Combinatorial Thom - Hirzebruch Characteristic Classes.

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1 Introduction

This paper is part of a joint project by A. Connes, D. Sullivan and N. Teleman [CST]. The content of this paper was presented at the Colloque "Algebres de Fonctions et Espaces Singulaires" - Luminy, November 1996. A first draft of it was written in July 1994.

In this paper we study combinatorial manifolds M endowed with a flat combinatorial Riemannian metric. With any such structure, we associate a family of measures τ_t^d on the $(d+1)$-power of the manifold M. The measure τ_t^d is given by an integrable top degree differential form whose value at each point depends on the geometry on the t-neighborhood of the point. The measure τ_t^d is a cycle in the Alexander - Spanier homology of the manifold. As t approaches zero, τ_t^d converges to a simplicial cycle τ^d of degree d on M. The cycle τ^d represents the Thom - Hirzebruch characteristic class of M. The construction of the measure τ_t^d depends also on the choice of a real valued functiom of real variable used as a cut-off function.

Our construction produces an explicite realization of the Thom - Hirzebruch measure τ^d, introduced in [CST], by a careful usage of the additional geometrical structure available while passing from quasi conformal data to combinatorial data. This is done by performing constructions which use solely the local geometry seen on the t-neighborhood of each point.

The bibliography indicated in this paper specifies only those papers on which it depends.

The reading of this paper requires familiarity with [CST].

2 The Main result

Let $\rho : [0,\infty)\to[0,1]$ be an arbitrary monotone smooth or piecewise linear cut-off function which is identically 1 on the interval [0,1/3] and identically zero on the complement.

Theorem 2.1 *(Main Theorem)*

1. *There is an explicite analytical construction which for any data consisting of: a closed, oriented, even dimensional combinatorial manifold M^{2l}, a flat combinatorial Riemannian metric g on M^{2l} and a cut-off function ρ, associates a one-parameter family of local Hodge decompositions H_t, $t > 0$ on the Hilbert space of L_2 forms of degree l, with grading γ induced by g and the orientation.*

2. *Let $k_t(\gamma)$ denote the operator kernel*

   (on $M \times M$) of the operator

   $$K_t^\gamma \doteq H_t \gamma H_t + \gamma$$

   and associate the corresponding measures

   $$\tau_t^d(\gamma) \doteq trace(\wedge^{d+1} k_t(\gamma))$$

   as defined in [CST], Theorem 2. Then, the measure

   $$\tau_t^d \doteq \frac{1}{2}(\tau_t^d(\gamma) - \tau_t^d(-\gamma))$$

   converges, as $t \searrow 0$, to a simplicial cycle L_d of dimension d on M. The convergence is in the sense of distributions (continuous functionals on the $(d+1)$-fold projective tensor product of the algebra A of piecewise differentiable functions on M^{2l}).

3. *The cycle L_d is locally determined by the metric g, the cut-off function ρ, and the orientation*

4. *The homology class of the cycle L_d is equal to $\lambda_d(L^{2l-d} \cap [M]$, where L^{2l-d} is the Hirzebruch-Thom L-class and where $\lambda_d = 2^{2m+1}(2\pi i)^{-m}\frac{m!}{(2m)!}$, $2m = d$.*

3 Flat Combinatorial Metrics and Distances.

A flat combinatorial Riemannian metric g consists of a collection of flat Riemannian metrics on the simpleces of the manifold, which are required to agree on the common faces.

The combinatorial distance $r(x,y)$ between two points x, y in M is by definition the minimal lenght of polygonal curves joining x and y.

For any closed simplex σ in M^{2l} and for any point x belonging to its interior, let $T_x(M)$ denote the tangent cone to M at the point x. Clearly, the tangent cone contains the $Star(\Sigma)$. It is combinatorially equivalent to a simplicial cone decomposition of R^{2l}. The metric g restricted first to the $Star(\sigma)$ and extended next, radially, all along the tangent cone will be denoted by g_σ.

The tangent cone $T_x(M)$ decomposes into the direct orthogonal product of $T(\sigma)$, the tangent cone to the open simplex σ, and its orthogonal complement, $T(\sigma)^\perp$.

Consequently, the distance function $r(x,y)$ between any two points x, y in the interior of $Star(\Sigma)$, and sufficienly close to σ, satisfies

$$r(x,y)^2 = r_\sigma(u(x), u(y))^2 + {r_\sigma}^\perp(v(x), v(y))^2,$$

where u, resp. v, denote the projections onto $T(\sigma)$, resp. $T(\sigma)^\perp$ and r_σ is the Euclidean distance in $T(\sigma)$, while ${r_\sigma}^\perp$ is the combinatorial distance on the orthogonal complement.

