The Resolvent of Closed Extensions of Cone Differential Operators

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Abstract. We study an elliptic differential operator on a manifold with conical singularities, acting as an unbounded operator on a weighted L_p-space. Under suitable conditions we show that the resolvent $(\lambda - A)^{-1}$ exists in a sector of the complex plane and decays like $1/|\lambda|$ as $|\lambda| \to \infty$. Moreover, we determine the structure of the resolvent with enough precision to guarantee existence and boundedness of imaginary powers of A.

As an application we treat the Laplace-Beltrami operator for a metric with straight conical degeneracy and establish maximal regularity for the Cauchy problem $\dot{u} - \Delta u = f$, $u(0) = 0$.

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1. Introduction

Understanding the resolvent of elliptic differential operators is of central interest for many questions in pde. Following the approach suggested by Seeley, it is crucial for the analysis of the heat operator or of complex powers. In his classical paper [**23**], he showed how the parametrix to an elliptic operator on a closed manifold can be constructed as a parameter-dependent pseudodifferential operator and how the structure of the parametrix determines the essential properties of the complex powers. He subsequently extended his methods to cover also boundary value problems [**24**] and proved the boundedness of the purely imaginary powers [**25**]. His results have attracted new interest in connection with modern methods in nonlinear evolution equations, where one requires maximal regularity for the generator of the associated semigroup, which in turn is implied by the boundedness of its purely imaginary powers.

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In the present paper we study an elliptic differential operator A on a manifold B with conical singularities (a 'cone differential operator'). The investigation of these operators started with the work of Cheeger [**2**]. Important contributions to the index theory were made in particular by Brüning & Seeley [**1**] and Lesch [**12**]; associated pseudodifferential calculi were devised by Melrose [**15**], Plamenevskij [**17**], and Schulze [**22**].

While the picture of the conical singularity helps the intuition, one prefers to perform the actual analysis on a manifold $\mathbb{B}$ with boundary, thought of as the blow-up of B. A cone differential operator of order μ by definition is an operator that can be written in the form

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j$$

in a neighborhood of the boundary. Here t is a boundary defining function and a_j a smooth family of differential operators of order $\mu - j$ on $\partial\mathbb{B}$.

We consider A as an unbounded operator acting in a (weighted) L_p-space. Our goal is to find conditions which ensure the existence of the resolvent $(A-\lambda)^{-1}$ in a sector of the complex plane with decay like $1/|\lambda|$ as $|\lambda| \to \infty$ and to determine its structure with enough precision to construct complex powers and to show their boundedness for purely imginary exponents. We work with a variant of Schulze's cone calculus because the concept of meromorphic Mellin symbols makes it easy to describe the connection between operators and function spaces with asymptotics.

A cone differential operator in general has many closed extensions, see e.g. Lesch [**12**, Section 1.3]. While, *a priori*, there is no preference for any of these from the analytical point of view, it is obvious that only for few of them the resolvent will have good properties. One basic problem therefore is to determine all possible choices. Our Theorem 2.8 completes Lesch's results in that we obtain an explicit formula for the domain of the maximal extension in the general situation.

Extending Theorem 3.14 from [**20**], we next clarify the structure of the inverse of a (bijective) closed extension of A. In Theorem 3.4 we show how A^{-1} can be decomposed as the sum of two operators in usual cone calculi for different weight data.

We then turn to the analysis of the resolvent. In order to keep the exposition short and the proofs transparent we restrict ourselves to the case where the coefficients a_j of the operator A are constant for small t. The general case will be treated in a subsequent publication.

Following a standard technique, we replace the spectral parameter λ by η^μ, where μ is the order of A, and η varies in a corresponding sector of $\mathbb{C}$. In close analogy to the above statement on the structure of the inverse we prove in Theorem 3.5 that $(A-\eta^\mu)^{-1}$ is the sum of two parameter-dependent cone operators; the parameter space is the new sector. In order to establish this fact we have to make assumptions which are restrictive but nevertheless seem natural in this context: Clearly, we have to ask for the invertibility of the principal pseudodifferential symbol of $A-\eta^\mu$ in the sector, with a certain uniformity as one approaches the singularity. Moreover, we require the invertibility of $\widehat{A}-\eta^\mu$, where $\widehat{A}$ is the 'model cone operator' associated to A. It reflects the behavior of A near the singular point and acts on a domain linked to that of A. As $\widehat{A}-\eta^\mu$ can be considered the analog of an edge principal symbol for $A-\eta^\mu$, its invertibility appears to be necessary for the above result. Finally, we assume for technical reasons that the domain of A (or more precisely the associated domain of $\widehat{A}$) be invariant under dilations ('saturated' in the language of Gil and Mendoza [**10**]).

It follows from Theorem 5.1 and Remark 5.5 in [**3**] that the structure of the resolvent we obtain from Theorem 3.5 is precisely that required for the construction of complex powers and implies the boundedness of the purely imaginary powers; we can hence extend the results of that paper as well as those in [**4**] to this new class of operators.

The idea of analyzing the resolvent of a cone differential operator in terms of a suitable pseudodifferential calculus is not new. In fact, writing the resolvent as a parameter-dependent cone operator can be seen as a special case of the edge parametrix construction, see Schulze [**6**, Section 9.3.3, Theorem 6]. Moreover, Gil [**7**], [**8**], and Loya [**13**], [**14**], also in joint work [**9**], used this technique to derive results on heat invariants, complex powers, and noncommutative residues. While these are important theorems, there is one deficiency: In all articles, the authors rely on a special form of the above ellipticity condition, namely the invertibility of $\widehat{A}-\eta^\mu$, acting between weighted Mellin Sobolev spaces. One can show, however, that this assumption fails in many cases, e.g. for the Laplace-Beltrami operator in dimensions ≤ 4, acting in L^2 with respect to any metric that has a straight conical singularity. Roughly speaking, this approach works only for the minimal (and hence by duality for the maximal) extension. The new point here is that we can now treat all closed extensions with dilation invariant (saturated) domains, opening the way for the analysis of larger classes of operators.

As an application we study the Laplacian in weighted L_p-spaces, $1<p<\infty$. Combining our analysis with techniques of Gil and Mendoza [**10**], we show in Theorems 5.6 and 5.7 how one can always choose the domain in such a way that the above ellipticity conditions are fulfilled. This yields maximal regularity for the Cauchy problem $\dot{u}-\Delta u=f$ on $]0,T[$, $u(0)=0$, which is the starting point for many results in nonlinear evolution equations.

2. Cone differential operators and their closed extensions

2.1. Operators on $\mathbb{B}$. Let $\mathbb{B}$ be a smooth, compact manifold with boundary. A μ-th order differential operator A with smooth coefficients acting on sections of a vector bundle over the interior of $\mathbb{B}$ is called a *cone differential operator* if, near the boundary, it has the form

$$A=t^{-\mu}\sum_{j=0}^{\mu}a_j(t)(-t\partial_t)^j,\qquad a_j\in\mathcal{C}^\infty([0,1[,\mathrm{Diff}^{\mu-j}(\partial\mathbb{B})). \tag{2.1}$$

Besides the standard pseudodifferential principal symbol $\sigma_\psi^\mu(A)\in\mathcal{C}^\infty(T^*\mathrm{int}\,\mathbb{B}\setminus 0)$, we associate with A two other symbols: First, there is the *rescaled symbol* $\widetilde{\sigma}_\psi^\mu(A)\in\mathcal{C}^\infty((T^*\partial\mathbb{B}\times\mathbb{R})\setminus 0)$ which, in local coordinates, is given by

$$\widetilde{\sigma}_\psi^\mu(A)(x,\xi,\tau)=\sum_{j=0}^{\mu}\sigma_\psi^{\mu-j}(a_j)(0,x,\xi)(-i\tau)^j.$$

Secondly, we have the *conormal symbol* $\sigma_M^\mu(A)$ defined by

$$\sigma_M^\mu(A)(z)=\sum_{j=0}^{\mu}a_j(0)z^j,\qquad z\in\mathbb{C}.$$

It is a polynomial in z of degree at most μ with values in differential operators on $\partial\mathbb{B}$ of order at most μ. In particular, $\sigma_M^\mu(A)\in\mathcal{A}(\mathbb{C},\mathcal{L}(H_p^s(\partial\mathbb{B}),\mathcal{L}(H_p^{s-\mu}(\partial\mathbb{B})))$, where $\mathcal{A}(\mathbb{C},E)$ denotes the holomorphic, E-valued functions on $\mathbb{C}$.

Let us define here two notions we shall frequently use throughout this paper.

DEFINITION 2.1. a) *A is called $\mathbb{B}$-elliptic if both $\sigma_\psi^\mu(A)$ and $\widetilde{\sigma}_\psi^\mu(A)$ are pointwise invertible.*
b) *A is said to have t-independent coefficients near the boundary if the functions a_j in* (2.1) *are constant in t.*

The operator A induces continuous actions

$$A:\mathcal{H}_p^{s,\gamma}(\mathbb{B})\longrightarrow\mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B}),\qquad s,\gamma\in\mathbb{R},\ 1<p<\infty, \tag{2.2}$$

in a scale of Sobolev spaces which is defined as follows:

Definition 2.2. *Let $s \in \mathbb{N}_0$. The space of all distributions $u \in H^s_{p,loc}(\operatorname{int}\mathbb{B})$ with*

$$t^{\frac{n+1}{2}-\gamma}(t\partial_t)^k \partial_x^\alpha(\omega u)(t,x) \in L_p([0,1[\times\partial\mathbb{B}, \tfrac{dt}{t}dx) \qquad \forall\, k+|\alpha| \le s$$

is denoted by $\mathcal{H}^{s,\gamma}_p(\mathbb{B})$. Here, $\omega \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ denotes an arbitrary cut-off function.

This definition extends to real s, yielding a scale of Banach spaces (Hilbert spaces in case $p=2$) with two properties we want to mention explicitly: The embedding $\mathcal{H}^{s',\gamma'}_p(\mathbb{B}) \hookrightarrow \mathcal{H}^{s,\gamma}_p(\mathbb{B})$ is continuous for $s' \ge s$, $\gamma' \ge \gamma$, and compact if $s' > s$, $\gamma' > \gamma$; the scalar-product of $\mathcal{H}^{0,0}_2(\mathbb{B})$ induces an identification of the dual space $(\mathcal{H}^{s,\gamma}_p(\mathbb{B}))'$ with $\mathcal{H}^{-s,-\gamma}_{p'}(\mathbb{B})$, where p' is the dual number to p, i.e. $\frac{1}{p}+\frac{1}{p'}=1$.

Instead of considering A as a continuous operator in the Sobolev spaces, cf. (2.2), we shall now study the closed extensions of the unbounded operator

$$A : \mathcal{C}^\infty_{\mathrm{comp}}(\operatorname{int}\mathbb{B}) \subset \mathcal{H}^{0,\gamma}_p(\mathbb{B}) \longrightarrow \mathcal{H}^{0,\gamma}_p(\mathbb{B}) \tag{2.3}$$

(without any difference we could also consider A on $\mathcal{C}^{\infty,\infty}(\mathbb{B})$, cf. Section 6.1). A natural assumption we from now on pose on A is its $\mathbb{B}$-ellipticity. In the upcoming Sections 2.2, 2.3 we shall give an explicit description of all possible closed extensions of A.

In the following let $\omega \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ denote a cut-off function (that can be chosen arbitrarily).

2.2. The minimal extension. Let us describe the domain of the closure. The following result was shown in [**10**], Proposition 3.6. We give here a short proof, using some results of [**12**].

Proposition 2.3. *The domain of the closure $A_{\min} = A^{\gamma,p}_{\min}$ of A from (2.3) is*

$$\begin{aligned} \mathcal{D}(A_{\min}) &= \mathcal{D}(A_{\max}) \cap \bigcap_{\varepsilon>0} \mathcal{H}^{\mu,\gamma+\mu-\varepsilon}_p(\mathbb{B}) \\ &= \Big\{ u \in \bigcap_{\varepsilon>0} \mathcal{H}^{\mu,\gamma+\mu-\varepsilon}_p(\mathbb{B}) \mid t^{-\mu}\sum_{j=0}^{\mu} a_j(0)(-t\partial_t)^j(\omega u) \in \mathcal{H}^{0,\gamma}_p(\mathbb{B}) \Big\}. \end{aligned} \tag{2.4}$$

In particular,

$$\mathcal{H}^{\mu,\gamma+\mu}_p(\mathbb{B}) \hookrightarrow \mathcal{D}(A_{\min}) \hookrightarrow \mathcal{H}^{\mu,\gamma+\mu-\varepsilon}_p(\mathbb{B}) \qquad \forall\, \varepsilon>0.$$

We have $\mathcal{D}(A_{\min}) = \mathcal{H}^{\mu,\gamma+\mu}_p(\mathbb{B})$ if and only if the conormal symbol $\sigma^\mu_M(A)(z)$ is invertible for all z with $\operatorname{Re} z = \frac{n+1}{2}-\gamma-\mu$.

Proof. According to [**12**], Proposition 1.3.12, we may assume that A has t-independent coefficients near the boundary. Now let $u \in \mathcal{D}(A_{\min})$, i.e. there exists a sequence of functions $u_n \in \mathcal{C}^\infty_{\mathrm{comp}}(\operatorname{int}\mathbb{B})$ such that $u_n \to u$ and $Au_n \to Au$ with convergence in $\mathcal{H}^{0,\gamma}_p(\mathbb{B})$. If $\omega,\widetilde\omega \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ are cut-off functions with $\omega\widetilde\omega = \omega$, and if $B = \widetilde\omega \operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(\sigma^\mu_M(A)^{-1})\, t^\mu$ with arbitrarily small $\varepsilon>0$ (and $\varepsilon=0$ in case of the invertibility of the conormal symbol), it follows that

$$\omega u \xleftarrow{n\to\infty} \omega u_n = BA(\omega u_n) \xrightarrow{n\to\infty} BA(\omega u) \tag{2.5}$$

with convergence in $\mathcal{H}^{0,\gamma}_p(\mathbb{B})$. By usual elliptic regularity we know that $u \in H^\mu_{p,\mathrm{loc}}(\operatorname{int}\mathbb{B})$, hence $(1-\omega)u$ belongs to $\mathcal{H}^{\mu,\gamma+\mu}_p(\mathbb{B}) \subset \mathcal{D}(A_{\max})$. Therefore $\omega u \in \mathcal{D}(A_{\max})$, i.e. $A(\omega u) \in \mathcal{H}^{0,\gamma}_p(\mathbb{B})$. From (2.5) we thus deduce that

$$\mathcal{D}(A_{\min}) \subset \mathcal{D}(A_{\max}) \cap \bigcap_{\varepsilon>0} \mathcal{H}^{\mu,\gamma+\mu-\varepsilon}_p(\mathbb{B}) =: V.$$

Since A has constant coefficients, $\mathcal{D}(A_{\max}) = \mathcal{D}(A_{\min}) \oplus \mathcal{E}$, where $\mathcal{E}$ has zero intersection with $\bigcap_{\varepsilon>0} \mathcal{H}^{\mu,\gamma+\mu-\varepsilon}_p(\mathbb{B})$, see [**12**], Proposition 1.3.11. From this we immediately obtain $\mathcal{D}(A_{\min}) = V$. $\square$

2.3. The maximal extension. Before characterizing the domain of the maximal extension, we shall discuss a certain type of operators necessary for formulating this characterization, namely operators of the form

$$G = \omega\left(\mathrm{op}_M^{\gamma_1-\frac{n}{2}}(g) - \mathrm{op}_M^{\gamma_2-\frac{n}{2}}(g)\right) : \mathcal{C}^\infty_{\mathrm{comp}}(\partial\mathbb{B}^\wedge) \longrightarrow \mathcal{C}^\infty(\mathrm{int}\,\mathbb{B}). \tag{2.6}$$

Here, $\partial\mathbb{B}^\wedge := \mathbb{R}_+ \times \partial\mathbb{B}$. Moreover, g is a meromorphic Mellin symbol with asymptotic type P as in Section 6.5; for the definition of $\mathrm{op}_M^{\gamma_j-n/2}$ see (6.12). Note that we could replace g by $g+h$ for any $h \in M_O^\mu(\partial\mathbb{B})$ without changing G; also we could take as domain $\mathcal{S}^\infty(\partial\mathbb{B}^\wedge)$ or $\mathcal{C}^\infty_{\mathrm{comp}}(]a,b[\times\partial\mathbb{B})$ with arbitrary $0<a<b<\infty$ without changing the image of G). Let

$$\sum_{k=0}^{n_p} R_{pk}(z-p)^{-(k+1)}, \qquad R_{pk} \in L^{-\infty}(\partial\mathbb{B}), \tag{2.7}$$

denote the principal part of g around $p \in \pi_{\mathbb{C}}P$. Recall that the R_{pk} are finite rank operators.

Lemma 2.4. *Let G be as in* (2.6) *with $\gamma_1<\gamma_2$, and R_{pk} as in* (2.7). *Then G is of finite rank and, for $u \in \mathcal{C}^\infty_{\mathrm{comp}}(\partial\mathbb{B}^\wedge)$,*

$$(Gu)(t,x) = \omega(t) \sum_{\substack{p\in\pi_{\mathbb{C}}P \\ -\gamma_2<\mathrm{Re}\,p-\frac{n+1}{2}<-\gamma_1}} \sum_{l=0}^{n_p} \zeta_{pl}(u)(x)\, t^{-p}(\log t)^l$$

with the linear maps $\zeta_{pl} : \mathcal{C}^\infty_{\mathrm{comp}}(\partial\mathbb{B}^\wedge) \to \mathrm{im}\,R_{pl} + \ldots + \mathrm{im}\,R_{pn_p} \subset \mathcal{C}^\infty(\partial\mathbb{B})$ given by

$$\zeta_{pl}(u)(x) = \sum_{k=l}^{n_p} \frac{(-1)^l}{l!(k-l)!} R_{pk} \frac{\partial^{k-l}}{\partial z^{k-l}}(\mathcal{M}u)(p,x),$$

where $\mathcal{M} = \mathcal{M}_{t\to z}$ denotes the Mellin transform.

The proof is a straightforward consequence of the residue theorem, since, by definition of op_M^δ,

$$(Gu)(t,x) = \Big(\int_{\Gamma_{\frac{n+1}{2}-\gamma_1}} - \int_{\Gamma_{\frac{n+1}{2}-\gamma_2}}\Big) t^{-z} g(z)(\mathcal{M}u)(z,x)\,đz = \int_{\mathcal{C}} t^{-z} g(z)(\mathcal{M}u)(z,x)\,đz$$

with a path $\mathcal{C}$ simply surrounding the poles of g in the strip $\frac{n+1}{2}-\gamma_2 < \mathrm{Re}\,z < \frac{n+1}{2}-\gamma_1$. For the detailed calculations, and also an expression for $\mathrm{rank}\,G$ see [**12**].

Remark 2.5. *Let γ with $\gamma_1<\gamma<\gamma_2$ be given. If $G_j = \omega\left(\mathrm{op}_M^{\gamma_j-\frac{n}{2}}(g) - \mathrm{op}_M^{\gamma-\frac{n}{2}}(g)\right)$, then $G = G_1 - G_2$ and*

$$\mathrm{im}\,G = \mathrm{im}\,G_1 \oplus \mathrm{im}\,G_2.$$

In fact, by the previous lemma, the images on the right-hand side have trivial intersection, and G_2u_2 only depends on finitely many Taylor coefficients of the Mellin transform $\mathcal{M}u_2$ in the poles of g lying in the strip $\frac{n+1}{2}-\gamma_2 < \mathrm{Re}\,z < \frac{n+1}{2}-\gamma$. The analogous statement holds for G_1u_1. Then the result follows from the following observation: Given finitely many points $p_1,\ldots,p_N \in \mathbb{C}$ and, in each of these, a finite number of Taylor coefficients, there exists a $u \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+)$ such that the Taylor expansion of $\mathcal{M}u$ in each p_j starts with these prescribed values.

