Localization of Hochschild Homology.

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1 Introduction

In a previous paper, Connes [C.1] proved that the continuous Hochschild homology of the algebra of smooth functions on a *compact* smooth manifold M, with coefficients in itself, is isomorphic to the space of differential forms on that manifold. Connes proves this result by producing a projective resolution of the algebra.

This result is important for at least two reasons. First, because it provides a conceptual procedure for asssociating with any associative algebra the analogue of the de Rham complex, and second, it shows that although the $k-chains$ of the Hochschild complex are smooth functions on M_{k+1}, the $k+1$ power of M and, hence, non local objects on M, its homology of degree k consists of local objects on M.

In this paper we present a general procedure which allows one to compute the Hochschild homology of a potentially large class of function algebras. Our procedure is based on a homotopy operator which concentrates (localizes) the Hochschild homology onto the diagonal.

The localization is performed in steps. The first step localizes the Hochschild homology to *germs* of functions at the diagonal; it already explains the local character of the Hochschild homology. The second step micro-localizes the homology to the level of *jets* along the diagonal. This step requires a limiting process and, hence, it offers the possibility of understanding the interply between the regularity properties of functions and the Hochschild homology. The third step is of *algebraic-combinatorial* nature, as it consists of computing the Hochschild homology of the diagonal complex of infinite jets.

In a subsequent joint paper with J. P. Brasselet and A. Legrand we will show how these techniques allow one to compute the Hochschild homology -1) of the algebra of piecewise differentiable functions on simplicial complexes, and -2) the algebra of smooth functions on the real line, whose derivatives, of any order, are bounded. This algebra is relevant in the study of stratified spaces.

In this paper we illustrate the procedure on the algebra of smooth functions on an arbitrary paracompact manifold. This already drops Connes' requirement of compactness.

Our procedure is based on an explicite homotopy operator, depending of a positive parameter $t \to 0$ which deforms the Hochschild chains onto chains which live on a t-tubular neighborhood of the main diagonal in the various powers of the manifold. The final homotopy operator is obtained by making the parameter t converge to 0. Its existence is reduced to a converging problem involving the regularity properties of the functions. The limit homotopy operator deforms the Hochschild chains onto objects whose support lies in the main diagonal.

In the subsection 6.2. we show that the homotopy operator depends only on the $1 - jets$ of functions along the main diagonal. A differential operator containing second order *mixed* partial derivatives in the direction of one of the factors of M_{k+1} (which, in the case of functions with just *one* derivative, as, e.g., Lipschitz functions, involves distributions) is used to explain directly why the Hochschild homology of such commutative algebras depends on the skew symmetric part of the $1 - jets$. This procedure can be used, in particular, to compute the Hochschild homology of the algebra of Lipschitz functions.

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2 General Definitions.

The Hochschild homology complex $\{C_*(A), \delta\}$ of the associative real algebra A with coefficients in A, seen as a bimodule over A, or as a left module over $A \otimes A^{op}$, is defined by $C_k = \otimes_{\mathbf{R}}^{k+1} A$, $0 \leq k$, and

$$\delta_k : C_k \longrightarrow C_{k-1}$$

where

$$\delta_k = \sum_{i=0}^{k} (-1)^i \delta_{k,i}$$

is expressed in terms of the *face operators*

$$\delta_{k,i}(f_0 \otimes f_1 \otimes .... \otimes f_k) = f_0 \otimes f_1 \otimes ... \otimes f_i f_{i+1} \otimes ..... \otimes f_k$$

if $0 \leq i \leq k-1$, and

$$\delta_{k,k}(f_0 \otimes f_1 \otimes .... \otimes f_k) = f_k f_0 \otimes f_1 \otimes .... \otimes f_{k-1}.$$

The homology of the Hochschild complex is by definition the Hochschild homology, $HH_*(A)$, of the algebra A. If A is a locally convex algebra we consider the projective tensor products instead of the algebraic tensor products. The resulting complex, resp.homology, will be by definition the continuous Hochschild complex, resp. homology, of the algebra .

If A is the algebra of smooth functions on the manifold M, then the continuous Hochschild chains of degree k will consist of smooth functions on M_{k+1}, the $(k+1)^{th}$-power of M.

Let, for any $0 \leq r \leq k-1$, denote by ι_r the embedding of M_{k-1} onto the r^{th} diagonal of M_k

$$\iota_r : M_{k-1} \longrightarrow M_k$$

given by

$$\iota_r(x_0, x_1, ..., x_{k-1}) =$$

$$(x_0, x_1, ..., x_{r-1}, x_r, x_r, x_{r+1}, ..., x_{k-1}), \quad for \quad 0 \leq r \leq k-1$$

$$(x_0, x_1, ..., x_{k-1}, x_0), \quad for \quad r = k.$$

Then

$$\delta_{k,r} = (\iota_r)^*$$

and the boundary operator is the alternating sum of these diagonal maps.