The direct product of any isometry of $T(\sigma)$ with the identity on the orthogonal complement, $T(\sigma)^\perp$, is an isometry of the tangent cone $T_x(M)$. This obvious geometrical property will be reflected, analytically, into local symmetry properties of the operators we are going to construct.

4 Partitions of Unity

We rescale the cut-off function ρ by introducing

$$\rho_t(r) = \rho(\frac{r}{t}).$$

For any simplex σ in M, let $r^\sigma(x)$ denote the distance from the point x to σ, for x in the star of σ and sufficiently close to the simplex. Let $\rho_t^\sigma(x) = \rho_t(r^\sigma(x))$ be the rescaled cut-off distance function to the simplex σ.

By an inductive procedure on the increasing dimension of the scheleta, we are going to construct two specific families of functions λ_t^σ and ψ_t^σ, whose supports are contained in the star of each simplex σ of the combinatorial manifold, with t a small positive parameter. The first family of functions will constitute a partition of unity on M, while the second family will provide functions which are identically 1 on the support of the corresponding functions from the first family.

For σ, a simplex of dimension zero, define

$$\lambda_t^\sigma = \rho_t^\sigma,$$

and

$$\psi_t^\sigma = \lambda_{2t}^\sigma.$$

Suppose the functions λ_t^σ, ψ_t^σ have been constructed for any simplex σ of dimension $\leq k$. Let τ be a $k+1$-simplex in M. There exists a natural number a_{k+1} such that the functions, so defined

$$\lambda_t^\tau = \rho_{t/a_{k+1}}^\tau \cdot (1 - \sum_{dim.\sigma \leq k} \lambda_t^\sigma)$$

and

$$\psi_t^\tau = \rho_{2t/a_{k+1}}^\tau.(1 - \sum_{dim.\sigma \leq k} \lambda_{t/2}^\sigma)$$

will have their supports disjoint from the boundary of $Star(\tau)$. In this formula we agree to assume that if $k+1 = dimM$, $\rho_t^\tau = 1$. These functions will be extended by zero on the complement of the $Star(\sigma)$. The sequence of numbers a_k depends on the local combinatorial data and the flat combinatorial metric. If the simplices are equilateral, the numbers a_k will be independent of the combinatorial data of the manifold.

Proposition 4.1 *The functions λ_t^σ, ψ_t^σ have the following properties*

1. *they are produced by an explicite construction completely determined by the combinatorial metric, the cut-off function ρ, and some positive constants a_k which depend on the combinatorial metric. If the simplices of the combinatorial manifold are equilateral, the constants a_k depend only on k.*

2. *the support of each of the functions λ_t^σ, ψ_t^σ is contained in a t-neighborhood of the simplex σ; in particular, when t converges to zero, their support converges to σ*

3. *each of the functions λ_t^σ, restricted to a small neighborhood of any point x, sufficiently close to σ, is invariant under all local isometries of the tangent cone at x*

4. *the functions λ_t^σ constitute a partition of unity by Lipschitz functions on M*

5. *the function ψ_t^σ is identically 1 on the support of the function λ_t^σ.*

5 Combinatorial Hodge Decompositions

5.1 Hodge Structures on Vector Spaces.

Recall that a conformal structure μ on the n-dimensional ($n = 2l$) vector space E is a field of conformal structures defined on the fibres of the l^{th}-exterior power of the cotangent bundle, $\Lambda^l T^\star(E)$. The comparizon between the self-dual and anti-self-dual spaces of the conformal structure, at each point, with the corresponding spaces, associated with a flat conformal structure μ_0 on E, i.e. defined by an Euclidean structure, leads to a field of automorphisms of $\Lambda^l T^\star(E)$. This field of automorphisms characterizes the conformal structure itself. It is lecit then to indicate it by the same symbol. If the field of automorphisms μ is measurable and its pointwise ∞-norms are bounded by a constant < 1, then the conformal structure μ is called measurable and bounded.

Given a bounded measurable conformal structure μ on E, consider the bounded operator S_μ on the Hilbert space of l-differential forms on E with coefficients in $L_2(E)$ defined by the property that it is the identity on the d-exact forms, and $(-1) \times$ *identity* on the δ-exact forms. This operator was called *Hodge decomposition* on E, v. [CST].

