C^*-Algebras of SIO's with Oscillating Symbols

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Abstract

For a domain $\mathcal{D} \subset \mathbb{R}^n$ with singular points on the boundary and a weight function w infinitely differentiable away from the singular points in $\bar{\mathcal{D}}$, we consider a C^*-algebra $\mathcal{A}(\mathcal{D}; w)$ of operators acting in the weighted space $L^2(\mathcal{D}, w)$. It is generated by the operators of multiplication by continuous functions on $\bar{\mathcal{D}}$ and the operators $\chi_{\mathcal{D}}\, F^{-1}\sigma F\, \chi_{\mathcal{D}}$ where σ is a homogeneous function. We show that the techniques of limit operators apply to define a symbol algebra for $\mathcal{A}(\mathcal{D}; w)$. When combined with the local principle, this leads to describing the Fredholm operators in $\mathcal{A}(\mathcal{D}; w)$.

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1 Introduction

Let D be a bounded domain in $\mathbb{R}^n$ whose boundary is smooth everywhere with the exception of a finite number of singular points. We write $\operatorname{sing} \partial D$ for the singular set and allow rather general singularities close to which the boundary may oscillate.

We consider a C^*-algebra $\mathcal{A}(D; w)$ of operators in the space $L^2(D, w)$ with a weight w, generated by singular integral operators on D with symbols discontinuous at the points of $\operatorname{sing} \partial D$. The weight functions w under study are supposed to be continuous away from the set $\operatorname{sing} \partial D$ in the closure of D, the behaviour of w close to $\operatorname{sing} \partial D$ being controlled. We construct a symbolic calculus of operators in $\mathcal{A}(D; w)$ and give a criterion for an operator to be Fredholm.

In the book [Pla89] Plamenevskii considered a class of pseudodifferential operators with discontinuous symbols on manifolds with conical singularities, the operators acting in spaces with power weights. This investigation was continued in a series of his papers jointly with Senichkin [PS87, PS90, PS95], where non-reducible representations for several C^*-algebras of pseudodifferential operators were described. We refer the reader to these papers for the complete bibliography.

The representation of pseudodifferential operators near singular points as Mellin operators with stabilising operator-valued symbols plays a crucial role in the mentioned papers. Since the symbols need not stabilise in our case, we invoke a limit operator approach [Rab98, RRS98, RRS00] which reduces the investigation of local invertibility of operators in $\mathcal{A}(D; w)$ to that for operators in the algebra of [Pla89].

The question on the boundedness of SIO's on $\mathbb{R}^n$ in the weighted L^p-spaces was studied by Hunt, Muckenhoupt and Widom, see [Muc79] and the references given there. However, as far as the authors know, the Fredholm property for multidimensional SIO's in spaces with rather general weights has not been treated anywhere.

In the present paper we introduce a new class of multidimensional SIO's and show the Fredholm property in spaces with general weight functions. The method relies on a very efficient notion of symbols generated in terms of limit operators. We systematically study pseudodifferential operators based on the Mellin transform, with operator-valued symbols and analyticity in a strip of the complex plane, here, with oscillating coefficients. Concerning Mellin pseudodifferential techniques in general we refer the reader to [Sch91, Sch98].

The Fredholm theory of multidimensional SIO's in domains with smooth boundary was first constructed in the papers of Simonenko [Sim65a, Sim65b] by means of his local theory. Later Vishik and Eskin studied general pseudodifferential problems in domains with smooth boundary, cf. the book [Esk73].

Boutet de Monvel [BdM71] restricted himself to pseudodifferential operators with transmission property in a smooth domain to form the well-known algebra of boundary value problems. Boundary value problems for pseudodifferential operators on singular domains, e.g., with conical points and edges, were investigated by Schulze [Sch98]. One-dimensional SIO's on curves which have oscillating type singularities were studied by the first author jointly with Böttcher and Karlovich, cf. [BKR96, BKR98, BKR00].

Wide classes of differential operators in domains with point singularities, edges and corners, close to which the boundary may oscillate and degenerate, are studied by the authors in [RST00, RST98, RST99].

The structure of the paper is the following. In Sections 2, 3, 4 and 5 we give necessary material concerning the local principle of Simonenko, Mellin pseudodifferential operators and limit operators. In Section 6 we specify the singularities of the boundary and the weight functions we deal with. In Section 7 we recall the Mellin representation of SIO's of [Pla89]. In Section 8 we introduce local C^*-algebras $\mathcal{A}(\mathcal{C}; w)$, $\mathcal{C}$ being an oscillating cone with singular points at 0 and ∞. We apply limit operators to derive a criterion of local invertibility at 0 for operators in $\mathcal{A}(\mathcal{C}; w)$. This leads to a construction of local symbols at singular points, presented in Section 9. The limit operator approach still works to describe local symbols at regular points, which is the subject of Section 10. In Section 11 we make use of the local principle to characterise the Fredholm operators on oscillating cones. Finally, in Section 12 we construct a global C^*-algebra $\mathcal{A}(D; w)$ in the domain D, and give Fredholm criteria in the algebra.

2 Local principle

In this section we give a slight modification of the local principle of Simonenko, cf. [Sim65a, Sim65b].

Let X be a Hausdorff topological space endowed with a non-negative measure m compatible with the topology. As usual, we assume that m is σ-additive and σ-finite, i.e., all Borel sets in X are measurable.

Given any $1 < p < \infty$, we write $L^p(X, m)$ for the Lebesgue space of functions u on X with a finite norm

$$\|u\|_{L^p(X,m)} = \left(\int_X |u(x)|^p \, dm \right)^{1/p}.$$

A bounded operator A in $L^p(X, m)$ is said to be of *local type* provided $\chi_{\sigma_2} A \chi_{\sigma_1}$ is a compact operator in $L^p(X, m)$ for all measurable sets $\sigma_1, \sigma_2 \subset X$ with disjoint closures. Here, χ_σ is thought of as the multiplication operator by the characteristic function of $\sigma \subset X$.

We say that an operator A in $L^p(X,m)$ is *locally invertible* at a point $x^0 \in X$ if there are a neighbourhood O of x^0 and operators $B'_{x^0}, B''_{x^0} \in \mathcal{L}(L^p(X,m))$ such that

$$\begin{array}{rcl} B'_{x^0} A \chi_O & = & \chi_O, \\ \chi_O A B''_{x^0} & = & \chi_O. \end{array}$$

Suppose m has no component with a support in a point. Then the local principle reads as follows.

Theorem 2.1 *Let A be a local type operator in $L^p(X,m)$. Then A is Fredholm if and only if it is locally invertible at each point $x^0 \in X$.*

3 Mellin operators

Let V be a Hilbert space, and let $\mathcal{L}(V)$ denote the space of all bounded operators acting in V.

Set $dm = dr/r$ and write $H = L^2(\mathbb{R}_+, m; V)$ for the Hilbert space of all V-valued functions on the semiaxis with a finite norm

$$\|u\|_H = \left(\int_{\mathbb{R}_+} \|u(r)\|_V^2 \, dm \right)^{1/2}.$$

Let further $\mathcal{S}(1_V)$ stand for the class of all C^∞ functions $a(r,\varrho)$ on $\mathbb{R}_+ \times \mathbb{R}$ with values in $\mathcal{L}(V)$, such that

$$\sup_{\mathbb{R}_+ \times \mathbb{R}} \|(rD_r)^\alpha D_\varrho^\beta a(r,\varrho)\|_{\mathcal{L}(V)} < \infty$$

for all $\alpha, \beta \in \mathbb{Z}_+$.

The elements of $\mathcal{S}(1_V)$ are called operator-valued *Mellin symbols.* A symbol $a \in \mathcal{S}(1_V)$ is said to *vary slowly* at $r = 0$ if

$$\lim_{r \to 0+} \sup_{\varrho \in \mathbb{R}} \|(rD_r)^\alpha D_\varrho^\beta a(r,\varrho)\|_{\mathcal{L}(V)} = 0 \qquad (3.1)$$

for all $\alpha \in \mathbb{Z}_+$, $\alpha \neq 0$, and $\beta \in \mathbb{Z}_+$.

Analogously we define Mellin symbols that vary slowly at $r = \infty$. Note that under the change of variables $R = 1/r$ the point $R = \infty$ corresponds to $r = 0$, and

$$rD_r\,(u\,(1/r)) = -\,((RD_R)\,u)\,(1/r)$$

for any $u(R)$ defined near $R = \infty$. Moreover, the change $R = \exp t$ takes $t = \infty$ to $R = \infty$, and

$$D_t\,(u\,(\exp t)) = ((RD_R)\,u)\,(\exp t)$$

for any $u(R)$ defined near $R = \infty$. Thus, our definition of symbols slowly varying at $r = \infty$ agrees with that of [RST00].