Let Δ_k denote the main diagonal in M_k

$$\Delta_k = \{(x, x, ..., x) \mid x \in M\};$$

in the sequel, by the diagonal we will mean the main diagonal. Let d be a Riemannian distance function $d : M \times M \longrightarrow [0, \infty)$ on M. If M is $\mathbf{R^n}$, then d will denote the Euclidean distance.

Choose a decreasing smooth function

$$\lambda : [0, \infty) \longrightarrow [0, 1]$$

having support in the interval $[0, 1]$, and which is identically 1 on the interval $[0, 1/2]$.

Let λ_t denote the rescaled function λ

$$\lambda_t(x) = \lambda(\frac{x}{t}), \quad for \quad 0 < t.$$

The function $\rho_k : M_{k+1} \longrightarrow [0, 1]$, defined by the formula

$$\rho_k(x_0, x_1, ..., x_k) = d^2(x_0, x_1) + d^2(x_1, x_2) + ... + d^2(x_{k-1}, x_k) + d^2(x_k, x_0)$$

measures the square of the distance to the diagonal.

The set

$$U_t := \{(x_0, x_1, ..., x_k) \mid \rho_k(x_0, x_1, ..., x_k) < t\}$$

will be called *t-neighborhood of the diagonal.*

The functions

$$\rho_{k,t} := \lambda_t \circ \rho_k$$

have three important properties:

1. • multiplication by $\rho_{*,t}$ commutes with the Hochschild boundary, i.e. they form a chain homomorhism of the Hochschild complex, or a *cyclic homorphism*, see Connes [C.2].

2. • the function $\rho_{k,t}$ has support in the t-neighborhood of the diagonal

3. • the function $\rho_{k,t}$ is identically 1 on the $\frac{t}{2}$-neighborhood of the diagonal.

Proposition 2.1 *(Localization on the first factor.) For any smooth function ξ on M, the multiplication of the elements in C_k by the functions ξ_k, $0 \leq k$*

$$\xi_k = \xi \otimes 1 \otimes 1 ...... \otimes 1$$

form a chain homomorphism.

We start the localization procedure towards the diagonal in two steps: first at the level of germs and next at the level of jets. For the first step, remark that any Hochschild cycle $f \in C_k$ may be split into a sum of two cycles

$$f = \rho_{k,t}.f + (1 - \rho_{k,t})f;$$

the first cycle has support in a t-neighborhood of the diagonal (with t arbitrarily small), whilst the second cycle is identically zero on the $\frac{t}{2}-$neighborhood of the diagonal.

In the next section we are going to show that all cycles which vanish on a uniform neighborhood of the diagonal are homologous to zero.

3 The Localization to Germs.

Let C_*^t denote the subcomplex of C_* consisting of those functions f which vanish on the t-neighborhood of the diagonal, and let C_*^0 denote the inductive limit of C_*^t as t decreases to 0.

We are going to construct on C_*^0 a chain homotopy between the identity and the zero homomorphism. For, we begin by producing the *extension operators*

$$E_t : C_k \longrightarrow C_{k+1}$$

given by the formula

$$E_t(f))(x_0 ..., x_{k+1}) = \lambda_t(d^2(x_0, x_1)).f(x_1 ..., x_{k+1}).$$

Proposition 3.1 *For any $s > 0$*

$$E_t : C_k^s \longrightarrow C_{k+1}^{\frac{s}{4}}.$$

Proof. Consider an arbitrary point $(x_0..., x_{k+1})$ such that

$$d^2(x_0, x_1) + d^2(x_1, x_2) + ... + d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x_0) < \frac{s}{4}.$$

Then, a fortiori, one has

$$d^2(x_1, x_2) + d^2(x_2, x_3) + ... + d^2(x_k, x_{k+1}) < \frac{s}{4}.$$

as well as

$$d^2(x_0, x_1) + d^2(x_{k+1}, x_0) < \frac{s}{4}, \quad d^2(x_0, x_1) < \frac{s}{4}, \quad and \quad d^2(x_{k+1}, x_0) < \frac{s}{4}.$$

¿From these inequalities one gets, in the Euclidean case, that

$$d^2(x_{k+1}, x_1) \leq d^2(x_{k+1}, x_0) + 2d(x_{k+1}, x_0).d(x_0, x_1) + d^2(x_0, x_1) < \frac{3s}{4}.$$

The first group of inequalities, together with this last one, give

$$d^2(x_1, x_2) + d^2(x_2, x_3) + \quad .... + d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x_1) < s.$$

This shows that if f vanishes on the s-neighborhood of the diagonal, then $E_t f$ vanishes on the $\frac{s}{4}-$ neighborhood of the diagonal.