The operator S_μ can be expressed in terms of the operator S_0 associated with the flat conformal structure μ_0, by the formula

$$S_\mu = (1-\mu)^{-1}(S_0-\mu)S_0(S_0-\mu)^{-1}(1-\mu).$$

From this formula one derives easily that for any bounded measurable function f one has

$$[S_\mu, f] = -(1+\mu)(S_0-\mu)^{-1}[S_0, f](S_0-\mu)^{-1}(1-\mu).$$

The operators S_μ have the following analytical properties.

Proposition 5.1 *1. S_μ is self adjoint on $L_2(E)$ and $S_\mu^2 = 1$*

2. if f is a Lipschitz function on E with compact support, then $[S_0, f]$ belongs to the Lorentz space $\mathcal{L}^{(2l,\infty)}$, v. [CST].

Proof.

1. Obvious.

2. The operator S_0 is a translation invariant Calderon - Zygmund singular operator with smooth symbol.

3. The commutator $[S_0, f]$ belongs to $\mathcal{L}^{(2l,\infty)}$ if and only if $f \in W^{1,2l}$, where $W^{1,2l}$ is the Sobolev space of all locally integrable functions on $\mathbf{R}^{2l}$ whose first order distributional partial derivatives belong to $L^{2l}(\mathbf{R}^{2l})$, see Theorem Appendix [CST].

5.2 Combinatorial Hodge Decompositions on Manifolds.

In this subsection we are going to carefully construct an one-parameter family of local Hodge structures on combinatorial manifolds. The main idea of the construction consists of producing operators which at any point reflect to the maximum the combinatorial geometry on a t-neighborhood of that point.

The homological combinatorial information will be extracted by making t go to zero.

Consider the combinatorial manifold M with combinatorial metric g. For any simplex σ in M we associate the tangent cone $T_x(M)$ at any interior point x of σ. As seen before, the metric g extends combinatorially to a metric on this tangent cone. Its underlying conformal structure is bounded, measurable and, hence, it defines a Hodge decomposition, S^σ.

Let

$$r_t^\sigma(x,y) \doteq \rho_t \circ \rho^\sigma(x,y)$$

be the rescaled cut-off distance function on the tangent cone.

We cut-off the operator S^σ by multiplying its operator kernel by the function r_t^σ; the resulting operator will be denoted by S_t^σ.

Finally, we produce the one-parameter family of geometric operators on M

$$S_t \doteq \sum_{\sigma \in M} \psi_t^\sigma . S_t^\sigma . \lambda_t^\sigma .$$

As in the general case treated in [CST], these operators satisfy all properties of a local Hodge decomposition, except for the property that $S_t\gamma + \gamma S_t$ is trace class; this graded commutator belongs merely to the Lorentz space $\mathcal{L}^{(2l,\infty)}$.

We apply onto the geometric operator S_t the algebraic regularizing procedure discussed in [CST]-section 4 to manufacture the desired local Hodge decompositions H_t. While we are going to recall that procedure, we will sharpen it.

Define $\theta_t = S_t^2 - 1$.

Proposition 5.2 *θ_t belongs to the Lorentz space $\mathcal{L}^{(2l,\infty)}$.*

Let $q(x)$ be the unique polynomial of degree n such that

$$(1+x)^{-1/2} = q(x) + O(x^{n+1}),$$

for x small and

$$p(x) \doteq (1+x)q(x)^2 - 1 = O(x^{n+1}).$$

Finally, the operator

$$H_t = \gamma p(\theta_t) + [1 - \frac{1+\gamma}{2} p(\theta_t)] q(\theta_t) S_t.$$

is the desired regularized operator S_t.

The operator

$$K_t^\gamma \doteq H_t \gamma H_t + \gamma$$

contains the desired homological information.

Theorem 5.3 *The operator K_t^γ is trace class, and hence, its kernel $k_t^\gamma(x, y)$ is integrable.*

Proof. A direct computation, see [CST], shows

$$\gamma K_t^\gamma = \gamma H_t + H_t \gamma = 2p(\theta_t).$$

The kernel k_t^γ is an (l, l) bi-form on $M \times M$. Specifically, the operator K_t^γ and its kernel are related by the formula

$$(K_t^\gamma \omega)(x) = \int_{M(y)} k_t^\gamma(x, y) \wedge \omega(y),$$

for any L_2-form ω of degree l.