We write $\mathcal{S}_{\mathrm{sv}}(1_V)$ for the space of those Mellin symbols which vary slowly both at $r = 0$ and at $r = \infty$.

We also distinguish the subclass $\mathcal{I}_0(1_V)$ in $\mathcal{S}(1_V)$, consisting of those $a(r, \varrho)$ for which the equality (3.1) holds for all $\alpha, \beta \in \mathbb{Z}_+$, thus including $\alpha = 0$. Defining $\mathcal{I}_\infty(1_V)$ analogously, we set

$$\mathcal{I}(1_V) = \mathcal{I}_0(1_V) \cap \mathcal{I}_\infty(1_V).$$

Let $a \in \mathcal{S}(1_V)$. The operator $A = \mathrm{op}(a)$ given by

$$Au(r) = \frac{1}{2\pi} \int_{\mathbb{R}} d\varrho \int_{\mathbb{R}_+} \left(\frac{r}{r'}\right)^{i\varrho} a(r, \varrho)\, u(r')\, \frac{dr'}{r'},$$

for $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+, V)$, is called the *Mellin pseudodifferential operator* with symbol a. Following [RST00], the class of such operators is denoted $\mathcal{OP}\,\mathcal{S}(1_V)$. The notation $\mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(1_V)$, $\mathcal{OP}\,\mathcal{I}_0(1_V)$, $\mathcal{OP}\,\mathcal{I}_\infty(1_V)$, $\mathcal{OP}\,\mathcal{I}(1_V)$, etc., has obvious meaning. For $A = \mathrm{op}(a)$, with $a \in \mathcal{S}(1_V)$, we will occasionally write $a = \sigma_A$.

Mellin pseudodifferential operators are often defined by *double symbols*. By this is meant any C^∞ function $a(r, r', \varrho)$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ with values in $\mathcal{L}(V)$, which satisfies

$$\sup_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}} \|(rD_r)^\alpha (r'D_{r'})^{\alpha'} D_\varrho^\beta a(r, r', \varrho)\|_{\mathcal{L}(V)} < \infty$$

for all $\alpha, \alpha', \beta \in \mathbb{Z}_+$. Denote $\mathcal{S}_d(1_V)$ the space of operator-valued functions with these properties.

We say that a double symbol $a(r, r', \varrho)$ *varies slowly* at $r = 0$ if

$$\lim_{r \to 0+} \sup_{(r', \varrho) \in K \times \mathbb{R}} \|(rD_r)^\alpha (r'D_{r'})^{\alpha'} D_\varrho^\beta a(r, rr', \varrho)\|_{\mathcal{L}(V)} = 0, \tag{3.2}$$

for each compact set $K \subset \mathbb{R}_+$, each $\alpha, \alpha' \in \mathbb{Z}_+$ with $\alpha + \alpha' \neq 0$, and each $\beta \in \mathbb{Z}_+$. Let $\mathcal{S}_{d,\mathrm{sv}}(1_V)$ stand for the space of double symbols that vary slowly both at $r = 0$ and at $r = \infty$.

Let us summarise the properties of Mellin pseudodifferential operators that we need in the sequel.

Theorem 3.1

1) Any operator $A = \mathrm{op}(a)$, $a \in \mathcal{S}(1_V)$, is bounded in $H = L^2(\mathbb{R}_+, m; V)$, and

$$\|A\|_{\mathcal{L}(H)} \le c \sum_{\alpha+\beta \le N} \sup_{\mathbb{R}_+ \times \mathbb{R}} \|(rD_r)^\alpha D_\varrho^\beta a(r, \varrho)\|_{\mathcal{L}(V)},$$

where the constants $c > 0$ and $N \in \mathbb{Z}_+$ are independent of A.

2) If $A \in \mathcal{OP}\,\mathcal{S}(1_V)$ is invertible in $\mathcal{L}(H)$, then the inverse belongs to $\mathcal{OP}\,\mathcal{S}(1_V)$, too.

3) If $A, B \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(1_V)$, then $BA \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(1_V)$ and the symbol of BA is of the form

$$\sigma_{BA}(r,\varrho) = \sigma_B(r,\varrho)\sigma_A(r,\varrho) + s(r,\varrho),$$

where $s \in \mathcal{I}(1_V)$.

4) If $A \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(1_V)$, then the adjoint of A in $\mathcal{L}(H)$ belongs to $\mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(1_V)$, too, and

$$\sigma_{A^*}(r,\varrho) = (\sigma_A(r,\varrho))^* + s(r,\varrho),$$

where $s \in \mathcal{I}(1_V)$.

5) If $A \in \mathcal{OP}\,\mathcal{S}_{d,\mathrm{sv}}(1_V)$ is an operator with double symbol $a(r,r',\varrho)$, then $A \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(1_V)$ and

$$\sigma_A(r,\varrho) = a(r,r,\varrho) + s(r,\varrho),$$

where $s \in \mathcal{I}(1_V)$.

Proof. Cf. Chapter 2 in [RST00].

□

4 Analytical symbols

Pick a C^∞ function $w(r)$ on $\mathbb{R}_+$. Roughly speaking, the weighted space H_w is defined by

$$H_w = \frac{1}{w}H,$$

i.e., $u \in H_w$ just amounts to $wu \in H$. The norm in H_w is therefore given by $\|u\|_{H_w} = \|wu\|_H$.

Recall that $H = L^2(\mathbb{R}_+, m; V)$ whence $H_w = L^2(\mathbb{R}_+, |w|^2 m; V)$. In what follows we take $w(r) = \exp\Delta(r)$ where Δ satisfies

$$\sup_{\mathbb{R}_+} |\,(rD_r)^\alpha\,\Delta(r)| < \infty \tag{4.1}$$

for all $\alpha = 1, 2, \ldots$. To explain the estimates (4.1) we observe that

$$(rD_r)^\alpha w(r) = \left(\sum_{1i_1+\ldots+\alpha i_\alpha=\alpha} n_{i_1,\ldots,i_\alpha} ((rD_r)\Delta(r))^{i_1} \ldots ((rD_r)^\alpha\Delta(r))^{i_\alpha} \right) w(r)$$

for all α, where $n_{i_1,\ldots,i_\alpha}$ are non-negative integers. Hence (4.1) just amounts to saying that the quotients

$$\frac{(rD_r)^\alpha w(r)}{w(r)}$$

are bounded uniformly in $r \in \mathbb{R}_+$.

We may regard $w(r)$ as a C^∞ function on $\mathbb{R}_+$ with values in $\mathcal{L}(V)$, the value of w at r being identified with the multiplication operator by $w(r)$. By the above, the hypothesis is that $w(r)$ fulfills the symbol estimates of the class $\mathcal{S}(w, 1_V)$, cf. [RST98]. Thus, the concept of slowly varying symbols applies to $w(r)$.

We say that $w(r)$ is a slowly varying *weight function* on $\mathbb{R}_+$ if (4.1) holds and

$$\begin{aligned} \lim_{r\to 0} (rD_r)^2 \Delta(r) &= 0, \\ \lim_{r\to\infty} (rD_r)^2 \Delta(r) &= 0. \end{aligned}$$

For $\alpha = 1$, (4.1) shows that there is a finite interval $(c, d) \ni 0$ with the property that

$$c < \inf_{\mathbb{R}_+} r\Delta'(r) \le \sup_{\mathbb{R}_+} r\Delta'(r) < d. \tag{4.2}$$

Let us denote by $\mathcal{W}(c, d)$ the set of all slowly varying weight functions $w(r)$ on $\mathbb{R}_+$, that satisfy (4.2) [1].

Definition 4.1 *Let $\Theta = \mathbb{R} + i(c, d)$. The symbol space $\mathcal{S}(1_V; \Theta)$ is defined to consist of all $a \in \mathcal{S}(1_V)$ that extend analytically in ϱ to the strip Θ and satisfy*

$$\sup_{\mathbb{R}_+\times\mathbb{R}} \|(rD_r)^\alpha D_\varrho^\beta a(r, \varrho + i\gamma)\|_{\mathcal{L}(V)} < \infty$$

uniformly in γ on compact subsets of (c, d).

The operators of $\mathcal{OP}\,\mathcal{S}(1_V; \Theta)$ are said to be Mellin pseudodifferential operators with analytical symbols.

In an evident way we also introduce the class $\mathcal{S}_{\mathrm{sv}}(1_V; \Theta)$ of slowly varying analytical symbols, the analyticity always referring to the covariable ϱ.