A straightforward computation shows that

$$\delta E_t + E_t \delta = 1 - N_t$$

where

$$(N_t f)(x_0, x_1, ..., x_k) =$$
$$(-1)^k \lambda_t(d^2(x_0, x_1))[f(x_1, x_2, .., x_k, x_0) - f(x_1, x_2, .., x_k, x_1)].$$

Proposition 3.2

$$[N_t]^k \mid (C_k^{(k+k^2).t}) = 0.$$

Proof. A direct computation shows that

$$[N_t]^k f(x_0, x_1, ..., x_k) = \prod_{i=0}^{k-1} \lambda_t(d^2(x_i, x_{i+1})) \cdot (\mathcal{F} f)(x_0, x_1, ..., x_k),$$

where $\mathcal{F}f$ is a linear combination of functions involving f, its restrictions to certain diagonals and permutations of the arguments. In order for the first factor not to vanish at an arbitrary fixed point $(x_0, x_1, ..., x_k)$, it is necessary that

$$d^2(x_i, x_{i+1}) < t, \quad for \quad 0 \leq i \leq k-1.$$

The triangle inequality implies then $d(x_0, x_k) < k.t^{\frac{1}{2}}$. At such a point, necessarily

$$\sum_{i=0}^{k} d^2(x_i, x_{i+1}) < kt + k^2 t.$$

But, if the function f vanishes on the $(k+k^2).t-$ neighborhood of the diagonal, then $[N_t]^k f(x_0, x_1, ..., x_k) = 0$.

Lemma 3.3 $\delta N_t = N_t \delta$.

Proof of the Lemma 3.3 One has

$$\delta N_t = \delta(1 - [\delta E_t + E_t \delta]) = \delta - \delta E_t \delta = (1 - [\delta E_t + E_t \delta])\delta = N_t \delta.$$

We know from the Proposition 3.2 that $N_t^k = 0$ on $C_k^{(k+k^2).t}$. Therefore

$$1 = 1 - N_t^k = (1 - N_t). \sum_{r=0}^{r=k-1} [N_t]^r = (\delta E_t + E_t \delta). \sum_{r=0}^{r=k-1} [N_t]^r =$$

$$= \delta E_t. \sum_{r=0}^{r=k-1} [N_t]^r + E_t. \sum_{r=0}^{r=k-1} [N_t]^r \delta.$$

The operator

$$K_t := E_t. \sum_{r=0}^{r=k-1} [N_t]^r,$$

satisfies the homotopy identity

$$\delta K_t + K_t \delta = 1.$$

Proposition 3.4 *The operator K_t satisfies the properties*

1.
$$K_t : C_k^{(k+k^2).t} \longrightarrow C_{k+1}^{(k+k^2).4^{-(k+1)}.t}$$

2. K_t establishes a homotopy between the inclusion

$$C_k^{(k+k^2).t} \longrightarrow C_k^{(k+k^2)4^{-(k+1)}.t}$$

and the zero homomorphism

$$\delta K_t + K_t \delta = 1.$$

Proof. The first property follows from the Proposition 3.1. along with the observation that K_t is a linear combination of operators of the forms

$$E_t(\delta E_t)^r \quad \textit{and} \quad E_t(E_t\delta)^r, \quad \textit{with} \quad 0 \leq r \leq k.$$

The second property follows from the homotopy identity proven above.

Remark. To prove the acyclicity of C_*^t at a fixed arbitrary degree k, it is sufficient to have a pair of operators $K_{k-1,t}$ and $K_{k,t}$ satisfying $(\delta K_{k-1,t} + K_{k,t}\delta)f_k = f_k$. We should not need to keep the same operators $K_{k,t}$ for all k.

These arguments prove the following

Theorem 3.5 *The complex C_*^0 is acyclic.*

Proof. The theorem follows from the above proposition by induction limit.

Definition 3.6 An algebra of functions on a metric space is called *fine* iff for any open covering there exists a partition of unity with functions belonging to the algebra and with supports contained in the sets of the covering.

We have proved the following result.

Theorem 3.7 *For any fine algebra A of functions on a metric space M, its Hochschild homology depends only on the germs of functions on the main diagonal of the powers of M.*

4 The Micro-localization.

By cutting-off progressively toward the main diagonal the support of chains defined on M_k, we intend to show by a limiting argument that the homology depends only on the $\infty-$jet of the germ at the diagonal and, ultimately, that, in fact, what counts is its skew-symmetrized 1-jet.

