Analytic Index Formulas for Elliptic Corner Operators

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Abstract

Spaces with corner singularities, locally modelled by cones with base spaces having conical singularities, belong to the hierarchy of (pseudo-) manifolds with piecewise smooth geometry. We consider a typical case of a manifold with corners, the so-called "edged spindle", and a natural algebra of ΨDOs on it with special degeneracy in the symbols, the "corner algebra". There are three levels of principal symbols in the corner algebra, namely the interior, edge and corner conormal symbols. An operator is called elliptic if all the three principal symbols are invertible. Elliptic corner operators possess the Fredholm property in appropriate Sobolev spaces. We derive an analytic index formula for such operators containing two terms of different nature: the interior and corner contributions.

Introduction

Manifolds with corners belong to the category of stratified spaces, where singularities are modelled by iteratively forming cones and wedges, starting

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from a closed C^∞ manifold as the base of the first model cone. Special structure of singular charts on such manifolds gives rise to differential operators with typical degeneracy in symbols. Algebraic operations with typical symbols generate specific pseudodifferential algebras, [Sch01]. Ellipticity in these algebras is determined by the bijectivity of components of a hierarchy of principal symbols. This entails the existence of parametrices within the algebras and (for compact spaces) the Fredholm property in adequate scales of Sobolev spaces.

Since the Atiyah-Patodi-Singer index theorem [APS75] one tries to explicitly express the index of elliptic operators on singular spaces in terms of symbol hierarchy. For conical singularities and Fuchs-type operators there are many results in the literature, cf. [FS96, FST99] and the references given there, while for edge singularities explicit answers specifically depend on the choice of the operator algebra. Such a choice becomes more and more "non-canonical" the higher we admit the orders of geometric singularities. An analytic index formula for the edge algebra in the sense of [Sch91] has been given in [FST98], cf. also [Roz00].

In the present paper we consider *corners* as the next step in the hierarchy of singularities. A corner c may be viewed as a cone whose base Y is itself a manifold with conical points. Each conical point $y_c \in Y$ gives rise to a one-dimensional edge, these edges meet together at the vertex c of the corner. We consider an algebra of pseudodifferential operators with special degeneracy properties near edges and corners, the so-called *corner algebra* constructed by the second author [Sch92]. There are three levels of principal symbols in the corner algebra, each corner operator A gives rise to a triple

$$\sigma(A) = \{\sigma_{\mathrm{int}}(A), \sigma_\wedge(A), \sigma_c(A)\} \tag{0.1}$$

of the so-called *interior, edge* and *corner principal symbols*. If all the entries of (0.1) are invertible then the operator A is called *elliptic*; with some precautions it has Fredholm property in appropriate *Sobolev spaces*.

We derive an index formula for elliptic corner operators in the spirit of [FS96, FST98]. A convenient iterative representation of operators allows us to employ a similar machinery. We consider a model manifold with corners which we call "edged spindle,"

$$M = [-1,1] \times Y \,/\, (\{-1\} \times Y) \cup (\{1\} \times Y). \tag{0.2}$$

Here Y is an $(n+1)$-dimensional manifold with one conical point y_c, $c_\pm$ are two corners, the curve $[-1,1] \times y_c$ is an edge. The variable $t \in [-1,1]$ is called the corner-axis variable.

The "edged spindle" is the simplest *compact* manifold with corners. In contrast to [FS96, FST98] we have chosen here a compact manifold as a more realistic model allowing one to treat pseudodifferential operators of any order without order reduction. The non-compact case of a pure corner $\mathbb{R}_+ \times Y \,/\, \{0\} \times Y$ is also included in our model. In this case we take operators of order zero with symbols stabilizing to 1 in a neighborhood of c_- and having a corner degeneracy at c_+.

The change of variables

$$t' = \ln \frac{1+t}{1-t}$$

reduces the spindle to an infinite cylinder

$$M = \mathbb{R} \times Y, \tag{0.3}$$

the corners $c_\pm$ become "cylindrical ends" $\pm\infty$ (cf. [FST99]). The corner algebra becomes a version of the *edge algebra* [ES97, FST98] with a special behavior at $\pm\infty$.

There is a vast literature concerning the index theory on singular manifolds (see, e.g., [FST99]). The authors had to solve a difficult problem: what material to include in the present paper without enlarging its volume enormously. After some hesitation we decided that the results of the *cone* and *edge theory* should be simply referred to. On the contrary, the results of the *corner theory* should be presented here in detail. The reason is that the edge theory is a well-developed branch while for the corner theory we have essentially only one reference [Sch92]. Besides, our approach developed here is slightly different from that in [Sch92]. We introduce special classes Σ^m of operator-valued symbols based on special families of norms $\|\cdot\|_\tau$ in Sobolev spaces on manifolds with edges. Such an approach will be called a *passive* one. It turns out to be equivalent to the usual *active* approach based on operator-valued symbol classes $\mathcal{S}^m$ and the group action κ_λ. The terminology comes from a passive and active approach to the change of variables. From the passive point of view, the geometrical points remain fixed, we change only coordinate systems, while in the active approach the system remains fixed and we move the points.

The passive approach developed here seems to be more convenient. It allows us to reduce easily parameter-dependent operator-valued symbols to standard integral operators in L^2 spaces, so that the calculus of operator-valued symbols becomes quite similar to that of scalar-valued symbols. We hope that such an approach will be useful when considering higher singularities.

An interior part of the corner operator on the manifold M, cf. (0.3), is represented by an edge symbol $a(t,\tau)$ which is an operator-valued symbol on the plane $(t,\tau) \in \mathbb{R}^2$ whose values are cone pseudodifferential operators on the fiber Y. It stabilizes to $a(\pm\infty,\tau)$ for large t. Ellipticity implies that this function is almost invertible, that is a fiberwise parametrix $r_0(t,\tau)$ exists. The latter is an edge symbol stabilizing to $r_0(\pm\infty,\tau)$ for large t, such that

$$\begin{gathered} 1 - r_0(t,\tau)a(t,\tau), \\ 1 - a(t,\tau)r_0(t,\tau) \end{gathered}$$

are trace class operators vanishing for τ large. This implies that the differential 2-form

$$\operatorname{tr}(dr_0 + r_0\, da\, r_0) \wedge da$$

has a compact support in $\mathbb{R}^2$, so, the integral

$$\frac{1}{2\pi i} \int_{\mathbb{R}^2} \operatorname{tr}(dr_0 + r_0\, da\, r_0) \wedge da$$

exists. It gives one part of the index formula which we call an *interior contribution*. The other parts are given by the so-called *corner contributions* at $t = \pm\infty$.

Let us describe the corner contribution at $t = +\infty$. Besides the function $a(+\infty, \tau)$ which is a limit of the edge symbol $a(t, \tau)$ and in general is not invertible for all τ but only for τ large enough, a corner operator possesses a so-called *corner symbol* $\sigma_c(A)$ which is an operator-valued symbol $a_+(\tau)$ with values in the cone operators on the base Y *invertible* everywhere on the corner weight line $\Im\tau = \gamma_+$. The corner contribution may be thought of as a kind of a "logarithmic residue"

$$\frac{1}{2\pi i} \int_{\Im\tau = \gamma_+} \operatorname{tr}_{\mathrm{reg}} a_+^{-1}(\tau) a_+'(\tau)\, d\tau.$$

Unfortunately, the logarithmic derivative does not belong to the trace class, so the trace should be defined via some regularization procedure. To this end we compare the logarithmic derivative $a_+^{-1}(\tau)a_+'(\tau)$ with the function $r_0(+\infty, \tau)a'(+\infty, \tau)$. These two functions are defined on different horizontal lines in the complex plane τ. The first one is defined on the weight line $\Im\tau = \gamma_+$, the second one on the real axis $\Im\tau = 0$. Nevertheless, the function

$$f(\tau) = r_0(+\infty, \tau) a'(+\infty, \tau)$$

may be formally shifted to the weight line by means of the formal Taylor series

$$\sum_{k=0}^{\infty} f^{(k)}(\tau - i\gamma_+)\, \frac{(i\gamma_+)^k}{k!}, \tag{0.4}$$

here τ belongs to the weight line $\Im\tau = \gamma_+$. Were $f(\tau)$ an entire function, this series would be convergent and would give us the restriction of $f(\tau)$ to the weight line. We use the series (0.4) to regularize the logarithmic derivative. Namely, we define the corner contribution to be

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\Im\tau = \gamma_+} \operatorname{tr} \left\{ (a_+^{-1} a_+')(\tau) - \sum_{k=0}^{N} (r_0 a')^{(k)}(+\infty, \tau - i\gamma_+) \frac{(i\gamma_+)^k}{k!} \right\} d\tau \\ &\quad = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \operatorname{tr} \left\{ \sum_{k=0}^{N} (a_+^{-1} a_+')^{(k)}(\tau + i\gamma_+) \frac{(-i\gamma_+)^k}{k!} - (r_0 a')(+\infty, \tau) \right\} d\tau. \end{aligned} \tag{0.5}$$

This integral converges for N large enough and is independent of N provided N is large. The second expression in (0.5) follows from the first one by a formal complex shift by $-i\gamma_+$ in the integrand. Clearly, this shift does not change the value of the integral. Indeed, for a function $b(\tau)$ decreasing rapidly on the line $\Im\tau = \gamma$ we have

$$\int_{-\infty}^{\infty} b(\tau + i\gamma)d\tau = 0,$$

for any $k > 0$, so that

$$\int_{\Im\tau=\gamma} = \int_{-\infty}^{\infty} \sum_{k=0}^{N} b^{(k)}(\tau + i\gamma)\frac{(-i\gamma)^k}{k!}.$$

For $t = -\infty$ we have a similar corner contribution. The full index formula has the form

$$\begin{aligned}
\operatorname{ind} A = {} & \frac{1}{2\pi i}\int_{\mathbb{R}^2} \operatorname{tr}\, (dr_0 + r_0\, da\, r_0) \wedge da \\
& + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \operatorname{tr}\left\{ \sum_{k=0}^{N} (a_+^{-1}a_+')^{(k)}(\tau + i\gamma_+)\frac{(-i\gamma_+)^k}{k!} - (r_0 a')(+\infty,\tau)\right\} d\tau \\
& - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \operatorname{tr}\left\{ \sum_{k=0}^{N} (a_-^{-1}a_-')^{(k)}(\tau - i\gamma_-)\frac{(i\gamma_-)^k}{k!} - (r_0 a')(-\infty,\tau)\right\} d\tau.
\end{aligned} \tag{0.6}$$

The orientation of the plane $\mathbb{R}^2$ is defined by the form $dt\wedge d\tau$. The sign "$-$" in the corner contribution at $t = -\infty$ comes from the change of orientation (the proper orientation of the t-axis near a corner point corresponds to t increasing to $+\infty$ when approaching the corner point). Like the index formulas in [FS96, FST98], this formula has the same drawback: it does not express the index in terms of principal symbols (0.1) only as one could expect.

There are special cases when corner contributions vanish, for example, if $\gamma_+ = 0$ and $a_+(\tau) \equiv a(+\infty,\tau)$. This case was considered in [FS96] for the cone elliptic operators. In general, however, all the contributions are present, so, we will not try to classify these special cases.

Let us describe briefly the contents of the paper. In Section 1 we introduce the Sobolev spaces $H^{s,\delta}(Y)$ on fibers with a family of norms $\|\cdot\|_\tau$ depending on a real or complex parameter τ. They play a distinguished role in our passive approach. We also use them to define the corner Sobolev spaces $H^{s,\delta,\gamma_+,\gamma_-}(M)$ which in our approach are L^2 spaces for scalar functions $\|\widehat{u}(\tau)\|_\tau$. We consider embedding properties for these spaces, especially trace class embeddings.

Section 2 may be viewed as an introduction to our passive approach in the theory of operator-valued symbols. Although we consider here a more simple situation when the fiber manifold Y is smooth, the method remains the same for more general cases when Y itself has singularities. In addition to the simplest case considered in Section 2 we need to define proper fiber norms $\| \cdot \|_\tau$ and investigate fiber-wise properties of the operator-valued symbols. After that the theory goes as in the simplest scalar-valued case. We discuss also a necessary modification of the Fredholm property of elliptic operators using a kernel cut-off procedure.

Section 3 is the most important one. Here we introduce the classes $\Sigma^m(\delta, \delta - l)$ of edge symbols which serve for the definition of corner operators. Starting with the well-known parametrix construction in the edge algebra, we construct a parametrix for a corner elliptic operator and derive a coarse index formula. This is done similarly to the case of a cone [FST99] using our passive approach.

Finally, in Section 4 we transform the coarse index formula to the final form (0.6) following [FS96] and using our passive approach.

As was already mentioned, we combine here the methods of [FS96, FST98, FST99]. Unfortunately, some essential technical changes are required. For the reader convenience we have gathered a necessary auxiliary material in the last Section 5 which may be regarded as an appendix.

1 Sobolev Spaces on the Edged Spindle

We introduce the spaces $H^{s,\delta,\gamma_+,\gamma_-}(M)$ on the edged spindle M represented as a product (0.3). They are similar to the corner and wedge Sobolev spaces introduced by the second author [Sch92]. The parameters s, δ, and $\gamma_\pm$ are called *smoothness*, *the cone weight*, and *corner weights* at $\pm\infty$, respectively, all the parameters being real.

The definition uses essentially the product structure (0.3). We treat a function u on M as a function $u(t)$ on $\mathbb{R}$ with values in a functional space on the fiber Y. Thus, first we need to describe the spaces $H^{s,\delta}(Y)$ on Y which is a manifold with a conical point y_c. A neighborhood of this point may be identified with an *infinite stretched* cone $X^\wedge$ and we will need the Sobolev spaces $K^{s,\delta}(X^\wedge)$ on an infinite cone $X^\wedge$. For the reader's convenience we recall the definition and the properties of these spaces following [FST98], see also [Sch92, ES97].

In the sequel we will use various cut-off functions: $\rho(y)$, $y \in Y$, with compact support or $\omega(t)$, $t \in \mathbb{R}$, with a support in $\mathbb{R}_+$. They usually appear with corresponding *covering functions* $\widetilde{\rho}(y)$, $\widetilde{\omega}(t)$ which are equal identically to 1 on the support of the cut-off function ρ, ω and vanish identically outside a neighborhood of finite radius of $\operatorname{supp} \rho$, $\operatorname{supp} \omega$, respectively. For a given cut-off function the corresponding covering function will be always denoted by tilde.

1.1 The spaces $K^{s,\delta}(X^\wedge)$

By a *stretched cone* $X^\wedge$ with a base manifold X we mean a Cartesian product $\mathbb{R}_+ \times X$ with the action of the group $\mathbb{R}_+$,

$$\lambda\,(r,x) = (\lambda r, x)$$

for $\lambda \in \mathbb{R}_+$ and $(r,x) \in \mathbb{R}_+ \times X$. The base X is supposed to be a smooth compact n-dimensional manifold without boundary. For a coordinate neighborhood $U \subset X$ we denote by $U^\wedge = \mathbb{R}_+ \times U$ the stretched conical neighborhood in $X^\wedge$. We use the notation $V^\Delta \subset \mathbb{R}^{n+1}$ for the geometrical conical neighborhood corresponding to a coordinate neighborhood $V \subset \mathbb{S}^n$ on the unit sphere in $\mathbb{R}^{n+1}$. The group $\mathbb{R}_+$ acts on V^Δ by homotheties. By a conical coordinate diffeomorphism $\chi : U^\wedge \to V^\Delta$ we mean a diffeomorphism which commutes with the action of $\mathbb{R}_+$. The inverse diffeomorphism

$$\chi^{-1} : \; V^\Delta \in \widetilde{x} \mapsto (r,x) \in V^\wedge \tag{1.1}$$

may be thought of as a passage to polar coordinates.

There are several modifications of the Sobolev spaces adopted to the conical structure.

1. The spaces $H^s(X^\wedge)$. Let $C_0^\infty(X^\wedge)$ denote the space of smooth functions with compact supports in an open stretched cone $X^\wedge = \mathbb{R}_+ \times X$. For a function $u \in C_0^\infty(X^\wedge)$, take its push-forward

$$(\chi_* u)\,(\widetilde{x}) = u\left(\chi^{-1}(\widetilde{x})\right)$$

under a conical coordinate diffeomorphism (1.1) and define

$$\|u\|_{H^s(X^\wedge)} = \|\chi_* u\|_{H^s(\mathbb{R}^{n+1})}.$$

The general case may be reduced to the special one considered above by taking a finite coordinate covering U_i of X and a subordinate partition of unity $\rho_i(x)$. For stretched conical neighborhoods $U_i^\wedge$, we take conical coordinate diffeomorphisms $\chi_i \colon U_i^\wedge \to V_i^\wedge$ and set

$$\|u\|^2_{H^s(X^\wedge)} = \sum_i \|\chi_{i*}\rho_i u\|^2_{H^s(\mathbb{R}^{n+1})}. \tag{1.2}$$

The norm (1.2) is independent of the covering, partition of unity and coordinate diffeomorphisms up to equivalence.