Theorem 4.2

1) Let w satisfy (4.1) and (4.2), and $a \in \mathcal{S}(1_V; \Theta)$. Then $w\,\mathrm{op}(a)\,w^{-1}$ lies in $\mathcal{OP}\,\mathcal{S}_d(1_V)$. In fact,

$$w\,\mathrm{op}(a)\,w^{-1} = \mathrm{op}\left(a(r, \varrho + i\sigma(r, r'))\right)$$

where $\sigma(r, r') = \int_0^1 \left(r^\theta r'^{1-\theta}\right) \Delta' \left(r^\theta r'^{1-\theta}\right) d\theta$.

2) Let $w \in \mathcal{W}(c, d)$ and $a \in \mathcal{S}_{\mathrm{sv}}(1_V; \Theta)$. Then $w\,\mathrm{op}(a)\,w^{-1} \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(1_V)$ and

$$w\,\mathrm{op}(a)\,w^{-1} = \mathrm{op}\left(a(r, \varrho + ir\Delta'(r))\right) + S,$$

where $S \in \mathcal{OP}\,\mathcal{I}(1_V)$.

[1] To extend the notation of [RST98] we should write $\Lambda'(V; (c, d))$ for this class.

Proof. We only sketch the proof. For $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+, V)$, an easy computation gives

$$\begin{aligned}
\left(w \operatorname{op}(a)\, w^{-1} u\right)(e^t) &= \frac{1}{2\pi} \int_{\mathbb{R}} d\varrho \int_{\mathbb{R}} e^{i(t-t')\varrho} e^{\Delta(e^t)-\Delta(e^{t'})} a\left(e^t, \varrho\right) u(e^{t'})\, dt' \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} d\varrho \int_{\mathbb{R}} e^{i(t-t')(\varrho - iv(t,t'))} a\left(e^t, \varrho\right) u(e^{t'})\, dt' \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} d\tau \int_{\mathbb{R}} e^{i(t-t')\tau} a\left(e^t, \tau + iv(t,t')\right) u(e^{t'})\, dt'
\end{aligned}$$

where

$$v(t,t') = \int_0^1 e^{\theta t + (1-\theta)t'} \Delta'(e^{\theta t + (1-\theta)t'})\, d\theta,$$

the third equality being a consequence of the analiticity of a and the Cauchy-Poincaré Theorem. Changing the variables by

$$\begin{aligned}
t &= \log r, \\
t' &= \log r'
\end{aligned}$$

yields

$$\left(w \operatorname{op}(a)\, w^{-1} u\right)(r) = \frac{1}{2\pi} \int_{\mathbb{R}} d\tau \int_{\mathbb{R}_+} \left(\frac{r}{r'}\right)^{i\tau} a\left(r, \tau + iv(\log r, \log r')\right) u(r')\, \frac{dr'}{r'},$$

and obviously $v(\log r, \log r') = \sigma(r, r')$, which explains the theorem. See [Rab95a] for more details.

□

Clearly, the condition (4.2) yields $\sigma(r,r') \in (c,d)$ for all $(r,r') \in \mathbb{R}_+ \times \mathbb{R}_+$, hence $a(r, \varrho + i\sigma(r,r'))$ is well defined.

Corollary 4.3 *Suppose w satisfy (4.1) and (4.2), and $a \in \mathcal{S}(1_V; \Theta)$. Then the operator $A = \operatorname{op}(a)$ is bounded in H_w.*

Proof. Indeed,

$$\begin{aligned}
\|Au\|_{H_w} &= \|w\, Au\|_H \\
&\le \|w\, A\, w^{-1}\|_{\mathcal{L}(H)} \|wu\|_H \\
&= \|w\, A\, w^{-1}\|_{\mathcal{L}(H)} \|u\|_{H_w},
\end{aligned}$$

and $w\, A\, w^{-1}$ is a bounded operator in H, which follows easily from Theorems 3.1 and 4.2.

□

5 Limit operators

Let $(\kappa_\lambda)_{\lambda>0}$ be the group of unitary dilatation operators on $H = L^2(\mathbb{R}_+, m; V)$, i.e.

$$(\kappa_\lambda u)(r) = u\left(\frac{r}{\lambda}\right)$$

for all $u \in H$.

Definition 5.1 *Let $A \in \mathcal{L}(H)$ and let $\Lambda = (\lambda_\nu)$ be a sequence in $\mathbb{R}_+$ tending to 0. The limit*

$$A_\Lambda = s\!-\lim_{\nu\to\infty} \kappa_{\lambda_\nu}^{-1} A \kappa_{\lambda_\nu},$$

if exists in the strong topology of $\mathcal{L}(H)$, is called the limit operator of A defined by the sequence Λ.

In the same manner we introduce limit operators of A defined by sequences $\Lambda = (\lambda_\nu)$ in $\mathbb{R}_+$ tending to ∞.

Denote $\mathrm{LO}_0(A)$ and $\mathrm{LO}_\infty(A)$ the sets of all limit operators of A defined by various sequences $\Lambda = (\lambda_\nu)$ in $\mathbb{R}_+$ tending to 0 or ∞, respectively. Basic properties of limit operators are summarised in the following theorem.

Theorem 5.2 *Suppose $\Lambda = (\lambda_\nu)$ is a sequence in $\mathbb{R}_+$ tending to 0 (or ∞). Then:*

1) If $A \in \mathcal{L}(H)$ and the limit operator A_Λ exists, then

$$\|A_\Lambda\|_{\mathcal{L}(H)} \le \|A\|_{\mathcal{L}(H)}.$$

2) If $A, B \in \mathcal{L}(H)$ and the limit operators A_Λ, B_Λ exist, then $(A+B)_\Lambda$ exists, too, and

$$(A+B)_\Lambda = A_\Lambda + B_\Lambda.$$

3) If $A, B \in \mathcal{L}(H)$ and the limit operators A_Λ, B_Λ exist, then $(BA)_\Lambda$ exists, too, and

$$(BA)_\Lambda = B_\Lambda A_\Lambda.$$

4) If $A \in \mathcal{L}(H)$ and the limit operator A_Λ exists, then $(A^*)_\Lambda$ exists, too, and

$$(A^*)_\Lambda = (A_\Lambda)^*.$$

5) If a sequence $A_\nu \in \mathcal{L}(H)$ converges to $A \in \mathcal{L}(H)$ in $\mathcal{L}(H)$ and the limit operators $(A_\nu)_\Lambda$ exist for ν large enough, then A_Λ exists, too, and

$$A_\Lambda = \lim_{\nu\to\infty} (A_\nu)_\Lambda.$$

Proof. The proof is immediate by Definition 5.1.

□

Let us denote by $C_b^\infty(\mathbb{R}_+)$ the space of all C^∞ functions $a(r)$ on $\mathbb{R}_+$ that satisfy

$$\sup_{r\in\mathbb{R}_+} |\,(rD_r)^\alpha\, a(r)| < \infty$$

for all $\alpha \in \mathbb{Z}_+$.

Definition 5.3 *The space $\mathcal{A}(H)$ is defined to consist of all $A \in \mathcal{L}(H)$ such that*

$$\lim_{h\to 1} \|[A, \kappa_\lambda a_h]\|_{\mathcal{L}(H)} = 0$$

uniformly with respect to $\lambda \in \mathbb{R}_+$, for each function $a \in C_b^\infty(\mathbb{R}_+)$, where $a_h(r) = a\left(r^{\log h}\right)$.

Note that the diffeomorphism $r \mapsto \delta(r) = \log r$ pulls back the additive group structure of $\mathbb{R}$ to the multiplicative group structure on $\mathbb{R}_+$. On the other hand, the multiplicative structure of the real axis is pulled back to $\mathbb{R}_+$ as

$$\delta^{-1}(\delta(h)\delta(r)) = r^{\log h},$$

for $h, r \in \mathbb{R}_+$.

Theorem 5.4 *$\mathcal{A}(H)$ is a C^*-algebra.*

Proof. Let $A, B \in \mathcal{A}(H)$. Obviously, we have $A+B \in \mathcal{A}(H)$. Furthermore, it follows from the equality

$$[BA, \kappa_\lambda a_h] = B[A, \kappa_\lambda a_h] + [B, \kappa_\lambda a_h]A$$

that

$$\lim_{h\to 1} \|[BA, \kappa_\lambda a_h]\|_{\mathcal{L}(H)} = 0$$

uniformly with respect to $\lambda \in \mathbb{R}_+$, for each function $a \in C_b^\infty(\mathbb{R}_+)$. Thus, $BA \in \mathcal{A}(H)$. In the same way the formula

$$[A^*, \kappa_\lambda a_h] = [\kappa_\lambda a_h, A]^*$$

implies $A^* \in \mathcal{A}(H)$. Finally, $\mathcal{A}(H)$ is obviously a closed subalgebra of $\mathcal{L}(H)$, which completes the proof.

□

Let us give an example of operators in $\mathcal{A}(H)$ which are especially important for us.