We have seen in the previous section that any Hochschild cycle may be localized with respect to the first variable and that its homology class depends only on its restriction to a small neighborhood of the diagonal. For this reason, we may localize to a coordinate chart, and hence, reduce the problem to the case of $M = \mathbf{R}^n$.

Denote by J_k^∞ the vector space of ∞ -jets of smooth functions on M_{k+1} in the direction of the last k variables $x_1, x_2, ..., x_k$ along the main diagonal. An element of this space can be expressed as a formal power series in the variables $x_j^i - x_0^i$, with $1 \leq i \leq n$ and $1 \leq j \leq k$, whose coefficients are smooth functions in $x_0 = (x_0^1, .., x_0^n)$. Let j_k^∞ denote the infinite jet homomorphism with respect to all variables, except for the variable x_0, along the main diagonal and let J_k^0 be its kernel.

We have the short exact sequence of complexes

$$0 \longrightarrow J_*^0 \longrightarrow C_* \longrightarrow J_*^\infty \longrightarrow 0.$$

Remark 4.0. The complex J_*^∞ is isomorphic to the equivariant Hochschild complex

$$C_{C^\infty(M),*}(J^\infty(M), C^\infty(M));$$

here, $C^\infty(M)$ is seen as a $J^\infty(M)-$ module, where the module action is given by the multiplication with the 0^{th}-order components of jets.

Proposition 4.1 *The complex J_*^0 is acyclic.*

Proof. The theorem referring to the localization about the diagonal tells us that the inclusion of the subcomplex consisting of the chains with support contained in the $t-$ neighborhood of the diagonal, $t > 0$, induces isomorphisms in homology.

For any $f \in C_k$ and $0 < t$, and for any constant C, the support of $\frac{d}{dt}(\lambda_{C.t} \circ \rho_k).f$ is contained in the annulus of radii $\frac{Ct}{2}$ and Ct about the main diagonal. On the other hand, the theorem referring to the localization about the diagonal tells us that the inclusion of the subcomplex consisting of all functions with support contained in an arbitrarily small neighborhood of the diagonal, into the whole complex, induces isomorphism in the Hochschild homology. Therefore, if we introduce the constants $C(k) := 2k(k+1)$, the homotopy formula proven in the previous section holds

$$\frac{d}{dt}\{(\lambda_{C(k).t} \circ \rho_k).f\} =$$

$$[K_t\delta_k + \delta_{k+1}K_t]\frac{d}{dt}\{(\lambda_{C(k).t} \circ \rho_k).f\}.$$

We integrate with respect to t on the interval $[\epsilon, 1]$, and we suppose that the support of f is contained in the $\frac{1}{2C(k)}-$neighborhood of the diagonal. We get

$$f - (\lambda_{C(k).\epsilon} \circ \rho_k).f =$$

$$(\lambda_{C(k)} \circ \rho_k).f - (\lambda_{C(k).\epsilon} \circ \rho_k).f =$$

$$\int_\epsilon^1 \frac{d}{dt}(\lambda_{C(k).t} \circ \rho_k).f dt =$$

$$\int_\epsilon^1 K_t\delta_k\frac{d}{dt}(\lambda_{C(k).t} \circ \rho_k).f dt + \int_\epsilon^1 \delta_{k+1}K_t\frac{d}{dt}(\lambda_{C(k).t} \circ \rho_k).f dt.$$

As δ commutes with the multiplier $\frac{d}{dt}(\lambda_{C(k).t} \circ \rho_k)$, we get

$$f - (\lambda_{C(k).\epsilon} \circ \rho_k).f = \{\delta_{k+1}\mathcal{K}(\epsilon) + \mathcal{K}(\epsilon)\delta_k\}$$

where

$$\mathcal{K}(\epsilon)f = \int_\epsilon^1 K_t\frac{d}{dt}(\lambda_{C(k).t} \circ \rho_k)f dt.$$

The next lemma completes the proof of the Proposition 4.1.

Lemma 4.2 *1. For any $f \in J_k^0$*

$$Lim_{t\to 0}(\lambda_{C(k).t} \circ \rho_k).f = 0$$

2.

$$Kf \doteq Lim_{\epsilon\to 0}\mathcal{K}(\epsilon)f \quad \in J_{k+1}^0$$

3.

$$f = (\delta_{k+1}\mathcal{K} + \mathcal{K}\delta_k)f.$$

The limits here are intended in the space of smooth functions endowed with the Fréchet topology.