2. The weighted spaces $H^{s,\delta}(X^\wedge)$. For a function $u(r,x) \in C_0^\infty(X^\wedge)$ supported in a conical neighborhood $U^\wedge$, set

$$v(z,x) = u(e^{-z}, x)$$

and define a norm

$$\|u\|^2_{H^{s,\delta}(X^\wedge)} = \int_{\mathbb{R}^n} d\xi \int_{\Im\zeta=\delta} |\langle \Re\zeta, \xi\rangle^s\, \widehat{v}(\zeta,\xi)|^2 d\zeta. \tag{1.3}$$

Here $\widehat{v}(\zeta,\xi)$ is the Fourier transform of $v(z,x) \in C_0^\infty(\mathbb{R}\times X)$. It is an entire function in ζ, so the integral (1.3) makes sense. The function $\eta \mapsto \langle\eta\rangle$, for $\eta = (\Re\zeta, \xi)$, is a *smooth norm function*, that is a smooth function satisfying $\langle\eta\rangle \geq 1$ everywhere and equal to $|\eta|$ for $|\eta| \geq C > 0$. Its concrete form does not affect the norm (1.3) up to equivalence. Clearly, we have

$$\langle\eta\rangle \sim (1+|\eta|^2)^{1/2}$$

where $\sim$ means that two-sided estimates hold

$$0 < C_1 \leq \frac{\langle\eta\rangle}{(1+|\eta|^2)^{1/2}} \leq C_2.$$

The general case may be reduced to this special one by means of a partition of unity similarly to (1.2).

3. The cone spaces $K^{s,\delta}(X^\wedge)$. Take a cut-off function $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1 near $r=0$ and set

$$\|u\|^2_{K^{s,\delta}(X^\wedge)} = \|\omega(r)u\|^2_{H^{s,\delta}(X^\wedge)} + \|(1-\omega(r))u\|^2_{H^s(X^\wedge)}. \tag{1.4}$$

So, the space $K^{s,\delta}$ is a "mixture" of the weighted space $H^{s,\delta}$ near $r=0$ and the usual Sobolev space $H^s(X^\wedge)$ near $r=+\infty$. The choice of $\omega(r)$ does not affect the norm (1.4) up to equivalence.

Let us recall some properties of these spaces.

1. For any fixed cut-off function $\omega_0(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ equal to 1 near $r=0$, we have four bounded multiplication operators

$$\begin{array}{rcl} K^{s,\delta} & \xrightarrow{\omega_0} & H^{s,\delta}, \\ H^{s,\delta} & \xrightarrow{\omega_0} & K^{s,\delta}; \end{array} \tag{1.5}$$

$$\begin{array}{rcl} K^{s,\delta} & \xrightarrow{1-\omega_0} & H^{s}, \\ H^{s} & \xrightarrow{1-\omega_0} & K^{s,\delta}. \end{array} \tag{1.6}$$

2. The group $\mathbb{R}_+$ acts on any of the spaces $H^s(X^\wedge)$, $H^{s,\delta}(X^\wedge)$, $K^{s,\delta}(X^\wedge)$ since it acts on the cone $X^\wedge$. It is convenient to modify this action by a factor, namely

$$\kappa_\lambda : \; u(r,x) \to \lambda^{(n+1)/2}\, u(\lambda r, x). \tag{1.7}$$

A general result concerning strongly continuous actions of $\mathbb{R}_+$ on Banach spaces consists in the following estimate of the norm of κ_λ:

$$\|\kappa_\lambda\| \leq C \left(\max\left(\lambda, \frac{1}{\lambda}\right)\right)^k \tag{1.8}$$

for some $C, k > 0$ (see [Hir90]).

3. There are continuous embeddings

$$\begin{array}{rcl} H^{s_1} & \hookrightarrow & H^{s_2}, \\ H^{s_1,\delta_1} & \hookrightarrow & H^{s_2,\delta_2}, \\ K^{s_1,\delta_1} & \hookrightarrow & K^{s_2,\delta_2} \end{array}$$

for $s_1 \geq s_2$, $\delta_1 \geq \delta_2$.

The following lemma is quite similar to [FST98, Lemma 1.2].

Lemma 1.1 *Let $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ satisfy $\omega \equiv 1$ near $r = 0$. Let $M(\lambda)$, $\lambda \geq 1$, be a multiplication operator followed by embedding*

$$M(\lambda): \ K^{s,\delta}(X^\wedge) \stackrel{\omega(r/\lambda)}{\longrightarrow} K^{s-N,\delta-\varepsilon}(X^\wedge)$$

where $N, \varepsilon \geq 0$. If $N > (n+1)/2$ and $\varepsilon > 0$, then $M(\lambda)$ is a Hilbert-Schmidt operator and the following estimate holds for its Hilbert-Schmidt norm

$$\|M(\lambda)\|_{\mathrm{HS}} \leq C \ \lambda^{(n+1)/2}.$$

For the proof we refer the reader to [FST98, Lemma 1.1, Lemma 1.2].

1.2 The spaces $H^{s,\delta}(Y)$

Our next goal is to define weighted Sobolev spaces on a fiber Y which is an $(n+1)$-dimensional manifold with one conical point y_c. We define a family of norms $\|\cdot\|_\tau$ on these spaces depending on a parameter $\tau \in \mathbb{R}$ (of course, these norms depend on s and δ but we will not indicate this dependence explicitly). We also admit complex values of τ setting by definition $\|u\|_\tau := \|u\|_{\Re\tau}$.

Fix a finite covering $\{U_i\}$, U_c of Y where U_i are coordinate neighborhoods not containing the conical point y_c, while U_c is a neighborhood of y_c. We identify U_c with a neighborhood of the vertex of the cone $X^\wedge$, so that $y \in U_c$ is represented as $(r, x) \in X^\wedge$ with $r < 1$. Let $\{\rho_i(y)\}$, $\rho_c(y)$ be a subordinate partition of unity. We decompose any function $v(y) \in C_0^\infty(Y \setminus \{y_c\})$ into the sum

$$\begin{array}{rcl} v & = & \sum_i v_i + v_c \\ & = & \sum_i \rho_i v + \rho_c v \end{array} \tag{1.9}$$

and define norms $\|\cdot\|_\tau$ for each summand. For the smooth neighborhoods U_i, we set

$$\|v_i\|_\tau^2 = \int_{\mathbb{R}^{n+1}} |\langle\tau, \eta\rangle^s \, \widehat{v}_i(\eta)|^2 \, d\eta, \tag{1.10}$$

while for the conical neighborhood U_c

$$\|v_c\|_\tau = \langle\tau\rangle^s \, \|\kappa_{\langle\tau\rangle}^{-1} v_c\|_{K^{s,\delta}(X^\wedge)}, \tag{1.11}$$

with κ_λ defined by (1.7). Finally, using decomposition (1.9), define

$$\|v\|_\tau^2 = \sum_i \|v_i\|_\tau^2 + \|v_c\|_\tau^2. \tag{1.12}$$

We will often need the space $H^{s,\delta}(Y) \oplus \mathbb{C}^N$ as a fiber space rather than $H^{s,\delta}(Y)$ itself. In this case we define

$$\|v\|_\tau^2 = \|v_1(y)\|_\tau^2 + \|v_2\|_\tau^2. \tag{1.13}$$

Here $v_1(y) \in H^{s,\delta}(Y)$ and $v_2 \in \mathbb{C}^N$ are direct summands, the norm $\|v_1(y)\|_\tau$ is defined by (1.12) while the norm of the vector $v_2 \in \mathbb{C}^N$ is

$$\|v_2\|_\tau := \langle\tau\rangle^s \, \|v_2\|_{\mathbb{C}^N}.$$

We will say that two families of norms $\|\cdot\|_{1,\tau}$ and $\|\cdot\|_{2,\tau}$ are *uniformly equivalent* if for the ratio the following two-sided uniform estimates hold

$$0 < C_1 \leq \frac{\|v\|_{1,\tau}}{\|v\|_{2,\tau}} \leq C_2$$

for any $v \in C_0^\infty(Y \setminus \{y_c\})$ and $\tau \in \mathbb{R}$.

Lemma 1.2 *The norm* (1.12) *is correctly defined up to uniform equivalence.*

Proof. It is sufficient to show that for a function v with a support in $U_i \cap U_j$ or in $U_i \cap U_c$ different expressions (1.10), (1.11) give uniformly equivalent norms. We will consider the only non-trivial case when the support of v belongs to $U_i \cap U_c$. Using conical coordinates $y = (r, x) \in X^\wedge$, we have

$$\kappa_{\langle\tau\rangle}^{-1} v(y) = \langle\tau\rangle^{-(n+1)/2} \, v\left(\frac{r}{\langle\tau\rangle}, x\right).$$

Since $v(r,x)$ vanishes for $0 < r < r_0$, the function $\kappa_{\langle\tau\rangle}^{-1} v(y)$ also vanishes for all $0 < r < r_0$ and $\tau \in \mathbb{R}$. Thus, in virtue of (1.6) we have

$$\|\langle\tau\rangle^s \, \kappa_{\langle\tau\rangle}^{-1} v\|_{K^{s,\delta}(X^\wedge)} \sim \|\langle\tau\rangle^s \kappa_{\langle\tau\rangle}^{-1} v\|_{H^s(X^\wedge)}$$

where $\sim$ means uniform equivalence. Now, the Fourier transform of the function $\langle\tau\rangle^{-(n+1)/2} \, v(y/\langle\tau\rangle)$ is equal to

$$\langle\tau\rangle^{(n+1)/2} \, \widehat{v}(\langle\tau\rangle\eta),$$

so the norm (1.11) may be written as

$$\int_{\mathbb{R}^{n+1}} |\langle\tau\rangle^s \langle\eta\rangle^s \, \widehat{v}(\langle\tau\rangle\eta)|^2 \, \langle\tau\rangle^{n+1} d\eta = \int_{\mathbb{R}^{n+1}} |\langle\tau\rangle^s \left\langle \frac{\eta}{\langle\tau\rangle} \right\rangle^s \widehat{v}(\eta)|^2 d\eta$$

up to uniform equivalence. Next, by (1.11)

$$\begin{aligned}
\langle\tau\rangle \left\langle \frac{\eta}{\langle\tau\rangle} \right\rangle &\sim (1+\tau^2)^{1/2} \left(1 + \left|\frac{\eta}{(1+\tau^2)^{1/2}}\right|^2\right)^{1/2} \\
&\sim (1+\tau^2+|\eta|^2)^{1/2} \\
&\sim \langle\tau,\eta\rangle,
\end{aligned} \tag{1.14}$$

giving the norm

$$\int_{\mathbb{R}^{n+1}} |\langle \tau, \eta \rangle^s \, \widehat{v}(\eta)|^2 \, d\eta$$

which coincides with (1.10).

$\square$

For different values of τ the norms $\|\cdot\|_\tau$ are equivalent but this equivalence is not uniform.

Lemma 1.3 *There exist constants $C, q > 0$ (depending on s, δ) such that*

$$\frac{\|v\|_{\tau_1}}{\|v\|_{\tau_2}} \leq C \, \langle \tau_1 - \tau_2 \rangle^q. \tag{1.15}$$

Proof. Using the uniform equivalence (1.14), we have for $|\tau_1| \geq |\tau_2|$

$$\frac{\langle \tau_1, \eta \rangle}{\langle \tau_2, \eta \rangle} \sim \left(\frac{1 + \tau_1^2 + |\eta|^2}{1 + \tau_2^2 + |\eta|^2} \right)^{1/2} \leq \left(\frac{1 + \tau_1^2}{1 + \tau_2^2} \right)^{1/2} \sim \frac{\langle \tau_1 \rangle}{\langle \tau_2 \rangle},$$

implying for the norms (1.10)

$$\frac{\|v_i\|_{\tau_1}}{\|v_i\|_{\tau_2}} \leq \left(\frac{\langle \tau_1 \rangle}{\langle \tau_2 \rangle} \right)^{|s|}.$$

For the norm (1.11) we make use of (1.8), so that

$$\frac{\|v_c\|_{\tau_1}}{\|v_c\|_{\tau_2}} \sim C \, \left(\frac{\langle \tau_1 \rangle}{\langle \tau_2 \rangle} \right)^{k+|s|}$$

with C, k from (1.8). This gives the estimate

$$\frac{\|v\|_{\tau_1}}{\|v\|_{\tau_2}} \leq C \left(\max \left(\frac{\langle \tau_1 \rangle}{\langle \tau_2 \rangle}, \frac{\langle \tau_2 \rangle}{\langle \tau_1 \rangle} \right) \right)^q$$

with $q = k + |s|$. This estimate implies (1.15) in virtue of Peetre's inequality

$$\frac{\langle \tau_1 \rangle^\alpha}{\langle \tau_2 \rangle^\alpha} \leq C \, \langle \tau_1 - \tau_2 \rangle^{|\alpha|}.$$

$\square$

Definition 1.4 *The space $H^{s,\delta}(Y)$ is a completion of $C_0^\infty(Y \setminus \{y_c\})$ with respect to any norm $\|\cdot\|_\tau$, for a fixed $\tau \in \mathbb{R}$.*

Clearly, we have an embedding

$$\imath : \; H^{s,\delta}(Y) \hookrightarrow H^{s-N,\delta-\varepsilon}(Y) \tag{1.16}$$

for each $N, \varepsilon \geq 0$. Supposing that both spaces are equipped with the norms $\|\cdot\|_\tau$ with the same $\tau \in \mathbb{R}$, we come to the following lemma.

Lemma 1.5 *If $N > (n+1)/2$ and $\varepsilon > 0$, the embedding operator $\imath$ belongs to the Hilbert-Schmidt class and*

$$\|\imath\|_{\mathrm{HS}} \leq C \, \langle \tau \rangle^{(n+1)/2-N}. \tag{1.17}$$

Proof. It is sufficient to prove the estimate (1.17) for each multiplication operator $\rho_i(y)$ or $\rho_c(y)$ followed by embedding (1.16). Denoting $v_i = \rho_i v$ and assuming that $\operatorname{supp} v \subset U_i$, we have for a smooth coordinate neighborhood U_i

$$\widehat{v}_i(\eta) = \int_{\mathbb{R}^{n+1}} \widehat{\rho}_i(\eta - \eta_1) \, \widehat{v}(\eta_1) \, d\eta_1$$

or

$$\langle \tau, \eta \rangle^{s-N} \, \widehat{v}_i(\eta) = \int_{\mathbb{R}^{n+1}} \frac{\langle \tau, \eta \rangle^{s-N}}{\langle \tau, \eta_1 \rangle^{s}} \widehat{\rho}_i(\eta - \eta_1) \, \langle \tau, \eta_1 \rangle^{s} \, \widehat{v}(\eta_1) d\eta_1.$$

Thus, we need to estimate the L^2 norm of the kernel

$$\begin{aligned} K(\eta, \eta_1) &= \frac{\langle \tau, \eta \rangle^{s-N}}{\langle \tau, \eta_1 \rangle^{s-N}} \, \widehat{\rho}_i(\eta - \eta_1) \, \langle \tau, \eta_1 \rangle^{-N} \\ &= O(\langle \eta - \eta_1 \rangle^{-\infty}) \, \langle \tau, \eta_1 \rangle^{-N}. \end{aligned}$$

We have made use of the Peetre inequality

$$\frac{\langle \tau, \eta \rangle^{s-N}}{\langle \tau, \eta_1 \rangle^{s-N}} \leq C \, \langle \eta_1 - \eta \rangle^{|s|+N}$$

with C independent of τ. Multiplying by a rapidly decreasing function $\widehat{\rho}_1(\eta - \eta_1)$, we obtain for the product an estimate $O(\langle \eta - \eta_1 \rangle^{-\infty})$. Thus,

$$\begin{aligned} \|\imath \rho_i\|^2_{\mathrm{HS}} &\leq C \int_{\mathbb{R}^{n+1}} \langle \tau, \eta_1 \rangle^{-2N} \, d\eta_1 \\ &\leq C \, \langle \tau \rangle^{-2N+(n+1)} \end{aligned}$$

giving the desired estimate (1.17).

Now, for the conical neighborhood U_c, we write

$$v_c(y) = \rho_c(y) v(y)$$

assuming that all the functions are defined on the infinite cone $X^\wedge$. Applying to both sides the operator $\kappa^{-1}_{\langle \tau \rangle}$ and multiplying by $\langle \tau \rangle^{s-N}$, we rewrite this equality in the form

$$\langle \tau \rangle^{s-N} \kappa^{-1}_{\langle \tau \rangle} \, v_c = \langle \tau \rangle^{-N} \rho_c \left(\frac{y}{\langle \tau \rangle} \right) \langle \tau \rangle^{s} \kappa^{-1}_{\langle \tau \rangle} v.$$

We need to estimate the Hilbert-Schmidt norm of the multiplication operator

$$\langle \tau \rangle^{-N} \rho_c \left(\frac{y}{\langle \tau \rangle} \right) : \; K^{s,\delta}(X^\wedge) \to K^{s-N,\delta-\varepsilon}(X^\wedge).$$

This may be done by Lemma 1.2, and we come to the desired estimate (1.17).

□

Corollary 1.6 *If $N > n+1$ and $\varepsilon > 0$ the embedding $\imath$ is a trace class operator.*

Proof. Decompose $\imath$ into the product

$$H^{s,\delta}(Y) \xrightarrow{\imath_1} H^{s-N/2,\delta-\varepsilon/2}(Y) \xrightarrow{\imath_2} H^{s-N,\delta-\varepsilon}(Y),$$

both embeddings $\imath_1$, $\imath_2$ being Hilbert-Schmidt operators by Lemma 1.5.