Let p be an interior point of the simplex $\sigma \in M$. Let $s = (s^1, ..., s^q)$ be a system of orthogonal coordinates on the Euclidean space $T_p(\sigma)$ and let $u = (u^1, ..., u^{2l-q})$ let be a system of piecewise linear system of coordinates on the orthogonal complement to σ at p. Then, for $t << dist(p, \sigma)$, and for points $(s_0, u_0), (s_1, u_1)$ sufficiently close to p, the kernel $k_t^\gamma((s_0, u_0), (s_1, u_1))$, depends only on the geometric data in a small neighborhood of p. As the geometric data is invariant under small translations and orientation preserving rotations along the simplex σ, in the neighbourhood of the point p, the corresponding operator K_t^γ is invariant at the same transformations; in fact, this operator may be extended all over the tangent cone to M at p in such a way that the extension will satisfy the same symmetry properties on the whole tangent space, without modifying its kernel in a small neighborhood of the point p. The translation invariance implies that

$$k_t^\gamma((s_0, u_0), (s_1, u_1)) = k_t^\gamma((0, u_0), (s_1 - s_0, u_1) \doteq \tilde{k}_t^\gamma(s_1 - s_0 \mid u_0, u_1),$$

while the rotation invariance will imply that for any $R \in SO(q)$

$$\tilde{k}_t^\gamma(R(s) \mid u_0, u_1) = \tilde{k}_t^\gamma(s \mid u_0, u_1).$$

For any even natural number d

$$\tau_t^d(\gamma) \doteq trace(\bigwedge^{d+1} k_t^{\gamma}),$$

is an absolutely continuous measure with respect to the Lebesque class on on $M^{(d+1)}$, the $(d+1)$ power of M. It can be seen also as a top degree differential form on $M^{(d+1)}$, along with the orientation $\mathcal{O} \times \mathcal{O} \times .... \times \mathcal{O}$, where $\mathcal{O}$ is the chosen orientation on M.

The main result of [CST] states that $\tau_t^d(\gamma)$ is an Alexander-Spanier homology cycle. This cycle represents the Poincare' dual to the Hirzebruch class L^{2l-d}, up to a constant factor.

Notice that the grading operator $\gamma \doteq i^l \star$, depends on the orientation $\mathcal{O}$ of M^{2l}; the reversal of the orientation has the effect of multiplying γ by -1.

From this we get also that, passing to homology classes

$$[\tau_t^d(-\gamma)] = -[\tau_t^d(\gamma)].$$

and, hence, the Alexander-Spanier homology cycle

$$\tau_t^d(\mathcal{O}) \doteq \frac{1}{2}\{\tau_t^d(\gamma) - \tau_t^d(-\gamma)\}$$

represents the same homology class as $\tau_t^d(\gamma)$. This measure has the important additional property that

$$\tau_t^d(-\mathcal{O}) = -\tau_t^d(\mathcal{O}).$$

The next theorem summarizes the basic properties of the measures $\tau_t^d(\mathcal{O})$ needed in the discussion of the distributional limit as t approaches 0.

Theorem 5.4 *The measure $\tau_t^d(\mathcal{O})$ has the properties:*

1. *The support of $\tau_t^d(\mathcal{O})$ is contained in the tubular neighbourhood of radius $Const.t$ about the main diagonal in M^{d+1}, where $Const.$ is a constant which depends only on the dimension of M, the metric g and the constants a_k.*

2. *Let p be a point in the interior of the simplex σ. Then, for t sufficiently small, depending on p, the measure $\tau_t^d(\mathcal{O})$ depends only on the geometric data in a small neighborhood of p.*

3.
$$\tau_t^d(-\mathcal{O}) = -\tau_t^d(\mathcal{O}).$$

4. Let $s = (s^1, ..., s^q)$ be a system of orthogonal coordinates on the Euclidean space $T_p(\sigma)$ and let $(u^1, ..., u^{2l-q})$ be a system of piecewise linear system of coordinates on the orthogonal complement to σ at p. Then

$$\tau_t^d(\mathcal{O})((s_0, u_0), (s_1, u_1), ..., (s_d, u_d)) =$$

$$\frac{1}{2}\{\tilde{k}_t^\gamma(s_1 - s_0 \mid u_0, u_1).\tilde{k}_t^\gamma(s_2 - s_1 \mid u_1, u_2)......\tilde{k}_t^\gamma(s_0 - s_d \mid u_d, u_0) -$$
$$\tilde{k}_t^{-\gamma}(s_1 - s_0 \mid u_0, u_1).\tilde{k}_t^{-\gamma}(s_2 - s_1 \mid u_1, u_2)......\tilde{k}_t^{-\gamma}(s_0 - s_d \mid u_d, u_0)\}$$

5. The measure $\tau_t^d(\mathcal{O})$ satisfies, locally, the scaling property

$$\tau_t^d(\mathcal{O})(x_0, x_1, ...x_d) = \frac{1}{t^{(d+1).2l}}.\tau_t^d(\mathcal{O})(\frac{1}{t}x_0, \frac{1}{t}x_1, \frac{1}{t}x_2, ..., \frac{1}{t}x_d)$$

(here, we assume that $p = 0$).