Theorem 5.5 $\mathcal{OP}\,\mathcal{S}(1_H) \subset \mathcal{A}(H)$

Proof. The proof easily follows from the composition formula for Mellin pseudodifferential operators.

□

Denote $\mathcal{A}'(H)$ the subalgebra of $\mathcal{A}(H)$ consisting of all operators A with the property that each sequence $\Lambda = (\lambda_\nu)$ tending to zero or infinity has a subsequence $\Lambda' = \left(\lambda_{\nu_j}\right)$ which defines a limit operator $A_{\Lambda'}$ of A.

Example 5.6 Let $A \in \mathcal{A}(H)$ and $\kappa_\lambda A = A \kappa_\lambda$ for all $\lambda \in \mathbb{R}_+$. Then $A \in \mathcal{A}'(H)$ and all limit operators of A coincide with A.

□

Example 5.7 Write M_a for the multiplication operator by any function $a \in C_b^\infty(\mathbb{R}_+)$, acting in H. Then $M_a \in \mathcal{A}'(H)$. Indeed, pick a sequence $\Lambda = (\lambda_\nu)$ in $\mathbb{R}_+$ tending to 0. We have

$$\kappa_\lambda^{-1} M_a \kappa_\lambda = M_{\kappa_\lambda^{-1} a}$$

for all $\lambda \in \mathbb{R}_+$. The sequence

$$\kappa_{\lambda_\nu}^{-1} a = a\left(\lambda_\nu r\right)$$

is bounded in the topology of $C_{\mathrm{loc}}^\infty(\mathbb{R}_+)$. By the Heine-Borel property of the space $C_{\mathrm{loc}}^\infty(\mathbb{R}_+)$, there is a subsequence $a\left(\lambda_{\nu_j} r\right)$ which converges in $C_{\mathrm{loc}}^\infty(\mathbb{R}_+)$ to a function $a_{\Lambda'}(r)$. It is easy to verify that $a_{\Lambda'} \in C_b^\infty(\mathbb{R}_+)$ and the limit operator of M_a relative to $\Lambda' = \left(\lambda_{\nu_j}\right)$ is $M_{a_{\Lambda'}}$.

□

An operator $A \in \mathcal{L}(H)$ is said to be *locally invertible* at the point $r = 0$ if there are an $\varepsilon > 0$ and operators $B', B'' \in \mathcal{L}(H)$ such that

$$\begin{aligned} B' A \chi_\varepsilon &= \chi_\varepsilon, \\ \chi_\varepsilon A B'' &= \chi_\varepsilon, \end{aligned}$$

where χ_ε is the multiplication operator by the characteristic function of the interval $(0, \varepsilon]$.

Theorem 5.8 *An operator $A \in \mathcal{A}'(H)$ is locally invertible at the point $r = 0$ if and only if all limit operators $A_\Lambda \in \mathrm{LO}_0(A)$ are uniformly invertible, that is*

$$\sup_{A_\Lambda \in \mathrm{LO}_0(A)} \|A_\Lambda^{-1}\|_{\mathcal{L}(H)} < \infty.$$

Proof. Cf. [RRS00].

□

In the same manner we define the local invertibility of operators in $\mathcal{L}(H)$ at the point $r = +\infty$. Theorem 5.8 still extends to this case.

6 Oscillating cones

Recall that a *standard cone* with vertex at x^0 in $\mathbb{R}^n$ is

$$\mathcal{C}_{x^0} = \{x \in \mathbb{R}^n : \frac{x - x^0}{|x - x^0|} \in \Omega\},$$

where Ω is a domain on $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$.

Fix a domain Ω on $\mathbb{S}^{n-1}$. Let $S(r,\omega)$, $r > 0$, be a family of diffeomorphisms of Ω onto Ω_r, a domain on $\mathbb{S}^{n-1}$, such that

1) $S(r,\cdot)$ oscillates about a diffeomorphism of Ω, when $r \to 0$;
2) $(rD_r)^\alpha S(r,\cdot)$, $\alpha \in \mathbb{Z}_+$, are bounded uniformly in $r \in \mathbb{R}_+$;
3) $(rD_r)\, S(r,\omega) \to 0$ uniformly in $\omega \in \Omega$, when $r \to 0$.

The condition 1) requires some explanation. Namely, by this we mean that any sequence (r_ν) tending to 0 has a subsequence $R' = \left(r_{\nu_j}\right)$ with the property that $S(r_{\nu_j},\cdot)$ converges to a diffeomorphism $S_{R'}$ of Ω onto a domain $\Omega_{R'}$ on $\mathbb{S}^{n-1}$.

Example 6.1 To encompass the definitions of [RST00, RST98, RST99], take $S(r,\omega) = h\left(f(r)h^{-1}(\omega)\right)$ where h is a diffeomorphism of a starlike domain in $\mathbb{R}^{n-1}$ onto a domain $\Omega \subset \mathbb{S}^{n-1}$.

□

By an *oscillating cone* with vertex at x^0 in $\mathbb{R}^n$ is meant the set

$$\mathcal{C}_{x^0,S} = \{x = x^0 + rS(r,\omega) :\ r \in \mathbb{R}_+,\ \omega \in \Omega\},$$

$S(r,\cdot)$, $r > 0$, being a family as above.

7 Mellin representation of SIO's

Following [Pla89] we introduce the operators

$$\begin{aligned}
\left(E(\lambda)u\right)(\omega') &= \frac{\Gamma(i\lambda + \frac{n}{2})}{(2\pi)^{\frac{n}{2}}}\, e^{i\frac{\pi}{2}(i\lambda+\frac{n}{2})} \int_{\mathbb{S}^{n-1}} \frac{1}{(-\omega'\omega + i0)^{i\lambda+\frac{n}{2}}}\, u(\omega)\, d\omega,\\
\left(E(\lambda)^{-1}f\right)(\omega) &= \frac{\Gamma(-i\lambda + \frac{n}{2})}{(2\pi)^{\frac{n}{2}}}\, e^{i\frac{\pi}{2}(-i\lambda+\frac{n}{2})} \int_{\mathbb{S}^{n-1}} \frac{1}{(\omega\omega' + i0)^{-i\lambda+\frac{n}{2}}}\, f(\omega')\, d\omega'
\end{aligned}$$

acting on functions $u, f \in C^\infty(\mathbb{S}^{n-1})$.

It is well known that the operator-valued function $E(\lambda)$ is analytic in the entire complex plane with the exception of $\lambda = i(k + n/2)$, $k \in \mathbb{Z}_+$. In particular, $E(\lambda) \in \mathcal{L}(C^\infty(\mathbb{S}^{n-1}))$ is analytic everywhere in the lower half-plane $\mathbb{R} + i(-\infty, n/2)$.

On the other hand, $E(\lambda)^{-1}$ is an analytic operator-valued function in all of $\mathbb{C}$ outside the points $\lambda = -i(k+n/2)$, with $k \in \mathbb{Z}_+$. In particular, the function $E(\lambda)^{-1} \in \mathcal{L}(C^\infty(\mathbb{S}^{n-1}))$ is analytic everywhere in the upper half-plane $\mathbb{R} + i(-n/2, \infty)$.

For $\lambda \neq \pm i(k+n/2)$, $k \in \mathbb{Z}_+$, the operators $E(\lambda)$ and $E(\lambda)^{-1}$ are known to be inverse to each other, cf. [Pla89, p. 18].

Let $\sigma(x,\xi)$ be a C^∞ function on $(\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$, homogeneous of order 0 in both x and ξ. Consider the SIO

$$\left(F^{-1}\sigma F u\right)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \sigma(x,\xi)\, Fu(\xi)\, d\xi \tag{7.1}$$

defined for $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n \setminus \{0\})$, where $Fu(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x'\rangle} u(x')dx'$ is the Fourier transform of u.

In polar coordinates $(r,\omega) \in \mathbb{R}_+ \times \mathbb{S}^{n-1}$ the operator (7.1) admits the representation

$$\left(F^{-1}\sigma F u\right)(r,\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \int_{\mathbb{R}_+} \left(\frac{r}{r'}\right)^{i(\lambda + i\frac{n}{2})} a_\sigma(\lambda)\, u(r')\, \frac{dr'}{r'},$$

for $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+, C^\infty(\mathbb{S}^{n-1}))$, where

$$a_\sigma(\lambda) u = E(\lambda)^{-1}_{\omega' \mapsto \omega}\, \sigma(\omega,\omega')\, E(\lambda)_{\omega \mapsto \omega'} u, \tag{7.2}$$

cf. [Pla89, p. 40]. Moreover, the function $a_\sigma(\lambda)$ with values in $\mathcal{L}(L^2(\mathbb{S}^{n-1}))$ is analytic in the strip $\mathbb{R} + i(-n/2, n/2)$ and satisfies the estimates

$$\sup_{\mathbb{R}} \|D_\varrho^\beta a_\sigma(\varrho + i\gamma)\|_{\mathcal{L}(L^2(\mathbb{S}^{n-1}))} < \infty$$

for all $\beta \in \mathbb{Z}_+$, uniformly in γ on compact subsets of $(-n/2, n/2)$, cf. *ibid.*, p. 66.