□

1.3 The spaces $H^{s,\delta,\gamma_+,\gamma_-}(M)$

We are now ready to define weighted Sobolev spaces on the edged spindle M. Using a product structure (0.3), we consider a function

$$u(t,y) \in C_0^\infty(\overset{\circ}{M})$$

as a function on the real axis $\mathbb{R}$ with values in $C_0^\infty(Y \setminus \{y_c\})$. For $b > a > 0$, consider a covering

$$\begin{array}{rcl} U_- & = & (-\infty, -a), \\ U_0 & = & (-b, b), \\ U_+ & = & (a, +\infty) \end{array}$$

and subordinate partition of unity $\rho_-(t)$, $\rho_0(t)$, $\rho_+(t)$. Given a function $u(t) \in C_0^\infty(\mathbb{R}, C_0^\infty(Y \setminus \{y_c\}))$, decompose it into a sum

$$\begin{array}{rcl} u(t) & = & u_+(t) + u_0(t) + u_-(t) \\ & = & \rho_+(t)u(t) + \rho_0(t)u(t) + \rho_-(t)u(t) \end{array}$$

and define a norm

$$\|u\|^2 = \int_{-\infty}^{\infty} \|\widehat{u}_0(\tau)\|_\tau^2 \, d\tau + \int_{-\infty}^{\infty} \|\widehat{u}_+(\tau + i\gamma_+)\|_\tau^2 \, d\tau + \int_{-\infty}^{\infty} \|\widehat{u}_-(\tau - i\gamma_-)\|_\tau^2 \, d\tau. \tag{1.18}$$

Here $\widehat{u}_0$, $\widehat{u}_+$, $\widehat{u}_-$ mean the Fourier transforms with respect to t. They are entire functions since $u \in C_0^\infty$, so the complex shifts by $\pm i\gamma_\pm$ make sense. For a fixed τ, the function $\widehat{u}_0(\tau, y)$ (as well as $\widehat{u}_\pm$) is considered as an element of $H^{s,\delta}(Y)$ with the norm $\|\cdot\|_\tau$ defined by (1.12) or as an element of $H^{s,\delta}(Y) \oplus \mathbb{C}^N$ with the norm (1.13).

The norm (1.18) is a "mixture" of four different types of norms. For functions with supports away from the edge and corners, it coincides with the usual Sobolev norm H^s on the smooth part of M. For functions with supports near the edge but away from the corners, we recover the wedge Sobolev spaces [ES97]. For functions with supports away from the edge but near the corner, we recover the weighted Sobolev spaces $H^{s,\gamma_\pm}$ as in the case of the conical point $c_\pm$. Finally, for functions with supports near the edge and the corner, we obtain corner Sobolev spaces (see [ST00]).

Clearly, we have an embedding

$$\imath : \; H^{s_1,\delta_1,\gamma_{1,+},\gamma_{1,-}}(M) \hookrightarrow H^{s_2,\delta_2,\gamma_{2,+},\gamma_{2,-}}(M) \tag{1.19}$$

for $s_1 \geq s_2$, $\delta_1 \geq \delta_2$, $\gamma_{1,+} \geq \gamma_{2,+}$, $\gamma_{1,-} \geq \gamma_{2,-}$.

Lemma 1.7 *If $s_1 > s_2 + (n+2)/2$, $\delta_1 > \delta_2$, $\gamma_{1,+} > \gamma_{2,+}$, $\gamma_{1,-} > \gamma_{2,-}$ then $\imath$ is a Hilbert-Schmidt operator. If in addition $s_1 > s_2 + (n+2)$ then $\imath$ is a trace class operator.*

Proof. We need to show that the three multiplication operators by $\rho_0(t)$, $\rho_+(t)$, $\rho_-(t)$ are Hilbert-Schmidt ones in the corresponding spaces. Consider first the multiplication by $\rho_0(t)$, that is

$$u_0(t) = \rho_0(t)\, u(t),$$

or

$$\widehat{u}_0(\tau) = \int_{-\infty}^{\infty} \widehat{\rho}_0(\tau - \tau_1)\widehat{u}(\tau_1)\, d\tau_1. \tag{1.20}$$

The function $\widehat{\rho}_0(\tau - \tau_1)$ is an entire function rapidly decreasing on any horizontal line, that is

$$\widehat{\rho}_0(\tau - \tau_1) = O\left(\langle \tau - \tau_1 \rangle^{-\infty}\right) \tag{1.21}$$

for $\Im\tau = \Im\tau_1 = \mathrm{const}$.

The value $\widehat{u}(\tau_1)$ is an element of the space $H^{s,\delta}(Y)$ equipped with the norm $\| \cdot \|_{\tau_1}$, while the value of the integrand is regarded as an element of $H^{s-N,\delta-\varepsilon}(Y)$ with the norm $\| \cdot \|_{\tau}$, the latter obtained from the former as a sequence of embedding operators

$$H^{s,\delta}(Y, \| \cdot \|_{\tau_1}) \xrightarrow{\imath_1} H^{s,\delta}(Y, \| \cdot \|_{\tau}) \xrightarrow{\imath_2} H^{s-N,\delta-\varepsilon}(Y, \| \cdot \|_{\tau})$$

with subsequent multiplication by a constant (1.21). The operator $\imath_1$ is bounded with a norm estimate

$$\|\imath_1\| \leq C\, \langle \tau - \tau_1 \rangle^k$$

(cf. Lemma 1.3), $\imath_2$ is a Hilbert-Schmidt operator and

$$\|\imath_2\|_{\mathrm{HS}} \leq C\, \langle \tau \rangle^{N-(n+1)/2}$$

(cf. Lemma 1.5). Thus, we have

$$\|\widehat{\rho}_0(\tau - \tau_1)\, \imath_2 \imath_1\|_{\mathrm{HS}} \leq \langle \tau \rangle^{-N+(n+1)/2}\, O\left(\langle \tau - \tau_1 \rangle^{-\infty}\right).$$

Then the square of the Hilbert-Schmidt norm of the integral operator (1.20) is equal to

$$\int \|\widehat{\rho}_0(\tau - \tau_1)\imath_2\imath_1\|^2_{\mathrm{HS}}\, d\tau d\tau_1 \leq C \int_{-\infty}^{\infty} \langle \tau \rangle^{-2N+n+1}\, d\tau,$$

the integral converges for $2N > n+2$.

Consider now the summand

$$u_+(t) = \rho_+(t)u(t).$$

We would like to rewrite this equation in terms of the Fourier transform similarly to (1.20). The difference is that τ_1 varies along the horizontal line $\Im\tau_1 = \gamma_+$ while τ belongs to the line $\Im\tau = \gamma_+ - \varepsilon$, where $\varepsilon > 0$. The function

$$\widehat{\rho}_+(\tau) = \int_{-\infty}^{\infty} e^{-i\tau t}\rho_+(t)\,dt$$

is defined for $\Im\tau < 0$. Integrating by parts we obtain

$$\widehat{\rho}_+ = \frac{1}{i\tau}\int_{-\infty}^{\infty} e^{-i\tau t}\rho_+'(t)\,dt.$$

The integral on the right-hand side is an entire function rapidly decreasing on any horizontal line $\Im\tau = \gamma$, so that $\widehat{\rho}_+(\tau)$ has the only first order pole $\tau = 0$. The relation (1.20) becomes now

$$\widehat{u}_0(\tau) = \int_{\Im\tau_1=\gamma_+} \widehat{\rho}_+(\tau-\tau_1)\,\widehat{u}(\tau_1)\,d\tau_1$$

where τ belongs to the line $\Im\tau = \gamma_+ - \varepsilon$ with fixed $\varepsilon > 0$. Again we have

$$\widehat{\rho}_+(\tau-\tau_1) = O\left(\langle\Re\tau - \Re\tau_1\rangle^{-\infty}\right),$$

so we can repeat the previous arguments to obtain that the operator $u \mapsto u_+$ is a Hilbert-Schmidt one in the corresponding spaces

$$H^{s,\delta,\gamma_+,\gamma_-}(M) \to H^{s,\delta,\gamma_+-\varepsilon,\gamma_-}(M).$$

The case of $u_- = \rho_- u$ is treated similarly.

$\square$

Corollary 1.8 *If* $s_1 > s_2 + (n+2)$, $\delta_1 > \delta_2$ *and* $\gamma_{1,+} > \gamma_{2,+}$, $\gamma_{1,-} > \gamma_{2,-}$, *then the embedding* (1.19) *belongs to the trace class.*

Proof. Taking

$$\begin{aligned} s' &= (s_1+s_2)/2, \\ \delta' &= (\delta_1+\delta_2)/2, \\ \gamma'_\pm &= (\gamma_{1,\pm}+\gamma_{2,\pm})/2, \end{aligned}$$

we represent the embedding $\imath$ as a product of two Hilbert-Schmidt embeddings

$$H^{s_1,\delta_1,\gamma_{1,+},\gamma_{1,-}}(M) \to H^{s',\delta',\gamma'_+,\gamma'_-}(M) \to H^{s_2,\delta_2,\gamma_{2,+},\gamma_{2,-}}(M).$$

$\square$

We will also need a modification of the spaces $H^{s,\delta,\gamma_+,\gamma_-}(M)$ obtained by replacing the fiber space $H^{s,\delta}(Y)$ by $H^{s,\delta}(Y) \oplus \mathbb{C}^N$ with the families of norms (1.13). As a result we obtain a direct sum $H^{s,\delta,\gamma_+,\gamma_-}(M) \oplus H^{s,\gamma_+,\gamma_-}(\mathbb{R})$ where the summand $H^{s,\gamma_+,\gamma_-}(\mathbb{R})$ is a Sobolev space on the edge $\{y_c\} \times \mathbb{R}$ with weights γ_+, γ_- at $t = \pm\infty$. The norm in the direct sum is defined again by (1.18) with the fiber norm $\| \cdot \|_\tau$, cf. (1.13). We preserve the notation $H^{s,\delta,\gamma_+,\gamma_-}(M)$ for this modified space.

2 Operators on a Smooth Spindle

Before considering a pseudodifferential operator algebra on an edged spindle we give a brief review of the algebra on a smooth spindle. It is, of course, a particular case of a cone algebra considered in [FST99], but here we demonstrate a passive approach based on families of norms $\| \cdot \|_\tau$ on fibers. So, the cone algebra on a spindle serves here as a simplified model of the corner algebra on the edged spindle considered further in Section 3. We also discuss how to get rid of the holomorphy condition which for the corner algebra seems too restrictive.

In this section we deal with a smooth manifold

$$M = Y \times \mathbb{R} \tag{2.1}$$

where the fiber Y is a smooth compact manifold of dimension $n+1$. The product structure makes it natural to use operator-valued symbols, that is functions $a(t,\tau)$ on the plane $(t,\tau) \in \mathbb{R}^2$ whose values are pseudodifferential operators on the fiber Y. It is not so clear, however, how to introduce proper classes of operator-valued symbols reflecting the most properties of the classical symbol classes $\mathcal{S}^m(M)$ on the product manifold M. We mention two possibilities:

- parameter-dependent theory when τ considered as a parameter is included into symbol estimates on Y;

- one considers a group action on fibers (e.g., an action (1.7)) and symbols estimates depending on this action.

We propose another approach considering the norms on fiber spaces $H^s(Y)$ varying with τ, the symbol estimates involve these families of norms. So, the symbol remains unchanged, only the norms vary. That is why this approach is called passive in contrast to the active one where the norm is constant while the symbol is transformed (e.g., by the group action). Before passing to precise definitions let us introduce some general notation concerning ΨDO's and their symbols.

We will consider operator-valued symbols on a line $t \in \mathbb{R}$, that is functions $a(t,\tau)$, $(t,\tau) \in \mathbb{R}^2$, whose values are ΨDO's on Y. Treating a function

$u \in C_0^\infty(M)$ as a function $u(t)$ on $\mathbb{R}$ with values in $C^\infty(Y)$, we define a ΨDO $\mathrm{Op}(a)$ in a standard way

$$\mathrm{Op}(a)u\,(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\tau} a(t,\tau)\widehat{u}(\tau)\,d\tau.$$

Here the integration line is the real axis $\Im\tau = 0$. We also will need ΨDO's with a complex integration line $\Im\tau = \gamma$, the so-called weight line

$$\mathrm{Op}^\gamma(a)u\,(t) = \frac{1}{2\pi}\int_{\Im\tau=\gamma} e^{it\tau} a(t,\tau)\widehat{u}(\tau)\,d\tau. \tag{2.2}$$

Clearly, the symbol is defined now for $t \in \mathbb{R}$ and $\Im\tau = \gamma$. As for the Fourier transform $\widehat{u}(\tau)$, it is an entire function since $u(t) \in C_0^\infty(\mathbb{R})$. Directly from definitions the relation follows

$$\mathrm{Op}^\gamma(a(t,\tau)) = e^{-\gamma t}\,\mathrm{Op}(a(t,\tau+i\gamma))\,e^{\gamma t}. \tag{2.3}$$

For two operator-valued symbols $a(t,\tau)$ and $b(t,\tau)$ defined on the same weight line $\Im\tau = \gamma$, we define their *Leibniz product* to be a formal series

$$a \circ b = \sum_{k=0}^{\infty} \frac{(-ih)^k}{k!}\,\partial_\tau^k a\,\partial_t^k b. \tag{2.4}$$

Here h is a formal parameter, its powers serve to order the terms of the series. The Leibniz product extends by linearity to formal power series in h whose coefficients are operator-valued symbols, the so-called *formal symbols*. One easily checks the associativity of the Leibniz product. We use the notation

$$a \circ b\,|_N = \sum_{k=0}^{N-1} \frac{(-i)^k}{k!}\,\partial_\tau^k a\,\partial_t^k b$$

for partial sums of the series (2.4) at $h = 1$.

A funny relation arises if we take the Leibniz product of symbols in (2.3),

$$e^{-\gamma t} \circ a(t,\tau+i\gamma) \circ e^{\gamma t} = \sum_{k=0}^{\infty} \partial_\tau^k a(t,\tau+i\gamma)\,\frac{(-ih\gamma)^k}{k!}.$$

One recognizes the formal Taylor expansion for $a(t,\tau+i\gamma-ih\gamma)$ in powers of $-ih\gamma$. We will also use an abbreviation $(a(\tau))_\gamma$ for such a formal complex shift $a(t,\tau-ih\gamma)$. Were a an entire function in τ the series $(a(\tau))_\gamma$ would be convergent at $h = 1$ to $a(t,\tau-i\gamma)$. Although it is a formal series, its partial sums

$$(a(t,\tau))_\gamma\,|_N = \sum_{k=0}^{N-1} \partial_\tau^k a(t,\tau)\,\frac{(-i\gamma)^k}{k!} \tag{2.5}$$

are useful when comparing ΨDO's with different weight lines.

2.1 The cone algebra

We are now going to introduce the symbol classes Σ^m of operator-valued symbols. The fiber spaces $H^s(Y)$ are the usual Sobolev spaces on a smooth compact manifold Y, but we equip them with a family of norms $\|\cdot\|_\tau$. As in Subsection 1.2 we take a fixed coordinate covering U_i and a subordinate partition of unity $\rho_i(y)$ (now it does not contain a singular chart U_c) and set

$$\|v\|_\tau^2 = \sum_i \|\rho_i v\|_\tau^2 \tag{2.6}$$

with the norm $\|v_i\|_\tau = \|\rho_i v\|_\tau$ defined by (1.10). Having the norms (2.6), we can define the spaces $H^{s,\gamma_+,\gamma_-}(M)$ on the spindle (2.1) similarly to Subsection 1.3 with the norm given by (1.18).

Lemma 2.1 *There is an embedding*

$$\imath : \ H^{s_1}(Y) \hookrightarrow H^{s_2}(Y)$$

if $s_1 \geq s_2$. The norm of this embedding fulfills an estimate

$$\|\imath\|_\tau \leq C \langle\tau\rangle^{s_2-s_1}$$

where the subscript τ means that both spaces $H^{s_1}(Y)$ and $H^{s_2}(Y)$ are equipped with the norms $\|\cdot\|_\tau$ with the same τ.

Proof. For a function v_i supported in the coordinate chart U_i we have

$$\begin{aligned} \|v_i\|_{H^{s_2},\tau}^2 &= \int |\langle\tau,\eta\rangle^{s_2}\, \widehat{v}(\eta)|^2\, d\eta \\ &= \int |\langle\tau,\eta\rangle^{s_1}\, \widehat{v}(\eta)|^2\, \langle\tau,\eta\rangle^{2(s_2-s_1)}\, d\eta. \end{aligned}$$

Now, taking into account that $s_2 - s_1 \leq 0$, we obtain

$$\begin{aligned} \langle\tau,\eta\rangle^{s_2-s_1} &\sim (1+\tau^2+|\eta|^2)^{(s_2-s_1)/2} \\ &\leq (1+\tau^2)^{(s_2-s_1)/2} \\ &\sim \langle\tau\rangle^{s_2-s_1}, \end{aligned}$$

so that

$$\|v_i\|_{H^{s_2},\tau}^2 \leq C\, \langle\tau\rangle^{2(s_2-s_1)}\, \|v_i\|_{H^{s_1},\tau}^2$$

proving the lemma.