6. For any $R \in O(q)$,

$$\tau_t^d(\mathcal{O})(R(s_0), u_0), (R(s_1), u_1), ..., (R(s_d, u_d)) =$$
$$|R|.\tau_t^d(\mathcal{O})((s_0, u_0), (s_1, u_1), ..., (s_d, u_d)),$$

where $|R| = \pm 1$, provided that R is preserving, resp. reversing, the orientation.

Although the next property is not used in the sequel, it is worth mentioning.

7. If the regularizing procedure used in the construction of the operators H_t in [CST]-section 4 is slightly modified, then $\tau_t^d(\mathcal{O})$ could be made differentiable of any fixed finite order with respect to the variables s.

Proof.

1. This property is clear.
2. Idem.

3. The measure $\tau_t^d(\mathcal{O})$ is obtained from $\tau_t^d(\gamma)$ by antisymmetrization with respect to γ.

4. Obvious consequence of the invariancy of the kernel k_t^γ with respet to translations.

5. The measure $\tau_t^d(\mathcal{O})$ depends on the following structures: - the conformal structure μ used in the construction of the Hodge decompositions S^σ on the tangent cones; in our construction $\mu \doteq [g]$ - the Riemannian metric g used in the construction of the partition of unity and the cut-off functions - an orientation $\mathcal{O}$.

   Let
   $$h : (M_1, \mu_1, g_1, \mathcal{O}_1) \longrightarrow (M_2, \mu_2, g_2,\ O_2,)$$
   be a combinatorial homeomorphism from the manifold M_1 to the manifold M_2, each equipped with structures as before, respectively. If h preserves all these structures, we will say that h is a *Hodge equivalence*. Clearly, a Hodge equivalence preserves also the corresponding measures τ_t^d. We are going to use this statement on tangent cones. Specifically, we restrict our attention to a tangent cone $T_x(M)$ equipped with the induced combinatorial Riemannian metric g and orientation $\mathcal{O}$. Let $h : T_x(M) \longrightarrow T_x(M)$ be the homothety of modulus $s > 0$, $z \mapsto \frac{1}{s}z$, with center at $x = 0$, for s small. Then
   $$h : (T_x(M), [g], g, \mathcal{O}) \longrightarrow (T_x(M), [g], s^2 g, \mathcal{O})$$
   is a Hodge equivalence (Note $[s^2 g] = [g]$. This property, along with the fact that the operators S depend only on the conformal structure, are responsible for the scaling property exibited below.

   If k_t^i, i=1,2, denote the kernels of the operators H_t^i associated with these structures, we have, for any test form ω of degree l, and any point $x \in T_x(M)$
   $$(H_t^1(h^*\omega))(x) = (h^* H_t^2(\omega))(x),$$
   or
   $$\int_y k_t^1(x, y) \wedge \frac{1}{s^l}\omega(\frac{y}{s}) = \int_u \frac{1}{s^l} k_t^2(\frac{x}{s}, u) \wedge \omega(u).$$

Changing the variable $u = \frac{y}{s}$ in the second integral gives

$$k_t^1(x,y) = \frac{1}{s^n} k_t^2(\frac{x}{s}, \frac{y}{s}).$$

We take in this last formula $s = t$. The operators H_t^i are manufactured by an explicite algebraic procedure which involves the geometric operators $S_{[g]} = S_{[s^2 g]}$ and the cut-off rescaled distance functions determined by the Riemanniam distances (the fixed cut-off function ρ is the same in both constructions). As the distance functions on the target space of the homothety h are multiplied by t, the rescaled cut-off distance functions on the target space are precisely the cut-off unscaled ($t = 1$) distance functions defined on the domain of the homothety. Therefore

$$k_t^1(x,y) = \frac{1}{s^{2l}} k_t^2(\frac{x}{t}, \frac{y}{t}) = \frac{1}{s^{2l}} k_1^1(\frac{x}{t}, \frac{y}{t}).$$

6. This property can be derived as above by considering the Hodge equivalence
$$h : (T_x(M), [g], g, \mathcal{O}) \longrightarrow (T_x(M), [g], g, |R|\mathcal{O}),$$
where $h(s,u) = (R(s), u)$, combined with the property 3.