Corollary 7.1 *Suppose $w \in \mathcal{W}(-n/2, n/2)$. Then $A = F^{-1}\sigma F$ extends to a continuous mapping of $L^2(\mathbb{R}^n, w)$. Moreover, $WAW^{-1} \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(1_V)$ where $V = L^2(\mathbb{S}^{n-1})$ and $W = wr^{n/2}$, and*

$$WAW^{-1} = \mathrm{op}\left(a_\sigma(\varrho + ir\Delta'(r))\right) + S,$$

where $S \in \mathcal{OP}\,\mathcal{I}(1_V)$.

Proof. Indeed, $L^2(\mathbb{R}^n, w)$ is easily identified with $L^2(\mathbb{R}_+, |W|^2 m; V)$. Furthermore,

$$\begin{aligned} WAW^{-1} &= W\left(r^{-\frac{n}{2}}\,\mathrm{op}(a_\sigma)\, r^{\frac{n}{2}}\right) W^{-1} \\ &= w\,\mathrm{op}(a_\sigma)\, w^{-1}, \end{aligned}$$

and it remains to make use of Theorem 4.2.

$\square$

8 Local invertibility at the vertex

Let $w \in \mathcal{W}(-n/2, n/2)$. Denote $\mathcal{A}(\mathcal{C}_{0,S}; w)$ the C^*-algebra in $\mathcal{L}(L^2(\mathcal{C}_{0,S}, w))$ generated by the operators $\chi_{\mathcal{C}_{0,S}} A \, \chi_{\mathcal{C}_{0,S}}$, where A is a SIO of the form (7.1) and $\chi_{\mathcal{C}_{0,S}}$ the characteristic function of $\mathcal{C}_{0,S}$, and by the multiplication operators M_a, where $a(x)$ is a continuous function on the one-point compactification of $\mathbb{R}^n$.

We write $C(\hat{\mathbb{R}}^n)$ for the space of all continuous functions on the one-point compactification of $\mathbb{R}^n$.

We are aimed at describing the limit operators of any $A \in \mathcal{A}(\mathcal{C}_{0,S}; w)$ at the singular points 0 and ∞. In terms of the limit operators we give conditions for local invertibility of A at these points. To this end, we first assign the set of limit operators to each generator of the C^*-algebra $\mathcal{A}(\mathcal{C}_{0,S}; w)$. Note that this algebra is unitary equivalent to the algebra $w \, \mathcal{A}(\mathcal{C}_{0,S}; w) \, w^{-1}$ of operators acting on $L^2(\mathcal{C}_{0,S})$.

Let $a(x) \in C(\hat{\mathbb{R}}^n)$. It is evident that all limit operators of M_a at $x = 0$ amount to $M_{a(0)}$, the multiplication operator by the constant $a(0)$. Similarly, all limit operators of M_a at $x = \infty$ coincide with $M_{a(\infty)}$.

We now consider the limit operators at $x = 0$ for the multiplication operator by the characteristic function of the oscillating cone $\mathcal{C}_{0,S}$. We have

$$\chi_{\mathcal{C}_{0,S}}(r, \omega) = \begin{cases} 1 & \text{if} \quad \omega \in S(r, \Omega); \\ 0 & \text{if} \quad \omega \notin S(r, \Omega), \end{cases}$$

where Ω is a domain on $\mathbb{S}^{n-1}$ with smooth boundary, and $S(r, \Omega) = \Omega_r$ the image of Ω by S. Pick a sequence (λ_ν) tending to 0. By the above, it has a subsequence $\Lambda' = \left(\lambda_{\nu_j}\right)$ such that $S(\lambda_{\nu_j} r, \cdot)$ converges to a diffeomorphism $S_{\Lambda'}$ of Ω onto a domain $\Omega_{\Lambda'}$ on $\mathbb{S}^{n-1}$. For abbreviation, we write $\chi_{\mathcal{C}_{0,S}}$ instead of $M_{\chi_{\mathcal{C}_{0,S}}}$, thus obtaining

$$\begin{aligned} s-\lim_{j \to \infty} \kappa^{-1}_{\lambda_{\nu_j}} \chi_{\mathcal{C}_{0,S}} \kappa_{\lambda_{\nu_j}} &= s-\lim_{j \to \infty} \chi_{\mathcal{C}_{0,S}}(\lambda_{\nu_j} r, \omega) \\ &= \chi_{\mathcal{C}_{0,S_{\Lambda'}}}(r, \omega), \end{aligned}$$

the limit being in the strong topology of $L^2(\mathbb{R}^n, w)$. Since $S_{\Lambda'}$ does not depend on r, we conclude that

$$\mathcal{C}_{0,S_{\Lambda'}} = \{x \in \mathbb{R}^n : \frac{x}{|x|} \in \Omega_{\Lambda'}\},$$

i.e., $\mathcal{C}_{0,S_{\Lambda'}}$ is a standard cone with vertex at the origin. In the same way we describe the limit operators of $M_{\chi_{\mathcal{C}_{0,S}}}$ at $x = \infty$.

Finally, we look at a SIO $A = F^{-1} \sigma F$ in the space $L^2(\mathbb{R}^n, w)$, where $\sigma(x, \xi)$ is a C^∞ function on $(\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$, homogeneous in both x and ξ, i.e.,

$$\sigma(\lambda_1 x, \lambda_2 \xi) = \sigma(x, \xi)$$

for all $\lambda_1, \lambda_2 > 0$. Letting $W = wr^{n/2}$, we consider the operator $W^{-1}AW$. By Corollary 7.1,

$$WAW^{-1} = \mathrm{op}\,(a_\sigma(\varrho + iF(r))) + \mathrm{op}\,(s)$$

where $s(r, \varrho) \in \mathcal{I}(1_{L^2(\mathbb{S}^{n-1})})$ and $F(r) = r\Delta'(r)$. Fix a sequence $\Lambda = (\lambda_\nu)$ tending either to 0 or to ∞. An easy verification shows that

$$\kappa_{\lambda_\nu}^{-1}\left(WAW^{-1}\right)\kappa_{\lambda_\nu} = \mathrm{op}\,(a_\sigma(\varrho + iF(\lambda_\nu r))) + \mathrm{op}\,(s(\lambda_\nu r, \varrho))\,.$$

Let $\Lambda' = \left(\lambda_{\nu_j}\right)$ be a subsequence of Λ with the property that the sequence $F(\lambda_{\nu_j} r)$ converges in the topology of $C^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ to a function $F_{\Lambda'}(r)$. Since $\lambda_{\nu_j} r$ converges either to 0 or to ∞, for each fixed $r > 0$, and $w(r)$ is a slowly varying weight function on $\mathbb{R}_+$, we easily deduce that the limit function $F_{\Lambda'}$ is actually constant. As is shown in [Pla89, pp. 77–78],

$$\lim_{j\to\infty}\ \sup_{(r,\varrho)\in K\times\mathbb{R}} \|(rD_r)^\alpha D_\varrho^\beta\left(a_\sigma(\varrho + iF(\lambda_{\nu_j} r)) - a_\sigma(\varrho + iF_{\Lambda'})\right)\|_{\mathcal{L}(L^2(\mathbb{S}^{n-1}))} = 0 \tag{8.1}$$

for each compact set $K \in \mathbb{R}_+$, and all $\alpha, \beta \in \mathbb{Z}_+$. This equality yields, by Theorem 3.1, 1), that

$$s\!-\!\lim_{j\to\infty} \mathrm{op}\left(a_\sigma(\varrho + iF(\lambda_{\nu_j} r))\right) = \mathrm{op}\,(a_\sigma(\varrho + iF_{\Lambda'}))\,.$$

On the other hand, since $s(r, \varrho) \in \mathcal{I}(1_{L^2(\mathbb{S}^{n-1})})$ we get

$$\lim_{r\to 0}\sup_{\varrho\in\mathbb{R}} \|(rD_r)^\alpha D_\varrho^\beta s(r, \varrho)\|_{\mathcal{L}(L^2(\mathbb{S}^{n-1}))} = 0$$

whence

$$s\!-\!\lim_{j\to\infty} \mathrm{op}\left(s(\lambda_{\nu_j} r, \varrho)\right) = 0.$$

We have thus proved that the limit operators at the points $r = 0$ and $r = \infty$ for the generators of the C^*-algebra $W\,\mathfrak{A}(\mathcal{C}_{0,S}; w)\,W^{-1}$ are:

1) The multiplication operators by $a(0)$ and $a(\infty)$, for a function a of $C(\hat{\mathbb{R}}^n)$.