Proof. To check the first two statements, note that when $t \to 0$, the integrands are of order $O(\frac{1}{t})$ at the main diagonal, whose divergence is absorbed by the infinite order vanishing of f along the main diagonal. For which concerns the partial derivatives of order r of the integrands, notice that they are of order $O(\frac{1}{t^{r+1}})$ while approaching the diagonal; this divergence is again absorbed by the infinite order vanishing of f along the main diagonal.

The third statement follows from the homotopy formula between 1 and $\lambda_t \circ \rho_k$ given above.

Corollary 4.3 *For the algebra $A = C^\infty(M, \mathbf{R})$ the homomorphism*

$$j^\infty : C_* \to J_*^\infty$$

induces isomorphism in homology.

The induced boundary operator in J_*^∞ acts pointwisely.

Corollary 4.4

$$HH_k(C^\infty(M)) = H_k(J_*^\infty(M)).$$

5 The Hochschild Homology of the Algebra of Jets.

The two localization theorems already tell us that the Hoschschild homology of the algebra of smooth functions is local. Its calculation is reduced to the pointwise computation of the Hochschild homology of the algebra of formal power series in n indeterminates $(x^1, ..., x^n)$

$$\mathcal{S}_n \doteq \mathbf{R}[[x^1, ..., x^n)]].$$

Once the first objective of the paper, the localization, was reached, we intend to discuss the second and the third objectives of the paper, that is, to explain both why the Hochschild homology depends only on the skew symmetric part of the 1− jets, as well as to isolate the interplay beween the analytical properties of functions and the combinatorics of the Hochschild complex.

The computation of the homology of the algebra $\mathcal{S}_n$ of infinite jets is a purely algebraic problem.

6 The Homology of the Diagonal Complex.

6.1 The Homology of J^{∞}_*.

In this subsection we compute the equivariant Hochschild homology of the algebra of infinite jets on M with values in the module of smooth functions on M. Let $(x_0, x_1, ..., x_k) \in (\mathbf{R}^n)^{k+1}$ and let $(x_i^1, ..., x_i^n)$ denote the coordinates in the i^{th}-factor $\mathbf{R}^n$ of this product. Let J^{∞}_k denote the vector space of infinite jets of smooth functions on $(\mathbf{R}^n)^{k+1}$ along the main diagonal, in the direction of the factors $i \neq 0$. Any element of J^{∞}_k may be represented as a formal power series in $x_i^j - x_0^j$, whose coefficients are smooth functions in x_0.

Proposition 6.1 *1.*

$$H_k(J^{\infty}_*) = HH_{C^{\infty}(M),k}(J^{\infty}(M), C^{\infty}(M)),$$

where $J^{\infty}(M)$ is seen as a $J^{\infty}(M)$ - module given by the multiplication of functions with the 0^{th} - order component of the jet

2.

$$H_k(J^{\infty}_*) = \{ \sum_{1 \leq j_1, ..., j_k \leq n} a_{j_1, j_2, ..., j_k}(x_0) \bigwedge (x_1^{j_1} - x_0^{j_1}) ... (x_k^{j_k} - x_0^{j_k}) \},$$

where $a_{j_1, j_2, ..., j_k}(x_0)$ are smooth functions and $\bigwedge$ indicates anti symmetrization of the variables $x_i - x_0$, for $i > 0$

Proof.

Definition 6.2 Consider a monomial

$$a_j^i(x_0)(x_{i_1}^{j_1} - x_0^{j_1})^{\alpha_1} ... (x_{i_r}^{j_r} - x_0^{j_r})^{\alpha_k},$$

where $\alpha_1, ..., \alpha_k$ are multi indices; k will be called its *homological* degree, while $\sum_i |\alpha_i|$ will be called its *algebraic degree.* A monomial with the property that $\alpha_i = 0$ for at least one index i does not depend on one or more of the vector variables $x_1, ..., x_k$; it will be called *lacunary.* All lacunary monomials form the *lacunary* subcomplex $J_*^{\infty,L} \subset J_*^{\infty}$. Let $J_*^{\infty,T}$ denote the subcomplex generated by all remaining monomials.

Lemma 6.3

$$J_*^{\infty} = J_*^{\infty,L} \oplus J_*^{\infty,T}$$

Lemma 6.4 *The lacunary subcomplex is acyclic.*

Proof.

Definition 6.5 A maximal string of consecutive missing vector variables in a lacunary monomial will be called a *lacuna.* The number of missing variables in a lacuna is its *lenght* and the index of the first missing vector variable in a lacuna is its *index.* The lacunae in a monomial are ordered by the natural order of the missing vector variables composing them.