□

Definition 2.2 *A function $a(t,\tau)$ on $\mathbb{R}^2$ whose values are pseudodifferential operators on Y is called an operator-valued symbol of order m (notation: $a \in \Sigma^m = \Sigma^m(\mathbb{R}^2)$) if for any integers $\alpha, \beta \geq 0$ and $s \in \mathbb{R}$ the operators*

$$\partial_t^\alpha \partial_\tau^\beta a(t,\tau) : \ H^s(Y) \to H^{s-m+\beta}(Y) \tag{2.7}$$

are bounded uniformly with respect to τ, that is

$$\|\partial_t^\alpha \partial_\tau^\beta a(t,\tau)\|_\tau \le C_{\alpha,\beta} \tag{2.8}$$

with $C_{\alpha,\beta}$ independent of τ.

As above, the subscript τ in (2.8) means that both spaces in (2.7) are equipped with the norm $\|\cdot\|_\tau$ with the same τ. This convention will be used from now on unless the contrary is specified. The definition remains meaningful if τ varies along the complex weight line $\Im\tau = \gamma$.

First let us discuss trivial properties of the symbols $a \in \Sigma^m$ following directly from the definition and Lemma 2.1. It is evident that the differentiation with respect to t does not change the class Σ^m while $\partial_\tau^k a \in \Sigma^{m+k}$.

The variables t, τ may be multidimensional, that is $(t,\tau) \in \mathbb{R}^{2n}$. The definition remains meaningful for the special case when Y is a point and $H^s(Y) \cong \mathbb{C}^N$ if we define the norm $\|v\|_\tau$ for $v \in \mathbb{C}^N$ by

$$\|v\|_\tau = \langle\tau\rangle^s \|v\|_{\mathbb{C}^N}.$$

In this latter case our definition gives the usual class $\mathcal{S}^m$ of matrix-valued symbols.

On the other hand, a classical pseudodifferential operator $a(t,\tau)$ on Y of order m with a parameter τ (in the sense of the parameter-dependent theory of ΨDO's) defines an operator-valued symbol from Σ^m. More precisely, let Q denote a fixed *quantization procedure* on the manifold Y. This is a way to construct a ΨDO $Q(a(y,\eta))$ on Y starting with a smooth function $a(y,\eta) \in C^\infty(T^*Y)$ which behaves regularly for large η. If $Y = \mathbb{R}^{n+1}$ then there is a standard construction $Q(a) = \operatorname{Op}(a)$. In general, we can use a coordinate covering U_i, subordinate partition of unity $\rho_i(y)$, and "covering" functions $\widetilde{\rho}_i(y)$ defining

$$Q(a) = \sum_i \rho_i(y) \operatorname{Op}_i(a(y,\eta))\, \widetilde{\rho}_i(y) \tag{2.9}$$

where Op_i means a standard ΨDO in local coordinates on U_i.

Having fixed a quantization map (2.9), we will consider operator-valued symbols $a(t,\tau)$ of the form

$$a(t,\tau) = Q(b(t,y,\tau,\eta)) \tag{2.10}$$

where $b(t,y,\tau,\eta) \in \mathcal{S}^m_{\mathrm{cl}}$ is a classical symbol of order m on the manifold $M = \mathbb{R}\times Y$. A standard consequence of the theory of ΨDO's is the following proposition.

Proposition 2.3 *The operator-valued symbol* (2.10) *belongs to* Σ^m.

The following estimates playing an important role in parameter-dependent theory are simple consequences of Lemma 2.1.

Corollary 2.4 *Let $a(t,\tau) \in \Sigma^m$. If $m-\beta \le 0$ then the operator*

$$\partial_t^\alpha \partial_\tau^\beta a(t,\tau) : \; H^s(Y) \to H^s(Y)$$

is bounded and its norm satisfies an estimate

$$\|\partial_t^\alpha \partial_\tau^\beta a(t,\tau)\|_\tau \le C \, \langle\tau\rangle^{m-\beta}.$$

The same operator considered in the spaces

$$\partial_t^\alpha \partial_\tau^\beta a(t,\tau) : \; H^s(Y) \to H^{s-m}(Y)$$

satisfies an estimate

$$\|\partial_t^\alpha \partial_\tau^\beta a(t,\tau)\|_\tau \le C \, \langle\tau\rangle^{-\beta}.$$

All these simple properties show that parameter-dependent norms $\|\cdot\|_\tau$ are appropriate tools for operator-valued symbols.

We will consider ΨDO's of the form

$$A = \rho(t) \, \mathrm{Op}(a(t,\tau)) \, \widetilde{\rho}(t) \tag{2.11}$$

or

$$A = \rho_+(t) \, \mathrm{Op}^\gamma(a_+(\tau)) \, \widetilde{\rho}_+(t). \tag{2.12}$$

Here $a(t,\tau)$, $a_+(\tau)$ belong to Σ^m, $a_+(\tau)$ is defined on the weight line $\Im\tau = \gamma$ and does not depend on t. The cut-off function ρ has compact support, ρ_+ is supported in $\mathbb{R}_+$, the functions $\widetilde{\rho}$, $\widetilde{\rho}_+$ are covering functions (see Section 1).

To illustrate how the symbol classes Σ^m work, we prove here a boundedness property for operators (2.11), (2.12).

Lemma 2.5 *The operator*

$$A : \; H^{s,\gamma_+,\gamma_-}(M) \to H^{s-m,\gamma_+,\gamma_-}(M) \tag{2.13}$$

given by (2.11) *or* (2.12) *is bounded.*

Proof. Similarly to (1.5) and (1.6) we have bounded multiplication operators

$$\begin{array}{rcl} H^{s,\gamma_+,\gamma_-} & \xrightarrow{\widetilde{\rho}(t)} & H^s, \\ H^s & \xrightarrow{\rho(t)} & H^{s,\gamma_+,\gamma_-}, \\ H^{s,\gamma_+,\gamma_-} & \xrightarrow{\widetilde{\rho}_+(t)} & H^{s,\gamma_+}, \\ H^{s,\gamma_+} & \xrightarrow{\rho_+(t)} & H^{s,\gamma_+,\gamma_-}. \end{array}$$

The space $H^s(M)$ is defined by the norm

$$\|u\|^2 = \int_{-\infty}^{\infty} \|\widehat{u}(\tau)\|_\tau^2 \, d\tau$$

and the space H^{s,γ_+} by the norm

$$\|u\|^2 = \int_{\Im\tau=\gamma_+} \|\widehat{u}(\tau)\|_\tau^2 \, d\tau.$$

In virtue of these properties the lemma reduces to the following two statements:

1. for an operator-valued symbol $a(t,\tau) \in \Sigma^m$ vanishing for $|t|$ large enough, the operator $\operatorname{Op}(a(t,\tau))$ is bounded from $H^s(M)$ to $H^{s-m}(M)$;

2. for an operator-valued symbol $a(\tau) \in \Sigma^m$ defined on the line $\Im\tau = \gamma_+$, the map
$$H^{s,\gamma_+}(Y) \ni \widehat{u}(\tau) \mapsto a(\tau)\widehat{u}(\tau) \in H^{s-m,\gamma_+}(Y)$$
defines a bounded operator from $H^{s,\gamma_+}(M)$ to $H^{s-m,\gamma_+}(M)$.

Both these statements are almost evident. The second one follows directly from the definition of the operator-valued symbol from Σ^m. Indeed, we have

$$\begin{aligned} \|a(\tau)\widehat{u}(\tau)\|_{H^{s-m}(Y),\tau} &\leq \|a(\tau)\|_\tau \|\widehat{u}(\tau)\|_{H^s(Y),\tau} \\ &\leq C\,\|\widehat{u}(\tau)\|_{H^s(Y),\tau}, \end{aligned}$$

so, the same inequality holds between the L^2 norms of the left- and right-hand sides on the line $\Im\tau = \gamma_+$.

To prove the first statement, we pass to the Fourier representation

$$\widehat{v}(\tau_1) = \int_{-\infty}^{\infty} \widehat{a}(\tau_1 - \tau, \tau)\widehat{u}(\tau) \, d\tau$$

where $\widehat{a}(\sigma,\tau) = F_{t\to\sigma}(a(t,\tau))$ is an operator-valued symbol acting from $H^s(Y)$ to $H^{s-m}(Y)$ with the norm estimate

$$\|\widehat{a}(\sigma,\tau)\|_\tau = O\left(\langle\sigma\rangle^{-\infty}\right). \tag{2.14}$$

This estimate means that for any N the function $\langle\sigma\rangle^N \|\widehat{a}(\sigma,\tau)\|_\tau$ is bounded uniformly in σ and τ. Now, using Lemma 1.3

$$\begin{aligned} \|\widehat{v}(\tau_1)\|_{H^{s-m}(Y),\tau_1} &\leq \int_{-\infty}^{\infty} \|\widehat{a}(\tau_1 - \tau, \tau)\,\widehat{u}(\tau)\|_{\tau_1} \, d\tau \\ &\leq C\int_{-\infty}^{\infty} \langle\tau_1 - \tau\rangle^q \,\|\widehat{a}(\tau_1 - \tau, \tau)\,\widehat{u}(\tau)\|_\tau \, d\tau \\ &\leq C\int_{-\infty}^{\infty} \langle\tau_1 - \tau\rangle^q \,\|\widehat{a}(\tau_1 - \tau, \tau)\|_\tau \,\|\widehat{u}(\tau)\|_{H^s(Y),\tau} \, d\tau. \end{aligned}$$

Now, by (2.14) we have

$$\|\widehat{v}(\tau_1)\|_{H^{s-m}(Y),\tau_1} \leq C\int_{-\infty}^{\infty} O\left(\langle\tau_1 - \tau\rangle^{-\infty}\right) \|\widehat{u}(\tau)\|_{H^s(Y),\tau} \, d\tau,$$

implying the boundedness in L^2-spaces.

□

Let us briefly discuss trace properties of the operators (2.11), (2.12) supposing that the function ρ_+ also has compact support. It is clear that for $m < -n-1$ the values of the symbol $a(t,\tau) \in \Sigma^m$ are trace class operators in fiber spaces $H^s(Y)$ since we have

$$H^s(Y) \xrightarrow{a(t,\tau)} H^{s-m}(Y) \hookrightarrow H^s(Y)$$

where the embedding operator is of trace class. Thus, the fiberwise trace $\operatorname{tr} a(t,\tau)$ exists.

Lemma 2.6 *Let $m < -n-2$ and the functions ρ, ρ_+ have compact support. Then the operators* (2.11) *and* (2.12) *are of trace class in the spaces H^{s,γ_+,γ_-} and*

$$\operatorname{Tr} A = \frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{tr} \rho(t)a(t,\tau)\, d\tau dt \tag{2.15}$$

or

$$\operatorname{Tr} A = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_{\Im\tau=\gamma_+} \operatorname{tr} \rho(t)a(\tau)\, d\tau. \tag{2.16}$$

Proof. Consider a sequence

$$H^{s,\gamma_+,\gamma_-}(M) \xrightarrow{A} H^{s-m,\gamma_+,\gamma_-}(M) \xrightarrow{\widetilde{\rho}} H^{s,\gamma_+,\gamma_-}(M).$$

Here $\widetilde{\rho}$ is a covering function with compact support for ρ or ρ_+. The first operator is bounded by Lemma 2.6, the second operator is of trace class by Corollary 1.8. Note that the weights γ_+, γ_- in the last space may be taken arbitrarily since $\widetilde{\rho}$ has compact support.

The trace formulas (2.15), (2.16) follow in a standard way from representation of the trace class operator as a product of two Hilbert-Schmidt operators.

□

The other properties of ΨDO's with symbols from Σ^m in the more general case when Y has a conical point can be found in the last Section 5.

We introduce now an algebra $\Psi(M)$ of pseudodifferential operators on a smooth spindle M. We also use the name *cone algebra* for it since the only singularities of M are two conical points at $t = \pm\infty$. An operator $A \in \Psi^m(M)$ of order m is defined by a triple

$$\{a_-(\tau), \quad a(t,\tau), \quad a_+(\tau)\}$$

of operator-valued symbols from Σ^m. The function $a(t,\tau)$ is defined for τ real, the functions $a_\pm(\tau)$ are defined on horizontal *weight lines* $\Im\tau = \pm\gamma_\pm$, and on these lines they are operator-valued symbols in $\Re\tau$ (independent of t) from Σ^m.

These symbols are assumed to satisfy the following hypothesis:

1. *stabilization conditions*: for $|t|$ large enough, $a(t,\tau)$ "stabilizes" to $a(\pm\infty,\tau)$;

2. *compatibility conditions*: the symbols $a(+\infty,\tau)$ and $a_+(\tau+i\gamma)$ are compatible in the sense that

$$\begin{aligned} a_N &:= a(+\infty,\tau) - (a(\tau+i\gamma_+))_{\gamma_+}|_N \\ &= a(+\infty,\tau) - \sum_{k=0}^{N-1} a_+^{(k)}(\tau+i\gamma_+)(-i\gamma_+)^k/k! \end{aligned} \tag{2.17}$$

   belongs to Σ^{m-N}. The same compatibility condition is fulfilled for $a_0(-\infty,\tau)$ and $a_-(\tau-i\gamma_-)$.

An operator $A\in\Psi^m(M)$ corresponding to this triple is defined on functions $u(t)\in H^{s,\gamma_+,\gamma_-}(M)$ by the formula

$$\begin{aligned} Au &= \rho_-(t)\,\mathrm{Op}^{-\gamma_-}(a_-(\tau))\,\widetilde{\rho}_-(t) \\ &\quad+\rho(t)\,\mathrm{Op}(a(t,\tau))\,\widetilde{\rho}(t) \\ &\quad+\rho_+(t)\,\mathrm{Op}^{\gamma_+}(a_+(\tau))\,\widetilde{\rho}_+(t). \end{aligned} \tag{2.18}$$

Here ρ_-, ρ, ρ_+ form a partition of unity on $\mathbb{R}$, $\widetilde{\rho}_-$, $\widetilde{\rho}$, $\widetilde{\rho}_+$ are covering functions. By Lemma 2.5 this operator is bounded from $H^{s,\gamma_+,\gamma_-}(M)$ to $H^{s-m,\gamma_+,\gamma_-}(M)$.

Let us explain the role of the compatibility condition. It makes the definition (2.18) correct, that is independent of the partition of unity and covering functions up to trace class operators. First of all, another choice of covering functions is irrelevant by pseudolocality (see Section 5). Further, if we change the partition of unity the difference will be of the form

$$\begin{aligned} &\Delta\rho_-(t)\,\mathrm{Op}^{-\gamma_-}(a_-(\tau))\,\widetilde{\rho}_-(t) \\ &+\Delta\rho(t)\,\mathrm{Op}(a(t,\tau))\,\widetilde{\rho}(t) \\ &+\Delta\rho_+(t)\,\mathrm{Op}^{\gamma_+}(a_+(\tau))\,\widetilde{\rho}_+(t). \end{aligned}$$

The functions $\Delta\rho_-$, $\Delta\rho$, $\Delta\rho_+$ have compact support and their sum is identically zero. Thus, we may take one and the same covering function $\widetilde{\rho}$ and replace $\Delta\rho$ by $-\Delta\rho_- - \Delta\rho_+$, so that the difference become

$$\begin{aligned} &\Delta\rho_-\left\{e^{\gamma_- t}\,\mathrm{Op}(a_-(\tau-i\gamma_-))\,e^{-\gamma_- t} - \mathrm{Op}(a(-\infty,\tau))\right\}\widetilde{\rho} \\ +\;&\Delta\rho_+\left\{e^{-\gamma_+ t}\,\mathrm{Op}(a_+(\tau+i\gamma_+))\,e^{\gamma_+ t} - \mathrm{Op}(a(+\infty,\tau))\right\}\widetilde{\rho}. \end{aligned} \tag{2.19}$$

Here we have made use of the relation (2.3) and replaced $a(t,\tau)$ by its limit on the supports of $\Delta\rho_\pm$. Further, by Lemma 5.5

$$e^{-\gamma_+ t}\,\mathrm{Op}(a_+(\tau+i\gamma_+))\,e^{\gamma_+ t} = \mathrm{Op}\left((a(\tau+i\gamma_+)_{\gamma_+}|_N\right)$$

up to a trace class operator, and now, the compatibility condition implies that the difference (2.19) is of trace class.

2.2 Ellipticity and index formula

In this subsection we consider cone operators $A \in \Psi(M)$ of the form (2.18) where the operator-valued symbol $a(t,\tau)$ has the form (2.10). On the contrary, the operator-valued symbols $a_\pm(\tau)$ corresponding to conical points are not assumed to have the form (2.10), they may be defined independently of the quantization map (2.9).

Definition 2.7 *The homogeneous component* $b_m(t,y,\tau,\eta)$ *of order* m *of the function* $b(t,y,\tau,\eta)$ *in* (2.10) *is called the principal interior symbol of the operator* A*. The operator-valued symbols* $a_\pm(\tau)$ *are called the principal conormal symbols of* A *at the conical points* $\pm\infty$*.*

Clearly, the principal symbols do not depend on the concrete choice of the quantization map.