Now let A be as in (2.1) and set

$$f_l(z) = \frac{1}{l!}\sum_{j=0}^{\mu} (d_t^l a_j)(0) z^j, \qquad k = 0,\ldots,\mu-1. \tag{2.8}$$

In particular, $f_0 = \sigma_M^\mu(A)$ is the conormal symbol of A. Due to the $\mathbb{B}$-ellipticity of A, f_0 is meromorphically invertible and f_0^{-1} can be written as the sum of a meromorphic Mellin symbol and a holomorphic symbol in $M_O^{-\mu}(\partial\mathbb{B})$ (see Theorem 6 in Section 2.3.1 of [**22**]). We now define recursively

$$g_0 = f_0^{-1}, \qquad g_l = -(T^{-l}f_0^{-1})\sum_{j=0}^{l-1}(T^{-j}f_{j-l})g_j, \qquad l = 1,\ldots,\mu-1, \tag{2.9}$$

with T^σ, $\sigma \in \mathbb{R}$, acting on meromorphic functions by $(T^\sigma f)(z) = f(z+\sigma)$.

Moreover, choose an $\varepsilon > 0$ subject to the following condition: If p is a pole of one of the symbols $g_0, \ldots, g_{\mu-1}$, then p either lies on one of the lines $\Gamma_{\frac{n+1}{2}-\gamma-\mu+k}$, $k = 0, \ldots, \mu$, or has a distance to all of these lines which is larger then ε.

Definition 2.6. *Let $g_0, \ldots, g_{\mu-1}$ be as in* (2.9) *and $\varepsilon > 0$ as described before. Then we set*

$$\mathcal{E} = \mathcal{E}_A^\gamma = \operatorname{im} G_0 + \ldots + \operatorname{im} G_{\mu-1},$$

where the operators $G_k = \sum_{l=0}^{k} G_{kl} : \mathcal{C}^\infty_{\mathrm{comp}}(\partial\mathbb{B}^\wedge) \to \mathcal{C}^{\gamma,\infty+\varepsilon}(\mathbb{B})$ are defined by

$$G_0 = G_{00} = \omega \left(\mathrm{op}_M^{\gamma+\mu+\varepsilon-1-\frac{n}{2}}(g_0) - \mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(g_0) \right),$$

and if $1 \le k \le \mu-1$, $0 \le l \le k$,

$$G_{kl} = \omega\, t^l \left(\mathrm{op}_M^{\gamma+\mu+\varepsilon-k-1-\frac{n}{2}}(g_l) - \mathrm{op}_M^{\gamma+\mu+\varepsilon-k-\frac{n}{2}}(g_l) \right).$$

The space $\mathcal{E}$ is a finite-dimensional subspace of $\mathcal{C}^{\infty,\gamma+\varepsilon}(\mathbb{B})$ and consists of functions of the form

$$u(t,x) = \omega(t) \sum_{j=0}^{N} \sum_{k=0}^{l} u_{jk}(x)\, t^{-q_j} \log^k t \tag{2.10}$$

with complex numbers q_j with $\frac{n+1}{2} - \gamma - \mu \le \operatorname{Re} q_j < \frac{n+1}{2} - \gamma$ and smooth functions $u_{jk} \in \mathcal{C}^\infty(\partial\mathbb{B})$.

Note that in case A has constant coefficients we have, due to Remark 2.5,

$$\mathcal{E} = \operatorname{im} G_{00} \oplus \ldots \oplus \operatorname{im} G_{(\mu-1)0} = \operatorname{im} \omega \left(\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1}) - \mathrm{op}_M^{\gamma+\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1}) \right). \tag{2.11}$$

Proposition 2.7. *For any $0 \le k \le \mu-1$ let $u_{k1}, \ldots, u_{kn_k} \in \mathcal{C}^\infty_{\mathrm{comp}}(\partial\mathbb{B}^\wedge)$ be chosen such that $\{G_{k0}u_{kj} \mid 1 \le j \le n_k\}$ is a basis of $\operatorname{im} G_{k0}$. Then*

$$\{G_k u_{kj} \mid 0 \le k \le \mu-1,\, 1 \le j \le n_k\} \subset \mathcal{E}$$

is a set of linearly independent functions such that

$$\operatorname{span}\{G_k u_{kj} \mid 0 \le k \le \mu-1,\, 1 \le j \le n_k\} \cap \mathcal{D}(A_{\min}) = \{0\}.$$

In particular,

$$\dim \mathcal{E} \ge \dim \operatorname{im} \omega \left(\mathrm{op}_M^{\gamma+\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1}) - \mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1}) \right).$$

We have equality at least in the cases where A has constant coefficients near the boundary, or $\sigma_M^\mu(A)^{-1}$ has no pole on the line $\operatorname{Re} z = \frac{n+1}{2} - \gamma - \mu$.

Proof. Let $\alpha_{jk} \in \mathbb{C}$ and

$$\sum_{k=0}^{\mu-1} \sum_{j=1}^{n_k} \alpha_{jk} G_k u_{kj} = u \in \mathcal{D}(A_{\min}) \subset \mathcal{H}_p^{0,\gamma+\mu-\varepsilon}(\mathbb{B}).$$

Setting $l = \mu - 1$, we obtain

$$\sum_{j=1}^{n_l} \alpha_{lj} G_{l0} u_{lj} = u - \sum_{k=0}^{l-1} \sum_{j=1}^{n_k} \alpha_{jk} G_k u_{kj} - \sum_{j=1}^{n_l} \alpha_{lk} \widetilde{G}_l u_{lj},$$

where we have set $\widetilde{G}_l = G_l - G_{l0}$. The right-hand side belongs to $\mathcal{H}_p^{0,\gamma+\mu-\varepsilon}(\mathbb{B}) + \mathcal{H}_p^{0,\gamma+1+\varepsilon}(\mathbb{B})$. The intersection of this space with $\operatorname{im} G_{l0}$ is trivial, hence $\sum_{j=1}^{n_l} \alpha_{lj} G_{l0} u_{lj} = 0$. Therefore $\alpha_{lj} = 0$ for all $1 \le j \le n_l$, since the $G_{l0}u_{lj}$ are linearly independent, by assumption. Iterating this process (i.e. taking $l = \mu-2$, $l = \mu-3$, etc.), we see that all α_{jk} must equal zero.

The estimate on the dimension follows from (2.11), as well as the equality in case A has constant coefficients. The remaining claim we shall obtain as a by-product of the following theorem. $\square$

We are now in the position to describe all closed extensions of A.

THEOREM 2.8. *The domain of the maximal extension $A_{\max} = A^{\gamma,p}_{\max}$ of A from* (2.3) *is*

$$\mathcal{D}(A_{\max}) = \mathcal{D}(A_{\min}) + \mathcal{E}$$

with $\mathcal{E}$ from Definition 2.6. *The sum is direct at least in the cases where A has constant coefficients near the boundary, or $\sigma_M^\mu(A)^{-1}$ has no pole on the line* $\operatorname{Re} z = \frac{n+1}{2} - \gamma - \mu$. *In any case,*

$$\mathcal{D}(A_{\min}) \cap \mathcal{E} \subset \operatorname{im}\omega\left(\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1}) - \operatorname{op}_M^{\gamma+\mu+\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1})\right).$$

Consequently, any closed extension $\underline{A} = \underline{A}^{\gamma,p}$ is given by the action of A on a domain

$$\mathcal{D}(\underline{A}) = \mathcal{D}(A_{\min}) + \underline{\mathcal{E}}, \qquad \underline{\mathcal{E}} \text{ subspace of } \mathcal{E}. \tag{2.12}$$

PROOF. Choosing a cut-off function $\widetilde{\omega} \in \mathcal{C}^\infty_{\text{comp}}([0,1[)$ supported sufficiently close to zero, the operator $\widetilde{A} = \widetilde{\omega}\, t^{-\mu} \sum_{j=0}^{\mu} a_j(0)(-t\partial_t)^j + (1-\widetilde{\omega})A$ is still $\mathbb{B}$-elliptic, and $\mathcal{D}(\widetilde{A}_{\min}) = \mathcal{D}(A_{\min})$ by [**12**], Proposition 1.3.12. Moreover,

$$\mathcal{D}(\widetilde{A}_{\max}) = \mathcal{D}(\widetilde{A}_{\min}) \oplus \widetilde{\mathcal{E}}$$

with $\widetilde{\mathcal{E}}$ given by the right-hand side of (2.11), and

$$\dim \mathcal{D}(A_{\max})/\mathcal{D}(A_{\min}) = \dim \mathcal{D}(\widetilde{A}_{\max})/\mathcal{D}(\widetilde{A}_{\min}) = \dim \widetilde{\mathcal{E}}.$$

The latter statements are due to [**12**], Proposition 1.3.11, Corollary 1.3.17. By Proposition 2.7 it therefore suffices to show that $\mathcal{E} \subset \mathcal{D}(A_{\max})$. In fact, we shall show that $\operatorname{im} G_k$ belongs to $\mathcal{D}(A_{\max})$ for any k. Since this is easy to see for $k = 0$, we shall only consider the case $k \geq 1$. If f_n are the holomorphic Mellin symbols from (2.8), then

$$A = \widetilde{\omega}\, t^{-\mu} \sum_{j=0}^{\mu-1} t^j \operatorname{op}_M(f_j) + t^\mu A'$$

with a μ-th order cone differential operator A'. Taking into account that $\operatorname{im} G_{kl}$ is a subset of $\mathcal{C}^{\infty,\gamma+\mu+\varepsilon-k+l-1}(\mathbb{B})$ we thus obtain for $u \in \mathcal{S}^\infty(\partial\mathbb{B}^\wedge)$

$$A(G_k u) \in \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \iff \widetilde{\omega} \sum_{j=0}^{k} \sum_{l=0}^{k-j} t^j \operatorname{op}_M(f_j)(G_{kl} u) \in \mathcal{H}_p^{0,\gamma+\mu}(\mathbb{B}).$$

Choosing $\widetilde{\omega}$ with $\widetilde{\omega}\omega = \widetilde{\omega}$, using the elementary rule

$$\operatorname{op}_M(f) t^\sigma \operatorname{op}_M^\delta(g) = t^\sigma \operatorname{op}_M^\delta((T^{-\sigma} f) g),$$

and rearranging the order of summation, we see that $A(G_k u) \in \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ if and only if

$$\widetilde{\omega} \sum_{j=0}^{k} t^j \sum_{l=0}^{j} \left(\operatorname{op}_M^{\gamma+\mu+\varepsilon-k-1}((T^{-l} f_{j-l}) g_l) - \operatorname{op}_M^{\gamma+\mu+\varepsilon-k}((T^{-l} f_{j-l}) g_l)\right)(u) \in \mathcal{H}_p^{0,\gamma+\mu}(\mathbb{B}).$$

However, this expression actually equals zero, since by definition of the symbols g_l, cf. (2.9), we have

$$\sum_{l=0}^{j} (T^{-l} f_{j-l}) g_l = \delta_{0j}, \qquad 0 \leq j \leq k,$$

with δ_{j0} denoting the Kronecker symbol. This shows the claim.

Let us turn to the remaining claims of the theorem. If A has constant coefficients near the boundary the intersection of $\mathcal{D}(A_{\min})$ and $\mathcal{E}$ is zero by (2.11). Using the description of elements u from $\mathcal{E}$ given in (2.10), and the fact that $\mathcal{D}(A_{\min}) \subset \mathcal{H}^{0,\gamma+\mu-\delta}(\mathbb{B})$ for any positive δ, we see that if $u \in \mathcal{D}(A_{\min}) \cap \mathcal{E}$ then u is of the form (2.10) with $\operatorname{Re} q_j = \frac{n+1}{2} - \gamma - \mu$. Since also $\omega \operatorname{op}_M(\sigma_M^\mu(A))(u)$ is contained in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$, u must be an element of $\operatorname{im}\omega\left(\operatorname{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1}) - \operatorname{op}_M^{\gamma+\mu+\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1})\right)$. In particular, the intersection is trivial if $\sigma_M^\mu(A)^{-1}$ has no pole on the line $\operatorname{Re} z = \frac{n+1}{2} - \gamma - \mu$ (this then also proves the last claim of Proposition 2.7 as we announced in the previous proof).

Since $\mathcal{E}$ is finite-dimensional all the operators $\underline{A}$ are in fact closed extensions of A. $\square$

2.4. The model cone operator. Freezing the coefficients of A at the boundary leads to a differential operator

$$\widehat{A} = t^{-\mu} \sum_{j=0}^{\mu} a_j(0)(-t\partial_t)^j \tag{2.13}$$

on the infinite half-cylinder $\partial\mathbb{B}^\wedge := \mathbb{R}_+ \times \partial\mathbb{B}$. We shall refer to this operator as the *model cone operator* of A. Let us first introduce a suitable scale of Sobolev spaces the operator $\widehat{A}$ acts in.

To this end let $\partial\mathbb{B} = X_1 \cup \ldots \cup X_J$ be an open covering of $\partial\mathbb{B}$; let $\kappa_j : X_j \to U_j$ be coordinate maps and $\{\varphi_1, \ldots, \varphi_J\}$ a subordinate partition of unity.

Given a function $u = u(t,x)$ on $\mathbb{R} \times \partial\mathbb{B}$, we shall say that $u \in H^s_{p,\text{cone}}(\mathbb{R} \times \partial\mathbb{B})$ provided that, for each j, the function

$$v(t,y) = \varphi_j(x)u(t,x), \quad x = \kappa_j^{-1}(y/\langle t\rangle),$$

is an element of $H^s_p(\mathbb{R} \times \mathbb{R}^n)$ (we consider the right-hand side to be zero for $x \notin U_j$). In other words: $\varphi_j u$ is the pull-back of a function in $H^s_p(\mathbb{R}^{n+1})$ under the composition of the maps

$$\text{id} \times \kappa_j : \mathbb{R} \times X_j \ni (t,x) \mapsto (t, \kappa_j(x)) \in \mathbb{R} \times U_j$$

and

$$\chi : \mathbb{R} \times U_j \ni (t,y) \mapsto (t, \langle t\rangle y) \in \mathbb{R}^{n+1},$$

so that the definition extends to distributions in the usual way for $s \in \mathbb{R}$, $1 < p < \infty$.

Definition 2.9. *$\mathcal{K}^{s,\gamma}_p(\partial\mathbb{B}^\wedge)$ is the space of all distributions $u \in H^s_{p,\text{loc}}(\mathbb{R}_+ \times \partial\mathbb{B})$ such that, for an arbitrary cut-off function ω,*

$$\omega u \in \mathcal{H}^{s,\gamma}_p(\mathbb{B}) \quad \text{and} \quad (1-\omega)u \in H^s_{p,\text{cone}}(\mathbb{R} \times \partial\mathbb{B}).$$

$\widehat{A}$ acts continuously in this scale of Sobolev spaces,

$$\widehat{A} : \mathcal{K}^{s,\gamma}_p(\partial\mathbb{B}^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}_p(\partial\mathbb{B}^\wedge), \qquad s,\gamma \in \mathbb{R},\ 1 < p < \infty.$$

We shall now consider the model cone operator as an unbounded operator, namely

$$\widehat{A} : \mathcal{C}^\infty_{\text{comp}}(\partial\mathbb{B}^\wedge) \subset \mathcal{K}^{0,\gamma}_p(\partial\mathbb{B}^\wedge) \longrightarrow \mathcal{K}^{0,\gamma}_p(\partial\mathbb{B}^\wedge).$$

If A satisfies condition (1) of Section 3.2 below, the domains of the closed extensions of $\widehat{A}$ can be read off from A. In analogy to Theorems 2.3 and 2.8 we have:

Proposition 2.10. *If A satisfies condition* (1) *of Section* 3.2*, then*

$$\mathcal{D}(\widehat{A}_{\min}) = \Big\{ u \in \bigcap_{\varepsilon>0} \mathcal{K}^{\mu,\gamma+\mu-\varepsilon}_p(\partial\mathbb{B}^\wedge) \mid \widehat{A}u \in \mathcal{K}^{0,\gamma}_p(\mathbb{B}) \Big\}.$$

This simplifies to $\mathcal{D}(\widehat{A}_{\min}) = \mathcal{K}^{\mu,\gamma+\mu}_p(\partial\mathbb{B}^\wedge)$ if and only if $\sigma^\mu_M(A)$ is invertible on the line $\Gamma_{\frac{n+1}{2}-\gamma-\mu}$.

If $\mathcal{E}$ is the space defined in (2.11)*, then*

$$\mathcal{D}(\widehat{A}_{\max}) = \mathcal{D}(\widehat{A}_{\min}) \oplus \mathcal{E}.$$

Hence, any closed extension $\underline{\widehat{A}}$ of $\widehat{A}$ is given by the action of $\widehat{A}$ on a domain

$$\mathcal{D}(\underline{\widehat{A}}) = \mathcal{D}(\widehat{A}_{\min}) \oplus \underline{\mathcal{E}}, \qquad \underline{\mathcal{E}} \text{ subspace of } \mathcal{E}.$$

Proof. We let $\tilde{P} = \sum_{j=0}^{\mu} a_j(x, D_x)(-\partial_t)^j - \eta^\mu$. This is a non-degenerate parameter-dependent differential operator with coefficients independent of t. It follows from assumption (1) that the parameter-dependent principal symbol

$$\tilde{p}_0(x,\xi,\tau,\eta) = \sum_{j=0}^{\mu} \sigma^{\mu-j}_\psi(a_j)(x,\xi)(-i\tau)^j - \eta^\mu$$

of $\tilde{P}$ is parameter-elliptic. Hence there exist parameter-dependent symbols $\tilde{q}_0$ of order $-\mu$ and r of order -1 such that

$$\tilde{p}_0\tilde{q}_0 = 1 + r. \tag{2.14}$$

The operator $P = t^\mu(\widehat{A} - \eta^\mu)$ has the principal symbol

$$\sigma_\psi^\mu(P)(x,t,\xi,\tau,\eta) = \tilde{p}_0(x,\xi,t\tau,t\eta).$$

Under the push-forward induced by $T := \chi \circ (\mathrm{id} \times \kappa_j)$ the operator P transforms into a weighted SG-pseudodifferential operator of order (μ,μ); modulo terms of order $(\mu-1,\mu-1)$ its SG-symbol is given by the push-forward of $\sigma_\psi^\mu(P)$. Indeed, for a differential operator this is a simple calculation, a proof for the general pseudodifferential case is given in [**19**], Theorem 3.8; details on SG-symbols can be found in [**18**]. Now equation (2.14) implies that the push-forward of $\sigma_\psi^\mu(P)$ is SG-parameter-elliptic if we restrict to a subset of $\mathbb{R}_+ \times \partial\mathbb{B}$ away from the boundary, say to $\{t \geq 1\}$. Hence, on this set, there is a parameter-dependent SG-parametrix S of order $(-\mu,-\mu)$ to the push-forward T_*P of P (i.e. we have $S \circ T_*P = I + R$, where R is an integral operator with a rapidly decreasing kernel). As the operator of multiplication by t^μ remains unchanged under the push-forward and is an SG-operator of order $(0,\mu)$, $S \circ t^\mu$ is an SG-parametrix of order $(-\mu, 0)$ to $A - \eta^\mu$.