2) The multiplication operators by the characteristic function of a cone $\mathcal{C}_{0,S_{\Lambda'}}$ cutting off a domain $\Omega_{\Lambda'} = S_{\Lambda'}(\Omega)$ on the unit sphere, $S_{\Lambda'}$ being a partial limit of $S(r, \omega)$ when r tends to 0 or ∞.

3) The SIO's that have the form $\mathrm{op}\,(a_\sigma(\varrho + iF_{\Lambda'}))$ in the Mellin realisation, where $F_{\Lambda'}$ is a partial limit of $F(r)$ when r tends to 0 or ∞. They act continuously in spaces with power-like weight functions as

$$r^{F_{\Lambda'}}\,\mathrm{op}\,(a_\sigma(\varrho))\,r^{-F_{\Lambda'}}:\ L^2(\mathbb{R}^n, |x|^{-F_{\Lambda'}-\frac{n}{2}}) \to L^2(\mathbb{R}^n, |x|^{-F_{\Lambda'}-\frac{n}{2}})$$

where $|F_{\Lambda'}| < n/2$.

Suppose an operator $A \in W\,\mathcal{A}(\mathcal{C}_{0,S}; w)\,W^{-1}$ has the form

$$A = \sum_{i=1}^{I} \prod_{j=1}^{J} A_{ij}$$

where A_{ij} is an operator of one of the above three types. Let (λ_ν) be a sequence of $\mathbb{R}_+$ tending to zero or ∞. Then there is a subsequence $\Lambda' = \left(\lambda_{\nu_j}\right)$ defining a limit operator $(A_{ij})_{\Lambda'}$, for each indices i and j. We may now invoke Theorem 5.2 to deduce that

$$A_{\Lambda'} = \sum_{i=1}^{I} \prod_{j=1}^{J} (A_{ij})_{\Lambda'}$$

and

$$\|A_{\Lambda'}\|_{\mathcal{L}(H)} \leq \| \sum_{i=1}^{I} \prod_{j=1}^{J} (A_{ij})_{\Lambda'} \|_{\mathcal{L}(H)},$$

where $H = L^2(\mathbb{R}_+, m; L^2(\mathbb{S}^{n-1}))$. Moreover, for any $A \in W\,\mathcal{A}(\mathcal{C}_{0,S}; w)\,W^{-1}$ and a sequence (λ_ν) tending to zero or infinity, there is a subsequence $\Lambda' = \left(\lambda_{\nu_j}\right)$ defining a limit operator $A_{\Lambda'}$ of A. Another way of stating this property is to say:

$$W\,\mathcal{A}(\mathcal{C}_{0,S}; w)\,W^{-1} \hookrightarrow \mathcal{A}'(H).$$

The limit operators A_Λ belong to the C^*-algebra generated by the operators of multiplication by constants, the multiplication operator by the characteristic function of the cone $\mathcal{C}_{0,S_\Lambda}$, and SIO's of the form $|x|^{F_\Lambda + n/2} F^{-1} \sigma F |x|^{-F_\Lambda - n/2}$ on $\mathbb{R}^n$.

As is shown in [Pla89], this latter algebra is unitary equivalent to a subalgebra of the C^*-algebra

$$C_b(\mathbb{R} + iF_\Lambda, \mathcal{L}^0(\Omega_\Lambda))$$

of bounded continuous functions on the line $\mathbb{R} + iF_\Lambda$ with values in the C^*-algebra generated by pseudodifferential operators of order zero acting in the space $L^2(\Omega_\Lambda)$, where $\Omega_\Lambda = S_\Lambda(\Omega)$ is a subdomain of $\mathbb{S}^{n-1}$ with smooth boundary.

Denote $\Sigma(0)$ and $\Sigma(\infty)$ the sets of all sequences $\Lambda = (\lambda_\nu)$ converging to zero and infinity, respectively, such that the limits

$$\begin{array}{rcl} \lim\limits_{\nu \to \infty} S(\lambda_\nu, \cdot) & = & S_\Lambda(\cdot), \\ \lim\limits_{\nu \to \infty} F(\lambda_\nu) & = & F_\Lambda \end{array}$$

exist.

Theorem 8.1 *An operator $A \in \mathcal{A}(\mathcal{C}_{0,S}; w)$ acting in $L^2(\mathcal{C}_{0,S}, w)$ is locally invertible at the point* 0 *(or ∞) if and only if all limit operators*

$$A_\Lambda : \; L^2(\mathbb{R}^n, |x|^{-F_\Lambda - \frac{n}{2}}) \to L^2(\mathbb{R}^n, |x|^{-F_\Lambda - \frac{n}{2}}),$$

Λ belonging to $\Sigma(0)$ or $\Sigma(\infty)$, are uniformly invertible.

9 Local symbol at singular points

By the above, the invertibility of A_Λ is equivalent to that of an operator-valued function

$$\sigma(A_\Lambda)(\varrho + iF_\Lambda) : \ L^2(\Omega_\Lambda) \to L^2(\Omega_\Lambda)$$

in $C_b(\mathbb{R} + iF_\Lambda, \mathcal{L}^0(\Omega_\Lambda))$.

Given any $\Lambda = (\lambda_\nu)$ in $\Sigma(0)$, write

$$\sigma_\Lambda : \ \mathcal{Q}(\mathcal{C}_{0,S}; w) \to C_b(\mathbb{R} + iF_\Lambda, \mathcal{L}^0(\Omega_\Lambda))$$

for the morphism of C^*-algebras that assigns to $A \in \mathcal{Q}(\mathcal{C}_{0,S}; w)$ the operator-valued symbol $\sigma(A_\Lambda)$ of the limit operator of A.

Now we introduce *local symbols* at the points 0 and ∞ as the morphisms of C^*-algebras

$$\begin{array}{rcl} \mathrm{smb}_0 & : & \mathcal{Q}(\mathcal{C}_{0,S}; w) \to \bigoplus_{\Lambda \in \Sigma(0)} C_b(\mathbb{R} + iF_\Lambda, \mathcal{L}^0(\Omega_\Lambda)), \\ \mathrm{smb}_\infty & : & \mathcal{Q}(\mathcal{C}_{0,S}; w) \to \bigoplus_{\Lambda \in \Sigma(\infty)} C_b(\mathbb{R} + iF_\Lambda, \mathcal{L}^0(\Omega_\Lambda)) \end{array}$$

given by

$$\begin{array}{rcl} \mathrm{smb}_0 & = & \oplus_{\Lambda \in \Sigma(0)} \sigma_\Lambda, \\ \mathrm{smb}_\infty & = & \oplus_{\Lambda \in \Sigma(\infty)} \sigma_\Lambda. \end{array}$$

Using this notion we can equivalently reformulate Theorem 8.1 in the following way.

Theorem 9.1 *An operator $A \in \mathcal{Q}(\mathcal{C}_{0,S}; w)$ acting in $L^2(\mathcal{C}_{0,S}, w)$ is locally invertible at the point x^0, where $x^0 = 0$ or ∞, if and only if $\mathrm{smb}_{x^0}(A)$ is invertible in*

$$\bigoplus_{\Lambda \in \Sigma(x^0)} C_b(\mathbb{R} + iF_\Lambda, \mathcal{L}^0(\Omega_\Lambda)).$$

The invertibility of $\mathrm{smb}_{x^0}(A)$ in $\bigoplus_{\Lambda \in \Sigma(x^0)} C_b(\mathbb{R}+iF_\Lambda, \mathcal{L}^0(\Omega_\Lambda))$ just amounts to the existence of $\sigma_\Lambda(A)^{-1}$, for each $\Lambda \in \Sigma(x^0)$, the inverse satisfying the estimate

$$\sup_{\substack{\varrho \in \mathbb{R} \\ \Lambda \in \Sigma(x^0)}} \|\sigma_\Lambda(A)^{-1}(\varrho + iF_\Lambda)\|_{\mathcal{L}(L^2(\mathbb{S}^{n-1}))} < \infty. \tag{9.1}$$

Following [RRS98] we can relax the latter condition by requiring an estimate like (9.1) for every fixed $\Lambda \in \Sigma(x^0)$.

Denote by $\mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$ the two-sided ideal in the algebra $\mathcal{Q}(\mathcal{C}_{0,S}; w)$, generated by the multiplication operators M_a, where $a \in C_b(\mathbb{R}^n)$ satisfies

$$\lim_{x \to x^0} a(x) = 0.$$

Theorem 9.2 *An operator $A \in \mathcal{A}(\mathcal{C}_{0,S}; w)$ is locally invertible at the point $x^0 = 0$ or ∞ if and only if the coset $A + \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$ is invertible in the quotient algebra*

$$Q_{x^0} = \frac{\mathcal{A}(\mathcal{C}_{0,S}; w)}{\mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)}.$$

Proof. We give the proof for $x^0 = 0$. The same reasoning applies to the case $x^0 = \infty$.