Definition 2.8 *An operator* $A \in \Psi(M)$ *is called elliptic if*

1. *its principal interior symbol* $\sigma_{\rm int}(A)$ *is an invertible (matrix-valued) function on* $T^*M \setminus \{0\}$ *(interior ellipticity);*

2. *for each conical point* $\pm\infty$*, the conormal symbol* $\sigma_c(A) = a_\pm(\tau)$ *is an invertible operator on* Y *for all* τ *on the weight line* $\Im\tau = \pm\gamma_\pm$ *(conormal ellipticity).*

We will also assume that the conormal symbols $a_\pm(\tau)$ are holomorphic functions in some strip $|\Im(\tau \pm i\gamma_\pm)| < \varepsilon$ around weight lines and on each horizontal line in these strips they define operator-valued symbols of order m (uniformly in smaller strips). This will be referred to as a *holomorphy condition.*

The index theory of the elliptic cone operators may be summarized in the following theorems [FS96, FST99].

Theorem 2.9 *Let* A *be an elliptic cone operator of order* m *and the holomorphy condition be fulfilled. Then it has a parametrix*

$$R:\ H^{s-m,\gamma_+,\gamma_-}(M) \to H^{s,\gamma_+,\gamma_-}(M)$$

such that $1-RA$ *and* $1-AR$ *are trace class operators in* $H^{s,\gamma_+,\gamma_-}(M)$ *and* $H^{s-m,\gamma_+,\gamma_-}(M)$*, respectively.*

In other words, the operator (2.13) is Fredholm, and the ellipticity conditions (both interior and conormal) are sufficient for the Fredholm property. Moreover, in [Sch91] their necessity is proved.

For the index of this Fredholm operator we have actually the same formula (0.6).

Theorem 2.10 *Let the ellipticity condition be fulfilled. Then there exists an operator-valued symbol $r_0(t,\tau) \in \Sigma^{-m}$ such that*

$$\begin{array}{l} 1 - r_0(t,\tau)a(t,\tau), \\ 1 - a(t,\tau)r_0(t,\tau) \end{array}$$

are trace class operators in $H^s(Y)$, for any $t,\tau \in \mathbb{R}^2$, vanishing outside a compact set in $\mathbb{R}^2$. If, in addition, A satisfies holomorphy conditions, then the operator is Fredholm and formula (0.6) *holds.*

The proof will be given in Sections 3 and 4 in a more general situation. The reader may also consult [FS96, FST99].

Here we would like to attract the reader's attention to the following fact. The right-hand side of (0.6) which we call a *topological index* exists under ellipticity conditions only, the holomorphy condition is not required. On the other hand, the *analytical index* on the left-hand side of (0.6) does require the holomorphy condition. In the next subsection we propose a modification of the analytical index which does not require the holomorphy condition.

2.3 The kernel cut-off

Let $a(\tau) \in \Sigma^m$ be an operator-valued symbol on a weight line $\Im\tau = \gamma$. We will not assume that $a(\tau)$ has an analytic extension to some strip around $\Im\tau = \gamma$. We will construct a new operator-valued symbol $a_\varepsilon(\tau)$ depending on a parameter $\varepsilon > 0$, which is an entire function in τ and, for ε small enough, is sufficiently close to $a(\tau)$. The corresponding procedure known as a *kernel cut-off* was introduced by the second author [Sch91, ES97]. It also works in more general situations when the symbol $a(t,\tau)$ also depends on t varying on a compact set.

Let $\psi(t) \in C_0^\infty(\mathbb{R})$ and let $\psi(t) \equiv 1$ in a neighborhood of $t = 0$. The Fourier transform $\widehat{\psi}(\tau)$ is an entire function in τ rapidly decreasing on any fixed horizontal line. We also introduce a small parameter $\varepsilon > 0$ considering the function $\psi(\varepsilon t)$, and its Fourier transform

$$\widehat{\psi}_\varepsilon(\tau) = \frac{1}{\varepsilon}\,\widehat{\psi}\left(\frac{\tau}{\varepsilon}\right).$$

Now, we define a new symbol $a_\varepsilon(\tau)$ by convolution of $a(\tau)$ and $\widehat{\psi}_\varepsilon(\tau)$, namely

$$\begin{aligned} a_\varepsilon(\tau) &= a(\tau) * \widehat{\psi}_\varepsilon(\tau) \\ &= \int_{\Im\tau_1=\gamma} \widehat{\psi}_\varepsilon(\tau-\tau_1)\, a(\tau_1)\, d\tau_1 \\ &= \int_{\Im(\tau-\tau_2)=\gamma} \widehat{\psi}_\varepsilon(\tau_2)\, a(\tau-\tau_2)\, d\tau_2. \end{aligned} \tag{2.20}$$

It is clear from (2.20) that $a_\varepsilon(\tau)$ is an entire function in τ since $\widehat{\psi}_\varepsilon(\tau-\tau_1)$ is.

Lemma 2.11 *If $a(\tau)$ is a symbol of Σ^m on the horizontal line $\Im\tau = \gamma$ then $a_\varepsilon(\tau) \in \Sigma^m$ on any horizontal line $\Im\tau = \tau_0$. Moreover, if $a(\tau)$ is invertible, so that $a^{-1}(\tau) \in \Sigma^{-m}$ on the line $\Im\tau = \gamma$, then $a_\varepsilon(\tau)$ is also invertible in some strip*

$$|\Im\tau - \gamma| \le \delta,$$

and $a_\varepsilon^{-1}(\tau) \in \Sigma^{-m}$ on each horizontal line in the strip.

Proof. From (2.20) it follows that

$$a_\varepsilon^{(\beta)}(\tau) = \int_{\Im\tau_1=\gamma} \widehat{\psi}_\varepsilon(\tau-\tau_1)\, a^{(\beta)}(\tau_1)\, d\tau_1.$$

By Lemma 1.3,

$$\|a^{(\beta)}(\tau_1)v\|_\tau \le C\,\langle \Re\tau - \Re\tau_1\rangle^q\, \|a^{(\beta)}(\tau_1)v\|_{\tau_1},$$

the norms here are taken in the spaces $H^{s-m+\beta}(Y)$. Thus, if

$$\begin{array}{rcl} \|a^{(\beta)}(\tau_1)v\|_{\tau_1} & \le & C\,\|v\|_{\tau_1} \\ & \le & C\,\langle \Re(\tau-\tau_1)\rangle^q\,\|v\|_\tau \end{array}$$

for $v \in H^s(Y)$, then

$$\|a_\varepsilon^{(\beta)}(\tau)v\|_\tau \le C \int_{-\infty}^{\infty} O\left(\langle \Re\tau - \Re\tau_1\rangle^{-\infty}\right) d\Re\tau_1\ \|v\|_\tau$$

proving that $a_\varepsilon(\tau) \in \Sigma^m$.

To prove the second part, we consider τ on the line $\Im\tau = \gamma$, τ_2 on the real axis, and expand $a(\tau-\tau_2)$ in (2.20) by Taylor's formula

$$a(\tau-\tau_2) = \sum_{k=0}^{N-1} \frac{1}{k!} a^{(k)}(\tau)(-\tau_2)^k + \frac{(-\tau_2)^N}{(N-1)!} \int_0^1 a^{(N)}(\tau-\theta\tau_2)(1-\theta)^{N-1} d\theta.$$

Since

$$\int_{-\infty}^{\infty} \tau_2^k\, \widehat{\psi}_\varepsilon(\tau_2)\, d\tau_1 = \left(-i\frac{d}{dt}\right)^k \psi(\varepsilon t)\,|_{t=0},$$

that is this expression equals $\psi(0) = 1$ for $k = 0$ and is zero for $k > 0$, we obtain from (2.20)

$$\begin{array}{l} a_\varepsilon(\tau)\, a^{-1}(\tau) - 1 \\ \quad = \dfrac{1}{(N-1)!} \displaystyle\int_{-\infty}^{\infty} (-\tau_2)^N \widehat{\psi}_\varepsilon(\tau_2)\, d\tau_2 \int_0^1 a^{(N)}(\tau-\theta\tau_2)\, a^{-1}(\tau)(1-\theta)^{N-1} d\theta. \end{array}$$

The operator

$$a^{(N)}(\tau - \theta\tau_2)\, a^{-1}(\tau) : \ H^s(Y) \to H^{s+N}(Y)$$

admits a norm estimate

$$\|a^{(N)}(\tau - \theta\tau_2)\, a^{-1}(\tau)\|_\tau \le C\, \langle \tau_2 \rangle^{2l}$$

following from Lemma 1.3. Lemma 2.1 and Corollary 2.4 show that the same estimate is valid for this operator acting from $H^s(Y)$ to $H^s(Y)$. Thus,

$$\|a_\varepsilon(\tau)\, a^{-1}(\tau) - 1\|_\tau \le C \int_{-\infty}^{\infty} |\tau_2|^N\, |\widehat{\psi}_\varepsilon(\tau_2)|\, \langle \tau_2 \rangle^{2l}\, d\tau_2.$$

Since

$$\widehat{\psi}_\varepsilon(\tau_2) = \frac{1}{\varepsilon}\, \widehat{\psi}\left(\frac{\tau_2}{\varepsilon}\right),$$

the change of variables $\tau_2 = \varepsilon x$ yields

$$\varepsilon^N \int_{-\infty}^{\infty} |x^N \widehat{\psi}(x)|\, \langle \varepsilon x \rangle^{2l}\, dx \le C\, \varepsilon^N,$$

implying the invertibility of $a_\varepsilon(\tau)$ on the line $\Im\tau = \gamma$.

It remains to show the invertibility of $a_\varepsilon(\tau)$ in some strip $|\Im\tau - \gamma| < \delta$. But taking $\tau' = \tau + ib$ with a real b, $|b| < \delta$, we get

$$\begin{aligned} \|a_\varepsilon(\tau')\, a_\varepsilon^{-1}(\tau) - 1\|_\tau &\le \int_0^b \|a'_\varepsilon(\tau + is)\, a_\varepsilon^{-1}(\tau)\|_\tau\, ds \\ &\le C\, \delta\, \langle \Re\tau \rangle^{-1} \end{aligned}$$

since $a'_\varepsilon(\tau + is) a_\varepsilon^{-1}(\tau)$ belongs to Σ^{-1}.

□

Using this lemma, we can extend the notion of the analytical index to elliptic operators not necessarily satisfying the holomorphy condition. Given an elliptic operator A defined by the triple $\{a_+(\tau), a(t,\tau), a_-(\tau)\}$ with non-degenerate $a_+(\tau)$, $a_-(\tau)$ on the weight lines $\Im\tau = \gamma_+$, $\Im\tau = -\gamma_-$, respectively, we apply kernel cut-off to obtain a new operator

$$A_\varepsilon = \{a_{+,\varepsilon}(\tau), a_\varepsilon(t,\tau), a_{-,\varepsilon}(\tau)\}$$

which is elliptic for sufficiently small $\varepsilon > 0$, by Lemma 2.11, and satisfies the holomorphy conditions. Clearly, the analytical index of A_ε is constant for ε small enough, because of the stability of the index. We can define a *modified analytical* index of the operator A (which is not a Fredholm one without holomorphy condition) as $\operatorname{ind} A_\varepsilon$. The index formula (0.6) is still valid for this modified index.

3 The Corner Algebra

3.1 Corner-degenerate symbols

In this section the singular manifold M will be an edged spindle (0.3). As in Subsection 2.1 we denote by $\Psi(M)$ an algebra of ΨDO's A defined by a triple $\{a_+(\tau), a(t,\tau), a_-(\tau)\}$ of operator-valued symbols. The only (but essential) difference is that now the entries of this triple are ΨDO's on a fiber Y which itself has a conical singularity y_c.

We introduce classes of operator-valued symbols $\Sigma^m(\delta, \delta - l)$ with $l \geq m$, similar to the classes Σ^m, cf. Definition 2.2.

Definition 3.1 *An operator-valued symbol $a(t,\tau)$ stabilizing for large $|t|$ belongs to $\Sigma^m(\delta, \delta - l)$, $l \geq m$, if for any integers α, β, and $s \in \mathbb{R}$ the operator*

$$\partial_t^\alpha \partial_\tau^\beta a(t,\tau) : \begin{array}{c} H^{s,\delta}(Y) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} H^{s-m+\beta,\delta-l}(Y) \\ \oplus \\ \mathbb{C}^{N_+} \end{array} \tag{3.1}$$

is bounded in the norms $\|\cdot\|_\tau$, cf. (1.13), *uniformly with respect to τ.*

In this section we construct a very special realization of the classes $\Sigma^m(\delta, \delta - l)$ which leads to the so-called *edge algebra*. The boundedness relations (3.1) for edge symbols are fulfilled in a slightly sharper form (see Lemma 3.2 below). Adding to an edge symbol two symbols $a_\pm(\tau)$ defined on *corner weight lines* $\Im\tau = \pm\gamma_\pm$, and imposing compatibility conditions on the triple $\{a_+(\tau), a(t,\tau), a_-(\tau)\}$ similar to (2.17), we come to the *corner algebra*, the main object of our interest.

The edge symbols $a(t,\tau)$ (respectively $a_\pm(\tau)$) consist of three different components. In this subsection we consider a so-called *interior part*. It is given by a classical (matrix-valued) symbol of order m, that is a function $b(t,y,\tau,\eta)$ on $T^*\overset{\circ}{M}$ where

$$\overset{\circ}{M} = \mathbb{R} \times (Y \setminus \{y_c\})$$

is the smooth part of M, and $(t,\tau) \in \mathbb{R}^2$, $y \in Y \setminus \{y_c\}$, $\eta \in T_y^*\overset{\circ}{M}$.

We assume the following:

1. The symbol b admits an asymptotic expansion for $(\tau,\eta) \to \infty$ in homogeneous components

$$b \sim \sum_{j=0}^{\infty} b_{m-j}(t,y,\tau,\eta)$$

where b_{m-j} are homogeneous functions in τ, η of degree $m - j$.

2. *Edge degeneracy.* In the singular chart $U_c \subset Y$, let $y = (r, x)$ with $r \in \mathbb{R}_+$ and $x \in X$, so that U_c is identified with the subset $0 \le r < 1$ of the cone $X^\wedge = \mathbb{R}_+ \times X$. Write $\eta = (\theta, \xi) \in T^*_y(Y \setminus \{y_c\})$, with $\theta \in \mathbb{R}$, the covariable for r, and $\xi \in T^*_x X$. Then

$$b(t, y, \tau, \eta) = r^{-m}\,\widetilde{b}(t, x, r\tau, r\theta, \xi). \tag{3.2}$$

3. *Stabilization.* For t large enough, positive or negative, the symbol $b(t, y, \tau, \eta)$ stabilizes to $b(\pm\infty, y, \tau, \eta)$.

Such a function will be called a *corner-degenerate* classical complete symbol of order m. Its homogeneous component of the highest degree m is called the *principal interior symbol*,

$$\sigma_{\mathrm{int}}(A) = b_m(t, y, \tau, \eta). \tag{3.3}$$

In general, the function $\widetilde{b}$ on the right-hand side of (3.2) may depend explicitly on r,

$$\widetilde{b} = \widetilde{b}(t, r, x, \widetilde{\tau}, \widetilde{\theta}, \xi).$$

In this case it is supposed to be smooth up to $r = 0$. We confine ourselves to the simplest case when $\widetilde{b}$ does not depend on r explicitly but only in combinations $r\tau$ and $r\theta$. It is sufficient for the purposes of the index theory.

Such a function defines an operator-valued symbol via a quantization map

$$Q(b(y, \eta)) = \sum_i \rho_i(y)\,\mathrm{Op}_i(b(y, \eta))\,\widetilde{\rho}_i(y) \tag{3.4}$$

(cf. (2.9)), where the summation is taken over smooth charts U_i. This is an operator-valued symbol acting from $H^{s,\delta}(Y)$ to $H^{s-m,\delta_1}(Y)$, for any $\delta, \delta_1 \in \mathbb{R}$, since the functions $\rho_i(y)$ vanish in a heighborhood of the conical point y_c.

As in the smooth case the Definition 2.2 is satisfied for the operator-valued symbols (3.4) with obvious replacement of $H^s(Y)$ and $H^{s-m+\beta}(Y)$ by $H^{s,\delta}(Y)$ and $H^{s-m+\beta,\delta_1}(Y)$, respectively. It may also be viewed as an operator-valued symbol acting in the spaces (3.1). In this case the operator is a (2×2)-matrix with the left upper corner equal to (3.4), and with remaining entries equal to 0.

3.2 Mellin symbols

The next ingredient is the so-called *complete edge symbol.* It is defined in the vicinity of the edge $\mathbb{R} \times \{y_c\}$, so that the fiber Y may be viewed as a cone $X^\wedge$. The manifold M in the vicinity of the edge is viewed as a wedge $\mathbb{R} \times X^\wedge$ with coordinates t, r, x, where $t \in \mathbb{R}$, $r \in \mathbb{R}_+$ are defined globally while $x \in X$. We also replace the spaces $H^{s,\delta}(Y)$ on the fibers by the spaces $K^{s,\delta}(X^\wedge)$ on the infinite cone, with the norm family $\|\cdot\|_\tau$ given by (1.11).