To describe easily the face operators acting on a lacunary monomial in $J_k^{\infty,L}$ it is convenient to replace each of the missing vector variables by the factor 1. By doing so, a lacuna of lenght h will be replaced by a string of h consecutive factors 1. Then

• the last face operator acting on a nonomial which contains the factor $(x_{i_k}^{j_k} - x_0^{j_k})$ to a positive power is zero

• the sum of the Hochschild face operators involving the missing variables of a lacuna is either zero, if the lenght of the lacuna is odd, and has the effect of reducing the lenght of the lacuna by one, if the lenght of the lacuna is even.

For each monomial $m \in J_k^{\infty,L}$, whose last lacuna has index i, define $G_k(m) \in J_{k+1}^{\infty,L}$ to be the monomial $G_k(m) = (-1)^{i+1}.m'$ where m' is obtained from m by extending to the right the lenght of the last lacuna by one.

An easy computation shows that the operators G_k produce a chain homotopy between the identity and the zero homomorphism in $J_*^{\infty,L}$.

Lemma 6.6

$$H_k(J^{\infty,T}) = \{ \sum_{1 \leq j_1, ..., j_k \leq n} a_{j_1, j_2, ..., j_k}(x_0) \bigwedge (x_1^{j_1} - x_0^{j_1}) ... (x_k^{j_k} - x_0^{j_k}) \},$$

Proof. Notice that the the first and the last Hochschild face operators acting on monomials of $J^{\infty,T}$ are zero. For this reason, the complex $J^{\infty,T}$ factorizes

$$J^{\infty,T} \simeq C^\infty(\mathbf{R}^n) \otimes_{\mathbf{R}} B_\star(J_0^\infty(\mathbf{R}^n),$$

where $B_\star(J_0^\infty(\mathbf{R}^n))$ is the bar complex of the algebra $J_0^\infty(\mathbf{R}^n)$ of jets vanishing at the origin. As this algebra does not have a unit, its bar complex is not necessarily acyclic .

Lemma 6.7

$$H_r(B_\star(J_0^\infty(\mathbf{R}^n))) \simeq \wedge^r T^\star(\mathbf{R}^n).$$

Notice that the boundary b of the bar complex preserves the algebraic degree of monomials. Let J_0^r denote the subcomplex of $B_\star(J_0^\infty(\mathbf{R}^n)))$ generated by all monmials of total degree r; therefore, the complex $B_\star(J_0^\infty(\mathbf{R}^n)))$ splits up in the direct sum of complexes

$$B_\star(J_0^\infty(\mathbf{R}^n))) = \sum_r J_0^r.$$

The next lemma will complete the proof of the Proposition 6.1.

Lemma 6.8

$$H_r(J_0^r) = \wedge^r(T^\star(\mathbf{R}^n)).$$

Proof of Lemma. The proof of this lemma is done by increasing induction on r, combined with a spectral sequence argument. If $r=1$, the result is obviously true. Suppose now the lemma was checked for the complex J_0^{r-1} . We are going to compute the homology of the complex J_0^r .

Define F_p^r as being the subcomplex of J_0^r generated by all monomials

$$a_j^i(x_{i_1}^{j_1})^{\alpha_1}...(x_{i_r}^{j_r})^{\alpha_k},$$

with the property that $|\alpha_1| \geq r-p$. This produces an increasing filtration of the complex J_0^r .

Let $F_{p,q}^r \subset F_p^r$ consist of all chains of homological degree $p+q$.

The boundary operator b^0 on the term E^0 of the associated spectral sequence is the bar boundary operator minus the first face operator. The induction hypothesis implies that the only non zero terms of E^1 are

$$E_{p,1}^1 = \mathbf{R}_{r-p}[x^1,...,x^n] \otimes_{\mathbf{R}} \wedge^p T^\star(\mathbf{R}^n), \quad for \quad 1 \leq p \leq r-1,$$

where $\mathbf{R}_k[x^1, ..., x^n]$ is the vector space of homogeneous polynomials of degree k.

The elements of $E^1_{p,1}$ may be thought of as differential forms of degree p on $\mathbf{R}^n$ whose coefficients are homogeneous polynomials of degree $r-p$. The boundary operator b^1 on E^1 may be described in a more familiar fashion; indeed, it is nothing but the interior product i_X of forms with the tangent field

$$X = \sum_{k=1}^{n} x^k \frac{\partial}{\partial x^k}.$$

Let d denote the exterior derivative of forms. To compute E^2, introduce the operators

$$h_p = d : E^1_{p,1} \longrightarrow E^1_{p+1,1}, \quad for \quad 2 \leq p \leq r-1.$$