With this information we conclude the proof in the standard way: Given $u \in \mathcal{K}_p^{0,\gamma}$, a cut-off function ω equal to 1 in $\{t \leq 1\}$, and a function φ_j in the partition of unity on $\partial\mathbb{B}$,

$$S \circ t^\mu \circ T_* \Big((\widehat{A} - \eta^\mu)(1-\omega)\varphi_j u \Big) = T_*(1-\omega)\varphi_j u + RT_*(1-\omega)\varphi_j u \tag{2.15}$$

Assume additionally that u belongs to $\mathcal{D}(\widehat{A}_{\max})$, i.e. $\widehat{A}u \in \mathcal{K}_p^{0,\gamma}$. Standard elliptic regularity implies that $u \in H^\mu_{p,\mathrm{loc}}$. Hence $\widehat{A}(1-\omega)\varphi_j u \in \mathcal{K}_p^{0,\gamma}$, as it coincides with $(1-\omega)\varphi_j \widehat{A}u$ outside a compact set, and its push-forward via T belongs to $H_p^0(\mathbb{R}^{n+1})$. In view of the fact that $St^\mu : H_p^0(\mathbb{R}^{n+1}) \to H_p^\mu(\mathbb{R}^{n+1})$ is bounded, we deduce from (2.15) that $(1-\omega)u \in H^\mu_{p,\mathrm{cone}}(\mathbb{R} \times \partial\mathbb{B})$. On the other hand, we trivially have $u \in \mathcal{D}(\widehat{A}_{\min})$ for every u in $H^\mu_{p,\mathrm{cone}}(\mathbb{R} \times \partial\mathbb{B})$ supported in $\{t \geq 1\}$. As a consequence, the domains of all extensions of $\widehat{A}$ coincide with $H^\mu_{p,\mathrm{cone}}(\mathbb{R} \times \partial\mathbb{B})$ away from $\{t = 0\}$.

Close to $\{t = 0\}$, however, the analysis is the same as in the standard case. This completes the argument. ☐

3. Structure of the resolvent

Let us now come to the main objective of this paper. We shall consider a closed extension of a cone differential operator and give conditions that ensure that its resolvent exists in a given sector Λ (up to finitely many exceptional points). We shall describe the structure of this resolvent in terms of a class of parameter-dependent pseudodifferential operators. As an application we derive boundedness of purely imaginary powers.

Before considering the resolvent, we want to investigate the inverse of a given closed extension. This is a simpler problem but already illustrates some of the structures we shall see in the discussion of the resolvent.

3.1. The inverse of a closed extension.

Let A be a cone differential operator and assume that

$$\underline{A} : \mathcal{D}(\underline{A}) = \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B}) \oplus \underline{\mathcal{E}} \longrightarrow \mathcal{H}_p^{0,\gamma}(\mathbb{B})$$

is bijective (for a fixed p). We shall analyze the structure of its inverse. Since $\underline{\mathcal{E}}$ is finite-dimensional, $A : \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B}) \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is a Fredholm operator. In [**20**] we have shown that the Fredholm property is equivalent to the ellipticity of A, i.e. A is $\mathbb{B}$-elliptic and the conormal symbol is invertible on the line $\Gamma_{\frac{n+1}{2}-\gamma-\mu}$. In other words, A is an elliptic element of the cone algebra $C^\mu(\gamma+\mu,\gamma,k)$ for any $k \in \mathbb{N}$. The cone algebra on $\mathbb{B}$ was introduced by Schulze; for its definition we refer to [**21**] (and, concerning notation we use here, to the corresponding definitions of the parameter-dependent version given in the appendix).

Let us now set $\underline{\mathcal{F}} = A(\underline{\mathcal{E}})$. This space is finite-dimensional and $\mathcal{H}_p^{0,\gamma}(\mathbb{B}) = A(\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B})) \oplus \underline{\mathcal{F}}$.

Lemma 3.1. *There exists an asymptotic type $Q \in \mathrm{As}(\gamma, k)$, $k \in \mathbb{N}$ arbitrily large, such that $\underline{\mathcal{F}} = A(\underline{\mathcal{E}}) \subset \mathcal{C}_Q^{\infty,\gamma}(\mathbb{B})$.*

Proof. Let u be of the form (2.10), and choose a cut-off function $\tilde{\omega}$ with $\tilde{\omega}\omega = \tilde{\omega}$. Then, with A as in (2.1),

$$(Au)(t) = \tilde{\omega} t^{-\mu} \sum_{j=0}^{\mu} \sum_{q,k} a_j(t)(t\partial_t)^j (c_{qk} t^{-q} \log^k t) + (1 - \tilde{\omega})(t)(Au)(t).$$

The second term belongs to $\mathcal{C}_{\mathrm{comp}}^{\infty}(\operatorname{int}\mathbb{B})$. Now a Taylor expansion of the coefficients a_j in t at 0 shows the claim. □

Let us now denote by

$$\pi_{\underline{\mathcal{E}}} : \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B}) \oplus \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}, \qquad \pi_{\underline{\mathcal{F}}} : A(\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B})) \oplus \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{F}}$$

the canonical projections, and let

$$B = (1 - \pi_{\underline{\mathcal{E}}})\underline{A}^{-1} : \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \longrightarrow \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B})$$

be a left-inverse of $A : \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B}) \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$.

Lemma 3.2. $1 - AB = \pi_{\underline{\mathcal{F}}}$ *belongs to $C_G(\gamma, \gamma, k)$ for arbitrily large $k \in \mathbb{N}$.*

Proof. By construction, it is clear that $1 - AB = \pi_{\underline{\mathcal{F}}}$. Let $w_1, \ldots, w_N$ be a basis of $\underline{\mathcal{F}}$. Then we can write $\pi_{\underline{\mathcal{F}}}(\cdot) = \sum_{j=1}^{N} l_j(\cdot) w_j$ with functionals l_j on $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$. By duality there exist $v_j \in \mathcal{H}_{p'}^{0,-\gamma}(\mathbb{B})$ such that $l_j(\cdot) = \langle \cdot, v_j \rangle_{\mathcal{H}_2^{0,0}(\mathbb{B})}$. Then, for all $u \in \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B})$,

$$0 = \pi_{\underline{\mathcal{F}}}(Au) = \sum_{j=1}^{N} \langle Au, v_j \rangle_{\mathcal{H}_2^{0,0}(\mathbb{B})} w_j = \sum_{j=1}^{N} \langle u, A^* v_j \rangle_{\mathcal{H}_2^{0,0}(\mathbb{B})} w_j,$$

where A^* denotes the formal adjoint of A, which belongs to $C^{\mu}(-\gamma, -\gamma-\mu, k)$. Hence $v_j \in \ker A^*$ for $j = 1, \ldots, N$. Since with A also A^* is an elliptic cone operator, $\ker A^* \subset \mathcal{C}_{Q'}^{\infty,-\gamma}(\mathbb{B})$ for some asymptotic type $Q' \in \mathrm{As}(-\gamma, k)$ by elliptic regularity, cf. [**22**], Theorem 8 in Section 2.2.1. Thus $\pi_{\underline{\mathcal{F}}}$ has a kernel in $\underline{\mathcal{F}} \otimes \mathcal{C}_{Q'}^{\infty,-\gamma}(\mathbb{B})$, and therefore is a Green operator. □

Proposition 3.3. *B is an element of $C^{-\mu}(\gamma, \gamma+\mu, k)$ for arbitrarily large $k \in \mathbb{N}$.*

Proof. Since A is an elliptic element of $C^{\mu}(\gamma+\mu, \gamma, k)$ as we have shown above, there exists a parametrix $C \in C^{-\mu}(\gamma, \gamma+\mu, k)$, i.e.

$$AC - 1 = G_R \in C_G(\gamma, \gamma, k), \qquad CA - 1 = G_L \in C_G(\gamma+\mu, \gamma+\mu, k).$$

Multiplying these identities from the left, respectively from the right with B yields $B = C - BG_R$ and $B = CAB - G_L B = C - C(1 - AB) - G_L B$. Inserting the first equation into the second gives

$$B = C - C\pi_{\underline{\mathcal{F}}} - G_L C + G_L B G_R.$$

The third term on the right-hand side is a Green operator, since Green operators are an ideal in the cone algebra. The same is true for the second and fourth term in view of Lemma 3.2 and the continuity of B, respectively. □

Theorem 3.4. *$\underline{A}^{-1} = B + G$ for some $B \in C^{-\mu}(\gamma, \gamma+\mu, k)$ and $G \in C_G(\gamma, \gamma, k)$ with arbitrarily large $k \in \mathbb{N}$.*

PROOF. We decompose $\underline{A}^{-1} = (1-\pi_{\underline{\mathcal{E}}})A^{-1} + A^{-1}\pi_{\underline{\mathcal{F}}} = B + G$. By Proposition 3.3, B is as claimed. From the proof of Lemma 3.2 we know that $\pi_{\underline{\mathcal{F}}}$ has an integral kernel in $\underline{\mathcal{F}} \otimes \mathcal{C}_{Q'}^{\infty,-\gamma}(\mathbb{B})$ for some type $Q' \in \mathrm{As}(-\gamma,k)$. Therefore, $A^{-1}\pi_{\underline{\mathcal{F}}}$ has a kernel in $\underline{\mathcal{E}} \otimes \mathcal{C}_{Q'}^{\infty,-\gamma}(\mathbb{B})$, hence is a Green operator. □

As we shall explain in Section 4.1, this shows that the invertibility of $\underline{A} = \underline{A}_p$ is independent of $1 < p < \infty$.

Theorem 3.4 especially implies that $\underline{A}^{-1} \in C_O^{-\mu} + C_G(\gamma,\gamma,\varepsilon)$. The main result of the present paper says that for the resolvent of $\underline{A}$ there is a parameter-dependent analogue of this statement.

3.2. The ellipticity assumptions and statement of the main result. Let A be a μ-th order cone differential operator whose coefficients on $[0,1] \times \partial\mathbb{B}$ are independent of t. Let

$$\underline{A} : \mathcal{D}(\underline{A}) = \mathcal{D}(A_{\min}) \oplus \underline{\mathcal{E}} \subset \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \longrightarrow \mathcal{H}_p^{0,\gamma}(\mathbb{B})$$

be a closed extension of A as described in the Section 2.1. Let us now assume

(1) Both $\sigma_\psi^\mu(A)$ and $\tilde{\sigma}_\psi^\mu(A)$ have no spectrum in the sector Λ,
(2) With the above choice of $\underline{\mathcal{E}}$, the domain $\mathcal{D}(\underline{\widehat{A}}) = \mathcal{D}(\widehat{A}_{\min}) \oplus \underline{\mathcal{E}}$ of the model cone operator is invariant under dilations,
(3) The sector $\Lambda \setminus \{0\}$ contains no spectrum of the model cone operator

$$\underline{\widehat{A}} : \mathcal{D}(\underline{\widehat{A}}) \subset \mathcal{K}_p^{0,\gamma}(\partial\mathbb{B}^\wedge) \longrightarrow \mathcal{K}_p^{0,\gamma}(\partial\mathbb{B}^\wedge).$$

Condition (2) means the following: whenever $u = u(t,x)$ belongs to $\mathcal{D}(\underline{\widehat{A}})$, the same is true for the functions $u_\varrho(t,x) = u(\varrho t,x)$, $\varrho > 0$. It is easy to see that the domain $\mathcal{D}(\underline{\widehat{A}})$ is invariant under dilations if and only if this is true for $\mathcal{C}_{\mathrm{comp}}^\infty(\partial\mathbb{B}^\wedge) \oplus \underline{\mathcal{E}}$.

Note that condition (2) is always satisfied for the minimal extension, the maximal extension, and for extensions with domain equal to $\mathcal{D}(A_{\max}) \cap \mathcal{H}_p^{0,\sigma}(\mathbb{B})$ and $\gamma - \mu < \sigma \le \gamma$. For concrete examples see Section 5 below.

In (1) and (3), $\Lambda = \Lambda_\theta$ is a closed sector in the complex plane containing zero, i.e.

$$\Lambda_\theta = \{z \in \mathbb{C} \mid |\arg z| \ge \theta\} \cup \{0\},$$

where $0 \le \theta < \pi$ and $-\pi \le \arg z < \pi$. For notational convenience let us fix a sector Σ such that the mapping $\eta \mapsto \eta^\mu$ induces a bijection $\Sigma \to \Lambda$. Instead of working directly with Λ we shall use this sector Σ.

THEOREM 3.5. *Let* $\underline{A}$ *satisfy conditions* (1), (2), and (3) *with respect to the sector* Λ*. Then*

a) $\underline{A}$ *has at most finitely many spectral points in* Λ.
b) *There exists a parameter-dependent cone pseudodifferential operator*

$$c(\eta) \in C_O^{-\mu}(\Sigma) + C_G^{-\mu}(\Sigma;\gamma,\gamma,\varepsilon)$$

with a certain $\varepsilon > 0$*, such that* $(\eta^\mu - \underline{A})^{-1} = c(\eta)$ *for* $\eta \in \Sigma$ *with* $|\eta|$ *sufficiently large.*

For the notation used in part b) of this theorem we refer to the appendix (see Definition 6.6 and Definition 6.13). Note that part a) of the theorem follows from b): Since the domain of $\underline{A}$ is compactly embedded in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$, $\underline{A}$ has a compact resolvent, hence a discrete spectrum.

3.3. The proof of the main result. We shall use the material and notation given in the appendix. Let us write $a(\eta) = \eta^\mu - A$. Then

$$a(\eta) \in C_O^\mu(\Sigma) \subset C^\mu(\Sigma; \tilde{\gamma}, \tilde{\gamma} - \mu, k)$$

for any $\tilde{\gamma} \in \mathbb{R}$ and $k \in \mathbb{N}$. The ellipticity assumption (1) on A ensures that $a(\eta)$ is $\mathbb{B}$-elliptic with respect to the sector Σ, i.e. satisfies condition (E) from Sections 6.3 and 6.6. In particular, the conormal symbol of $a(\eta)$ respectively A is meromorphically invertible. Note that we do not require A to be conormal elliptic with respect to the weight $\gamma + \mu$, i.e. the inverted conormal symbol possibly has a pole on the line $\Gamma_{\frac{n+1}{2} - \gamma - \mu}$. By Theorem 6.15 we then get:

PROPOSITION 3.6. *Let $\varepsilon > 0$ be sufficiently small. Then there exists a parameter-dependent cone operator $b_L(\eta) \in C^{-\mu}(\Sigma; \gamma - \mu + \varepsilon, \gamma + \varepsilon, \mu)$ such that*

$$b_L(\eta)a(\eta) - 1 = g_L(\eta) \in C_G^0(\Sigma; \gamma + \varepsilon, \gamma + \varepsilon, \mu). \tag{3.1}$$

In this context, 'sufficiently small' means $0 < \varepsilon < \varepsilon_0$ such that the conormal symbol is holomorphically invertible in the strip $\frac{n+1}{2} - \gamma - \mu < \operatorname{Re} z < \frac{n+1}{2} - \gamma - \mu + \varepsilon_0$.

PROPOSITION 3.7. *To any sufficiently small $\varepsilon > 0$ there exists a parameter-dependent cone operator $b_\varepsilon(\eta) \in C^{-\mu}(\Sigma; \gamma - \varepsilon, \gamma + \mu - \varepsilon, \mu)$ such that $b_\varepsilon(\eta)|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})}$ is independent of the choice of ε for any $s \in \mathbb{R}$ and $1 < p < \infty$. Moreover,*

$$b_R(\eta) := b_\varepsilon(\eta)|_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} : \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \longrightarrow \mathcal{D}(A_{\min})$$

and, for a suitable asymptotic type $Q \in \operatorname{As}(\gamma + \varepsilon, 1 - 2\varepsilon)$,

$$a(\eta)b_R(\eta) - 1 = g_R(\eta) \in C_G^0(\Sigma; \gamma - \varepsilon, 1 - 2\varepsilon, O; \gamma + \varepsilon, 1 - 2\varepsilon, Q). \tag{3.2}$$

PROOF. Let us first view $a(\eta)$ as an element in $C_O^\mu(\Sigma)$. According to Theorem 6.8 there exists a flat parameter-dependent cone operator $\tilde{b}(\eta) \in C_O^{-\mu}(\Sigma)$ such that

$$a(\eta)\tilde{b}(\eta) - 1 \equiv \omega(t[\eta]) \operatorname{op}_M(\tilde{f})(\eta)\, \omega_0(t[\eta]) \qquad \bmod C_G^0(\Sigma)_\infty \tag{3.3}$$

with a holomorphic Mellin symbol $\tilde{f} \in M^{-\infty}(\partial\mathbb{B}; \Sigma)$. Setting $\tilde{f}_0(z) = \tilde{f}(z, 0)$, we have

$$\omega(t[\eta]) \left\{ \operatorname{op}_M^{\tilde{\gamma} - \frac{n}{2}}(\tilde{f})(\eta) - \operatorname{op}_M^{\tilde{\gamma} - \frac{n}{2}}(\tilde{f}_0) \right\} \omega_0(t[\eta]) \in C_G^0(\Sigma; \tilde{\gamma}, 1, O; \tilde{\gamma}, 1, O) \tag{3.4}$$

for any $\tilde{\gamma} \in \mathbb{R}$. In fact, by Taylor expansion, $\tilde{f}(z, t\eta) = \tilde{f}_0(z) + \sum_{i=1}^n (t\eta_i)\tilde{f}_i(z, t\eta)$ for suitable $\tilde{f}_i(z, \eta) \in \mathcal{C}^\infty(\Sigma, M_O^{-\infty}(\partial\mathbb{B}))$. Therefore, the operator-family in (3.4) pointwise has the mapping properties of Green symbols from $R_G^0(\Sigma; \tilde{\gamma}, 1, O; \tilde{\gamma}, 1, O)$, cf. Definition 6.11. Moreover, it is homogeneous of order 0 for large $|\eta|$ in the sense of (6.9) with respect to the group action of (6.10), hence also satisfies the required symbol estimates.