Thus, we are in the setting of the wedge algebra (see [ES97]). We fix a quantization map Q_X on X allowing one to construct a ΨDO on X by a function $a(x,\xi)$ on T^*X, for example,

$$Q_X\left(a(x,\xi)\right) = \sum_i \rho_i(x)\, \mathrm{Op}_i(a(x,\xi))\, \widetilde{\rho}_i(x). \tag{3.5}$$

Recall that we have a group action κ_λ on the spaces $K^{s,\gamma}(X^\wedge)$ given by (1.7), and the norm $\|\cdot\|_\tau$ is defined via this group action, cf. (1.11).

A *Mellin complete symbol* of order m is a function [1]

$$h(t,r,\tau,\zeta,\xi) = r^{-m}\, \widetilde{h}(t,x,r\tau,\zeta,\xi) \tag{3.6}$$

where ζ is a complex variable, and $\widetilde{h}(t,x,\widetilde{\tau},\zeta,\xi)$ is a holomorphic function in ζ belonging to a *cone weight strip*

$$S = \{|\Im\zeta - \delta| < \varepsilon\} \tag{3.7}$$

around the *cone weight line* $\Im\zeta = \delta$.

We assume that $\widetilde{h}$ is a classical symbol of order m on any horizontal line $\Im\zeta = \mathrm{const}$ inside the weight strip. This means that there is an asymptotic expansion for $\lambda \to \infty$

$$\widetilde{h}(t,x,\lambda\widetilde{\tau},\lambda\Re\zeta + i\delta,\lambda\xi) \sim \sum_{j=0}^{\infty} \widetilde{h}_{m-j}\,(t,x,\widetilde{\tau},\Re\zeta,\xi)\,\lambda^{m-j} \tag{3.8}$$

where $\widetilde{h}_{m-j}$ are homogeneous functions of degree $m-j$ in $(\widetilde{\tau},\Re\zeta,\xi)$.

Applying the quantization map Q_X to it, we obtain a function still denoted by

$$\widetilde{h}(t,\widetilde{\tau},\zeta) = Q_X(\widetilde{h}(t,x,\widetilde{\tau},\zeta,\xi)),$$

whose values are ΨDO's on X of order $\le m$. Further, we associate to it a Mellin ΨDO as follows. For a function $u(r)$ with values in $C^\infty(X)$, we first change variables $v(z) = f_* u = u(e^{-z})$, then apply to v a ΨDO with the symbol $\widetilde{h}(t,\widetilde{\tau},\xi)$,

$$\begin{aligned} w(z) &= \mathrm{Op}^\delta(\widetilde{h}(t,\widetilde{\tau},\zeta))\, v(z) \\ &= \frac{1}{2\pi}\int_{\Im\zeta=\delta} e^{i\zeta z}\, \widetilde{h}(t,\widetilde{\tau},\zeta)\, \widehat{v}(\zeta)\, d\zeta, \end{aligned}$$

and then make the inverse change of variables $z = -\ln r$. The result will be denoted by

$$\mathrm{Op}_M^\delta(\widetilde{h}(t,\widetilde{\tau},\zeta))\, u = \frac{1}{2\pi}\int_{\Im\zeta=\delta} e^{i\zeta z}\, \widetilde{h}(t,\widetilde{\tau},\zeta)\, F_{z\to\zeta}\left(u(e^{-z})\right) d\zeta\, \Big|_{z=-\ln r}. \tag{3.9}$$

[1] In general, $\widetilde{h}$ may depend explicitly on r being smooth up to $r=0$. Similarly to the interior corner-degenerate symbols we confine ourselves to a particular case when $\widetilde{h}$ does not depend on r explicitly.

The next assumption on the function $\widetilde{h}$ is its *compatibility* with the interior corner-degenerate symbol $\widetilde{b}$. This condition is quite similar to (2.17) with the cone weight line $\Im\zeta = \delta$ instead of the corner weight line in (2.17). It means that the operator (3.9) coincides with the ΨDO defined by the symbol $\widetilde{b} = \widetilde{b}(t, x, \widetilde{\tau}, r\theta, \xi)$ up to order $-\infty$. More precisely, applying the quantization map Q_X on X, cf. (3.5), we obtain a function still denoted by

$$\widetilde{b}(t, \widetilde{\tau}, \widetilde{\theta}) = Q_X(\widetilde{b}(t, x, \widetilde{\tau}, \widetilde{\theta}, \xi)),$$

whose values are ΨDO's on X of order m, and associate to it a ΨDO

$$\mathrm{Op}_F(\widetilde{b}(t, \widetilde{\tau}, r\theta))$$

which acts on functions $u(r) \in C_0^\infty(\mathbb{R}_+, C^\infty(X))$ by

$$\mathrm{Op}_F(\widetilde{b}(t, \widetilde{\tau}, r\theta))u\,(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\theta r}\,\widetilde{b}(t, \widetilde{\tau}, r\theta)\,\widehat{u}(\theta)\,d\theta. \tag{3.10}$$

The requirement is that both the operators (3.9) and (3.10) coincide up to order $-\infty$. In terms of the Mellin symbol $\widetilde{h}(t, x, \widetilde{\tau}, \zeta, \xi)$ and the interior corner-degenerate symbol $\widetilde{b}(t, x, \widetilde{\tau}, \widetilde{\theta}, \xi)$ given by (3.2), this condition looks as follows. We first consider a formal complex shift by $i\delta$ in the variable ζ applied to the function $\widetilde{h}(t, x, \widetilde{\tau}, \zeta, \xi)$, that is the formal Taylor series

$$\sum_{k=0}^{\infty}\frac{1}{k!}\,\partial_{\widetilde{\theta}}^{k}\widetilde{h}(t, \widetilde{\tau}, \widetilde{\theta} + i\delta)\,(-i\delta)^k \tag{3.11}$$

with the real variable $\widetilde{\theta}$. Next, we apply to (3.11) a change of variables in symbols generated by $z = -\ln r$. The result must coincide with the symbol $\widetilde{b}(t, x, \widetilde{\tau}, \widetilde{\theta}, \xi)$, that is

$$\begin{aligned}&\widetilde{b}(t, x, \widetilde{\tau}, \widetilde{\theta}, \xi)\\ &\sim \sum_{\alpha=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\alpha!}\frac{1}{k!}\,\partial_{\widetilde{\theta}}^{k+\alpha}\widetilde{h}(t, x, \widetilde{\tau}, \widetilde{\theta} + i\delta, \xi)\,\frac{(-i\delta)^k}{k!}\,\partial_s^\alpha \exp(-i\widetilde{\theta}(e^{-s} - 1 + s))\,|_{s=0}\end{aligned} \tag{3.12}$$

were $\sim$ means that the homogeneous components in $\widetilde{\tau}$, $\widetilde{\theta}$, ξ of the classical symbols in both sides coincide. Vice versa, if the symbol $\widetilde{b}(t, x, \widetilde{\tau}, \widetilde{\theta}, \xi)$ is given, then the symbol $\widetilde{h}(t, x, \widetilde{\tau}, \zeta, \xi)$ holomorphic in ζ and satisfying (3.12) may be found in two steps. First, we change the variables $r = e^{-z}$ in symbols defining

$$\widetilde{h}(t, x, \widetilde{\tau}, \sigma, \xi) \sim \sum_{\alpha=0}^{\infty}\frac{(-i)^{|\alpha|}}{\alpha!}\,\partial_\sigma^\alpha \widetilde{b}(t, x, \widetilde{\tau}, \sigma, \xi)\,\partial_s^\alpha e^{i\sigma(\ln(1+s)-s)}\,|_{s=0},$$

then apply the procedure of asymptotic summation to this formal series, and then the kernel cut-off procedure (Subsection 2.3) to obtain a Mellin symbol $\widetilde{h}(t,x,\widetilde{\tau},\zeta,\xi)$ which is an entire function in ζ. The relation (3.12) shows that such a Mellin symbol is defined uniquely up to a *smoothing Mellin symbol*, for which all the terms of the asymptotic expansion (3.8) are equal to 0.

A crucial observation for the edge and corner theory is that the symbol (3.6) and the corresponding operator (3.9) possess another homogeneity property, quite different from (3.8). It is easy to verify that the operator (3.9) is invariant under the κ_λ action (3.10). It means that the operator-valued symbol

$$b_M(t,\tau) = \mathrm{Op}_M^\delta(r^{-m}\widetilde{h}(t,r\tau,\zeta)) \tag{3.13}$$

is a homogeneous function in τ in the following sense

$$\kappa_\lambda^{-1}\, b_M(t,\lambda\tau)\,\kappa_\lambda \equiv \lambda^m\, b_M(t,\tau). \tag{3.14}$$

This property will be referred to as "twisted" homogeneity [2]. Notice that the factor r^{-m} in front of $\widetilde{h}$ ensures that the degree of "twisted" homogeneity coincides with the highest degree of homogeneity in (3.8).

Now, having a corner-degenerate symbol $b(t,r,x,\tau,\theta,\xi)$ and a Mellin symbol $h(t,r,x,\tau,\zeta,\xi)$ compatible with b, and using various cut-off functions, we construct an operator-valued symbol

$$b_{\mathrm{edge}}(t,\tau):\ H^{s,\delta}(Y) \to H^{s-m,\delta-m}(Y).$$

First, we have a function $\rho_c(y)=\rho_c(r)$ corresponding to a singular chart $U_c\subset Y$, and a covering function $\widetilde{\rho}_c(r)$. Next, we take a partition of unity $\varphi_0(r)$, $\varphi_\infty(r)$ on $\mathbb{R}_+$, with $\varphi_0(r)\in C^\infty(\overline{\mathbb{R}}_+)$ satisfying $\varphi_0(0)=1$, and $\varphi_\infty = 1-\varphi_0$, and let $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_\infty$ be covering functions for φ_0 and φ_∞. Then we set

$$b_{\mathrm{edge}}(t,\tau) = b_0(t,\tau) + b_\infty(t,\tau) \tag{3.15}$$

where

$$\begin{array}{rcl} b_0(t,\tau) &=& \rho_c(r)\,\varphi_0(r\langle\tau\rangle)\,\mathrm{Op}_M^\delta Q_X\,(h(t,r,x,\tau,\zeta,\xi))\,\widetilde{\varphi}_0(r\langle\tau\rangle)\,\widetilde{\rho}_c(r),\\ b_\infty(t,\tau) &=& \rho_c(r)\,\varphi_\infty(r\langle\tau\rangle)\,\mathrm{Op}_F Q_X\,(b(t,r,x,\tau,\theta,\xi))\,\widetilde{\varphi}_\infty(r\langle\tau\rangle)\,\widetilde{\rho}_c(r). \end{array} \tag{3.16}$$

Here Op_M^δ and Op_F denote Mellin and Fourier ΨDO's with respect to the variable r.

Lemma 3.2 *The operator-valued symbols $b_0(t,\tau)$ and $b_\infty(t,\tau)$ acting in the spaces*

$$K^{s,\delta}(X^\wedge) \to K^{s-m,\delta-m}(X^\wedge)$$

or, equivalently,

$$H^{s,\delta}(Y) \to H^{s-m,\delta-m}(Y)$$

[2]In the general case when $\widetilde{h}$ depends explicitly on r there is an asymptotical expansion in λ of the left-hand side of (3.14) whose leading term of degree m coincides with the right-hand side.

belong to the class $\Sigma^m(\delta, \delta - m)$. *Moreover, the relation* (3.1) *is fulfilled in a sharper form: the operator*

$$\partial_t^\alpha \partial_\tau^\beta b_{\mathrm{edge}}(t,\tau) : \ H^{s,\delta}(Y) \to H^{s-m+\beta,\delta-m+\beta}(Y).$$

is bounded in the norms $\|\cdot\|_\tau$ *uniformly with respect to* τ.

For the proof the reader is referred to [FST98, Lemma 2.6].

We next define a *principal edge symbol* $b_\wedge(t,\tau)$ of the operator-valued symbol (3.15) by

$$\begin{aligned} b_\wedge(t,\tau) &= b_{0,\wedge}(t,\tau) + b_{\infty,\wedge}(t,\tau) \\ &= \varphi_0(r|\tau|)\, \mathrm{Op}_M^\delta Q_X\left(h(t,r,x,\tau,\zeta,\xi)\right) \widetilde{\varphi}_0(r|\tau|) \\ &\quad + \varphi_\infty(r|\tau|)\, \mathrm{Op}_F Q_X\left(b(t,r,x,\tau,\theta,\xi)\right) \widetilde{\varphi}_\infty(r|\tau|), \end{aligned} \tag{3.17}$$

for $\tau \neq 0$. Let us explain the meaning of (3.17). Because of the special form of the Mellin symbol h (cf. (3.13)), the operator $\mathrm{Op}_M^\delta Q_X(h)$ is homogeneous of degree m (cf. (3.14)). In fact,

$$\begin{aligned} \lambda^{-m}\, \kappa_\lambda^{-1}\, \mathrm{Op}_M^\delta Q_X\left(h(t,r,\lambda\tau,\zeta,\xi)\right) \kappa_\lambda &= \mathrm{Op}_M^\delta Q_X\left(h\left(t,\frac{r}{\lambda},\lambda\tau,\zeta,\xi\right)\right) \\ &\equiv \mathrm{Op}_M^\delta Q_X\left(h(t,r,\tau,\zeta,\xi)\right) \end{aligned}$$

and

$$\begin{aligned} \kappa_\lambda^{-1} \varphi_0(r\langle\lambda\tau\rangle)\, \kappa_\lambda &= \varphi_0\left(\frac{r}{\lambda}\langle\lambda\tau\rangle\right) \\ &\equiv \varphi_0(r|\tau|) \end{aligned}$$

for $\lambda > 0$ large enough, and

$$\begin{aligned} \kappa_\lambda^{-1} \rho_c(r)\, \kappa_\lambda &= \rho_c\left(\frac{r}{\lambda}\right) \\ &\to \rho_c(0) \\ &= 1 \end{aligned}$$

as $\lambda \to \infty$. Similar relations hold for $\mathrm{Op}_F Q_X(b)$ and $\varphi_\infty(r\langle\tau\rangle)$. This means that (3.17) may be viewed as a limit

$$\lim_{\lambda \to +\infty} \lambda^{-m}\, \kappa_\lambda^{-1}\, b_{\mathrm{edge}}(t,\lambda\tau)\, \kappa_\lambda,$$

at least formally. For fixed t and $\tau \neq 0$, the operator

$$b(t,\tau) : \ K^{s,\delta}(X^\wedge) \to K^{s-m,\delta-m}(X^\wedge)$$

is a cone ΨDO, cf. [ES97], and as such has a *conormal symbol*

$$\sigma_M(b_\wedge(t,\tau)) := \widetilde{h}(t,0,\zeta). \tag{3.18}$$

3.3 Green symbols

The last ingredient of the edge algebra is the so-called Green operator-valued symbol.

Definition 3.3 *An operator-valued symbol*

$$b_G(t,\tau): \; K^{s,\delta}(X^\wedge) \to K^{s-m,\delta-l}(X^\wedge)$$

belonging to $\Sigma^m(\delta,\delta-l)$, $l \geq m$, *is called a Green symbol of order* m *if for some* $\varepsilon > 0$ *and any fixed* τ, *the operator* $\partial_t^\alpha \partial_\tau^\beta b_G(t,\tau)$ *can be extended as a bounded operator to*

$$K^{s_1,\delta}(X^\wedge) \to K^{s_2,\delta-l+\varepsilon}(X^\wedge),$$

for any $s_1, s_2 \in \mathbb{R}$.

Thus, this definition means that the operator b_G is smoothing one on fibers and besides gives a gain in the cone weight by some $\varepsilon > 0$. A typical example of the Green operator of order m is given by (3.13) where the Mellin symbol $\widetilde{h}(t,x,\widetilde{\tau},\zeta,\xi)$ belongs to $\mathcal{S}^{-\infty}$, that is decreases rapidly in $(\widetilde{\tau},\zeta,\xi)$ and besides its conormal symbol $\widetilde{h}(t,x,0,\zeta,\xi)$ is identically equal to 0. Another example of the Green symbol (of order $-\infty$) gives a difference of the symbols $b_{\mathrm{edge}}(t,\tau)$ and $b'_{\mathrm{edge}}(t,\tau)$ both obtained by (3.15), (3.16) with a different choice of cut-off functions, namely φ_0, φ_∞, $\widetilde{\varphi}_0$, $\widetilde{\varphi}_\infty$ for b_{edge} and φ'_0, φ'_∞, $\widetilde{\varphi}'_0$, $\widetilde{\varphi}'_\infty$ for b'_{edge}. In [GSS00] it is shown that the choice $\varphi_0 \equiv \widetilde{\varphi}_0 \equiv 1$ and $\varphi_\infty \equiv \widetilde{\varphi}_\infty \equiv 0$ is also possible up to a Green symbol of order $-\infty$.

The restriction $l \geq m$ ensures that $\Sigma^m(\delta,\delta-m) \in \Sigma(\delta,\delta-l)$, so that corner-degenerate and Mellin symbols of Subsections 3.1 and 3.2 belonging by construction to $\Sigma^m(\delta,\delta-m)$ are also included into $\Sigma(\delta,\delta-l)$ (see Section 5 for a more detailed discussion).