Suppose the coset $A + \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$ is invertible in the quotient algebra Q_{x^0}. Then there are operators $B_1, B_2 \in \mathcal{A}(\mathcal{C}_{0,S}; w)$ with the property that

$$\begin{array}{rcl} B_1 A & = & 1 - S_1, \\ A B_2 & = & 1 - S_2 \end{array}$$

where $S_1, S_2 \in \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$.

Choose $R > 0$ such that

$$\begin{array}{rcl} \|S_1 \chi_R\|_{\mathcal{L}(L^2(\mathcal{C}_{0,S}, w))} & \leq & 1/2, \\ \|\chi_R S_2\|_{\mathcal{L}(L^2(\mathcal{C}_{0,S}, w))} & \leq & 1/2 \end{array}$$

where χ_R is the characteristic function of the ball B_R with centre x^0 and radius R in $\mathbb{R}^n$. Let $R' < R$. Then

$$\begin{array}{rcl} B_1 A \, \chi_{R'} & = & (1 - S_1 \chi_R) \, \chi_{R'}, \\ \chi_{R'} \, A B_2 & = & \chi_{R'} \, (1 - \chi_R S_2), \end{array}$$

whence

$$\begin{array}{rcl} (1 - S_1 \chi_R)^{-1} \, B_1 A \, \chi_{R'} & = & \chi_{R'}, \\ \chi_{R'} \, A B_2 \, (1 - \chi_R S_2)^{-1} & = & \chi_{R'}. \end{array}$$

Conversely, suppose A is locally invertible at the point x^0. Then a priori estimates

$$\begin{array}{rcl} \|A \chi_R u\|_{L^2(\mathcal{C}_{0,S}, w)} & \geq & c \, \|\chi_R u\|_{L^2(\mathcal{C}_{0,S}, w)}, \\ \|A^* \chi_R f\|_{L^2(\mathcal{C}_{0,S}, w)} & \geq & c \, \|\chi_R f\|_{L^2(\mathcal{C}_{0,S}, w)} \end{array}$$

hold, with c a positive constant independent of u and f.

From the first estimate it follows that

$$|\, (\chi_R A^* A \chi_R u, \chi_R u) \,| \geq c^2 \, \|\chi_R u\|^2_{L^2(\mathcal{C}_{0,S}, w)}$$

for all $u \in L^2(\mathcal{C}_{0,S}, w)$, the scalar product being in $L^2(\mathcal{C}_{0,S}, w)$. Hence the operator $\chi_R A^* A \chi_R$ is invertible in $L^2(\mathcal{C}_{0,S} \cap B_R, w)$, and the inverse G belongs to $\mathcal{A}(\mathcal{C}_{0,S} \cap B_R; w)$. We thus get

$$G \, \chi_R A^* A \chi_R = \chi_R$$

whence $B_1 A = 1 - S_1$, with

$$\begin{array}{rcl} B_1 & = & G\,\chi_R A^*, \\ S_1 & = & (1-\chi_R) - G\,\chi_R A^* A\,(1-\chi_R) \end{array}$$

belonging to $\mathcal{A}(\mathcal{C}_{0,S}; w)$ and $\mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$, respectively.

In the same we construct a $B_2 \in \mathcal{A}(\mathcal{C}_{0,S}; w)$ satisfying $AB_2 = 1 - S_2$, where $S_2 \in \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$. This means that the coset $A + \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$ is invertible in Q_{x^0}.

□

Note that $\mathrm{smb}_{x^0}(A) = 0$ if $A \in \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$. Hence the symbol mapping smb_{x^0} extends to the quotient algebra Q_{x^0} to characterise the invertible cosets thereof, cf. Theorem 9.1.

10 Local symbol at regular points

In this section we define a local symbol of operators in the algebra $\mathcal{A}(\mathcal{C}_{0,S}; w)$ at the regular points of $\mathcal{C}_{0,S}$.

Let x^0 be an interior point of the cone $\mathcal{C}_{0,S}$. We are going to evaluate the symbol for

$$A = \sum_{i=1}^{I} \prod_{j=1}^{J} A_{ij} \tag{10.1}$$

where A_{ij} are operators described at the beginning of Section 8. Thus, we look for a symbol for the generators.

Let $A = M_a$ be the multiplication operator by a function $a \in C(\hat{\mathbb{R}}^n)$. We introduce polar coordinates (r, ω) around x^0 and pick a sequence $\Lambda = (\lambda_\nu)$ converging to zero. Then the limit operator A_Λ is easily seen to be $M_{a(x^0)}$. We write it simply $a(x^0)$ and set

$$\mathrm{smb}_{x^0}(A) = a(x^0).$$

Let $A = \chi_{\mathcal{C}_{0,S}}\, F^{-1}\sigma F\, \chi_{\mathcal{C}_{0,S}}$, where $F^{-1}\sigma F$ is a SIO of the form (7.1). Given any sequence $\Lambda = (\lambda_\nu)$ converging to zero, the limit operator of A defined by Λ is $A_\Lambda = F^{-1}\sigma(x^0, \xi) F$, which does not depend on Λ at all. We therefore introduce

$$\mathrm{smb}_{x^0}(A) = \sigma(x^0, \xi).$$

For A of the form (10.1), we set

$$\mathrm{smb}_{x^0}(A) = \sum_{i=1}^{I} \prod_{j=1}^{J} \mathrm{smb}_{x^0}(A_{ij}),$$

which is a homogeneous function on $\mathbb{R}^n$ continuous on the unit sphere. The well-known estimate

$$\sup_{\xi \in \mathbb{S}^{n-1}} |\mathrm{smb}_{x^0}(A)| \leq \|A\|_{\mathcal{L}(L^2(\mathcal{C}_{0,S},w))}$$

allows us to define $\mathrm{smb}_{x^0}(A)$ for each operator $A \in \mathcal{Q}(\mathcal{C}_{0,S}; w)$.

Denote by $\mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$ the ideal in $\mathcal{Q}(\mathcal{C}_{0,S}; w)$ generated by the multiplication operators M_a, where $a \in C(\hat{\mathbb{R}}^n)$ satisfies $\lim_{x \to x^0} a(x) = 0$. We actually have

$$\|A + \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)\|_{Q_{x^0}} = \sup_{\xi \in \mathbb{S}^{n-1}} |\mathrm{smb}_{x^0}(A)|$$

where Q_{x^0} is the quotient algebra $\mathcal{Q}(\mathcal{C}_{0,S}; w)/\mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$. Hence the mapping $\mathrm{smb}_{x^0} \colon Q_{x^0} \to C(\mathbb{S}^{n-1})$ is an isometrical embedding of C^*-algebras.

Theorem 10.1 *Suppose $A \in \mathcal{Q}(\mathcal{C}_{0,S}; w)$. For an interior point x^0 of $\mathcal{C}_{0,S}$, the coset $A + \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$ is invertible in Q_{x^0} if and only if $\mathrm{smb}_{x^0}(A)$ is invertible in $C(\mathbb{S}^{n-1})$.*

It remains to consider the case $x^0 \in \partial\mathcal{C}_{0,S} \setminus \{0\}$, i.e., regular boundary points which are, of course, also regarded as singular ones of the configuration.

We still set $\mathrm{smb}_{x^0}(A) = a(x^0)$ if $A = M_a$ is the multiplication operator by an $a \in C(\hat{\mathbb{R}}^n)$.

Let us look for the limit operators of $\chi_{\mathcal{C}_{0,S}}$ at x^0. Write (r, ω) for polar coordinates with centre x^0. If $\Lambda = (\lambda_\nu)$ is a sequence of positive numbers converging to zero then

$$\begin{aligned} \lim_{\nu \to \infty} \kappa_{\lambda_\nu}^{-1} \chi_{\mathcal{C}_{0,S}} \kappa_{\lambda_\nu} &= \lim_{\nu \to \infty} \chi_{\mathcal{C}_{0,S}}(\lambda_\nu r, \omega) \\ &= \lim_{r \to 0} \chi_{\mathcal{C}_{0,S}}(r, \omega) \\ &= \chi_{\mathbb{H}_{x^0}}, \end{aligned}$$

where $\mathbb{H}_{x^0}$ is the tangent cone (half-space) to $\mathcal{C}_{0,S}$ at the point x^0.