Next, we are going to modify $\tilde{b}(\eta)$ by a smoothing Mellin term, i.e. we set

$$b_\varepsilon(\eta) = \tilde{b}(\eta) + \omega(t[\eta])\, t^\mu \operatorname{op}_M^{\gamma - \varepsilon - \frac{n}{2}}(f)\, \omega_0(t[\eta]), \tag{3.5}$$

where we determine f in such a way that the conormal symbol of $a(\eta)b_\varepsilon(\eta) - 1$ vanishes, i.e.

$$0 = \sigma_M^0(ab_\varepsilon)(z) - 1 = \sigma_M^0(a\tilde{b})(z) + \sigma_M^\mu(a)(z - \mu)f(z) - 1 = \tilde{f}_0(z) + \sigma_M^\mu(a)(z - \mu)f(z).$$

Solving this identity for f yields the choice

$$f(z) = -\sigma_M^\mu(a)^{-1}(z - \mu)\tilde{f}_0(z) = \sigma_M^\mu(a)^{-1}(z - \mu) - \sigma_M^{-\mu}(\tilde{b})(z).$$

By the holomorphy of the conormal symbol of $a(\eta)$ in a strip $\frac{n+1}{2} - \gamma - \mu < \operatorname{Re} z < \frac{n+1}{2} - \gamma - \mu + \varepsilon_0$, the action of $b_\varepsilon(\eta)$ on $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is independent of $0 < \varepsilon < \varepsilon_0$, and by the description of $\mathcal{D}(A_{\min})$ given in Proposition 2.3, indeed $b_\varepsilon(\eta)$ maps $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ to the domain of $A_{\min}$. Choosing a cut-off function ω_1 with $\omega\omega_1 = \omega$ it follows from formulas (3.3) and (3.4) that

$$a(\eta)b_\varepsilon(\eta) - 1 \equiv \omega_1(t[\eta]) \operatorname{op}_M^{\gamma - \varepsilon - \frac{n}{2}}(\tilde{f}_0)\, \omega_0(t[\eta]) + \omega_1(t[\eta])\, a(\eta)\, \omega(t[\eta])\, t^\mu \operatorname{op}_M^{\gamma - \varepsilon - \frac{n}{2}}(f)\, \omega_0(t[\eta]) \tag{3.6}$$

modulo $C_G^0(\Sigma;\gamma-\varepsilon,1-2\varepsilon,O;\gamma+\varepsilon,1-2\varepsilon,O)$. Here we employed the fact that changing ω in (3.3) to ω_1 only causes a remainder in $C_G^0(\Sigma)_\infty$, and used Lemma 3.11 (applied to $\gamma-\varepsilon$ instead of γ). Since $a(\eta)=\eta^\mu-t^{-\mu}\mathrm{op}_M(\sigma_M^\mu(a))$ on $[0,1]\times\partial\mathbb{B}$, the second term on the right-hand side of (3.6) equals

$$-\omega_1(t[\eta])\,\mathrm{op}_M^{\gamma-\varepsilon-\frac{n}{2}}(\tilde{f}_0)\,\omega_0(t[\eta])-\omega(t[\eta])\,(t\eta)^\mu\,\mathrm{op}_M^{\gamma-\varepsilon-\frac{n}{2}}(f)\,\omega_0(t[\eta])-$$
$$-\omega_1(t[\eta])\,\mathrm{op}_M^{\gamma-\varepsilon-\frac{n}{2}}(T^{-\mu}\sigma_M^\mu(a))\,(1-\omega)(t[\eta])\,\mathrm{op}_M^{\gamma-\varepsilon-\frac{n}{2}}(f)\,\omega_0(t[\eta]).$$

Here, $T^{-\mu}$ is the operator of shifting by $-\mu$, cf. Theorem 6.7. The first term cancels with the first term of (3.6), the other two particularly belong to $C_G^0(\Sigma;\gamma-\varepsilon,1,O;\gamma+\varepsilon,1,Q)$, where Q is the asymptotic type induced by the meromorphic structure of $(T^{-\mu}\sigma_M^\mu(a)^{-1})^*$. □

Passing in (3.1), (3.2) to the principal edge symbols, cf. (6.15), and solving for $(\eta^\mu-\underline{\widehat{A}})^{-1}$ yields (for sufficiently small $\varepsilon>0$) the identity

$$(\eta^\mu-\underline{\widehat{A}})^{-1}=\sigma_\wedge^{-\mu}(b_R)(\eta)-\sigma_\wedge^{-\mu}(b_L)(\eta)\sigma_\wedge^0(g_R)(\eta)+\sigma_\wedge^0(g_L)(\eta)(\eta^\mu-\underline{\widehat{A}})^{-1}\sigma_\wedge^0(g_R)(\eta) \tag{3.7}$$

on $\mathcal{K}_p^{0,\gamma}(\partial\mathbb{B}^\wedge)$. Here we have set $\sigma_\wedge^{-\mu}(b_R)(\eta)=\sigma_\wedge^{-\mu}(b_\varepsilon)(\eta)\big|_{\mathcal{K}_p^{s,\gamma}(\partial\mathbb{B}^\wedge)}$. We are now going to show that the second and third term on the right-hand side of (3.7) are the principal edge symbol of a parameter-dependent Green operator. Let us set

$$\underline{\mathcal{E}}'=\underline{\mathcal{E}}\oplus\mathrm{im}\,\omega\left(\mathrm{op}_M^{\gamma+\mu-\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1})-\mathrm{op}_M^{\gamma+\mu+\varepsilon-\frac{n}{2}}(\sigma_M^\mu(A)^{-1})\right), \tag{3.8}$$

i.e. we add to $\underline{\mathcal{E}}$ the asymptotic terms coming from the poles of $\sigma_M^\mu(A)^{-1}$ on the line $\mathrm{Re}\,z=\frac{n+1}{2}-\gamma-\mu$.

Proposition 3.8. *There exists a Green symbol $g\in R_G^\mu(\Sigma;\gamma,\varepsilon;\gamma,\mu+\varepsilon)$ for some $\varepsilon>0$ such that*

$$\sigma_\wedge^{-\mu}(g)(\eta)=-\sigma_\wedge^{-\mu}(b_L)(\eta)\sigma_\wedge^0(g_R)(\eta)+\sigma_\wedge^0(g_L)(\eta)(\eta^\mu-\underline{\widehat{A}})^{-1}\sigma_\wedge^0(g_R)(\eta)$$

and the integral kernel k_g of g, cf. (6.1) *and Theorem* 6.12, *satisfies*

$$k_g(\eta,t,x,t',x')\in S_{cl}^\mu(\Sigma,\mathcal{S}_{\underline{\mathcal{E}}'}^\gamma(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\pi\,\mathcal{S}_\varepsilon^{-\gamma}(\partial\mathbb{B}^\wedge))=S_{cl}^\mu(\Sigma)\,\widehat{\otimes}_\pi\,\mathcal{S}_{\underline{\mathcal{E}}'}^\gamma(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\pi\,\mathcal{S}_{\varepsilon/2}^{-\gamma}(\partial\mathbb{B}^\wedge),$$

where we set $\mathcal{S}_{\underline{\mathcal{E}}'}^\gamma(\partial\mathbb{B}^\wedge)=\mathcal{S}_{\varepsilon/2}^{\gamma+\mu}(\partial\mathbb{B}^\wedge)\oplus\underline{\mathcal{E}}'$ and $\underline{\mathcal{E}}'$ is as described in (3.8).

Proof. For brevity let us write $\widehat{g}(\eta)=\sigma_\wedge^{-\mu}(g)(\eta)$, and analogously for the principal edge symbols of b, g_L, and g_R. By the previous proposition, we have

$$\mathcal{K}_2^{s,\gamma}(\partial\mathbb{B}^\wedge)\hookrightarrow\mathcal{K}_2^{s,\gamma-\varepsilon}(\partial\mathbb{B}^\wedge)\xrightarrow{\widehat{g}_R(\eta)}\mathcal{S}_Q^{\gamma+\varepsilon}(\partial\mathbb{B}^\wedge)\hookrightarrow\mathcal{S}_Q^{\gamma-\mu+\varepsilon}(\partial\mathbb{B}^\wedge),$$

where for the last embedding one considers $Q\in\mathrm{As}(\gamma+\varepsilon,1-2\varepsilon)$ as an asymptotic type $Q\in\mathrm{As}(\gamma-\mu+\varepsilon,\mu)$. By standard mapping properties of cone operators there exists an asymptotic type $Q'\in\mathrm{As}(\gamma+\varepsilon,\mu)$ such that $\widehat{b}_L(\eta):\mathcal{S}_Q^{\gamma-\mu+\varepsilon}(\partial\mathbb{B}^\wedge)\to\mathcal{S}_{Q'}^{\gamma+\varepsilon}(\partial\mathbb{B}^\wedge)\hookrightarrow\mathcal{S}_{Q'}^\gamma(\partial\mathbb{B}^\wedge)$, where for the last embedding we consider Q' as an element of $\mathrm{As}(\gamma,\mu+\varepsilon)$. Making similar considerations for the adjoint, we thus obtain $\widehat{b}_L(\eta)\widehat{g}_R(\eta)\in R_G^{(-\mu)}(\Sigma;\gamma,\varepsilon;\gamma,\mu+\varepsilon)$.

Next, $(\eta^\mu-\underline{\widehat{A}})^{-1}$ is a smooth function on $\Sigma\setminus\{0\}$ with values in $\mathcal{L}(\mathcal{K}_p^{0,\gamma}(\partial\mathbb{B}^\wedge),\mathcal{D}(\underline{\widehat{A}}))$. In view of assumption (2) on the scaling invariance of $\underline{\mathcal{E}}$, it is (twisted) homogeneous of order $-\mu$ in the sense of (6.9). In particular, $(\eta^\mu-\underline{\widehat{A}})^{-1}\in S^{(-\mu)}(\Sigma;\mathcal{K}_p^{0,\gamma}(\partial\mathbb{B}^\wedge),\mathcal{K}_p^{0,\gamma+\varepsilon}(\partial\mathbb{B}^\wedge)$ for sufficiently small $\varepsilon>0$. But then it is clear that $\widehat{g}_L(\eta)(\eta^\mu-\underline{\widehat{A}})^{-1}\widehat{g}_R(\eta)$ also belongs to $R^{(-\mu)}(\Sigma;\gamma,\varepsilon;\gamma,\mu+\varepsilon)$. If we now define

$$g(\eta)=\chi(\eta)\{-\widehat{b}_L(\eta)\widehat{g}_R(\eta)+\widehat{g}_L(\eta)(\eta^\mu-\underline{\widehat{A}})^{-1}\widehat{g}_R(\eta)\}$$

with an arbitraray zero excision function $\chi(\eta)$, then $g\in R_G^\mu(\Sigma;\gamma,\varepsilon;\gamma,\mu+\varepsilon)$ and the principal edge symbol is given by the formula stated in the proposition. It remains to investigate the kernel.

Since $g(\eta)$ is a Green symbol of the given class (and by the kernel characterization) there exists some asymptotic type $Q \in \mathrm{As}(\gamma, \mu + \frac{\varepsilon}{2})$ such that the integral kernel k_g of $g(\eta)$ belongs to $S^{\mu}_{cl}(\Sigma)\,\widehat{\otimes}_\pi\, \mathcal{S}^{\gamma+\varepsilon}_Q(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\pi\,\mathcal{S}^{-\gamma}_{\varepsilon/2}(\partial\mathbb{B}^\wedge)$. According to Definition 6.10,

$$\mathcal{S}^{\gamma+\varepsilon}_Q(\partial\mathbb{B}^\wedge) = \mathcal{S}^{\gamma+\varepsilon/2}_{\mu}(\partial\mathbb{B}^\wedge) \oplus \mathcal{E}_Q.$$

By possibly shrinking ε, we may assume that Q contains no triple (q,l,L) with $\mathrm{Re}\, q < \frac{n+1}{2} - \gamma - \mu$. By possibly enlarging Q we can assume that $\underline{\mathcal{E}}' \subset \mathcal{E}_Q$, and therefore $\mathcal{E}_Q = \underline{\mathcal{E}}' \oplus V$ for a certain finite-dimensional space V. Therefore k_g can be written as $k_g = k^0_g + k^1_g$ with k^0_g containing the contribution of $\underline{\mathcal{E}}'$, and k^1_g that of V. However, from identity (3.7) one sees that $g(\eta)$ maps into the domain of $\widehat{\underline{A}}$, and therefore k^1_g must equal 0. This then finishes the proof, since $\mathcal{S}^{\gamma+\varepsilon/2}_{\mu}(\partial\mathbb{B}^\wedge) = \mathcal{S}^{\gamma+\mu}_{\varepsilon/2}(\partial\mathbb{B}^\wedge)$. $\square$

With $b_R(\eta)$ from Proposition 3.7 and $g(\eta)$ from Proposition 3.8 let us now define for certain cut-off functions $\sigma, \sigma_0 \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$

$$b(\eta) = b_R(\eta) + \sigma\, g(\eta)\,\sigma_0 = \tilde{b}(\eta) + \omega(t[\eta])\, t^\mu\, \mathrm{op}_M^{\gamma-\varepsilon-\frac{n}{2}}(f)\,\omega_0(t[\eta]) + \sigma\, g(\eta)\,\sigma_0 \tag{3.9}$$

(as an operator-family $\mathcal{H}^{0,\gamma}_p(\mathbb{B}) \to \mathcal{D}(\underline{A})$). More detailed, by the construction of $b_R(\eta)$, cf. (3.5) and the definition of $C^\mu_O(\Sigma)$ in Definition 6.6, we have

$$b(\eta) = \sigma\left\{t^\mu\, \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) + \omega(t[\eta])\, t^\mu\, \mathrm{op}_M^{\gamma-\varepsilon-\frac{n}{2}}(f)\,\omega_0(t[\eta]) + g(\eta)\right\}\sigma_0 + (1-\sigma)\, p(\eta)\,(1-\sigma_1),$$

where h, p, and the cut-off functions $\sigma, \sigma_0, \sigma_1$ are as described in Definition 6.6 and the paragraph thereafter. Since, due to A having constant coefficients,

$$\sigma^{-\mu}_\wedge(b_R)(\eta) = t^\mu\, \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) + \omega(t|\eta|)\, t^\mu\, \mathrm{op}_M^{\gamma-\varepsilon-\frac{n}{2}}(f)\,\omega_0(t|\eta|),$$

and $\sigma^{-\mu}_\wedge(g)(\eta) = g(\eta)$ for large enough $|\eta|$ by construction of g, we obtain that

$$b(\eta) = \sigma\,(\eta^\mu - \widehat{\underline{A}})^{-1}\,\sigma_0 + (1-\sigma)\, p(\eta)\,(1-\sigma_1), \qquad |\eta| \geq R, \tag{3.10}$$

for a sufficiently large $R > 0$. Moreover, we observe that changing the cut-off functions in (3.10) only alters $b(\eta)$ by a nice remainder:

Lemma 3.9. *Let $\tilde{\sigma}, \tilde{\sigma}_0, \tilde{\sigma}_1 \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ be cut-off functions satisfying the conditions posed in Definition 6.6 and the subsequent paragraph. Then, using the notation from (3.10),*

$$b(\eta) = \tilde{\sigma}\,(\eta^\mu - \widehat{\underline{A}})^{-1}\,\tilde{\sigma}_0 + (1-\tilde{\sigma})\, p(\eta)\,(1-\tilde{\sigma}_1) + r(\eta), \qquad |\eta| \geq R,$$

with a remainder $r(\eta)$ having an integral kernel (with a certain $\varepsilon > 0$)

$$k_r(\eta, y, y') \in \mathcal{S}(\Sigma, \mathcal{C}^{\infty,\gamma}_{\underline{\mathcal{E}}'}(\mathbb{B})\,\widehat{\otimes}_\pi\, \mathcal{C}^{\infty,-\gamma}_{\varepsilon}(\mathbb{B})).$$

Here, we have set $\mathcal{C}^{\infty,\gamma}_{\underline{\mathcal{E}}'}(\mathbb{B}) = \mathcal{C}^{\infty,\gamma+\mu}_{\varepsilon} \oplus \underline{\mathcal{E}}'$. In particular, $r(\eta) \in \mathcal{S}(\Sigma, \mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{B}), \mathcal{D}(\underline{A})))$.

Proof. This statement is easily seen, if we use the representation of $b(\eta)$ in (3.9). Changing cut-off functions alters $\tilde{b}(\eta)$ only by a flat Green symbol in $C^{-\infty}_G(\Sigma)_\infty$, which has in particular a kernel of the mentioned structure. It remains to note that $\sigma g(\eta)\sigma_0 - \tilde{\sigma} g(\eta)\tilde{\sigma}_0 = \sigma g(\eta)(\sigma_0 - \tilde{\sigma}_0) + (\tilde{\sigma} - \sigma) g(\eta)\tilde{\sigma}_0$ and both these terms have the required structure (recall that they are Green symbols of order $-\infty$, since both $\tilde{\sigma} - \sigma$ and $\sigma_0 - \tilde{\sigma}_0$ belong to $\mathcal{C}^\infty_{\mathrm{comp}}(]0,1[)$). $\square$

Proposition 3.10. *If $b(\eta)$ is as in (3.10), then for $|\eta|$ large enough*

$$(\eta^\mu - \underline{A})b(\eta) - 1 = r_R(\eta), \qquad b(\eta)(\eta^\mu - \underline{A}) - 1 = r_L(\eta),$$

for certain remainders (with a certain $\varepsilon > 0$)

$$r_R(\eta) \in C^{-\infty}_G(\Sigma; \gamma, \gamma, \varepsilon), \qquad r_L(\eta) \in \mathcal{S}(\Sigma, \mathcal{L}(\mathcal{D}(\underline{A}))).$$

In particular, $(\eta^\mu - \underline{A}) : \mathcal{D}(\underline{A}) \to \mathcal{H}^{0,\gamma}_p(\mathbb{B})$ is invertible for sufficiently large $|\eta|$.

PROOF. Since A is a local operator, we can write $(\eta^\mu-\underline{A})=\sigma(\eta^\mu-\underline{A})\sigma_0+(1-\sigma)(\eta^\mu-A)(1-\sigma_1)$ with cut-off functions mentioned before. Then

$$(\eta^\mu-\underline{A})\,b(\eta)=\sigma\,(\eta^\mu-\underline{A})\,\sigma_0\,b(\eta)+(1-\sigma)\,(\eta^\mu-A)\,(1-\sigma_1)\,b(\eta).$$

To treat the first summand choose a representation of $b(\eta)$ as in Lemma 3.9 with $\tilde\sigma\sigma_0=\sigma_0$. Then

$$\sigma\,(\eta^\mu-\underline{A})\,\sigma_0\,b(\eta)=\sigma+\sigma\,(\eta^\mu-A)\,\sigma_0\,r(\eta),$$

where the second term belongs to $C_G^{-\infty}(\Sigma;\gamma,\gamma,\varepsilon)$ (recall Lemma 3.1). For the second summand we choose for $b(\eta)$ a representation with $\tilde\sigma_0$ such that $\sigma_1\tilde\sigma_0=\tilde\sigma_0$. Then

$$(1-\sigma)\,(\eta^\mu-\underline{A})\,(1-\sigma_1)\,b(\eta)\equiv(1-\sigma)+(1-\sigma)\,((\eta^\mu-A)p(\eta)-1)\,(1-\tilde\sigma_1)$$

modulo a remainder of the prescribed form. However, by (parametrix-)construction the term on the right-hand side of the last relation is of the form $(1-\sigma)a(\eta)(1-\tilde\sigma_1)$ with $a(\eta)\in L^{-\infty}(2\mathbb{B};\Sigma)$, hence is a remainder of the desired form. The considerations for $b(\eta)(\eta^\mu-\underline{A})$ are analogous. □

To finish the proof of Theorem 3.5.b), it remains to modify $b(\eta)$ in such a way that we obtain the inverse of $\eta^\mu-\underline{A}$. To do so, we may assume that $r_R(\eta)$ of Proposition 3.10 satisfies $\|r_R(\eta)\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))}\le\frac12$ for all $\eta\in\Sigma$ (otherwise we multiply $r_R(\eta)$ with a suitable zero excision function $\chi(\eta)$). But then $1+r_R(\eta)\in\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))$ is invertible for all $\eta\in\Sigma$ and

$$(1+r_R(\eta))^{-1}=1-r_R(\eta)+r_R(\eta)(1+r_R(\eta))^{-1}r_R(\eta)=:1+r(\eta).$$

Clearly, $r(\eta)$ belongs to $C_G^{-\infty}(\Sigma;\gamma,\gamma,\varepsilon)$, again. Hence, by Proposition 3.10 and (3.9),

$$(\eta^\mu-\underline{A})^{-1}=b(\eta)(1+r_R(\eta))^{-1}=b(\eta)+b(\eta)r(\eta)=\tilde b(\eta)+\tilde r(\eta)$$

for large $|\eta|$, with $\tilde r(\eta)\in C_G^{-\mu}(\Sigma;\gamma,\gamma,\varepsilon)$ (note that the smoothing Mellin term in (3.9) belongs to $C_{M+G}^{-\mu}(\Sigma;\gamma-\varepsilon,\gamma+\mu-\varepsilon,\mu)\subset C_G^{-\mu}(\Sigma;\gamma,\gamma,\varepsilon)$).