A direct computation shows that for any differential form of degree k

$$\omega = \sum_{\alpha} a_\alpha(x) dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge ... \wedge dx^{\alpha_k}$$

one has the identity

$$(i_X \circ d + d \circ i_X)(\omega) = k\omega + \sum_{\alpha} (X a_\alpha(x)) dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge ... \wedge dx^{\alpha_k}.$$

This identity along with the Euler identity imply that for $\omega \in E^1_{p,1}$ one has the homotopy formula

$$(i_X \circ d + d \circ i_X)(\omega) = (p + (r-p))\omega = r\omega.$$

The homotopy formula shows that as long as

$$d(E^1_{p,1}) \subset E^1_{p+1,1}$$

one has

$$E^2_{p,1} = 0.$$

Notice, however, that for example, if $r \leq n$

$$d(E^1_{r-1,1}) \ \textit{is not contained in} \ E^1_{r,0}$$

because the forms in $E^{p,1}$ should have as coefficients polynomials of degree at least one. Now, we are in a position to compute E^2. For, suppose first that $r \leq n$. If $2 \leq r - p$, the operator $h := d$ is well defined on $(E^1_{p,1})$ and the homotopy formula above shows that $E^2_{p,1} = 0$. Notice that

$$E^2_{r-1,1} = Ker\{E^1_{r-1,1} \to E^1_{r-2,1}\}.$$

as $E^1_{r,1} = 0$. If $\omega \in E^1_{r-1,1}$, the above homotopy formula gives us that

$$(b^1 \circ d + d \circ b^1)(\omega) = r\omega;$$

Therefore,

$$\omega \in Ker\{E^1_{r-1,1} \to E^1_{r-2,1}\}$$

iff

$$\omega \in (i_X \circ d)(\mathbf{R}_1 \otimes_{\mathbf{R}} \wedge^{r-1} T^\star(\mathbf{R}^n).$$

Notice that

$$d(\mathbf{R}_1 \otimes_{\mathbf{R}} \wedge^{r-1} T^\star(\mathbf{R}^n) = \wedge^r T^\star(\mathbf{R}^n)$$

This shows that

$$\omega \in i_X(\wedge^r T^\star(\mathbf{R}^n)),$$

which, corresponds, via the previous identifications

$$\{jets\} \longleftrightarrow \{differential \quad forms \quad with \quad polynomial \quad coefficients\}$$

to skew symmetric multilinear jets with r vector variables.

If $r > n$, then the homotopy formula applies and $E^2_{p,q} = 0$ for any p, q.

This shows that

$$E^2_{p,q} = E^\infty_{p,q},$$

which completes the proof of the Proposition 6.1.

These lead to the following extension of Connes' theorem [C.1]

Theorem 6.9 *For any paracompact manifold M*

$$HH_k(C^\infty(M)) = C^\infty(M, \wedge^k T^* M).$$

6.2 Hochschild Homology of Algebras of Functions with one Derivative.

In this susection we intend to analyze how the Hochschild homology is tied with the existance of one derivative. By doing so, our method will apply in much more general situations. For, we are going to exemplify onto the algebra of smooth functions how the procedure of micro-localization may be adapted to the case of algebras of functions having just one partial derivative. For this purpose, we are going to ignore that the smooth functions have derivatives of order higher than one.

To start with, we replace the ∞-jets by 1-jets, and we get the short exact sequence of complexes instead

$$0 \to C_*^0 \xrightarrow{i_*} C_* \xrightarrow{j_1} J_*^1 \to 0.$$

Proposition 6.10 *1. The operator K_k defined in section 5 is a well defined operator*

$$K_k : C_k^0 \longrightarrow C_{k+1}.$$

2. the complex C_*^0 is contractible in C_*.

Proof. Any $f \in C_k^0$ has a zero of order 1 with respect to all variables, except for x_0, along the diagonal and, for $t > 0$ small

$$\frac{d}{dt}(\lambda_t \circ \rho_k).f = O(\frac{1}{t}).$$

This proves the first part of the lemma. The operator K_k establishes a homotopy between the identity of C_*^0 and the zero homomorphism in C_*.

Corollary 6.11 *For any $0 \leq k$ there is an exact sequence in homology*

$$0 \to H_k(C_*) \xrightarrow{j_1} H_k(J_*^1) \xrightarrow{\partial_k} H_{k-1}(C_*^0) \to 0.$$

Corollary 6.12

$$H_k(C_*) = Ker \quad \partial_k.$$

The computation of all homologies involved may be reduced to local problems in $\mathbf{R}^n$. Let $x_i = (x_i^1, ..., x_i^n)$, for $1 \leq i \leq k$, be coordinates in the i^{th}-factor $\mathbf{R}^n$. Any element of J_k^1 may be represented as a polynomial in $\xi_i \doteq x_i - x_0$ of degree 0 or 1 in each of the factors ξ_i, whose coefficients are smooth functions of x_0.