7. It is sufficient to take as polynomial $q(t)$, given by formula (1) in [CST]-section 4, a sufficiently higher order approximation of the power series $(1+t)^{-\frac{1}{2}}$.)

6 *Proof of the Main Theorem 2.1*

The main technical device which leads to the proof of the theorem is the *Symmetry Principle:* The integral of a function against a measure, which is invariant at a compact group of transformations, is equal to the integral of its Haar - average over the group.

6.1 Review of Orthogonal Invariants

Let V^q be an Euclidean vector space of dimension q, and let $O(V)$ be the group of its orthogonal transformations. The diagonal action of the group

$O(q)$ on the direct sum $\oplus^d$ of d copies of V is defined by asking each orthogonal transformation to act simultaneously on each summand. A real polynomial function φ defined on $\oplus^d$ is called *odd invariant* over the orthogonal group iff for any $T \in O(V)$

$$\varphi(Tx_1, Tx_2, \ldots.., Tx_d) = |T|.\varphi(x_1, x_2, \ldots.., x_d),$$

where $|T| = \pm 1$, depending on whether T is orientation preserving, resp. reversing, transformation.

Theorem 6.1 *1. If $d < q$, then the only odd invariant polynomial is the zero polynomial.*

2. If $d \geq q$, than any odd, invariant polynomial belongs to the ideal generated by the polynomials

$$determinant[x_{i_1}, x_{i_2}, \ldots., x_{i_q}],$$

where $1 \leq i_1 < i_2 < i_3 < \ldots. < i_q \leq d$.

For a discussion of this theorem, see Weyl [W].

6.2 Proof of Theorem 2.1.(2).

Theorem 6.2 *For any test function F in the $d+1$-fold projective tensor product of the algebra of piecewise differentiable functions on M,*

$$\lim_{t \searrow 0} \int F d\tau_t^d =$$

$$\sum_{dim.\ \sigma = d} L(\sigma) \cdot \int_\sigma d_{x_1} d_{x_2} \ldots.. d_{x_d} F(x_0, x_1, x_2, \ldots.. x_d),$$

where $L(\sigma)$ is a constant.

Proof.

Take an arbitrary point $p \in M$. We intend to study the distributional limit of the measure $\tau_t^d(\mathcal{O})$, when t goes to zero, in a small neighbourhood of the point p. Let σ be the unique simplex of M which contains p in its interior. Let q be its dimension. Take F a piecewise differentiable test function on the $(d+1)$ power of M. As we are interested in the limit on a small neighbourhood of the point p, we may assume that the support of F is contained in the $d+1$ power of $Star(\sigma)$. Let $s^1, s^2, ..., s^q$ be a system of orthogonal linear coordinates along the simplex σ, and let $u^1, u^2, ..., u^{2l-q}$ be system of barycentric coordinates on the othogonal complement to σ at p.

Consider in $Star(\sigma)^{d+1}$ the direct product of these coordinates and denote them by s^i_j, u^h_j; here the bottom indices $j = 0, 1, 2, ..., d$ refer to the factors, while the top indices $i = 1, 2, ..., 2l$, $h = 1, 2, ..., 2l-q$ denote the coordinates in each factor.

We intend to show that the limit of the measure $\tau^d_t(\mathcal{O})$, acting on the test function F, involves only its jet along the diagonal . For this purpose, it is convenient to change the coordinates by introducing

$$\xi^i_0 = s^i_0$$

$$\xi^i_j = s^i_j - s^i_0, \quad for \quad i = 1, 2, ..., q, \quad j = 1, 2, ..., d,$$

and

$$\beta^h_0 = u^h_0, \quad \beta^h_j = u^h_j - u^h_0.$$

In these coordinates, the diagonal is given by $\xi^i_j = \beta^h_j = 0$. The infinite jet of F along the diagonal, $j_\infty(F)$, is a formal power series in the variables ξ^i_j, β^h_j whose coefficients are functions of s^i_0 , u^h_0.

Lemma 6.3 *If*

$$Lim_{t\searrow 0} \int F d\tau^d_t(\mathcal{O})$$

exists, it depends only on the $2l-$jet of F.