LEMMA 3.11. *Let $\gamma\in\mathbb{R}$ and $0<\varepsilon<\frac12$. Then*

$$C_G^0(\Sigma;\gamma,1,O;\gamma,1,O)\subset C_G^0(\Sigma;\gamma,1-2\varepsilon,O;\gamma+2\varepsilon,1-2\varepsilon,O).$$

PROOF. Let $g(\eta)\in C_G^0(\Sigma;\gamma,1,O;\gamma,1,O)$. Then, by Definition 6.13, we can write $g(\eta)=\sigma\,a(\eta)\,\sigma_0+r(\eta)$ with $a(\eta)\in R_G^0(\Sigma;\gamma,1,O;\gamma,1,O)$ and $r(\eta)\in C_G^{-\infty}(\Sigma;\gamma,1,O;\gamma,1,O)$. We now have to show that

$$a(\eta)\in R_G^0(\Sigma;\gamma,1-2\varepsilon,O;\gamma+2\varepsilon,1-2\varepsilon,O)\tag{3.11}$$

and that $r(\eta)\in C_G^{-\infty}(\Sigma;\gamma,1-2\varepsilon,O;\gamma+2\varepsilon,1-2\varepsilon,O)$. We restrict ourselves to the proof of (3.11), since the symbol $r(\eta)$ can be treated in an analogous, even simpler way. By Theorem 6.12 it suffices to show that

$$\begin{aligned}\mathcal{S}_1^\gamma(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\Gamma\,\mathcal{S}_1^{-\gamma}(\partial\mathbb{B}^\wedge)&\subset\mathcal{S}_{1-2\varepsilon}^{\gamma+2\varepsilon}(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\Gamma\,\mathcal{S}_{1-2\varepsilon}^{-\gamma}(\partial\mathbb{B}^\wedge)\\&=[\mathcal{S}_{1-2\varepsilon}^{\gamma+2\varepsilon}(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\pi\,\mathcal{S}_0^{-\gamma}(\partial\mathbb{B}^\wedge)]\cap[\mathcal{S}_0^{\gamma+2\varepsilon}(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\pi\,\mathcal{S}_{1-2\varepsilon}^{-\gamma}(\partial\mathbb{B}^\wedge)]\end{aligned}$$

(recall that we write $\mathcal{S}_\theta^\gamma=\mathcal{S}_O^\gamma$ if $O\in\mathrm{As}(\gamma,\theta)$ is the empty asymptotic type). By Proposition 4.5 of **[27]** (in a version for operators on $\partial\mathbb{B}^\wedge$) we have

$$\mathcal{S}_1^\gamma(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\Gamma\,\mathcal{S}_1^{-\gamma}(\partial\mathbb{B}^\wedge)=\bigcap_{0\le\sigma\le1}\mathcal{S}_\sigma^\gamma(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\pi\,\mathcal{S}_{1-\sigma}^{-\gamma}(\partial\mathbb{B}^\wedge).$$

Taking the parameter $\sigma=1$ yields

$$\mathcal{S}_1^\gamma(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\Gamma\,\mathcal{S}_1^{-\gamma}(\partial\mathbb{B}^\wedge)\subset\mathcal{S}_1^\gamma(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\pi\,\mathcal{S}_0^{-\gamma}(\partial\mathbb{B}^\wedge)=\mathcal{S}_{1-2\varepsilon}^{\gamma+2\varepsilon}(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\pi\,\mathcal{S}_0^{-\gamma}(\partial\mathbb{B}^\wedge),$$

where the last identity directly follows from the definition of the involved spaces. Inserting $\sigma=2\varepsilon+\delta$, $\delta>0$ small, yields

$$\mathcal{S}_1^\gamma(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\Gamma\,\mathcal{S}_1^{-\gamma}(\partial\mathbb{B}^\wedge)\subset\mathcal{S}_{2\varepsilon+\delta}^{\gamma}(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\pi\,\mathcal{S}_{1-2\varepsilon-\delta}^{-\gamma}(\partial\mathbb{B}^\wedge)\subset\mathcal{S}_0^{\gamma+2\varepsilon}(\partial\mathbb{B}^\wedge)\,\widehat\otimes_\pi\,\mathcal{S}_{1-2\varepsilon-\delta}^{-\gamma}(\partial\mathbb{B}^\wedge).$$

Passing to the intersection over all $\delta > 0$ gives

$$\mathcal{S}_1^{\gamma}(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\Gamma\,\mathcal{S}_1^{-\gamma}(\partial\mathbb{B}^\wedge) \subset \mathcal{S}_0^{\gamma+2\varepsilon}(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\pi\,\mathcal{S}_{1-2\varepsilon}^{-\gamma}(\partial\mathbb{B}^\wedge).$$

This finishes the proof. □

4. Spectral invariance and bounded imagainary powers

4.1. Independence on the choice of $1 < p < \infty$. The invertibility result of $\underline{A} = \underline{A}_p$ of Theorem 3.5 a priori holds for a given, fixed $p = p_0$. However, the inverse $c(\eta)$ of Theorem 3.5.b) induces continuous operators

$$c(\eta) : \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \longrightarrow \mathcal{D}(\underline{A}) = \mathcal{D}(\underline{A}_p)$$

for all $1 < p < \infty$. Moreover, $c(\eta)(\eta^\mu - \underline{A}_p) = 1$ on $\mathcal{C}^{\infty,\infty}(\mathbb{B}) \oplus \underline{\mathcal{E}}$ for each p, since this is true for $p = p_0$ and the left-hand side of the latter equation is independent of p on $\mathcal{C}^{\infty,\infty}(\mathbb{B}) \oplus \underline{\mathcal{E}}$. Similarly, $(\eta^\mu - \underline{A}_p)c(\eta) = 1$ on $\mathcal{C}^{\infty,\infty}(\mathbb{B})$ for all p. Thus a density argument shows:

REMARK 4.1. *If $\underline{A} = \underline{A}_p$ satisfies Theorem 3.5.b) for one p, then automatically for all $1 < p < \infty$.*

The constructions of Section 3.3, cf. (3.7), show that

$$(\eta^\mu - \widehat{\underline{A}}_p)^{-1} \in R^{(-\mu)}(\Sigma) + R_G^{(-\mu)}(\Sigma; \gamma, \gamma, \varepsilon),$$

where $R^{(-\mu)}(\Sigma)$ denotes the space of all principal edge symbols $\sigma_\wedge^{-\mu}(c)$ with $c \in C_O^{-\mu}(\Sigma)$. Thus, arguing similarly as above, we obtain

REMARK 4.2. *If $\underline{A} = \underline{A}_p$ satisfies ellipticity assumptions (1), (2), and (3) of Section 3.2 for one p, then automatically for all $1 < p < \infty$.*

4.2. Bounded imaginary powers. In the paper [**3**] we have shown that the closure of a cone differential operator – under ellipticity conditions (1) and (3) with $\underline{\mathcal{E}} = \{0\}$ – posseses bounded imaginary powers whose operator-norm in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ can be estimated by $c_p e^{\theta|z|}$, where θ is the angle determining $\Lambda = \Lambda_\theta$. We also had pointed out in Remark 5.5 of [**3**] that the validity of this result 'only' relies on the structure of the resolvent and not on the fact that we dealt with the closure of the operator. But Theorem 3.5 now states that the resolvent of a general closed extension has exactly this required structure (in [**3**] we described the resolvent $(\lambda - \overline{A})^{-1}$ in terms of anisotropic symbols, while here we described $(\eta^\mu - \underline{A})^{-1}$. However, both ways are obviously equivalent). Thus we have the following result:

THEOREM 4.3. *Let $\underline{A}$ be a closed extension of a cone differential operator A, satisfying the ellipticity assumptions (1), (2), and (3) with respect to $\Lambda = \Lambda_\theta$. Then there exists a constant $c \geq 0$ such that $\underline{A} + c$ has bounded imaginary powers and, for a certain constant $c_p \geq 0$,*

$$\|(\underline{A} + c)^{i\varrho}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \leq c_p\, e^{\theta|\varrho|} \qquad \forall\, \varrho \in \mathbb{R}.$$

Let us mention that the operator $A + c$ does not satisfy the assumption of constant coefficients near the boundary (since we have to write $c = t^{-\mu}(t^\mu c)$). However, the structure of the resolvent remains uneffected by the shift with a constant c.

In [**3**] we did not assume A to have t-independent coefficients near the singularity, but additionally required A to be conormal elliptic with respect to the weight $\gamma + \mu$, so that $\mathcal{D}(A_{\min}) = \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B})$, cf. Proposition 2.3. Using the method of proof in Section 3.3, this additional assumption is obsolete.

5. Example: The Laplace-Beltrami operator

Let the interior of $\mathbb{B}$ be equipped with a metric that coincides on $]0,1[\times\partial\mathbb{B}$ with dt^2+t^2g for some fixed metric g on $\partial\mathbb{B}$ (straight conical degeneracy). The associated Laplacian Δ is a second order cone differential operator, and

$$\Delta = t^{-2}\left\{(t\partial_t)^2+(n-1)t\partial_t+\Delta_\partial\right\},\qquad n=\dim\partial\mathbb{B},$$

near the boundary of $\mathbb{B}$, where Δ_∂ denotes the Laplacian on $\partial\mathbb{B}$ with respect to g.

Clearly, $-\Delta$ satisfies ellipticity condition (1) of Section 3.2 for any sector Λ not containing positive reals.

5.1. The conormal symbol. By the definition given in the beginning of Section 2.1, the conormal symbol of the Laplacian is

$$\sigma_M^2(\Delta)(z)=z^2+(n-1)z+\Delta_\partial.$$

Let us investigate the inverse of this function. To this end let $0=\lambda_0>\lambda_1>\ldots$ be the eigenvalues of Δ_∂ and $E_0,E_1,\ldots$ the corresponding eigenspaces. Moreover, let $\pi_j\in\mathcal{L}(L_2(\partial\mathbb{B}))$ denote the orthogonal projection onto E_j.

The *non*-bijectivity points of $\sigma_M^2(\Delta)$ are exactly the points $z=q_j^+$ and $z=q_j^-$ with

$$q_j^\pm=\tfrac{n-1}{2}\pm\sqrt{\left(\tfrac{n-1}{2}\right)^2-\lambda_j},\qquad j\in\mathbb{N}_0. \tag{5.1}$$

Note the symmetry $q_j^+=(n-1)-q_j^-$ and that $z^2-(n-1)z+\lambda_j=(z-q_j^+)(z-q_j^-)$. It is straightforward to calculate that in case $\dim\mathbb{B}\geq 3$

$$(z^2-(n-1)z+\Delta_\partial)^{-1}\equiv\pm\tfrac{1}{q_j^+-q_j^-}\,\pi_j\,(z-q_j^\pm)^{-1}\qquad\text{near } z=q_j^\pm$$

modulo holomorphic functions (respectively germs). In case $\dim\mathbb{B}=2$ the same formula holds near $z=q_j^\pm$ if $j\geq 1$ but

$$(z^2+\Delta_\partial)^{-1}\equiv\pi_0\,z^{-2}\qquad\text{near } z=0.$$

5.2. Maximal domain and dilation invariance. With $q_j^\pm$ we associate we associate the function space

$$\mathcal{E}_{q_j^\pm}=E_j\otimes\omega\,t^{-q_j^\pm}=\{e(x)\,\omega(t)\,t^{-q_j^\pm}\mid e\in E_j\},\qquad j\in\mathbb{N},$$

and for $q_0^\pm$ we set

$$\mathcal{E}_{q_0^\pm}=\begin{cases}E_0\otimes\omega+E_0\otimes\omega\log t & \dim\mathbb{B}=2\\ E_0\otimes\omega\,t^{q_0^\pm} & \dim\mathbb{B}\geq 3\end{cases}. \tag{5.2}$$

Note that $q_0^+=q_0^-=0$ in case $\dim\partial\mathbb{B}=1$. Furthermore, for $\gamma\in\mathbb{R}$, set

$$I_\gamma=\{q_j^\pm\mid j\in\mathbb{N}_0\}\cap\,]\tfrac{n+1}{2}-\gamma-2,\tfrac{n+1}{2}-\gamma[\,=\{q_j^\pm\mid j\in\mathbb{N}_0\}\cap\,]\tfrac{n-1}{2}-\gamma-1,\tfrac{n-1}{2}-\gamma+1[.$$

Applying Theorems 2.3 and 2.8 to $A=\Delta$, we get the following:

Proposition 5.1. *Consider Δ as an unbounded operator in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$. Then the domain of the maximal extension is*

$$\mathcal{D}(\Delta_{\max})=\mathcal{D}(\Delta_{\min})\oplus\bigoplus_{q\in I_\gamma}\mathcal{E}_p.$$

In case $q_j^\pm\neq\frac{n+1}{2}-\gamma-2$ for all j, the minimal domain is $\mathcal{D}(\Delta_{\min})=\mathcal{H}_p^{2,\gamma+2}(\mathbb{B})$.

Let us now describe the closed extensions $\underline{\Delta}$ of Δ satisfying condition (2) of Section 3.2. For convenience such extensions we shall call dilation invariant extensions. A straightforward calculation (or an application of Lemmas 5.11 and 5.12 of [**10**]) yields:

Proposition 5.2. *Consider Δ as an unbounded operator in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$. The dilation invariant extensions $\underline{\Delta}$ are precisely those with a domain of the form*

$$\mathcal{D}(\underline{\Delta}) = \mathcal{D}(\Delta_{\min}) \oplus \bigoplus_{q\in I_\gamma} \underline{\mathcal{E}}_q, \qquad \underline{\mathcal{E}}_q \text{ subspace of } \mathcal{E}_q, \tag{5.3}$$

where in case $\dim \mathbb{B} = 2$ *either* $\underline{\mathcal{E}}_0 = \{0\}$ *or* $\underline{\mathcal{E}}_0 = E_0 \otimes \omega$ *or* $\underline{\mathcal{E}}_0 = \mathcal{E}_0$, *cf.* (5.2).

Let us point out that in (5.3) the sum is taken over all $q \in I_\gamma$ and that the summand $\underline{\mathcal{E}}_q = \{0\}$ may occur several times.

5.3. Adjoint operators. Since the scalar-product $\langle\cdot,\cdot\rangle_{0,0}$ of $\mathcal{H}_2^{0,0}(\mathbb{B})$ yields an identification of the dual space of $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ with $\mathcal{H}_{p'}^{0,-\gamma}(\mathbb{B})$, we can associate with each extension $\underline{\Delta}$ in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ an adjoint operator $\underline{\Delta}^*$. This adjoint is then an unbounded operator in $\mathcal{H}_{p'}^{0,-\gamma}(\mathbb{B})$ which is given by the action of Δ on the domain

$$\mathcal{D}(\underline{\Delta}^*) = \{v \in \mathcal{H}_{p'}^{0,-\gamma}(\mathbb{B}) \mid \exists\, f \in \mathcal{H}_{p'}^{0,-\gamma}(\mathbb{B})\ \forall\, u \in \mathcal{D}(\underline{\Delta}): \quad \langle v, \Delta u\rangle_{0,0} = \langle f, u\rangle_{0,0}\}.$$

It is easy to see that $\Delta_{\min}^* = \Delta_{\max}$ and $\Delta_{\max}^* = \Delta_{\min}$.

The goal of this subsection is to describe explicitly the adjoints of dilation invariant extensions $\underline{\Delta}$. For an analysis of adjoints of general cone differential operators (in case $p = 2$) see the paper [**10**]. Define the pairing

$$\langle\cdot,\cdot\rangle : \mathcal{D}_p^{\gamma}(\Delta_{\max}) \times \mathcal{D}_{p'}^{-\gamma}(\Delta_{\max}) \longrightarrow \mathbb{C}, \quad \langle u, v\rangle = \langle \Delta u, v\rangle_{0,0} - \langle u, \Delta v\rangle_{0,0},$$

where the indices σ, r in $\mathcal{D}_r^{\sigma}$ now indicate that we consider the Laplacian in the Sobolev space $\mathcal{H}_r^{0,\sigma}(\mathbb{B})$. Then the domain of the adjoint operator $\underline{\Delta}^*$ is just the orthogonal space (with respect to this pairing) to the domain of $\underline{\Delta}$, i.e.

$$\mathcal{D}_{p'}^{-\gamma}(\underline{\Delta}^*) = \mathcal{D}_p^{\gamma}(\underline{\Delta})^{\perp}.$$

Since $\langle u, v\rangle = 0$ whenever u or v belong to the minimal domain, the crucial part for calculating the orthogonal space is to understand which elements of $\oplus_{q\in I_{-\gamma}} \mathcal{E}_q$ are orthogonal to a given element of $\oplus_{q\in I_\gamma} \mathcal{E}_q$.

Let $u = e\,\omega\, t^{-q}$ with $q = q_j^+$ or $q = q_j^-$ and $e \in E_j$ for some fixed $j \in \mathbb{N}_0$. If $v_\pm = f\,\omega\, t^{-q_k^\pm}$ with $f \in E_k$, an elementary calculation yields

$$(\Delta u)\overline{v_\pm} - u\,\overline{\Delta v_\pm} = 2(q_k^\pm - q)\, e\,\overline{f}\,\omega\omega' t^{-q-q_k^\pm-1},$$

hence $\langle u, v_\pm\rangle = 0$ if and only if $q_k^\pm = q$ or $\langle e, f\rangle_{L_2(\partial\mathbb{B})} = 0$.

If $\dim\mathbb{B} = 2$ and $u = c\omega + d\omega\log t$ with $c, d \in \mathbb{C}$ and $v_\pm = f\,\omega\, t^{-q_k^\pm}$ with $f \in E_k$ and $k \neq 0$ then

$$(\Delta u)\overline{v_\pm} - u\,\overline{\Delta v_\pm} = 2\overline{f}(d + cq_k^\pm + dq_k^\pm \log t)\omega\omega' t^{-q_k^\pm-1},$$

hence $\langle u, v_\pm\rangle = 0$, since $\langle 1, f\rangle_{L_2(\partial\mathbb{B})} = 0$. If u is as before and $u = c_0\omega + d_0\omega\log t$ with $c_0, d_0 \in \mathbb{C}$, then

$$(\Delta u)\overline{v} - u\,\overline{\Delta v} = 2(c_0 d - d_0 c)\omega\omega' t^{-3},$$

hence if both u and v are different from zero, $\langle u, v\rangle = 0$ if and only if v is a multiple of u.

From this we derive the following description of adjoints of dilation invariant extensions:

Theorem 5.3. *Let $\underline{\Delta}$ be a dilation invariant extension of Δ in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ with domain*

$$\mathcal{D}_p^{\gamma}(\underline{\Delta}) = \mathcal{D}_p^{\gamma}(\Delta_{\min}) \oplus \bigoplus_{q\in I_\gamma} \underline{\mathcal{E}}_q$$

as described in Proposition 5.2. *Then the domain of the adjoint $\underline{\Delta}^*$ is*

$$\mathcal{D}_{p'}^{-\gamma}(\underline{\Delta}^*) = \mathcal{D}_{p'}^{-\gamma}(\Delta_{\min}) \oplus \bigoplus_{q\in I_\gamma} \underline{\mathcal{E}}_q^{\perp},$$

where the spaces $\underline{\mathcal{E}}_{q_j^\pm}^{\perp}$ are defined as follows:

i) *If either* $q_j^{\pm} \neq 0$ *or* $\dim \mathbb{B} \geq 3$, *there exists a unique subspace* $\underline{E}_j \subset E_j$ *such that* $\underline{\mathcal{E}}_{q_j^{\pm}} = \underline{E}_j \otimes \omega\, t^{-q_j^{\pm}}$. *Then we set*

$$\underline{\mathcal{E}}^{\perp}_{q_j^{\pm}} = \underline{E}_j^{\perp} \otimes \omega\, t^{-q_j^{\mp}},$$

where $\underline{E}_j^{\perp}$ *is the orthogonal complement of* $\underline{E}_j$ *in* E_j *with respect to the* $L_2(\partial\mathbb{B})$*-scalar product.*

ii) *If* $\dim \mathbb{B} = 2$ *and* $q_j^{\pm} = 0$ *define* $\underline{\mathcal{E}}_0^{\perp} = \{0\}$ *if* $\underline{\mathcal{E}}_0 = \mathcal{E}_0$, $\underline{\mathcal{E}}_0^{\perp} = \mathcal{E}_0$ *if* $\underline{\mathcal{E}}_0 = \{0\}$, *and* $\underline{\mathcal{E}}_0^{\perp} = \underline{\mathcal{E}}_0$ *if* $\underline{\mathcal{E}}_0 = E_0 \otimes \omega$.