Up to now we considered the operators acting on the spaces $K^{s,\delta}(X^\wedge)$ or $H^{s,\delta}(Y)$. We will need however Green symbols acting in the spaces $K^{s,\delta}(X^\wedge) \oplus \mathbb{C}^N$. The norm $\|\cdot\|_\tau$ on the direct sum is defined by (1.13) and the classes $\Sigma^m(\delta,\delta-l)$ are defined with respect to this norm family (Definition 3.1). Thus, a (2×2)-matrix

$$b_G(t,\tau): \quad \begin{array}{c} K^{s,\delta}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \quad \to \quad \begin{array}{c} K^{s-m,\delta-l}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{array} \tag{3.19}$$

belonging to $\Sigma^m(\delta,\delta-l)$ is a Green symbol of order m if for some $\varepsilon > 0$ and any fixed τ, s_1, s_2, the operator

$$\partial_t^\alpha \partial_\tau^\beta b_G(t,\tau): \quad \begin{array}{c} K^{s_1,\delta}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \quad \to \quad \begin{array}{c} K^{s_2,\delta-l+\varepsilon}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

is bounded.

For the operators (3.19) the notion of "twisted" homogeneity is meaningful, the action of $\mathbb{R}_+$ on the direct sum $K^{s,\delta}(X^\wedge) \oplus C^N$ is defined as $\kappa_\lambda \oplus 1$. So, the Green symbol (3.19) defined for $\tau \neq 0$ is *homogeneous* of degree m if

$$\lambda^{-m}\left(\kappa_\lambda^{-1} \oplus 1\right) b_G(t, \lambda\tau)\left(\kappa_\lambda \oplus 1\right)$$

is independent of λ. It is clear that multiplying this symbol by an excision function $\chi(\tau)$ with $\chi \equiv 0$ in a neighborhood of $\tau = 0$, we obtain a Green symbol $\chi(\tau) b_G(t,\tau)$ of order m. This observation allows us to define *classical* Green symbols, namely

$$b_G(t,\tau) \sim \sum_{j=0}^{\infty} b_{G,m-j}(t,\tau) \tag{3.20}$$

where $b_{G,m-j}(t,\tau)$ are homogeneous Green symbols of degree $m-j$ and $\sim$ means an asymptotic summation. In other words, (3.20) means that the difference

$$b_G(t,\tau) - \chi(\tau) \sum_{j=0}^{N-1} b_{G,m-j}(t,\tau)$$

is a Green symbol of order $m - N$.

For a classical Green symbol we define its *principal edge symbol* by

$$\sigma_\wedge(b_G(t,\tau)) = b_{G,m}(t,\tau). \tag{3.21}$$

We also set by definition

$$\sigma_M\left(\sigma_\wedge(b_G(t,\tau))\right) = 0.$$

Now, we are in a position to describe the corner algebra $\Psi(M)$. A corner operator is defined by a triple

$$A = \{a_+(\tau), a_i(t,\tau), a_-(\tau)\} \tag{3.22}$$

(we denote here and further on the interior part by $a_i(t,\tau)$ instead of $a(t,\tau)$), it acts on the spaces $H^{s,\delta,\gamma_+,\gamma_-}(M)$ similarly to (2.18), namely,

$$\begin{aligned} Au \;=\; & \rho_-(t)\,\mathrm{Op}^{-\gamma_-}(a_-(\tau))\,\widetilde{\rho}_-(t) \\ & + \rho_i(t)\,\mathrm{Op}(a_i(t,\tau))\,\widetilde{\rho}_i(t) \\ & + \rho_+(t)\,\mathrm{Op}^{\gamma_+}(a_+(\tau))\,\widetilde{\rho}_+(t). \end{aligned} \tag{3.23}$$

Here we have also changed the notation $\rho(t)$, $\widetilde{\rho}(t)$ for $\rho_i(t)$, $\widetilde{\rho}_i(t)$. The following definition (unfortunately, rather cumbersome) summarizes the constructions of this section.

Definition 3.4 *A corner symbol* $A = \{a_+(\tau), a_i(t,\tau), a_-(\tau)\}$ *of order* m *consists of an operator-valued symbol*

$$a_i(t,\tau) = b_{\text{int}}(t,\tau) + b_{\text{edge}}(t,\tau) + b_G(t,\tau)$$

acting on the spaces (3.1), *where*

- b_{int} *is defined by a corner-degenerate symbol* (3.2) *via* (3.4)*;*
- b_{edge} *is defined by a Mellin symbol* h (3.6) *via* (3.15), (3.16)*;*
- b_G *is a classical Green symbol of order* m.

All these symbols are assumed to stabilize as $t \to \pm\infty$ *to symbols independent of* t,

$$a_i(\pm\infty,\tau) = b_{\text{int}}(\pm\infty,\tau) + b_{\text{edge}}(\pm\infty,\tau) + b_G(\pm\infty,\tau).$$

The components $a_\pm(\tau)$, $\Im\tau = \pm\gamma_\pm$, *are also operator-valued symbols of order* m *acting in the spaces* (3.1). *These symbols must be compatible with* $a_i(\pm\infty,\tau)$ *in the sense that*

$$a_N = a_i(\pm\infty,\tau) - \sum_{k=0}^{N-1} a_\pm^{(k)}(\tau \pm i\gamma_\pm)\,\frac{(\mp i\gamma_\pm)^k}{k!} \tag{3.24}$$

belongs to $\Sigma^{m-N}(\delta, \delta - l + \varepsilon)$ *with some* $\varepsilon > 0$ (*cf.* (2.17)).

The component a_i is actually an element of the edge algebra on an infinite cylinder $M = \mathbb{R} \times Y$, and the components $a_\pm$ serve to compactify this cylinder by two corners.

Definition 3.5 *For a corner operator there are three levels of principal symbols:*

- *the principal interior symbol* $\sigma_{\text{int}}(A)$ *which is a homogeneous function of degree* m *on* $T^*\mathring{M}$ *defined by* (3.3)*;*
- *the principal edge symbol*

  $$\sigma_\wedge(A)(t,\tau) = \sigma_\wedge(b_{\text{edge}}) + \sigma_\wedge(b_G)$$

  which is a "twisted" homogeneous function of degree m *in* τ *defined by* (3.17), (3.21)*;*
- *the principal corner symbol* $\sigma_c(A)$ *at each corner* $t = \pm\infty$. *This is simply the operator* $a_\pm(\tau)$, $\Im\tau = \pm i\gamma_\pm$.

Besides, there is a conormal symbol $\sigma_M\sigma_\wedge(A)$ whose left upper corner is given by (3.18) and the remaining entries of the (2×2)-matrix are 0.

The corner theory may be summarized in the following theorem.

Theorem 3.6

1. *A corner operator of order m is bounded as an operator*
$$A: \ H^{s,\delta,\gamma_+,\gamma_-}(M) \longrightarrow H^{s-m,\delta-l,\gamma_+,\gamma_-}(M).$$
2. *The corner operators form an algebra with the additivity of orders.*
3. *The symbol maps $\sigma_{\rm int}$, $\sigma_\wedge$, σ_c and $\sigma_M\sigma_\wedge$ are homomorphisms.*

The detailed proof is contained in [Sch92], see also [ST00], and [ES97, ST99] containing a similar theorem for the edge algebra.

3.4 Kernel cut-off and holomorphy

In the original definition of the corner algebra [Sch92] there was an additional requirement comparing to the Definition 3.4: the opertor-valued functions $a_\pm(\tau)$ were supposed to be meromorphic in τ. The poles gave rise to the so-called *corner asymptotics* which were one of the objects of study in [Sch92].

For the purposes of the index theory we don't need the full asymptotic information, so the meromorphy of $a_\pm(\tau)$ is no longer needed. It is sufficient to require a milder *holomorphy condition*: the functions $a_\pm(\tau)$ defined on the *corner weight lines* $\Im\tau = \gamma_\pm$ have analytic extensions into a *corner weight strip*
$$S_\pm = \{|\Im\tau \mp i\gamma_\pm| < \varepsilon\},$$
moreover,
$$a_\pm(\tau) \in \Sigma^m(\delta, \delta - l)$$
on any horizontal line in the strip. A similar condition with a cone weight strip (3.7) instead of $S_\pm$ was imposed on the Mellin symbol (3.6).

However, at the first glance even this milder condition seems to contradict the Definition 3.4. Indeed, because of the compatibility (3.24) it is natural to assume that $a_i(+\infty, \tau)$ satisfies the holomorphy condition as well. But because of the cut-off functions $\varphi_0(r\langle\tau\rangle)$, $\varphi_\infty(r\langle\tau\rangle)$ (cf. (3.16), which are by no means holomorphic in τ, the holomorphy of the whole operator $a_i(\tau)$ is doubtful. Quite surprisingly, the following lemma shows that the holomorphy property may be obtained by means of the kernel cut-off procedure. In addition to Lemma 2.11 the kernel cut-off procedure in the symbol classes $\Sigma^m(\delta, \delta - l)$, $l \geq m$, possesses the following property.

Lemma 3.7 *Let an operator-valued symbol*
$$a(\tau) = b_{\rm int}(\tau) + b_{\rm edge}(\tau) + b_G(\tau)$$
defined on a weight line $\Im\tau = \gamma$ belong to the symbol class $\Sigma^m(\delta, \delta - l)$. Then the operator-valued symbol
$$\begin{aligned} a_\varepsilon(\tau) &= a(\tau) * \widehat{\psi}_\varepsilon \\ &= \int_{\Im\tau=\gamma} \widehat{\psi}_\varepsilon(\tau - \tau_1)\, a(\tau_1)\, d\tau_1 \end{aligned}$$

also belongs to $\Sigma^m(\delta, \delta - l)$. *Moreover, the difference* $a_\varepsilon(\tau) - a(\tau)$ *is a Green symbol.*

Proof. Similarly to the proof of Lemma 2.11 we have by Taylor's formula

$$\begin{aligned} & a_\varepsilon(\tau) - a(\tau) \\ = & \frac{1}{(N-1)!} \int_{-\infty}^{\infty} (-\tau_2)^N \widehat{\psi}_\varepsilon(\tau_2)\, d\tau_2 \int_0^1 a^{(N)}(\tau - \theta\tau_2)\,(1-\theta)^{N-1} d\theta. \end{aligned}$$

By Proposition 5.1 the symbol $a^{(N)}(\tau)$ belongs to $\Sigma^{m-N}(\delta, \delta - l + \varepsilon_1)$ with some $\varepsilon_1 > 0$. Thus, $a_\varepsilon(\tau) - a(\tau)$ also belongs to $\Sigma^{m-N}(\delta, \delta - l + \varepsilon_1)$ since $\widehat{\psi}_\varepsilon(\tau_2)$ is rapidly decreasing on the real axis.

Since N may be taken arbitrarily large, it means that $a_\varepsilon(\tau) - a(\tau)$ is infinitely smoothing and gives a gain in the weight by ε_1. In other words, it is a Green operator.

□

As a consequence we see that for a corner operator

$$A = \{a_+(\tau), a_i(t, \tau), a_-(\tau)\}$$

we can take another corner operator

$$A_\varepsilon = \{a_{+,\varepsilon}(\tau), a_{i,\varepsilon}(t, \tau), a_{-,\varepsilon}(\tau)\} \tag{3.25}$$

which differs from A by a Green operator and satisfies the holomorphy condition. Moreover, we could introduce a subclass of corner operators satisfying the holomorphy condition in the spirit of the original definition in [Sch92], and restrict ourselves to this subalgebra. We prefer, however, to remain in a little wider class of Definition 3.4 using, if needed, a holomorphic approximation (3.25).

3.5 Ellipticity and parametrix

Having defined three principal symbols, cf. Definition 3.5, we introduce *elliptic corner operators* as those for which σ_{int}, $\sigma_\wedge$ and σ_c are invertible. More precisely, the *interior ellipticity* means that the function $\sigma_{\mathrm{int}} = b_m(t, \tau, y, \eta)$ on $T^*\overset{\circ}{M} \setminus \{0\}$ is an invertible homomorphism. Near the edge we actually have

$$b_m(t, y, \tau, \eta) = r^{-m}\, \widetilde{b}_m(t, x, r\tau, r\theta, \xi)$$

with a function $\widetilde{b}_m(t, x, \widetilde{\tau}, \widetilde{\theta}, \xi)$ called *compressed interior symbol.* We assume that it is an invertible homomorphism for $(\widetilde{\tau}, \widetilde{\theta}, \xi) \neq (0, 0, 0)$, including this assumption into the interior ellipticity.

Next, the *edge ellipticity* means that the principal edge symbol

$$\sigma_\wedge(A)(t, \tau) : \begin{array}{c} K^{s,\delta}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} K^{s-m,\delta-l}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

is an invertible operator for $\tau \neq 0$. Finally, the *corner ellipticity* at $t = +\infty$ means that the operator

$$a_+(\tau) : \begin{array}{c} H^{s,\delta}(Y) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} H^{s-m,\delta-l}(Y) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

is invertible for any τ belonging to the corner weight line $\Im\tau = \gamma_+$. A similar condition is imposed at $t = -\infty$.

A basic question of the elliptic theory is if the ellipticity implies the Fredholm property of a corner operator of order m

$$A : \; H^{s,\delta,\gamma_+,\gamma_-}(M) \to H^{s-m,\delta-l,\gamma_+,\gamma_-}(M)$$

as in the smooth case. A well-known way to answer this question is to construct a parametrix

$$R : \; H^{s-m,\delta-l,\gamma_+,\gamma_-}(M) \to H^{s,\delta,\gamma_+,\gamma_-}(M)$$

such that the operators $1 - RA$ and $1 - AR$ are compact (and even of trace class) in the spaces $H^{s,\delta,\gamma_+,\gamma_-}(M)$ and $H^{s-m,\delta-l,\gamma_+,\gamma_-}(M)$, respectively. In contrast to the smooth elliptic theory the invertibility of principal symbols is not sufficient for the parametrix to exist, one needs also a *holomorphy condition* for the corner symbols $a_\pm(\tau)$, that is, the symbols $a_\pm(\tau)$ as well as their inverses $a_\pm^{-1}(\tau)$ have analytic extensions into *corner weight strips*

$$S_\pm = \{|\Im\tau \mp i\gamma_\pm| < \varepsilon\},$$

moreover,

$$\begin{array}{rcl} a_\pm(\tau) & \in & \Sigma^m(\delta, \delta - l), \\ a_\pm^{-1}(\tau) & \in & \Sigma^{-m}(\delta - l, \delta) \end{array}$$

on any horizontal line in the strip. As was explained in the preceding subsection we may assume without loss of generality that both *cone* and *corner holomorphy conditions are fulfilled.* Under these assumptions we prove the following theorem.

Theorem 3.8 *Let*

$$A = \{a_+(\tau), a_i(t,\tau), a_-(\tau)\} \in \Psi^m(M) \tag{3.26}$$

be an elliptic operator satisfying the holomorphy conditions. Then there exists a parametrix

$$R = \{a_+^{-1}(\tau), r_i(t,\tau), a_-^{-1}(\tau)\} \in \Psi^{-m}(M) \tag{3.27}$$

such that both $1 - RA$ *and* $1 - AR$ *are trace class operators in the spaces* $H^{s,\delta,\gamma_+,\gamma_-}(M)$ *and* $H^{s-m,\delta-l,\gamma_+,\gamma_-}(M)$*, respectively.*

Thus,

$$A: \ H^{s,\delta,\gamma_+,\gamma_-}(M) \to H^{s-m,\delta-l,\gamma_+,\gamma_-}(M)$$

is a Fredholm operator. We also prove that its index may be expressed by the formula

$$\begin{aligned}
\operatorname{ind} A &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \{\operatorname{tr}(1 - r_i \circ a_i)\,|_N - \operatorname{tr}(1 - a_i \circ r_i)\,|_N\}\, dt d\tau \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \operatorname{tr}\left\{ \sum_{k=0}^{N-1} (a_+^{-1}a_+')^{(k)}(\tau + i\gamma_+)\frac{(-i\gamma_+)^k}{k!} - (r_i a_i')(+\infty,\tau) \right\} d\tau \\
&- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \operatorname{tr}\left\{ \sum_{k=0}^{N-1} (a_-^{-1}a_-')^{(k)}(\tau - i\gamma_-)\frac{(i\gamma_-)^k}{k!} - (r_i a_i')(-\infty,\tau) \right\} d\tau.
\end{aligned} \tag{3.28}$$

for any $N \geq n+1$. All the operators in curly brackets are of trace class in the spaces $H^{s,\delta}(Y)$ or $H^{s-m,\delta-l}(Y)$, and tr means the corresponding operator trace.

Proof. The operator-valued symbol $a_i(t,\tau)$ entering (3.26) is an edge symbol of order m stabilizing at $t \to \pm\infty$. We use a well-known parametrix construction in the edge algebra (see e.g. [ES97, FST98, ST99]) which gives us a symbol $r_i(t,\tau) \in \Sigma^{-m}(\delta - l, \delta)$ such that $1 - r_i \circ a_i\,|_N$ and $1 - a_i \circ r_i\,|_N$ are Green symbols of order $-N$ with respect to weight data (δ,δ) for $1 - r_i \circ a_i\,|_N$ and $(\delta - l, \delta - l)$ for $1 - a_i \circ r_i\,|_N$.

To pass further to corner algebra, we follow the scheme of [FST99, Theorem 3.1]. Actually, we simply repeat the calculations for the case of operator-valued symbols. Observe that the symbol variables t, τ are defined globally, the only transition being a complex shift in τ. The computations use different versions of the theorem on the regularized trace of a product, and a pseudolocality of pseudodifferential operators.