6.2.1 The Homology of J_*^1.

Proposition 6.13

$$H_k(J^1) = \{ \sum_{1 \leq j_1, ..., j_k \leq n} a_{j_1, j_2, ..., j_k}(x_0)(x_1^{j_1} - x_0^{j_1})...(x_k^{j_k} - x_0^{j_k}) \},$$

where $a_{j_1, j_2, ..., j_k}(x_0)$ *are smooth functions.*

Proof.

Definition 6.14 A monomial

$$a_j^i(x_0)(x_{i_1}^{j_1} - x_0^{j_1})...(x_{i_r}^{j_r} - x_0^{j_r})$$

with $r < k$ will not depend on one or more of the vector variables $x_1, ..., x_k$; as in the previous subsection, it will be called *lacunary*. All lacunary monomials form the *lacunary* subcomplex $J_*^{1,L} \subset J_*^1$. Let $J_*^{1,T}$ be the subcomplex whose chains of degree k are generated by the monomials of top degree k.

Lemma 6.15

$$J_*^1 = J_*^{1,L} \oplus J_*^{1,T}$$

Lemma 6.16

$$H_k(J^{1,T}) = \{ \sum_{1 \leq j_1, ..., j_k \leq n} a_{j_1, j_2, ..., j_k}(x_0)(x_1^{j_1} - x_0^{j_1})...(x_k^{j_k} - x_0^{j_k}) \},$$

Proof. In J^1, one has $(x_i^{j_1} - x_0^{j_1})(x_i^{j_2} - x_0^{j_2}) = 0$.
Remark. This relation shows that $H_*(J^1)$ is, essentially, at each point, the Hochschild homology of the unitary algebra $\mathbf{R}.1 \oplus \mathbf{R}^n$, with trivial multiplication on $\mathbf{R}^n$.

Lemma 6.17 *The lacunary subcomplex is acyclic.*

Proof. The proof of this lemma will proceed along the same lines as in the subsection 6.1, except for a new specific phenomenon:

• any quadratic factor in one of the vector variables, obtained by the action of a Hochschild face operator, is zero in J^1.

Second Proof of the Theorem 6.9. In view of the Corollary 6.12, it remains to show that

$$Ker\ \partial_k = \{Skew\ symmetric\, 1 - jets\}.$$

Lemma 6.16. allows us to represent any element of $H_k(J^1)$ by a homogeneous polynomial of degree k in the variables $x_i - x_0$, whose coefficients are smooth functions of x_0. Such an element may be thought of as an element of C_k. If such a polynomial is skew symmetric in the variables $x_1, ..., x_k$, then it is also skew symmetric in the variables $x_1 - x_0, ..., x_k - x_0$. Its Hochschild boundary is obviously zero. This shows that

$$\{Skew\ symmetric\, 1 - jets\} \subseteq Ker\ \partial_k.$$

To prove the opposite inclusion, suppose that $\sigma \in Ker\ \partial_k$. Then there exists $f \in C_k$ such that $j_1 f = \sigma$ and $\delta_k f = 0$. Let D be a differential operator of the form

$$D = \frac{\partial^{k+1}}{\partial x_1^{j_1} ....\partial x_{r-1}^{j_{r-1}} (\partial x_r^{j_r} \partial x_r^{j_{r+1}}) \partial x_{r+1}^{j_{r+2}} ....\partial x_{k-1}^{j_k}}.$$

A direct computation shows that for any smooth function f one has

$$D\delta f = \frac{\partial}{x_1^{j_1}} ... \frac{\partial}{x_{r-1}^{j_{r-1}}} (\frac{\partial}{x_r^{j_r}}.\frac{\partial}{x_{r+1}^{j_{r+1}}} + \frac{\partial}{x_r^{j_{r+1}}}.\frac{\partial}{x_{r+1}^{j_r}}).\frac{\partial}{x_{r+2}^{j_{r+2}}} ... \frac{\partial}{x_k^{j_k}} f.$$

As $\delta_k f = 0$, one has $D\delta_k f = 0$, and hence $j_1 f$ is skew symmetric. This completes the proof of the main theorem.

Remark. If f has integrable partial derivatives which involve, at most, one partial derivative with respect to each vector variable x_r, as e.g. in the case of Lipschitz functions, $D\delta f$ has to be thought in the sense of distributions.

Remark. The techniques outlined in this subsection may be used to compute the Hochschild homology of the algebra of Lipschitz functions.

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