Note that $\underline{\mathcal{E}}^{\perp}_{q_j^{\pm}}$ *is a subspace of* $\mathcal{E}_{q_j^{\mp}}$ *or, equivalently,* $\underline{\mathcal{E}}_q^{\perp}$ *is a subspace of* $\mathcal{E}_{(n-1)-q}$.

Corollary 5.4. *The selfadjoint dilation invariant extensions* $\underline{\Delta}$ *of* Δ *in* $\mathcal{H}_2^{0,0}(\mathbb{B})$ *are those with a domain of the form*

$$\mathcal{D}_2^0(\underline{\Delta}) = \mathcal{D}_2^0(\Delta_{\min}) \oplus \bigoplus_{q \in I_0} \underline{\mathcal{E}}_q$$

with $\underline{\mathcal{E}}_q^{\perp} = \underline{\mathcal{E}}_{(n-1)-q}$ *for all* $q \in I_0$ *(in particular* $\underline{\mathcal{E}}_0 = E_0 \otimes \omega$ *in case* $\dim \mathbb{B} = 2$*).*

Applying Theorems 8.3 and 8.12 of [**10**], the Friedrichs extension of Δ has the domain

$$\mathcal{D}_2^0(\underline{\Delta}) = \begin{cases} \mathcal{D}_2^0(\Delta_{\min}) \oplus \bigoplus\limits_{\substack{q \in I_0 \\ \operatorname{Re} q < 0}} \mathcal{E}_q \oplus (E_0 \otimes \omega) & \dim \mathbb{B} = 2 \\ \mathcal{D}_2^0(\Delta_{\min}) \oplus \bigoplus\limits_{\substack{q \in I_0 \\ \operatorname{Re} q \leq \frac{n-1}{2}}} \mathcal{E}_q & \dim \mathbb{B} \geq 3 \end{cases}.$$

In particular, the Friedrichs extension is dilation invariant.

Remark 5.5. *All the results of Sections* 5.2 *and* 5.3 *hold true in an analogous form for the model cone operator* $\widehat{\Delta}$ *considered as an unbounded operator in* $\mathcal{K}_p^{0,\gamma}(\partial\mathbb{B}^\wedge)$.

5.4. Elliptic extensions. Proposition 5.2 provides a complete description of the closed extensions $\underline{\Delta}$ of Δ such that $-\underline{\Delta}$ satisfies the ellipticity conditions (1) and (2) of Section 3.2. Now we discuss how extensions look like that also satisfy condition (3). We shall assume that $|\gamma| < \frac{1}{2}\dim \mathbb{B}$ (the choice of this range is connected to the scale of natural L_p-spaces on $\mathbb{B}$ as we shall explain below). Then to each given γ we find at least one extension having property (3), but in case $\dim \mathbb{B} \leq 3$ we find more than one. However, the extensions we describe might not represent all possible choices.

Theorem 5.6. *Consider* $-\Delta$ *as an unbounded operator in* $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ *and assume* $\dim \mathbb{B} \geq 4$. *Then conditions* (1), (2), *and* (3) *of Section* 3.2 *for any sector* $\Lambda \subset \mathbb{C} \setminus \mathbb{R}_+$ *are fulfilled by* $-\Delta_{\max}$ *in case* $0 \leq \gamma < \frac{1}{2}\dim \mathbb{B}$ *and by* $-\Delta_{\min}$ *in case* $-\frac{1}{2}\dim \mathbb{B} < \gamma \leq 0$.

The assumption on the dimension of $\mathbb{B}$ in the previous theorem ensures that Δ in $\mathcal{H}_2^{0,0}(\mathbb{B})$ is essentially self-adjoint or, in other words, the inverted conormal symbol has no pole in the interval I_0. We shall omit the proof of this theorem, since it is a simpler version of that for the following one (cf. also the proof of Theorem 7.1 in [**3**]).

Theorem 5.7. *Consider* $-\Delta$ *as an unbounded operator in* $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$, *assume* $\dim \mathbb{B} \leq 3$, *and let* $|\gamma| < \frac{1}{2}\dim \mathbb{B}$. *An extension* $-\underline{\Delta}$ *satisfies conditions* (1), (2), *and* (3) *of Section* 3.2 *for any sector* $\Lambda \subset \mathbb{C} \setminus \mathbb{R}_+$, *provided we choose its domain*

$$\mathcal{D}_p^\gamma(\underline{\Delta}) = \mathcal{D}_p^\gamma(\Delta_{\min}) \oplus \bigoplus_{q \in I_\gamma} \underline{\mathcal{E}}_q$$

according to the following rules:

(i) *If* $q \in I_\gamma \cap I_{-\gamma}$, *then* $\underline{\mathcal{E}}_q^{\perp} = \underline{\mathcal{E}}_{(n-1)-q}$.

(ii) *If* $\gamma \geq 0$ *and* $q \in I_\gamma \setminus I_{-\gamma}$, *then* $\underline{\mathcal{E}}_q = \mathcal{E}_q$.
(iii) *If* $\gamma \leq 0$ *and* $q \in I_{-\gamma} \setminus I_\gamma$, *then* $\underline{\mathcal{E}}_q = \{0\}$.

In particular, $\mathcal{D}_p^\gamma(\underline{\Delta}) = \mathcal{D}_p^\gamma(\Delta_{\max})$ *if* $\gamma \geq 1$ *and* $\mathcal{D}_p^\gamma(\underline{\Delta}) = \mathcal{D}_p^\gamma(\Delta_{\min})$ *if* $\gamma \leq -1$.

Proof. By Remark 4.2 we may assume that $p = 2$, and by duality it suffices to treat the case $\gamma \geq 0$. Let $\underline{\Delta}_0$ denote the selfadjoint extension of Δ in $\mathcal{H}_p^{0,0}(\mathbb{B})$ with $\mathcal{D}_2^\gamma(\underline{\Delta}) \subset \mathcal{D}_2^0(\underline{\Delta}_0)$. Such an extension always exists due to assumption (i) on the domain of $\underline{\Delta}$ and by Corollary 5.4; its domain is

$$\mathcal{D}_2^0(\underline{\Delta}_0) = \mathcal{D}_2^0(\Delta_{\min}) \oplus \bigoplus_{q \in I_0 \setminus I_{-\gamma}} \mathcal{E}_q \oplus \bigoplus_{q \in I_\gamma \cap I_{-\gamma}} \underline{\mathcal{E}}_q.$$

If we then pass to the associated model cone operators and use Remark 5.5, we get that

$$\lambda + \widehat{\underline{\Delta}} : \mathcal{D}_2^\gamma(\widehat{\underline{\Delta}}) \longrightarrow \mathcal{K}_2^{0,\gamma}(\partial\mathbb{B}^\wedge), \qquad \lambda \notin \overline{\mathbb{R}}_+, \tag{5.4}$$

is injective, since $\operatorname{spec}(-\widehat{\underline{\Delta}}_0) \subset \overline{\mathbb{R}}_+$ and $\mathcal{D}_2^\gamma(\widehat{\underline{\Delta}}) \subset \mathcal{D}_2^0(\widehat{\underline{\Delta}}_0)$.

By Theorem 5.3 (in the formulation for model cone operators), the adjoint $\widehat{\underline{\Delta}}^*$ of $\widehat{\underline{\Delta}}$ has the domain

$$\mathcal{D}_2^{-\gamma}(\widehat{\underline{\Delta}}^*) = \mathcal{D}_2^{-\gamma}(\widehat{\Delta}_{\min}) \oplus \bigoplus_{q \in I_\gamma \cap I_{-\gamma}} \underline{\mathcal{E}}_q.$$

Now let $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$ and $u \in \mathcal{D}_2^{-\gamma}(\widehat{\underline{\Delta}}^*)$ be an element of the kernel of $\lambda + \widehat{\underline{\Delta}}^*$, i.e. $(\lambda + \widehat{\Delta})u = 0$. We shall show now that this implies $u = 0$. To this end write $u = u_0 + u_1$ with $u_0 \in \mathcal{D}_2^{-\gamma}(\widehat{\Delta}_{\min})$ and $u_1 \in \bigoplus_{q \in I_\gamma \cap I_{-\gamma}} \underline{\mathcal{E}}_q$. Note that $u_0, u_1 \in \mathcal{K}_2^{0,0}(\partial\mathbb{B}^\wedge)$ by the assumption on the dimension of $\mathbb{B}$ and the structure of the domain of $\underline{\Delta}$. Since $\Delta u_1 \in \mathcal{C}_{\mathrm{comp}}^\infty(\operatorname{int}\mathbb{B})$ (as this is true for any linear combination of functions from the spaces $\mathcal{E}_q$), we obtain $(\lambda + \Delta)u_0 = -\lambda u_1 - \Delta u_1 \in \mathcal{K}_2^{0,0}(\partial\mathbb{B}^\wedge)$. But this means that

$$u_0 \in \mathcal{D}_2^0(\widehat{\Delta}_{\max}) \cap \mathcal{D}_2^{-\gamma}(\widehat{\Delta}_{\min}) = \mathcal{D}_2^0(\widehat{\Delta}_{\min}) \oplus \bigoplus_{q \in I_0 \setminus I_{-\gamma}} \mathcal{E}_q \subset \mathcal{D}_2^0(\widehat{\underline{\Delta}}_0).$$

The last inclusion is valid by construction of $\underline{\Delta}_0$. This yields $u \in \mathcal{D}_2^0(\widehat{\underline{\Delta}}_0)$ and $(\lambda + \widehat{\Delta})u = 0$, hence $u = 0$, since $\operatorname{spec}(-\widehat{\underline{\Delta}}_0) \subset \overline{\mathbb{R}}_+$.

This shows the bijectivity of (5.4), since there $\lambda + \widehat{\underline{\Delta}}$ is a Fredholm operator (this follows from [**12**], Proposition 1.3.16), hence has closed range. ☐

5.5. The Cauchy Problem. Let $1 < p < \infty$ and let $L_p(\mathbb{B})$ denote the L_p-space on $\operatorname{int}\mathbb{B}$ associated to the measure induced by the conical metric on $\operatorname{int}\mathbb{B}$. Then

$$L_p(\mathbb{B}) = \mathcal{H}_p^{0,\gamma_p}(\mathbb{B}), \qquad \gamma_p = (n+1)\big(\tfrac{1}{2} - \tfrac{1}{p}\big).$$

In fact, away from the boundary these spaces coincide by definition; thus it suffices to consider functions supported close to the boundary. But then, cf. Definition 2.2,

$$\|u\|^p_{\mathcal{H}_p^{0,\gamma_p}(\mathbb{B})} = \int_{[0,1]\times\partial\mathbb{B}} |t^{\frac{n+1}{2}-\gamma_p} u(t,x)|^p \, \tfrac{dt}{t} dx = \int_{[0,1]\times\partial\mathbb{B}} |u(t,x)|^p \, t^n dt dx = \|u\|^p_{L_p(\mathbb{B})}.$$

Clearly, $|\gamma_p| < \frac{n+1}{2} = \frac{1}{2}\dim\mathbb{B}$ when p ranges from 1 to ∞. Therefore the results of the previous Section 5.4 can be applied to the Laplacian in $L_p(\mathbb{B})$, $1 < p < \infty$.

Combining these results with Theorem 4.3 and the Dore-Venni theorem (Theorem 3.2 in [**5**]), one obtains maximal regularity for solutions of the Cauchy problem:

Theorem 5.8. *Consider* Δ *as an unbounded operator in* $L_p(\mathbb{B})$, $1 < p < \infty$. *If* $\underline{\Delta}$ *denotes any extension from Theorems* 5.6 *or* 5.7 *associated with* $\gamma = \gamma_p$, *the Cauchy problem*

$$\dot{u}(t) - \Delta u(t) = f(t) \quad \text{on} \quad 0 < t < T, \qquad u(0) = 0,$$

has for any $f \in L_q([0,T], L_p(\mathbb{B}))$, $1 < q < \infty$, *a unique solution*

$$u \in W_q^1([0,T], L_p(\mathbb{B})) \cap L_q([0,T], \mathcal{D}_p^{\gamma_p}(\underline{\Delta})).$$

6. Appendix: Parameter-dependent cone pseudodifferential operators

Let us review a calculus of parameter-dependent pseudodifferential operators on $\mathbb{B}$. It was introduced by Schulze [**21**], [**22**]. Our presentation follows [**26**] and [**11**]. While there the parameter-space was $\mathbb{R}^q$, we focus here on a subsector of the complex plane. The proofs pass over to this situation without any changes, and thus will be dropped here.

We split the presentation into two parts: In Sections 6.1 to 6.3 we describe a sub-calculus of *flat* operators. This is relatively simple to describe, and already contains $\eta^\mu - A$ for a μ-th order cone differential operator A and – under suitable ellipticity assumptions on A – a rough parametrix. To describe the resolvent $(\eta^\mu - A)^{-1}$ we need to enlarge this calculus. This shall be explained starting with Section 6.4.

In the following, Σ is a closed sector in the complex plane (identified with $\mathbb{R}^2$) containing zero, i.e.

$$\Sigma = \{\eta \in \mathbb{C} \mid \theta_1 \leq \arg\eta \leq \theta_2\} \cup \{0\}, \qquad -\pi \leq \theta_1, \theta_2 \leq \pi.$$

If E is a Fréchet space, we let $\mathcal{C}^\infty(\Sigma, E)$ denote the space of all continuous functions $\Sigma \to E$ that are smooth in the interior of Σ and whose derivatives have continuous extensions to the whole sector Σ. The space of rapidly decreasing functions $\mathcal{S}(\Sigma, E)$ refers to the decay of functions for $|\eta| \to \infty$.

6.1. Smoothing elements of the flat calculus.

The space $\mathcal{C}^{\infty,\infty}(\mathbb{B})$, consisting of all functions that are smooth in the interior of $\mathbb{B}$ and vanish to infinite order at the boundary, is Fréchet in a natural way. Taking the projective tensor product yields the space

$$\mathcal{C}^{\infty,\infty}(\mathbb{B} \times \mathbb{B}) = \mathcal{C}^{\infty,\infty}(\mathbb{B}) \widehat{\otimes}_\pi \mathcal{C}^{\infty,\infty}(\mathbb{B}).$$

Definition 6.1. *Let* $C_G^{-\infty}(\Sigma)_\infty$ *be the space of all operator-families* $r(\eta) : \mathcal{C}^{\infty,\infty}(\mathbb{B}) \to \mathcal{C}^{\infty,\infty}(\mathbb{B})$, $\eta \in \Sigma$, *such that*

$$[r(\eta)u](y) = \int_{\mathbb{B}} k_r(\eta, y, y')u(y')\, dy',$$

where dy' *is a measure induced by a conic metric on* $\mathbb{B}$, *and the kernel* $k_r \in \mathcal{S}(\Sigma, \mathcal{C}^{\infty,\infty}(\mathbb{B} \times \mathbb{B}))$ *depends rapidly decreasing on* $\eta \in \Sigma$.

Besides this kind of smoothing operators – which act globally on $\mathbb{B}$ and depend rapidly decreasing on the parameter – there appears another kind of smoothing operators that are localized near the boundary but have a non trivial dependence on $\eta \in \Sigma$.

To this end let $\mathcal{S}^\infty(\partial\mathbb{B}^\wedge)$ denote the space of smooth functions $\mathbb{R}_+ \times \partial\mathbb{B} \to \mathbb{C}$ that vanish to infinite order in $t = 0$ and decrease rapidly for $t \to \infty$. We then define

$$\mathcal{S}^\infty(\partial\mathbb{B}^\wedge \times \partial\mathbb{B}^\wedge) = \mathcal{S}^\infty(\partial\mathbb{B}^\wedge) \widehat{\otimes}_\pi \mathcal{S}^\infty(\partial\mathbb{B}^\wedge).$$

Definition 6.2. *Let* $R_G^\mu(\Sigma)_\infty$, $\mu \in \mathbb{R}$, *denote the space of all operator-families* $a(\eta) : \mathcal{S}^\infty(\partial\mathbb{B}^\wedge) \to \mathcal{S}^\infty(\partial\mathbb{B}^\wedge)$, $\eta \in \Sigma$, *such that*

$$[a(\eta)u](t,x) = [\eta]^{n+1} \int_{\partial\mathbb{B}^\wedge} k_a(\eta, t[\eta], x, t'[\eta], x')u(t',x')\, t'^{n/2} dt' dx', \tag{6.1}$$

with an integral kernel satisfying

$$k_a(\eta, t, x, t', x') \in S_{cl}^\mu(\Sigma, \mathcal{S}^\infty(\partial\mathbb{B}^\wedge \times \partial\mathbb{B}^\wedge)) := S_{cl}^\mu(\Sigma) \widehat{\otimes}_\pi \mathcal{S}^\infty(\partial\mathbb{B}^\wedge \times \partial\mathbb{B}^\wedge).$$

Using such operator-families, the so-called *flat Green symbols* or parameter-dependent *flat Green operators* are defined as follows:

Definition 6.3. *For $\mu \in \mathbb{R}$ let $C_G^\mu(\Sigma)_\infty$ denote the space of all operator-families $g(\eta): \mathcal{C}^{\infty,\infty}(\mathbb{B}) \to \mathcal{C}^{\infty,\infty}(\mathbb{B})$, $\eta \in \Sigma$, such that*

$$g(\eta) = \sigma\, a(\eta)\, \sigma_0 + r(\eta) \tag{6.2}$$

for some cut-off functions $\sigma, \sigma_0 \in \mathcal{C}^\infty([0,1[)$, $a \in R_G^\mu(\Sigma)_\infty$, and $r \in C_G^{-\infty}(\Sigma)_\infty$.

Note that if g is as in (6.2), then $g(\eta) = \tilde{\sigma}\, a(\eta)\, \tilde{\sigma}_0 + \tilde{r}(\eta)$ for any choice of cut-off functions $\tilde{\sigma}, \tilde{\sigma}_0 \in \mathcal{C}^\infty([0,1[)$ with a resulting $\tilde{r} \in C_G^{-\infty}(\Sigma)_\infty$. Moreover, the (pointwise) composition of such operator-families is again of the same type, i.e. the composition yields a map

$$C_G^{\mu_0}(\Sigma)_\infty \times C_G^{\mu_1}(\Sigma)_\infty \longrightarrow C_G^{\mu_0+\mu_1}(\Sigma)_\infty. \tag{6.3}$$

6.2. Holomorphic Mellin symbols. A holomorphic Mellin symbol of order $\mu \in \mathbb{R}$ is a function $h: \overline{\mathbb{R}}_+ \times \mathbb{C} \to L_{cl}^\mu(\partial\mathbb{B};\Sigma)$ depending smoothly on $t \in \overline{\mathbb{R}}_+$ and holomorphically on $z \in \mathbb{C}$. It has its values in the Frechét space of parameter-dependent pseudodifferential operators on the boundary of $\mathbb{B}$. Moreover we require that

$$c_l(\delta) := \sup_{t\ge 0} \langle t\rangle^l\, |||\partial_t^l h(t,\delta+i\tau)|||$$

is a locally bounded function of $\delta \in \mathbb{R}$ for any $l \in \mathbb{N}_0$ and any semi-norm $|||\cdot|||$ of $L_{cl}^\mu(\partial\mathbb{B};\mathbb{R}_\tau\times\Sigma)$. Let us denote the space of all such symbols by $M_O^\mu(\overline{\mathbb{R}}_+\times\partial\mathbb{B};\Sigma)$. The space of t-independent symbols is denoted by $M_O^\mu(\partial\mathbb{B};\Sigma)$.