Proof of Lemma 6.3. Let $Q_r \doteq \{(x_0, x_1, ..., x_d) \quad | \quad |(x_j| \leq r\}$ denote the cube of side r. Let

$$U_r = \{(x_0, x_1, ..., x_d)| \quad |x_j - x_0| \leq r\}$$

denote the tubular neighbourhood of radius r about the diagonal. We may assume that the support of $\tau^d_t(\mathcal{O})$ is contained in U_t. Let F be a test function; for simplicity, we assume that the support of F is contained in the cube Q_1. Then

$$\int_{Q_1} F(x_0,x_1,...,x_d)\tau_t^d(\mathcal{O})(x_0,x_1,...,x_d)|dx| =$$

$$\frac{1}{t^{(d+1).2l}}\int_{Q_1\cap U_t} F(x_0,x_1,...,x_d).\tau_1^d(\mathcal{O})(\frac{1}{t}x_0,\frac{1}{t}x_1,\frac{1}{t}x_2,........,\frac{1}{t}x_d)|dx| =$$

$$\int_{Q_{\frac{1}{t}}\cap U_1} F(t.y_0,t.y_1,...,t.y_d)\tau_1^d(\mathcal{O})(y_0,y_1,...,y_d)|dy|.$$

The absolute value of this integral is dominated by

$$Const.t^{-2l}.Sup_{U(r_0.t)}|F|.$$

Therefore, if the $2l-$ jet of F vanishes along the diagonal, then

$$Lim_{t\searrow 0}\int Fd\tau_t^d(\mathcal{O}) = 0,$$

which proves the lemma.

Although the measure $\tau_t^d(\mathcal{O})$ is odd invariant with respect to the diagonal action of $O(q)$, this action does not does not keep the diagonal pointwise fixed and, hence, it does not factorize to the space of jets of functions F with respect to the variables $x_j - x_0$. For this reason, we define on $T_p(M)^{d+1}$ the *semi-diagonal* action of $R \in O(q)$

$$R\circ((s_0,u_0),(s_1,u_1),...,(s_d,u_d)) \doteq$$

$$((s_0,u_0),(s_0+R(s_1-s_0),u_1),...,(s_0+R(s_d-s_0),u_d)).$$

Lemma 6.4 *$\tau_t^d(\mathcal{O})$ is odd invariant at the semi-diagonal action of $O(q)$, i.e. for any $R \in O(q)$*

$$\tau_t^d(\mathcal{O})(R\circ((s_0,u_0),(s_1,u_1),...,(s_d,u_d))) =$$

$$|R|.\tau_t^d(\mathcal{O})((s_0,u_0),(s_1,u_1),...,(s_d,u_d)).$$

Proof. It is a consequence of the properties 4. and 6. of the Proposition 5.4

$$\tau_t^d(\mathcal{O})(R\circ(s_0,u_0),(s_1,u_1),...,(s_d,u_d)) =$$

$$\frac{1}{2}\tilde{k}_t^{\gamma}(R(s_1 - s_0) \mid u_0, u_1).\tilde{k}_t^{\gamma}(R(s_2 - s_1) \mid u_1, u_2)......\tilde{k}_t^{\gamma}(R(s_0 - s_d) \mid u_d, u_0) -$$

$$\frac{1}{2}\tilde{k}_t^{-\gamma}(R(s_1 - s_0) \mid u_0, u_1).\tilde{k}_t^{-\gamma}(R(s_2 - s_1) \mid u_1, u_2)......\tilde{k}_t^{-\gamma}(R(s_0 - s_d) \mid u_d, u_0) =$$

$$|R|.\tau_t^d(\mathcal{O})((s_0, u_0), (s_1, u_1), ..., (s_d, u_d)).$$

Returning to the proof of the Theorem 6.2, the Lemma 6.1 along with the symmetry principle show that we may assume, additionally, that the function F and, hence, $j_\infty(F)$, are odd invariant at the semi-diagonal action of $O(q)$.

Proposition 6.5 *Suppose the support of F is contained in the interior of $Star(\sigma)^{d+1}$, $dim(\sigma) = q$.*

1. *If $d < q$, then*
$$\int F d\tau_t^d = o(t^\infty), \quad for \quad t \searrow 0.$$

2. *If $d = q$, then*
$$\lim_{t \searrow 0} \int F d\tau_t^d(\mathcal{O}) =$$
$$L(\sigma) \cdot \int_\sigma d_{x_1} d_{x_2} .....d_{x_d} F(x_0, x_1, x_2, ...x_d).$$
where $L(\sigma)$ is a constant

3. *If $d > q$, then*
$$\lim_{t \searrow 0} \int F d\tau_t^d(\mathcal{O}) = 0.$$

Proof.

