mathbb{R}$ by the continuous map $dF(k) : \mathcal{B} \to \mathbb{R}$. We can then apply the Baire's Theorem (see for e.g. [3]) which insures that $\bigcup_k \Sigma_k$ has empty interior. ☐

Lemma 11. *If the Hamiltonian $H = F \circ \pi$ is "Rüssmann" non-degenerate, then the space of functions $C^\infty(\mathcal{M})$ constant along the fibers equals to the space of functions $C^\infty(\mathcal{M})$ which Poisson-commute with H.*

Proof. Indeed, if a function $A \in C^{\infty}(\mathcal{M})$ satisfies $\{H, A\} = 0$, then one has $X_H(A) = 0$ and therefore A is constant along the trajectories of X_H. For each torus $\mathcal{M}_b$ on which the dynamics is ergodic, this implies that A is constant over this torus. Now, Lemma 10 insures that when H satisfies Condition "Rüssmann", then the set of ergodic tori is dense in $\mathcal{B}$. By continuity, this shows that F is constant along all the fibers $\mathcal{M}_b$. Conversely, if F is constant along all the fibers, then $\{H, F\} = X_H(F) = 0$ since X_H is tangent to the fibration. □

Corollary 12. *Let $H = F \circ \pi$ be a "Rüssmann" non-degenerate Hamiltonian and $A, B \in C^{\infty}(\mathcal{M})$ two functions. Then, the following holds*

$$\{A, H\} = \{B, H\} = 0 \Rightarrow \{A, B\} = 0.$$

Proof. Indeed, if A and B commute with H, Lemma 11 implies that A and B are constant along the fibers. Moreover, since the fibers are lagrangian, X_A is tangent to them and therefore $\{A, B\} = X_A(B) = 0$. □

Theorem 13. *If $\left(H, \mathcal{M} \xrightarrow{\pi} \mathcal{B}\right)$ is a "Rüssmann" non-degenerate C.I system, then $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ is the unique fibration such that H is constant along the fibers.*

Proof. Indeed, suppose $\mathcal{M} \xrightarrow{\pi'} \mathcal{B}'$ is another fibration such that H is constant along the fibers. Let $a_1, ..., a_d \in C^{\infty}\left(\mathcal{B}'\right)$ be a local coordinate system in an open subset $\mathcal{O}' \subset \mathcal{B}'$ and $A_j = a_j \circ \pi'$, $j = 1..d$, the pull-back functions. Since the differentials dA_j are linearly independent in $\pi^{-1}\left(\mathcal{O}'\right)$, the fibers $\left(\pi'\right)^{-1}$ are given by the level-sets of the A_j's. Moreover, we have $\{A_j, H\} = 0$ since by hypothesis H is constant along the lagrangian fibration π'. Now, Lemma 11 implies that the functions A_j must be constant along the fibers of the first fibration π, since H is non-degenerate. This means that the fibers π^{-1} are included in the level sets of the A_j's, and thus are included in the fibers of π'. Since the fibers of both fibrations have the same dimension, they must coincide. □

References

[1] V.I. Arnol'd. A theorem of Liouville concerning integrable dynamics. *Siberian Math. J.*, 4:471–474, 1963.

[2] A. D. Brjuno. *Local Methods in nonlinear differential equations*. Springer-Verlag, 1989.

[3] J. Dixmier. *Topologie Générale*. Presses Universitaires de France, 1981.

[4] J. J. Duistermaat. Global action-angle coordinates. *Comm. Pure App. Math.*, 32:687–706, 1980.

[5] A.N. Kolmogorov. On conservation of conditionally periodic motions for a small change in Hamilton's function. *Dokl. Akad. Nauk. SSSR*, 98(4):527–530, 1954.

[6] J. Liouville. Note sur l'intégration des équations différentielles de la dynamique. *J. Math. Pure Appl.*, 20:137–138, 1855.

[7] H. Mineur. Réduction des systèmes mécaniques à n degrès de liberté admettant n intégrales premières uniformes en involution aux systèmes à variables séparées. *J. Math. Pure Appl.*, 15:221–267, 1936.

[8] H. Rüssmann. Nondegeneracy in the perturbation theory of integrable dynamical systems. In *Number theory and dynamical systems (York, 1987)*, volume 134 of *London Math. Soc. Lecture Note Ser.*, pages 5–18. Cambridge Univ. Press, Cambridge, 1989.

[9] A. Weinstein. Symplectic manifolds and their lagrangian submanifolds. *Adv. in Math.*, 6:329–346, 1971.

[10] A. Weinstein. Lagrangian submanifolds and hamiltonian systems. *Ann. of Math.*, 98:377–410, 1973.