With $h \in M_O^\mu(\overline{\mathbb{R}}_+\times\partial\mathbb{B};\Sigma)$ we associate an operator-family $\mathcal{S}^\infty(\partial\mathbb{B}^\wedge) \to \mathcal{S}^\infty(\partial\mathbb{B}^\wedge)$ by

$$[\mathrm{op}_M(h)(\eta)u](t,x) = \int_\Gamma t^{-z} h(t,z,t\eta)(\mathcal{M}u)(z,x)\, đz, \qquad u \in \mathcal{S}^\infty(\partial\mathbb{B}^\wedge), \tag{6.4}$$

where Γ is an arbitrary vertical line in the complex plane (the choice of it is arbitrary due to Cauchy's integral formula). Note that on the right-hand side of (6.4) we do not use the symbol $h(t,z,\eta)$ itself, but the 'degenerate' one $h(t,z,t\eta)$. Operators of that kind we refer to as parameter-dependent Mellin pseudodifferential operators or, shortly, Mellin operators.

Remark 6.4. *If A is a cone differential operator as in* (2.1) *then, for any $\varphi \in \mathcal{C}^\infty([0,1[)$,*

$$\varphi(\eta^\mu - A) = \varphi\, t^{-\mu}\, \mathrm{op}_M(h)(\eta), \qquad h(t,z,\eta) = \eta^\mu - \sum_{j=0}^{\mu} a_j(t) z^j.$$

Mellin operators behave well under composition:

Theorem 6.5. *Let $h_j \in M_O^{\mu_j}(\overline{\mathbb{R}}_+\times\partial\mathbb{B};\Sigma)$ for $j=0,1$. Then*

$$(h_0\#h_1)(t,z,\eta) = \iint s^{i\tau} h_0(t,z+i\tau,\eta) h_1(st,z,s\eta)\, \frac{ds}{s}\, đ\tau \tag{6.5}$$

defines an element $h_0\#h_1 \in M_O^{\mu_0+\mu_1}(\overline{\mathbb{R}}_+\times\partial\mathbb{B};\Sigma)$, the so-called Leibniz-product, and

$$\mathrm{op}_M(h_0)(\eta)\mathrm{op}_M(h_1)(\eta) = \mathrm{op}_M(h_0\#h_1)(\eta) \qquad \forall\, \eta \in \Sigma.$$

The right-hand side of (6.5) is understood as an oscillatory integral in a suitable sense.

6.3. The calculus of flat cone operators. The operator-families we now consider are, roughly speaking, those which in the interior of $\mathbb{B}$ are usual parameter-dependent pseudodifferential operators, and which near the boundary are parameter-dependent Mellin operators. The global smoothing elements are flat Green symbols. Let us make this precise:

Definition 6.6. *Let $\mu \in \mathbb{R}$. Then $C_O^\mu(\Sigma)$ denotes the space of all operator-families $\mathcal{C}^{\infty,\infty}(\mathbb{B}) \to \mathcal{C}^{\infty,\infty}(\mathbb{B})$ of the form*

$$c(\eta) = \sigma\, t^{-\mu}\, \mathrm{op}_M(h)(\eta)\, \sigma_0 \;+\; (1-\sigma)\, p(\eta)\, (1-\sigma_1) \;+\; g(\eta), \tag{6.6}$$

where $\sigma, \sigma_0, \sigma_1 \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ are cut-off functions satisfying $\sigma\sigma_0 = \sigma$, $\sigma\sigma_1 = \sigma_1$, and

a) $h(t,z,\eta) \in M_O^\mu(\overline{\mathbb{R}}_+ \times \partial\mathbb{B};\Sigma)$ *is a holomorphic Mellin symbol, cf. Section* 6.2,
b) $p(\eta) \in L_{cl}^\mu(2\mathbb{B};\Sigma)$ *is a parameter-dependent pseudodifferential operator on* $2\mathbb{B}$,
c) $g(\eta) \in C_G^\mu(\Sigma)_\infty$ *is a Green symbol, cf. Definition* 6.2.

One can always achieve that for any choice of $0 < \varrho < 1$ the symbols h and p in the representation (6.6) are *compatible* in the sense that

$$\varphi\left\{t^{-\mu}\operatorname{op}_M(h)(\eta) - p(\eta)\right\}\psi \;\in\; C_G^{-\infty}(\Sigma)_\infty \qquad \forall\,\varphi,\psi \in \mathcal{C}^\infty_{\mathrm{comp}}(]\varrho,1[).$$

In order to formulate the calculus in a smooth way, we shall fix such a ϱ and shall always assume this compatibility relation to be satisfied. Moreover, we assume the involved cut-off functions σ,σ_0,σ_1 to be identically 1 in a neighborhood of $[0,\varrho]$. Occasionally, we shall write $c(\eta) = \operatorname{op}(h,p,g)$ if $c(\eta)$ is as in (6.6).

THEOREM 6.7. *The pointwise composition of operator-families yields a map*

$$C_O^{\mu_0}(\Sigma) \times C_O^{\mu_1}(\Sigma) \longrightarrow C_O^{\mu_0+\mu_1}(\Sigma).$$

More precisely, if $c_j(\eta) = \operatorname{op}(h_j,p_j,g_j)$ *for* $j=0,1$, *then*

$$c_0(\eta)c_1(\eta) = \operatorname{op}((T^{\mu_1}h_0)\#h_1, p_0p_1, \tilde{g})$$

with a resulting Green symbol $\tilde{g} \in C_G^{\mu_0+\mu_1}(\Sigma)_\infty$. *Moreover, the shift-operator* T^δ, $\delta \in \mathbb{R}$, *is defined by* $(T^\delta h)(t,z,\eta) = h(t,z+\delta,\eta)$.

The operator-families from $C_O^\mu(\Sigma)$ introduced above are a certain subclass of parameter-dependent pseudodifferential operators on the interior of $\mathbb{B}$. In particular, we can associate with them the usual homogeneous principal symbol

$$\sigma_\psi^\mu(c)(y,\varrho,\eta) \in \mathcal{C}^\infty((T^*\operatorname{int}\mathbb{B} \times \Sigma)\setminus 0) \tag{6.7}$$

with (y,ϱ) refering to variables of the cotangent bundle of $\operatorname{int}\mathbb{B}$. In the coordinates $y=(t,x)$ near the boundary with corresponding covariables $\varrho = (\tau,\xi)$, the principal symbol has the form

$$\sigma_\psi^\mu(c)(t,x,\tau,\xi,\eta) = t^{-\mu}\,p_{(\mu)}(t,x,t\tau,\xi,t\eta)$$

with a function $p_{(\mu)}(t,x,\tau,\xi,\eta)$, which is smooth in $(t,x) \in \overline{\mathbb{R}}_+ \times \mathbb{R}^n$ and $0 \neq (\tau,\xi,\eta) \in \mathbb{R}^{n+1} \times \Sigma$, and is positive homogeneous of order μ in (τ,ξ,η). Passing to the symbol $p_{(\mu)}(0,x,\tau,\xi,\eta)$ globally leads to the definition of the *rescaled* principal symbol

$$\tilde{\sigma}_\psi^\mu(c)(x,\tau,\xi,\eta) \in \mathcal{C}^\infty((T^*\partial\mathbb{B} \times \mathbb{R} \times \Sigma)\setminus 0). \tag{6.8}$$

Roughly speaking, this rescaled symbol describes the behaviour of the principal symbol in the conical singularity itself. We call $c(\eta) \in C_O^\mu(\Sigma)$ $\mathbb{B}$-*elliptic* if

(E) both the principal symbol $\sigma_\psi^\mu(c)$ and the rescaled symbol $\tilde{\sigma}_\psi^\mu(c)$ are pointwise everywhere invertible.

This condition allows the construction of a rough parametrix:

THEOREM 6.8. *Let* $c(\eta) = \operatorname{op}(h_0,p_0,g_0) \in C_O^\mu(\Sigma)$ *be* $\mathbb{B}$-*elliptic. Then there exists an operator-family* $b(\eta) = \operatorname{op}(h_1,p_1,g_1) \in C_O^{-\mu}(\Sigma)$ *such that*

$$b(\eta)c(\eta) = 1 + \omega(t[\eta])\operatorname{op}_M(f_L)(\eta)\,\omega_0(t[\eta]) + g_L(\eta)$$
$$c(\eta)b(\eta) = 1 + \omega(t[\eta])\operatorname{op}_M(f_R)(\eta)\,\omega_0(t[\eta]) + g_R(\eta)$$

with an arbitrary choice of cut-off functions $\omega,\omega_0 \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$, *Mellin symbols* $f_L,f_R \in M_O^{-\infty}(\overline{\mathbb{R}}_+ \times \partial\mathbb{B};\Sigma)$, *and flat Green symbols* $g_L,g_R \in C_G^0(\Sigma)_\infty$. *Moreover,* $f_L = (T^\mu h_1)\#h_0 - 1$ *and* $f_R = (T^{-\mu}h_0)\#h_1 - 1$ *on* $[0,1[$.

Hence, elliptic symbols can be inverted up to smoothing remainders. However, this parametrix is not quite satisfactory, since a smoothing Mellin term is present and the Green symbols still have order 0. To improve the quality of the remainder, one has to enlarge the calculus substantially (and has to pose additional ellipticity conditions). The elements of this enlarged calculus will be described in the next sections.

6.4. Green symbols with asymptotics. Let E^j, $j=0,1$, be Banach spaces and $\kappa^j = \{\kappa^j_\varrho \mid \varrho > 0\} \subset \mathcal{L}(E^j)$ a strongly continuous group on E^j, i.e. $\kappa^j_1 = 1$ and $\kappa^j_\varrho \kappa^j_\sigma = \kappa^j_{\varrho\sigma}$. We also refer to κ^j as the *group action* of E^j.

A function $a \in \mathcal{C}^\infty(\Sigma, \mathcal{L}(E^0, E^1))$ is said to be a symbol of order $\mu \in \mathbb{R}$, if

$$\|\kappa^1_{1/\langle\eta\rangle}\, \partial^\alpha_\eta a(\eta)\, \kappa^0_{\langle\eta\rangle}\|_{\mathcal{L}(E^0,E^1)} \le c_\alpha\ \langle\eta\rangle^{\mu-|\alpha|}$$

uniformly in $\eta \in \Sigma$ and for all multiindices α. Then we shall write $a \in S^\mu(\Sigma; E^0, E^1)$.

A function $a \in \mathcal{C}^\infty(\Sigma \setminus \{0\}, \mathcal{L}(E^0, E^1))$ is called *twisted homogeneous* of order $\mu \in \mathbb{R}$, if

$$a(\varrho\eta) = \varrho^\mu\, \kappa^1_\varrho\, a(\eta)\, \kappa^0_{1/\varrho} \qquad \forall\, \varrho > 0,\, \eta \neq 0. \tag{6.9}$$

The space of such functions we shall denote by $S^{(-\mu)}(\Sigma; E^0, E^1)$. Using this notion of homogeneity, the standard concept of classical (respectively polyhomogeneous) symbols having asymptotic expansions into homogeneous components, passes over to this more general situation. We then write $a \in S^\mu_{cl}(\Sigma; E^0, E^1)$.

As a straightforward modification, one also can admit E^1 to be a Fréchet space, which is the projective limit of Banach spaces, $E^1 = \varprojlim_{k\in\mathbb{N}} E^1_k$ with $E^1_1 \hookleftarrow E^1_2 \hookleftarrow \ldots$, such that the group action on E^1_1 induces (by restriction) the group actions on all E^1_k, $k \in \mathbb{N}$. Then we simply set

$$S^\mu_{(cl)}(\Sigma; E^0, E^1) = \bigcap_{k\in\mathbb{N}} S^\mu_{(cl)}(\Sigma; E^0, E^1_k).$$

General Green symbols now shall be first introduced as such operator valued symbols with a specific choice of Banach spaces. In a second step we shall see that they also can be described with integral kernels in the spirit of (6.1).

We now work with distribution spaces on $\partial\mathbb{B}^\wedge = \mathbb{R}_+ \times \partial\mathbb{B}$. The group action κ on the various spaces occuring is – as a rule – always that induced by

$$(\kappa_\varrho u)(t,x) = \varrho^{\frac{n+1}{2}} u(\varrho t, x), \qquad u \in \mathcal{C}^\infty_{\text{comp}}(\partial\mathbb{B}^\wedge). \tag{6.10}$$

Definition 6.9. *Let $\gamma, \theta \in \mathbb{R}$ and $\theta > 0$. An* asymptotic type *$Q \in \mathrm{As}(\gamma,\theta)$ is a finite set of triples (q,l,L), where q is a complex number with $\frac{n+1}{2} - \gamma - \theta < \mathrm{Re}\, q < \frac{n+1}{2} - \gamma$, $l \in \mathbb{N}_0$, and $L \subset \mathcal{C}^\infty(\partial\mathbb{B})$ is a finite-dimensional space of smooth functions. We shall write $Q = O$ if Q is the empty set.*

The conjugate type to Q is the set of triples $(\overline{q}, l, L)$, where $(q,l,L) \in Q$. This type we shall denote by $\overline{Q} \in \mathrm{As}(\gamma,\theta)$.

With an asymptotic type $Q = \{(q_j, l_j, L_j) \mid j = 0, \ldots, N\} \in \mathrm{As}(\gamma,\theta)$ we associate a finite-dimensional subspace of $\mathcal{K}^{\infty,\gamma}_p(\partial\mathbb{B}^\wedge)$ (respectively $\mathcal{H}^{\infty,\gamma}_p(\mathbb{B})$), namely

$$\mathcal{E}_Q = \Big\{(t,x) \mapsto \omega(t) \sum_{j=0}^{N} \sum_{k=0}^{l_j} u_{jk}(x)\, t^{-q_j}\, \log^k t \mid u_{jk} \in L_j\Big\}. \tag{6.11}$$

Here $\omega \in \mathcal{C}^\infty_{\text{comp}}([0,1[)$ is an arbitrary cut-off function.

Definition 6.10. *Let $s \in \mathbb{R}$ and $Q \in \mathrm{As}(\gamma,\theta)$ be an asymptotic type. Then define*

$$\mathcal{K}^{s,\gamma}_{p,Q}(\partial\mathbb{B}^\wedge) = \mathcal{E}_Q \oplus \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma+\theta-\varepsilon}_p(\partial\mathbb{B}^\wedge)$$

$$\mathcal{S}^\gamma_Q(\partial\mathbb{B}^\wedge) = \{u \in \mathcal{K}^{\infty,\gamma}_{2,Q}(\partial\mathbb{B}^\wedge) \mid (1-\omega)u \in \mathcal{S}(\partial\mathbb{B}^\wedge)\}.$$

In case $Q = O$ being the empty set, we agree to write $\mathcal{K}^{s,\gamma}_{p,\theta}(\partial\mathbb{B}^\wedge)$ and $\mathcal{S}^\gamma_\theta(\partial\mathbb{B}^\wedge)$, respectively.

Note that the spaces are independent of the choice of the involved cut-off function, Moreover, both spaces are Fréchet and can be written as a projective limit of Banach spaces. In particular, we can speak of operator-valued symbols in the above sense.

DEFINITION 6.11. *Let $Q \in \mathrm{As}(-\gamma,\theta)$, $Q' \in \mathrm{As}(\gamma',\theta')$ be given asymptotic types. Then let us denote by $R^\mu_G(\Sigma;\gamma,\theta,Q;\gamma',\theta',Q')$ the space of all functions $a:\Sigma\to\mathcal{L}(\mathcal{K}^{0,\gamma}_2(\partial\mathbb{B}^\wedge),\mathcal{K}^{0,\gamma'}_2(\partial\mathbb{B}^\wedge))$ with*

$$a \in \bigcap_{s\in\mathbb{R}} S^\mu_{cl}(\Sigma;\mathcal{K}^{s,\gamma}_2(\partial\mathbb{B}^\wedge),\mathcal{S}^{\gamma'}_{Q'}(\partial\mathbb{B}^\wedge)),\qquad a^* \in \bigcap_{s\in\mathbb{R}} S^\mu_{cl}(\Sigma;\mathcal{K}^{s,-\gamma'}_2(\partial\mathbb{B}^\wedge),\mathcal{S}^{-\gamma}_{Q}(\partial\mathbb{B}^\wedge)).$$

Here, the $*$ refers to the pointwise adjoint with respect to the scalar product of $\mathcal{K}^{0,0}_2(\partial\mathbb{B}^\wedge)$. Moreover, we set

$$R^\mu_G(\Sigma;\gamma,\theta;\gamma',\theta') = \bigcup_{Q,Q'} R^\mu_G(\Sigma;\gamma,\theta,Q;\gamma',\theta',Q')$$

and we agree to write $R^\mu_G(\Sigma;\gamma,\gamma',\theta)$ if $\theta=\theta'$. The corresponding spaces of (twisted) homogeneous functions of order μ we denote by $R^{(\mu)}_G(\Sigma;\ldots)$.

As an example, flat Green symbols from Definition 6.2 are very special symbols of that type, namely

$$R^\mu_G(\Sigma)_\infty = \bigcap_{\gamma,\gamma',\theta,\theta'} R^\mu_G(\Sigma;\gamma,\theta,O;\gamma',\theta',O).$$

The last definition is most convenient for checking wether a given function is a Green symbol. However, for other purpose it is also important to know that such symbols have integral kernels with a specific structure to be explained now. To do so, set

$$\mathcal{S}^\gamma_0(\partial\mathbb{B}^\wedge) = \{u\in\mathcal{K}^{\infty,\gamma}_2(\partial\mathbb{B}^\wedge)\mid (1-\omega)u\in\mathcal{S}(\partial\mathbb{B}^\wedge),(\log^k t)\omega u\in\mathcal{K}^{\infty,\gamma}_2(\partial\mathbb{B}^\wedge)\ \forall\, k\in\mathbb{N}_0\}.$$

THEOREM 6.12. *Let $a:\Sigma\to\mathcal{L}(\mathcal{K}^{0,\gamma}_2(\partial\mathbb{B}^\wedge),\mathcal{K}^{0,\gamma'}_2(\partial\mathbb{B}^\wedge))$ for given asymptotic types $Q\in\mathrm{As}(-\gamma,\theta)$ and $Q'\in\mathrm{As}(\gamma',\theta')$. Then $a\in R^\mu_G(\Sigma;\gamma,\theta,Q;\gamma',\theta',Q')$ if and only if a satisfies (6.1) with a kernel*

$$k_a \in S^\mu_{cl}(\Sigma)\,\widehat{\otimes}_\pi\,\mathcal{S}^{\gamma'}_{Q'}(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\Gamma\,\mathcal{S}^{-\gamma}_{\overline{Q}}(\partial\mathbb{B}^\wedge),$$

where we have set

$$\mathcal{S}^{\gamma'}_{Q'}(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\Gamma\,\mathcal{S}^{-\gamma}_{\overline{Q}}(\partial\mathbb{B}^\wedge) = [\mathcal{S}^{\gamma'}_{Q'}(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\pi\,\mathcal{S}^{-\gamma}_0(\partial\mathbb{B}^\wedge)]\cap[\mathcal{S}^{\gamma'}_0(\partial\mathbb{B}^\wedge)\,\widehat{\otimes}_\pi\,\mathcal{S}^{-\gamma}_{\overline{Q}}(\partial\mathbb{B}^\wedge)].$$

To define general Green symbols on $\mathbb{B}$ we need to introduce some function spaces on $\mathbb{B}$:

$$\begin{aligned}
\mathcal{C}^{\infty,\gamma}(\mathbb{B}) &= \{u\in\mathcal{C}^\infty(\mathrm{int}\,\mathbb{B})\mid \omega u\in\mathcal{S}^\gamma_0(\partial\mathbb{B}^\wedge)\},\\
\mathcal{C}^{\infty,\gamma}_Q(\mathbb{B}) &= \{u\in\mathcal{C}^\infty(\mathrm{int}\,\mathbb{B})\mid \omega u\in\mathcal{S}^\gamma_Q(\partial\mathbb{B}^\wedge)\},\\
\mathcal{H}^{s,\gamma}_{p,Q}(\mathbb{B}) &= \{u\in H^s_{p,loc}(\mathrm{int}\,\mathbb{B})\mid \omega u\in\mathcal{K}^{s,\gamma}_{p,Q}(\partial\mathbb{B}^\wedge)\}.
\end{aligned}$$

These are subspaces of $\mathcal{H}^{s,\gamma}_p(\mathbb{B})$, independent of the choice of the involved cut-off function ω. We shall write $\mathcal{C}^{\infty,\gamma}_\theta(\mathbb{B})$ and $\mathcal{H}^{s,\gamma}_\theta(\mathbb{B})$ if $Q=O\in\mathrm{As}(\gamma,\theta)$ is the empty asymptotic type.