Hence the limit operator of $A = \chi_{\mathcal{C}_{0,S}} F^{-1} \sigma F \chi_{\mathcal{C}_{0,S}}$ defined by a sequence Λ is

$$\chi_{\mathbb{H}_{x^0}} F^{-1} \sigma(x^0, \xi) F \chi_{\mathbb{H}_{x^0}},$$

which is actually independent of Λ. Similarly to [Esk73] we assign an operator-valued symbol to $\chi_{\mathbb{H}_{x^0}} F^{-1} \sigma(x^0, \xi) F \chi_{\mathbb{H}_{x^0}}$, which is a family of Hopf-Wiener operators on the half-line defined by the unit inward normal vector $\nu(x^0)$ to $\partial\mathcal{C}_{0,S}$ at x^0. Namely, set

$$\mathrm{smb}_{x^0}(A) = \chi_{\mathbb{N}^+_{x^0}} F^{-1}_{\varrho \mapsto r} \sigma(x^0, \eta, \varrho) F_{r \mapsto \varrho} \chi_{\mathbb{N}^+_{x^0}}$$

for $\eta \in \mathbb{S}^{n-2}$, where $\mathbb{N}^{\pm}_{x^0} = \{r\nu(x^0) : \pm r > 0\}$, $x = (y, r)$ is the splitting of coordinates corresponding to $\mathbb{H}_{x^0} = \partial\mathbb{H}_{x^0} \oplus \mathbb{N}^{+}_{x^0}$, and $\xi = (\eta, \varrho)$ the corresponding covariables.

For general A of the form (10.1), we set

$$\mathrm{smb}_{x^0}(A) = \sum_{i=1}^{I} \prod_{j=1}^{J} \mathrm{smb}_{x^0}(A_{ij}),$$

thus obtaining a family of Hopf-Wiener operators on $\mathbb{N}^{+}_{x^0}$, continuously parametrised by $\eta \in \mathbb{S}^{n-2}$. It is well known that

$$\sup_{\eta \in \mathbb{S}^{n-2}} \|\mathrm{smb}_{x^0}(A)\|_{\mathcal{L}(L^2(\mathbb{N}^{+}_{x^0}))} \leq \|A\|_{\mathcal{L}(L^2(\mathcal{C}_{0,S}, w))}$$

which allows one to extend the symbol mapping $\mathrm{smb}_{x^0}(A)$ to the entire algebra $\mathcal{A}(\mathcal{C}_{0,S}; w)$. In fact, we have

$$\|A + \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)\|_{Q_{x^0}} = \sup_{\eta \in \mathbb{S}^{n-2}} \|\mathrm{smb}_{x^0}(A)\|_{\mathcal{L}(L^2(\mathbb{N}^{+}_{x^0}))}$$

for all $A \in \mathcal{A}(\mathcal{C}_{0,S}; w)$. This just amounts to saying that the mapping

$$\mathrm{smb}_{x^0} : \; Q_{x^0} \to C(\mathbb{S}^{n-2}, \mathcal{L}(L^2(\mathbb{N}^{+}_{x^0})))$$

is an isometrical embedding of C^*-algebras.

Theorem 10.2 *Let $A \in \mathcal{A}(\mathcal{C}_{0,S}; w)$. Given any point $x^0 \in \partial\mathcal{C}_{0,S} \setminus \{0\}$, the coset $A + \mathcal{I}_{x^0}(\mathcal{C}_{0,S}; w)$ is invertible in Q_{x^0} if and only if* $\mathrm{smb}_{x^0}(A)$ *is invertible in $C(\mathbb{S}^{n-2}, \mathcal{L}(L^2(\mathbb{N}^{+}_{x^0})))$.*

An operator $A \in \mathcal{A}(\mathcal{C}_{0,S}; w)$ is said to be *uniformly elliptic* if $\mathrm{smb}_{x^0}(A)$ is invertible in the corresponding C^*-algebra, for each $x^0 \in \mathcal{C}_{0,S} \setminus \{0\}$, and the inverse is bounded uniformly in x^0. For uniformly elliptic operators A, condition (9.1) is equivalent to the invertibility of $\sigma_\Lambda(A)(\varrho + iF_\Lambda)$ for all $\varrho \in \mathbb{R}$ and $\Lambda \in \Sigma(x^0)$.

11 Fredholm property

Combining Theorems 2.1 and 5.8 we deduce that for an operator $A \in \mathcal{A}(\mathcal{C}_{0,S}; w)$ to be Fredholm it is necessary and sufficient that all of its limit operators be invertible and the norms of their inverses be uniformly bounded. We now invoke the symbol calculus of Sections 9 and 10 to characterise the Fredholm property.

Theorem 11.1 *An operator $A \in \mathcal{A}(\mathcal{C}_{0,S}; w)$ is Fredholm if and only if, for each $x^0 \in \mathcal{C}_{0,S} \cup \{\infty\}$, the local symbol $\mathrm{smb}_{x^0}(A)$ is invertible in the corresponding C^*-algebra.*

Proof. The proof follows immediately from the local principle of Simonenko, cf. Theorem 2.1.

□

Recall that a symbol calculus requires a Banach algebra $\mathcal{S}_{x^0}$ of operator-valued functions as well as a continuous algebra homomorphism smb from the Banach algebra of operators into $\mathcal{S}_{x^0}$.

12 Global algebra

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ whose boundary is smooth outside of a finite set of singular points $x^1, \ldots, x^N \in \partial\mathcal{D}$. We assume that every point x^ν has a neighbourhood O_{x^ν} such that $\mathcal{D} \cap O_{x^\nu} = \mathcal{C}_{x^\nu, S^\nu} \cap O_{x^\nu}$ where $\mathcal{C}_{x^\nu, S_\nu}$ is an oscillating cone with vertex at x^ν, and $S^\nu(r, \cdot)$, $r > 0$, is a family of diffeomorphisms on $\mathbb{S}^{n-1}$, as in Section 6. As usual, we write $\mathcal{C}_{x^\nu, S^\nu}$ for the shift of $\mathcal{C}_{0,S^\nu}$ by x^ν.

Denote $\mathcal{A}(\mathcal{D}; w)$ the C^*-algebra of operators in $L^2(\mathcal{D}, w)$ generated by the SIO's of the form $\chi_{\mathcal{D}}\, F^{-1}\sigma F\, \chi_{\mathcal{D}}$, and the operators of multiplication by continuous functions on $\bar{\mathcal{D}}$, the closure of $\mathcal{D}$. The weight function w is assumed to be C^∞ away from the singular points $x^1, \ldots, x^N$, and close to any x^ν, to coincide with some $w^\nu \in \mathcal{W}(-n/2, n/2)$. This ensures the boundedness of SIO's in $L^2(\mathcal{D}, w)$.

Let x^ν be a singular point. Given any operator $A \in \mathcal{A}(\mathcal{D}; w)$, the restriction of A to a sufficiently small neighbourhood O_{x^ν} of x^ν coincides with the restriction of an operator $A^\nu \in \mathcal{A}(\mathcal{C}_{x^\nu, S^\nu}; w^\nu)$ to O_{x^ν}. Hence we can define the mapping

$$\mathrm{smb}_{x^\nu} : \frac{\mathcal{A}(\mathcal{D}; w)}{\mathcal{I}_{x^\nu}(\mathcal{D}; w)} \to \bigoplus\nolimits_{\Lambda \in \Sigma(x^\nu)} C_b(\mathbb{R} + iF_\Lambda, \mathcal{L}^0(\Omega_\Lambda))$$

by $\mathrm{smb}_{x^\nu}(A) = \mathrm{smb}_{x^\nu}(A^\nu)$.

In a similar way we introduce symbol mappings

$$\mathrm{smb}_{x^0} : \frac{\mathcal{A}(\mathcal{D}; w)}{\mathcal{I}_{x^0}(\mathcal{D}; w)} \to C(\mathbb{S}^{n-1})$$

at the interior points x^0 of $\mathcal{D}$, and

$$\mathrm{smb}_{x^0} : \frac{\mathcal{A}(\mathcal{D}; w)}{\mathcal{I}_{x^0}(\mathcal{D}; w)} \to C(\mathbb{S}^{n-2}, \mathcal{L}(L^2(\mathbb{N}^+_{x^0})))$$

at the regular points x^0 of $\partial\mathcal{D}$.

By the above, an operator $A \in \mathfrak{A}(\mathcal{D}; w)$ is locally invertible at a point $x^0 \in \bar{\mathcal{D}}$ if and only if $\mathrm{smb}_{x^0}(A)$ is invertible in $\mathcal{S}_{x^0}$. Hence the following theorem holds.

Theorem 12.1 *For an operator $A \in \mathfrak{A}(\mathcal{D}; w)$ to be Fredholm it is necessary and sufficient that* $\mathrm{smb}_{x^0}(A)$ *be invertible in $\mathcal{S}_{x^0}$, for each $x^0 \in \bar{\mathcal{D}}$.*

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