1. Case $d < q$. The above discussion, combined with the Theorem 6.1.1, allow us to assume that $j_\infty(F) = 0$.

Remark. We have proved that if $\tau_t^d(\mathcal{O})$ is odd invariant at the semi-diagonal action of the group $O(q)$, $q > d$, for t sufficiently small, then

$$\lim_{t \searrow 0} d\tau_t^d(\mathcal{O}) = 0 \quad (as \quad measure).$$

The convergence to zero is more rapid than any power of t.

2. Case $d = q$. The domain of integration $Q_{\frac{1}{t}} \cap U_1 \cup Supp\tau_t^d(\mathcal{O})$ will be decomposed in two parts

$$Q_{\frac{1}{t}} \cap U_1 \cap Supp\tau_t^d(\mathcal{O}) = D^{d+} \cup D^d$$

where D^{d+}, resp. D^d, is the subdomain on which $d\tau_t^d(\mathcal{O})$ has $O(d+)$-symmetry, resp. $O(d)$-symmetry, with respect to the semi-diagonal action.

Then

$$\int_{Q_{\frac{1}{t}} \cap U_1} F.d\tau_t^d(\mathcal{O}) = \int_{D^{d+}} F.d\tau_t^d(\mathcal{O}) + \int_{D^d} F.d\tau_t^d(\mathcal{O}).$$

The domain D^{d+} is a finite disjoint union of domains on which $d\tau_t^d(\mathcal{O})$ has semi-diagonal $O(q')$ - symmetry, with $q' > d$. The Remark above gives

$$\lim_{t \searrow 0} \int_{D^{d+}} F.d\tau_t^d(\mathcal{O}) = O.$$

Let $(s, u) \in Star(\sigma)$, where $s \in \sigma$ and $u \in \sigma^{\perp}$.

Theorem 6.1.2 implies that on D^{d+} one has

$$F((s_0, u_0), (s_1, u_1), ..., (s_d, u_d)) =$$

$$\frac{\partial^q}{\partial s_1^{i_1} \partial s_2^{i_2} .... \partial s_q^{i_q}} F|_{(s_0,0),(s_0,0),...,(s_0,0)} .(s_1^{i_1} - s_0^{i_1}) \wedge (s_2^{i_2} - s_0^{i_2}) \wedge ...... \wedge (s_q^{i_q} - s_0^{i_q}).(1+O(t)).$$

From here we get

$$\int_{D^d} F.d\tau_t^d(\mathcal{O}) =$$

$$\int_{D^d} t^q \frac{\partial^q}{\partial s_1^{i_1} \partial s_2^{i_2} .... \partial s_q^{i_q}} F|_{(t.s_0,0),(t.s_0,0),...,(t.s_0,0)} .$$

$(s_1^{i_1} - s_0^{i_1}) \wedge (s_2^{i_2} - s_0^{i_2}) \wedge ... \wedge (s_q^{i_q} - s_0^{i_q}).(1+O(t)).\tau_1^d(\mathcal{O})(x_0, x_1, ..., x_d)|dx|.$

Notice that while $t \searrow 0$, the domain of integration D^d looks like a tube of constant radius about the tangent space to the simplex σ, contained in the diagonal, which elongates along the diagonal, having lenght $\frac{1}{t}$. As the measure $d\tau_t^d(\mathcal{O})$ is invariant at translations along this tube, for $t \searrow 0$, the first integral converges to

$$L(\sigma) \cdot \int_\sigma d_{x_1} d_{x_2} .....d_{x_d} F(x_0, x_1, x_2, ...x_d),$$

and the second integral converges to zero.

3. Case $d > q$. In this case, the domain of integration will be decomposed in a disjoint union

$$Q_{\frac{1}{t}} \cap U_1 \cap Supp\tau_t^d(\mathcal{O}) = D^{d+} \cup D^d \cup D^{d-},$$

where $D^{d-} = (Q_{\frac{1}{t}} \cap U_1 \cap Supp\tau_t^d(\mathcal{O})) - (D^{d+} \cup D^d)$ is a neighbourhood of the q'-scheleton, for $q' \leq d-1$. The skew-symmetry of the function F ensures that

$$F(tx) = O(t^d).$$

On the other hand, the analogue description of the domain of integration D^{d-} as a tube of constant radius about a scheleton of dimension $\leq (d-1)$ and lenght $\frac{1}{t}$, leads to the conclusion

$$Lim_{t\searrow} \int_{D^{d-}} F.d\tau_t^d(\mathcal{O}) = 0.$$

The integrals over the domains D^{d+}, D^d are computed as in the previous case. This completes the proof of the theorem.

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