Now we define $C^{-\infty}_G(\Sigma;\gamma,\theta,Q;\gamma',\theta',Q')$ as the space of all functions $r:\Sigma\to\mathcal{L}(\mathcal{H}^{0,\gamma}_2(\mathbb{B}),\mathcal{H}^{0,\gamma'}_2(\mathbb{B}))$ such that

$$r\in\bigcap_{s\in\mathbb{R}} S^{-\infty}(\Sigma;\mathcal{H}^{s,\gamma}_2(\mathbb{B}),\mathcal{C}^{\infty,\gamma'}_{Q'}(\mathbb{B})),\qquad r^*\in\bigcap_{s\in\mathbb{R}} S^{-\infty}(\Sigma;\mathcal{H}^{s,-\gamma'}_2(\mathbb{B}),\mathcal{C}^{\infty,-\gamma}_{Q}(\mathbb{B})),$$

where $*$ refers to the adjoint with respect to the scalar-product of $\mathcal{H}^{0,0}_2(\mathbb{B})$ and all spaces are eqipped with the trivial group action $\kappa\equiv 1$.

Such operator-families posses an integral kernel in analogy to Definition 6.1, but now the kernel satisfies

$$k_r\in\mathcal{S}(\Sigma)\,\widehat{\otimes}_\pi\,\mathcal{C}^{\infty,\gamma'}_{Q'}(\mathbb{B})\,\widehat{\otimes}_\Gamma\,\mathcal{C}^{\infty,-\gamma}_{\overline{Q}}(\mathbb{B}),$$

where, similar as above,

$$\mathcal{C}^{\infty,\gamma'}_{Q'}(\mathbb{B})\,\widehat{\otimes}_\Gamma\,\mathcal{C}^{\infty,-\gamma}_{\overline{Q}}(\mathbb{B}) = [\mathcal{C}^{\infty,\gamma'}_{Q'}(\mathbb{B})\,\widehat{\otimes}_\pi\,\mathcal{C}^{\infty,-\gamma}(\mathbb{B})]\cap[\mathcal{C}^{\infty,\gamma'}(\mathbb{B})\,\widehat{\otimes}_\pi\,\mathcal{C}^{\infty,-\gamma}_{\overline{Q}}(\mathbb{B})].$$

Taking the union over all possible asymptotic types leads to the spaces $C_G^{-\infty}(\Sigma;\gamma,\theta;\gamma',\theta')$ and $C_G^{-\infty}(\Sigma;\gamma,\gamma',\theta)$ if $\theta=\theta'$.

Definition 6.13. *Let $C_G^{\mu}(\Sigma;\gamma,\theta;\gamma',\theta')$ denote the space of all operator-families $g(\eta):\mathcal{C}^{\infty,\gamma}(\mathbb{B})\to\mathcal{C}^{\infty,\gamma'}(\mathbb{B})$, $\eta\in\Sigma$, such that*

$$g(\eta)=\sigma\, a(\eta)\,\sigma_0+r(\eta)$$

for some cut-off functions $\sigma,\sigma_0\in\mathcal{C}^\infty([0,1[)$, $a\in R_G^{\mu}(\Sigma;\gamma,\theta;\gamma',\theta')$, and $r\in C_G^{-\infty}(\Sigma;\gamma,\theta;\gamma',\theta')$.

Note that the (pointwise) composition of such operator-families is again of the same type, i.e. the composition yields a map

$$C_G^{\mu_0}(\Sigma;\gamma',\theta_1';\gamma'',\theta'')\times C_G^{\mu_1}(\Sigma;\gamma,\theta;\gamma',\theta')\longrightarrow C_G^{\mu_0+\mu_1}(\Sigma;\gamma,\theta;\gamma'',\theta'').$$

6.5. Meromorphic Mellin symbols. An *asymptotic type* for Mellin symbols P is a set of triples (p,n,N) with $p\in\mathbb{C}$, $n\in\mathbb{N}_0$, and N a finite-dimensional subspace of finite rank operators from $L^{-\infty}(\partial\mathbb{B})$. Moreover, we require that $\pi_{\mathbb{C}}P\cap\{z\in\mathbb{C}\mid -\delta\le\operatorname{Re}z\le\delta\}$ is a finite set for each $\delta>0$, where

$$\pi_{\mathbb{C}}P=\{p\in\mathbb{C}\mid (p,n,N)\in P \text{ for some } n,\ N\}.$$

We shall write $P=O$ if P is the empty set.

A meromorphic Mellin symbol with asymptotic type P is a meromorphic function $f:\mathbb{C}\to L^{-\infty}(\partial\mathbb{B})$ with poles at most in the points of $\pi_{\mathbb{C}}P$. Moreover it satisfies: If $(p,n,N)\in P$, then the principal part of the Laurent series of f in p is of the form $\sum\limits_{k=0}^{n}R_k(z-p)^{-k-1}$ with $R_k\in N$; if $\chi\in\mathcal{C}^\infty(\mathbb{C})$ is a $\pi_{\mathbb{C}}P$-excision function (i.e. identically zero in an ε-neighborhood around $\pi_{\mathbb{C}}P$ and identically 1 outside the 2ε-neighborhood), then $c(\delta)=|||(\chi f)(\delta+i\tau)|||$ is a locally bounded function in $\delta\in\mathbb{R}$ for each semi-norm of $L^{-\infty}(\partial\mathbb{B};\mathbb{R}_\tau)=\mathcal{S}(\mathbb{R}_\tau,L^{-\infty}(\partial\mathbb{B}))$.

As in (6.4) we can associate with meromorphic Mellin symbols a pseudodifferential operator. However, this operator will depend on the choice of Γ in (6.4). We shall define $\mathrm{op}_M^{\delta}(f)$ by

$$[\mathrm{op}_M^{\delta}(f)u](t,x)=\int_{\Gamma_{\frac12-\delta}}t^{-z}f(t,z)(\mathcal{M}u)(z,x)\,đz. \tag{6.12}$$

Of course, we also have to require that none of the poles of f lies on the chosen line; this we always shall assume implicitly.

Definition 6.14. *Let $\gamma,\mu\in\mathbb{R}$ and $k\in\mathbb{N}$. Then $C_{M+G}^{\mu}(\Sigma;\gamma,\gamma-\mu,k)$ denotes the space of all operator-families $\mathcal{C}^{\infty,\gamma}(\mathbb{B})\to\mathcal{C}^{\infty,\gamma-\mu}(\mathbb{B})$ of the form*

$$\omega(t[\eta])\Big(\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{j}t^{-\mu+j}\mathrm{op}_M^{\gamma_{j\alpha}-\frac{n}{2}}(f_{j\alpha})\eta^\alpha\Big)\tilde\omega(t[\eta])\ +\ g(\eta), \tag{6.13}$$

where $\omega,\tilde\omega\in\mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ are arbitrary cut-off functions, $g\in C_G^{\mu}(\Sigma;\gamma,\gamma-\mu,k)$, $f_{j\alpha}\in M^{-\infty}_{P_{j\alpha}}$ for certain asymptotic types $P_{j\alpha}$ and weights $\gamma_{j\alpha}\in\mathbb{R}$ with $\gamma-j\le\gamma_{j\alpha}\le\gamma$.

Changing the cut-off functions $\omega,\tilde\omega$ in (6.13) only yields remainders in $C_G^{\mu}(\Sigma;\gamma,\gamma-\mu,k)$.

6.6. The calculus of cone pseudodifferential operators. For $\gamma,\mu\in\mathbb{R}$ and $k\in\mathbb{N}$ let

$$C^{\mu}(\Sigma;\gamma,\gamma-\mu,k)=C_O^{\mu}(\Sigma)\ +\ C_{M+G}^{\mu}(\Sigma;\gamma,\gamma-\mu,k) \tag{6.14}$$

with $C_O^{\mu}(\Sigma)$ from Definition 6.6 and $C_{M+G}^{\mu}(\Sigma;\gamma,\gamma-\mu,k)$ from Definition 6.14. The elements of that space are operator-families $\mathcal{C}^{\infty,\gamma}(\mathbb{B})\to\mathcal{C}^{\infty,\gamma-\mu}(\mathbb{B})$ (that extend continuously to the Sobolev spaces). Pointwise composition induces a map

$$C^{\mu_0}(\Sigma;\gamma-\mu_1,\gamma-\mu_1-\mu_0,k)\times C^{\mu_1}(\Sigma;\gamma,\gamma-\mu_1,k)\longrightarrow C^{\mu_0+\mu_1}(\Sigma;\gamma,\gamma-\mu_1-\mu_0,k).$$

Now let $c(\eta) \in C^{\mu}(\Sigma; \gamma, \gamma-\mu, k)$ be given; then

$$c(\eta) = \sigma\, t^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\, \sigma_0 + (1-\sigma)\, p(\eta)\, (1-\sigma_1) + (m+g)(\eta),$$

where the first two terms are as in (6.6) and $(m+g)(\eta)$ is as in (6.13). Since $(m+g)(\eta)$ has, in particular, a smooth distributiontheoretical kernel, $c(\eta)$ is a parameter-dependent pseudodifferential operator on the interior of $\mathbb{B}$, and we can associate with it the principal symbol and rescaled symbol as in (6.7) and (6.8), respectively.

Let us furthermore introduce the *principal edge symbol* (the terminology 'edge' shows up here, since parameter-dependent cone operators serve as symbols for pseudodifferential operators on manifolds with edges)

$$\sigma_{\wedge}^{\mu}(c)(\eta) = t^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(h_0)(\eta) + \omega(t|\eta|)\Big(\sum_{j=0}^{k-1} \sum_{|\alpha|=j} t^{-\mu+j} \operatorname{op}_M^{\gamma_{j\alpha}-\frac{n}{2}}(f_{j\alpha})\eta^{\alpha}\Big)\tilde{\omega}(t|\eta|) + g_{(\mu)}(\eta), \tag{6.15}$$

where $h_0(z,\eta) = h(0,z,\eta)$ and $g_{(\mu)}(\eta)$ is the homogeneous principal symbol of $g(\eta) \in R_G^{\mu}(\Sigma; \gamma, \gamma-\mu, k)$. We consider the principal edge symbol as an operator-family

$$\sigma_{\wedge}^{\mu}(c)(\eta) : \mathcal{K}_p^{s,\gamma}(\partial\mathbb{B}^{\wedge}) \longrightarrow \mathcal{K}_p^{s-\mu,\gamma-\mu}(\partial\mathbb{B}^{\wedge}), \qquad \eta \neq 0,$$

for $s \in \mathbb{R}$ and $1 < p < \infty$. Finally, the *conormal symbol* of $c(\eta)$ is the meromorphic function

$$\sigma_M^{\mu}(c)(z) = h(0,z,0) + f_{00}(z) : H_p^s(\partial\mathbb{B}) \longrightarrow H_p^{s-\mu}(\partial\mathbb{B}), \qquad z \in \mathbb{C}, \tag{6.16}$$

respectively a meromorphic function with values in $L_{cl}^{\mu}(\partial\mathbb{B})$.

We shall call $c(\eta) \in C^{\mu}(\Sigma; \gamma, \gamma-\mu, k)$ *elliptic*, if

(E) both $\sigma_{\psi}^{\mu}(c)$ and $\tilde{\sigma}_{\psi}^{\mu}(c)$ are pointwise everywhere invertible (i.e. $c(\eta)$ is $\mathbb{B}$-elliptic),
($\mathrm{E}_{\wedge}$) the principal edge symbol $\sigma_{\wedge}^{\mu}(c)$ is pointwise everywhere invertible.

Here, the second condition initially is required to hold for some s and p; but then it holds for all. Note that if $c(\eta)$ satisfies only one of the conditions (E) or ($\mathrm{E}_{\wedge}$) then the conormal symbol is meromorphically invertible. It is bijective on the vertical line $\Gamma_{\frac{n+1}{2}-\gamma}$ in case $c(\eta)$ satisfies ($\mathrm{E}_{\wedge}$).

Theorem 6.15. *Assume $c(\eta) \in C^{\mu}(\Sigma; \gamma, \gamma-\mu, k)$ satisfies condition* (E) *and the conormal symbol is invertible on the line $\Gamma_{\frac{n+1}{2}-\gamma}$. Then there exists a $b(\eta) \in C^{-\mu}(\Sigma; \gamma-\mu, \gamma, k)$ such that*

$$b(\eta)c(\eta) - 1 \ \in\ C_G^{0}(\Sigma; \gamma, \gamma, k), \qquad c(\eta)b(\eta) - 1 \ \in\ C_G^{0}(\Sigma; \gamma-\mu, \gamma-\mu, k).$$

This (still rough) parametrix $b(\eta)$ is uniquely determined modulo $C_G^{-\mu}(\Sigma; \gamma-\mu, \gamma, k)$.

Theorem 6.16. *Let $c(\eta) \in C^{\mu}(\Sigma; \gamma, \gamma-\mu, k)$ be elliptic. Then there exists a $b(\eta) \in C^{-\mu}(\Sigma; \gamma-\mu, \gamma, k)$ such that*

$$b(\eta)c(\eta) - 1 \ \in\ C_G^{-\infty}(\Sigma; \gamma, \gamma, k), \qquad c(\eta)b(\eta) - 1 \ \in\ C_G^{-\infty}(\Sigma; \gamma-\mu, \gamma-\mu, k).$$

The parametrix $b(\eta)$ is uniquely determined modulo $C_G^{-\infty}(\Sigma; \gamma-\mu, \gamma, k)$.

References

[1] J. Brüning and R. Seeley. An index theorem for regular singular operators. *Amer. J. Math.* **110**:659-714, 1988.
[2] J. Cheeger. On the spectral geometry of spaces with cone-like singularities. *Proc. Nat. Acad. Sci. USA* **76**:2103-2106, 1979.
[3] S. Coriasco, E. Schrohe, J. Seiler. Bounded imaginary powers of cone differential operators. *Math. Z.*, to appear. (Preprint math.AP/0106008).
[4] S. Coriasco, E. Schrohe, J. Seiler. Differential operators on conic manifolds: Maximal regularity and parabolic equations. *Bull. Soc. Roy. Sci. Liège* **70**: 207-229, 2001.
[5] G. Dore, A. Venni. On the closednes of the sum of two operators. *Math. Z.* **196**: 189-201, 1987.
[6] Yu. Egorov, B.-W. Schulze. *Pseudodifferential Operators, Singularities, Applications.*, Birkhäuser Verlag 1997.
[7] J.B. Gil. Heat trace asymptotics for cone differential operators. PhD thesis, Potsdam 1998.
[8] J.B. Gil. Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators. To appear in *Math. Nachr.*

[9] J.B. Gil, P. Loya. On the noncommutative residue and the heat trace expansion on conic manifolds To appear in *Manuscripta Math.*

[10] J.B. Gil, G.A. Mendoza. Adjoints of elliptic cone operators. Preprint, Temple University, Philadelphia, 2001. To appear in *Amer. J. Math.*

[11] J.B. Gil, B.-W. Schulze, J. Seiler. Cone pseudodifferential operators in the edge symbolic calculus. *Osaka J. Math.* **37**: 221-260, 2000.

[12] M. Lesch. *Operators of Fuchs type, Conical Singularities, and Asymptotic Methods.* Teubner-Texte Math. **136**, Teubner-Verlag, 1997.

[13] P. Loya. Complex Powers of Differential Operators on Manifolds with Conical Singularities. To appear in *J. Anal. Math.*

[14] P. Loya. On the Resolvent of Differential Operators on Conic Manifolds. To appear in *Comm. Anal. Geom.*

[15] R. Melrose. Transformation of boundary value problems. *Acta Math.* **147**:149-236, 1981.

[16] E. Mooers. Heat kernel asymptotics on manifolds with conic singularities. *J. Anal. Math.* **78**: 1-36, 1999.

[17] B. Plamenevskij. *Algebras of pseudodifferential operators.* Nauka, Moscow 1986 (in Russian).

[18] E. Schrohe. Spaces of weighted symbols and weighted Sobolev spaces on manifolds. In H.O. Cordes, B. Gramsch, H. Widom (eds.), *Pseudodifferential operators*, Oberwolfach (1986), p. 360-377. LN Math. **1256**, Springer Verlag, 1987.

[19] E. Schrohe, B.W. Schulze. Edge-degenerate boundary value problems on cones. In *Evolution equations and their applications in physical and life sciences*, Bad Herrenalb (1998), p. 159-173. Marcel Dekker, 2001.

[20] E. Schrohe, J. Seiler. Ellipticity and invertibility in the cone algebra on L_p-Sobolev spaces. *Int. Eq. Oper. Th.* **41**: 93-114, 2001.

[21] B.-W. Schulze. The Mellin pseudo-differential calculus on manifolds with corners. In H. Triebel et al. (eds.), *Symposium 'Analysis on Manifolds with Singularities', Breitenbrunn 1990*, Teubner-Texte Math. **131**, Teubner-Verlag, 1992.

[22] B.-W. Schulze. *Pseudo-differential Operators on Manifolds with Singularities.* North-Holland, 1991.

[23] R. Seeley. Complex powers of an elliptic operator. In *Amer. Math. Soc. Proc. Symp. Pure Math.*, volume **10**: 288-307, 1967.

[24] R. Seeley. The resolvent of an elliptic boundary problem. *Amer. J. Math.* **91**: 889-920, 1969.

[25] R. Seeley. Norms and domains of the complex powers A_B^z. *Amer. J. Math.* **93**: 299-309, 1971.

[26] J. Seiler. *Pseudodifferential Calculus on Manifolds with Non-compact Edges.* PhD-thesis, Institut für Mathematik, Potsdam, 1997.

[27] J. Seiler. The cone algebra and a kernel characterization of Green operators. In J.B. Gil et al. (eds.), *Approaches to Singular Analysis*, Birkhäuser Verlag, 1-29, 2001.

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