We start with the well-known formula for the index of a Fredholm operator

$$\operatorname{ind} A = \operatorname{Tr}(1 - RA) - \operatorname{Tr}(1 - AR).$$

According to (3.26), (3.27), and (3.23) the operators A and R are given by the following expressions

$$\begin{aligned}
A &= A_- + A_i + A_+ \\
&= \rho_-(t)\operatorname{Op}^{-\gamma_-}(a_-(\tau))\,\widetilde{\rho}_-(t) \\
&\quad + \rho_i(t)\operatorname{Op}(a_i(t,\tau))\,\widetilde{\rho}_i(t) \\
&\quad + \rho_+(t)\operatorname{Op}^{\gamma_+}(a_+(\tau))\,\widetilde{\rho}_+(t),
\end{aligned}$$

and

$$\begin{aligned}
R &= R_- + R_i + R_+ \\
&= \rho_-(t)\operatorname{Op}^{-\gamma_-}(a_-^{-1}(\tau))\,\widetilde{\rho}_-(t) \\
&\quad + \rho_i(t)\operatorname{Op}(r_i(t,\tau))\,\widetilde{\rho}_i(t) \\
&\quad + \rho_+(t)\operatorname{Op}^{\gamma_+}(a_+^{-1}(\tau))\,\widetilde{\rho}_+(t),
\end{aligned}$$

The product RA contains nine summands, two of them R_-A_+ and R_+A_- being equal to 0 since $\widetilde{\rho}_-\rho_+ \equiv \widetilde{\rho}_+\rho_- \equiv 0$. Thus,

$$\begin{aligned} 1 - RA &= 1 - R_iA_i - R_-A_- - R_+A_+ \\ &\quad - R_-A_i - R_+A_i - R_iA_- - R_iA_+ \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} 1 - AR &= 1 - A_iR_i - A_-R_- - A_+R_+ \\ &\quad - A_iR_- - A_iR_+ - A_-R_i - A_+R_i. \end{aligned} \tag{3.30}$$

The corresponding summands in (3.29) and (3.30) differ only by the order of factors. Were these summands trace class operators, the difference of their traces would be equal to 0. We will transform the summands pairwise replacing them by equivalent pairs where the equivalence means the following. We say that two pairs AB, BA and $A'B'$, $B'A'$ are equivalent if the differences $AB - A'B'$ and $BA - B'A'$ are trace class operators and their traces coincide. For example, if $A - A'$ and $B - B'$ belong to the trace class, then

$$\begin{aligned} \operatorname{tr}(AB - A'B') &= \operatorname{tr}(A - A')B + \operatorname{tr} A'(B - B') \\ &= \operatorname{tr} B(A - A') + \operatorname{tr}(B - B')A' \\ &= \operatorname{tr}(BA - B'A'). \end{aligned}$$

In other words, if we change A or B adding to them trace class operators, we obtain an equivalent pair.

Consider first the corresponding pairs from the first lines of (3.29) and (3.30), for example

$$R_+A_+ = \rho_+ \operatorname{Op}^{\gamma_+}(a_+^{-1})\, \widetilde{\rho}_+\rho_+ \operatorname{Op}^{\gamma_+}(a_+)\, \widetilde{\rho}_+$$

and

$$A_+R_+ = \rho_+ \operatorname{Op}^{\gamma_+}(a_+)\, \widetilde{\rho}_+\rho_+ \operatorname{Op}^{\gamma_+}(a_+^{-1})\, \widetilde{\rho}_+.$$

We may drop here $\widetilde{\rho}_+$ since the operators

$$\begin{gathered} \rho_+ \operatorname{Op}^{\gamma_+}(a_+)\,(1 - \widetilde{\rho}_+), \\ \rho_+ \operatorname{Op}^{\gamma_+}(a_+^{-1})\,(1 - \widetilde{\rho}_+) \end{gathered}$$

belong to the trace class by pseudolocality, cf. Lemma 5.4. So, this pair is equivalent to the following one

$$\begin{gathered} \operatorname{Op}^{\gamma_+}(\rho_+a_+^{-1}) \operatorname{Op}^{\gamma_+}(\rho_+a_+), \\ \operatorname{Op}^{\gamma_+}(\rho_+a_+) \operatorname{Op}^{\gamma_+}(\rho_+a_+^{-1}), \end{gathered}$$

which in turn is equivalent to

$$\begin{gathered} \operatorname{Op}^{\gamma_+}\left(\rho_+a_+^{-1} \circ \rho_+a_+ \,|_N\right), \\ \operatorname{Op}^{\gamma_+}\left(\rho_+a_+ \circ \rho_+a_+^{-1} \,|_N\right) \end{gathered} \tag{3.31}$$

by the theorem on the regularized trace of a product (Theorem 5.6).

Similarly, remaining two pairs from the first lines are equivalent to

$$\begin{aligned}&\mathrm{Op}\left(\rho_i r_i \circ \rho_i a_i\,|_N\right),\\ &\mathrm{Op}\left(\rho_i a_i \circ \rho_i r_i\,|_N\right)\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\mathrm{Op}^{-\gamma_-}\left(\rho_- a_-^{-1} \circ \rho_- a_-\,|_N\right),\\ &\mathrm{Op}^{-\gamma_-}\left(\rho_- a_- \circ \rho_- a_-^{-1}\,|_N\right).\end{aligned} \tag{3.33}$$

Now, there are four pairs in the second lines of (3.29), (3.30). Consider one of them, say

$$R_i A_+ = \rho_i\,\mathrm{Op}(r_i)\,\widetilde{\rho}_i \rho_+\,\mathrm{Op}^{\gamma_+}(a_+)\,\widetilde{\rho}_+$$

and

$$A_+ R_i = \rho_+\,\mathrm{Op}^{\gamma_+}(a_+)\,\widetilde{\rho}_+ \rho_i\,\mathrm{Op}(r_i)\,\widetilde{\rho}_i.$$

Here the factors have different representations corresponding to different weight lines. First of all we need to pass to the same representation in both factors.

To this end we replace the cut-off functions ρ_+ and $\widetilde{\rho}_+$ by a pair ρ_1, $\widetilde{\rho}_1$ of compactly supported functions satisfying the following conditions:

1. $\widetilde{\rho}_i \rho_+ \equiv \widetilde{\rho}_i \rho_1$;
2. $\rho_1 \widetilde{\rho}_1 \equiv \rho_1$, so that $\widetilde{\rho}_1$ is a covering function for ρ_1.

This replacement gives us an equivalent pair of operators because of pseudolocality. Indeed, replacing first ρ_+ by ρ_1, we obtain a new pair of operators which differ from the previous one by

$$\begin{aligned}&\rho_i\,\mathrm{Op}(r_i)\,\widetilde{\rho}_i\,(\rho_+ - \rho_1)\,\mathrm{Op}^{\gamma_+}(a_+)\,\widetilde{\rho}_+ \equiv 0,\\ &(\rho_+ - \rho_1)\,\mathrm{Op}^{\gamma_+}(a_+)\,\widetilde{\rho}_+ \rho_i\,\mathrm{Op}(r_i)\,\widetilde{\rho}_i.\end{aligned}$$

The second operator is of trace class by pseudolocality, for $(\rho_+ - \rho_1)\widetilde{\rho}_+ \rho_i \equiv 0$. Its trace is equal to 0 because moving the factor $\widetilde{\rho}_+$ under the trace sign to the first place we get $\widetilde{\rho}_i(\rho_+ - \rho_1) \equiv 0$. Thus, A may be changed to

$$\rho_1\,\mathrm{Op}^{\gamma_+}(a_+)\,\widetilde{\rho}_+.$$

Here $\widetilde{\rho}_+$ may be replaced by $\widetilde{\rho}_1$, since the difference

$$\rho_1\,\mathrm{Op}^{\gamma_+}(a_+)\,(\widetilde{\rho}_+ - \widetilde{\rho}_1)$$

is of trace class because of pseudolocality. Finally, we obtain the operator

$$\rho_1\,\mathrm{Op}^{\gamma_+}(a_+)\,\widetilde{\rho}_1$$

instead of A_+.

Now, we use the compatibility condition which implies that the difference

$$\rho_1 \left(\mathrm{Op}^{\gamma_+}(a_+) - \mathrm{Op}(a_i)\right) \widetilde{\rho}_1$$

is of trace class. As a result we come to an equivalent pair

$$\begin{array}{l} \rho_i \, \mathrm{Op}(r_i) \, \widetilde{\rho}_i \rho_1 \, \mathrm{Op}(a_i) \, \widetilde{\rho}_1, \\ \rho_1 \, \mathrm{Op}(a_i) \, \widetilde{\rho}_1 \rho_i \, \mathrm{Op}(r_i) \, \widetilde{\rho}_i, \end{array}$$

which can be handled similarly to the pairs from the first lines. We drop $\widetilde{\rho}_i$ and $\widetilde{\rho}_1$ and then apply the theorem on the regularized trace of a product, obtaining an equivalent pair

$$\begin{array}{l} \mathrm{Op}\left(\rho_i r_i \circ \rho_+ a_i \left|_N\right.\right), \\ \mathrm{Op}\left(\rho_i a_i \circ \rho_+ r_i \left|_N\right.\right). \end{array} \tag{3.34}$$

In the final expression we have again replaced ρ_1 by ρ_i without changing the symbol.

Similarly, the remaining three pairs may be transformed to equivalent ones

$$\mathrm{Op}\left(\rho_i r_i \circ \rho_- a_i \left|_N\right.\right), \qquad \mathrm{Op}\left(\rho_- a_i \circ \rho_i r_i \left|_N\right.\right); \tag{3.35}$$

$$\mathrm{Op}\left(\rho_+ r_i \circ \rho_i a_i \left|_N\right.\right), \qquad \mathrm{Op}\left(\rho_i a_i \circ \rho_+ r_i \left|_N\right.\right); \tag{3.36}$$

$$\mathrm{Op}\left(\rho_- r_i \circ \rho_i a_i \left|_N\right.\right), \qquad \mathrm{Op}\left(\rho_i a_i \circ \rho_- r_i \left|_N\right.\right). \tag{3.37}$$

In (3.36) and (3.37) we have again used the compatibility condition implying that

$$\rho_1 \left(\mathrm{Op}^{\gamma_+}(a_+^{-1}) - \mathrm{Op}(r_i)\right) \widetilde{\rho}_1$$

is a trace class operator. Indeed, it can be rewritten as

$$\begin{array}{l} \rho_1 \, \mathrm{Op}^{\gamma_+}(a_+^{-1}) \left(1 - \mathrm{Op}(a_i)\mathrm{Op}(r_i)\right) \widetilde{\rho}_1 \\ + \rho_1 \, \mathrm{Op}^{\gamma_+}(a_+^{-1}) \left(\mathrm{Op}(a_i) - \mathrm{Op}^{\gamma_+}(a_+)\right) \mathrm{Op}(r_i) \widetilde{\rho}_1 \end{array}$$

since $\mathrm{Op}^{\gamma_+}(a_+(\tau))$ and $\mathrm{Op}^{\gamma_+}(a_+^{-1}(\tau))$ are mutually inverse. From this representation the trace class property is obvious.

The pairs (3.31) and (3.33) may be transformed further using the identities

$$\begin{array}{rcl} \rho_+ & \equiv & 1 - \rho_i - \rho_-, \\ \rho_+ \rho_- & \equiv & 0. \end{array}$$

Thus,

$$\begin{aligned} \mathrm{Op}^{\gamma_+}\left(\rho_+ a_+^{-1} \circ \rho_+ a_+ \left|_N\right.\right) &= \mathrm{Op}^{\gamma_+}\left(\rho_+ a_+^{-1} \circ a_+ \left|_N\right.\right) - \mathrm{Op}^{\gamma_+}\left(\rho_+ a_+^{-1} \circ \rho_i a_+ \left|_N\right.\right) \\ &= \rho_+ - \mathrm{Op}^{\gamma_+}\left(\rho_+ a_+^{-1} \circ \rho_i a_+ \left|_N\right.\right) \end{aligned}$$

since $a_+^{-1} \circ a_+ \equiv 1$ and $\mathrm{Op}^{\gamma_+}(\rho_+)$ is simply multiplication by $\rho_+(t)$. So, the pairs (3.31) and (3.33) become

$$\begin{array}{ll} \rho_+ - \mathrm{Op}^{\gamma_+}\left(\rho_+ a_+^{-1} \circ \rho_i a_+ \left|_N\right.\right), & \rho_+ - \mathrm{Op}^{\gamma_+}\left(\rho_+ a_+ \circ \rho_i a_+^{-1} \left|_N\right.\right); \\ \rho_- - \mathrm{Op}^{-\gamma_-}\left(\rho_- a_-^{-1} \circ \rho_i a_- \left|_N\right.\right), & \rho_- - \mathrm{Op}^{-\gamma_-}\left(\rho_- a_- \circ \rho_i a_-^{-1} \left|_N\right.\right). \end{array}$$

Now, gathering these two pairs and the previous ones (3.32), (3.34)–(3.37), we obtain

$$\begin{aligned} 1 - RA \;\sim\; & \operatorname{Op}\left(\rho_i(1 - r_i \circ a_i)\,|_N\right) \\ & + \operatorname{Op}^{\gamma_+}\left(\rho_+ a_+^{-1} \circ \rho_i a_+\,|_N\right) - \operatorname{Op}\left(\rho_+ r_i \circ \rho_i a_i\,|_N\right) \\ & + \operatorname{Op}^{-\gamma_-}\left(\rho_- a_-^{-1} \circ \rho_i a_-\,|_N\right) - \operatorname{Op}\left(\rho_- r_i \circ \rho_i a_i\,|_N\right) \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} 1 - AR \;\sim\; & \operatorname{Op}\left(\rho_i(1 - a_i \circ r_i)\,|_N\right) \\ & + \operatorname{Op}^{\gamma_+}\left(\rho_+ a_+ \circ \rho_i a_+^{-1}\,|_N\right) - \operatorname{Op}\left(\rho_+ a_i \circ \rho_i r_i\,|_N\right) \\ & + \operatorname{Op}^{-\gamma_-}\left(\rho_- a_- \circ \rho_i a_-^{-1}\,|_N\right) - \operatorname{Op}\left(\rho_- a_i \circ \rho_i r_i\,|_N\right). \end{aligned} \tag{3.39}$$

The summands in the first lines are obviously trace class operators and the difference of their traces yields the first line in (3.28).

Our next goal is to pass from the operators Op^γ to the operators Op. The following lemma gives us the needed rule.

Lemma 3.9 *Let $a(t,\tau)$ belong to $\Sigma^0(\delta,\delta)$, for $\Im\tau = \gamma$. Moreover, let it have a compact support in t. Then the following relation holds*

$$\begin{aligned} \operatorname{Op}^\gamma(a(t,\tau)) &= e^{-\gamma t} \operatorname{Op}\left(a(t,\tau+i\gamma)\right) e^{\gamma t} \\ &\sim \operatorname{Op}\left(e^{-\gamma t} a(t,\tau+i\gamma) \circ e^{\gamma t}\,|_N\right) \end{aligned} \tag{3.40}$$

where $A \sim B$ means that $A - B$ is a trace class operator with zero trace.

Proof. Let $\rho(t)$, $\widetilde{\rho}(t)$ have compact support, $\rho\, a(t,\tau) \equiv a(t,\tau)$ and $\widetilde{\rho}(t)\rho(t) \equiv \rho(t)$. By pseudolocality we may write

$$\begin{aligned} \operatorname{Op}^\gamma(a(t,\tau)) &\sim \rho \operatorname{Op}^\gamma(a(t,\tau))\,\widetilde{\rho} \\ &= \rho\, e^{-\gamma t} \operatorname{Op}(a(t,\tau+i\gamma))\, e^{-\gamma t}\,\widetilde{\rho}, \end{aligned}$$

the last equality being a simple consequence of the definition (2.2) of Op^γ. Now, all the symbols and functions have compact supports in t, so, we may apply the theorem on the regularized trace of a product to the operators

$$\begin{aligned} A &= \operatorname{Op}\left(\rho(t)\, e^{-\gamma t}\, a(t,\tau+i\gamma)\right), \\ B &= \operatorname{Op}\left(\widetilde{\rho}(t)\, e^{\gamma t}\right), \end{aligned}$$

resulting in

$$\begin{aligned} \operatorname{Tr}_N AB &= \operatorname{Tr}_N BA \\ &= 0 \end{aligned}$$

since the regularized trace of BA is obviously equal to 0. Dropping cut-off functions ρ, $\widetilde{\rho}$ from the final expressions, we obtain (3.40).

□

Observe that the formal series

$$e^{-\gamma t}a(t,\tau+i\gamma)\circ e^{\gamma t}=\sum_{k=0}^{\infty}\partial_\tau^k a(t,\tau+i\gamma)\,\frac{(-i\gamma)^k}{k!}$$

is a formal Taylor expansion for $a(t,(\tau+i\gamma)-i\gamma)$ in powers of $-i\gamma$ at $\tau+i\gamma$. Note also that for k large enough the operator $\mathrm{Op}(\partial_\tau^k a(t,\tau+i\gamma))$ is of trace class with zero trace. Indeed,

$$\begin{aligned}\mathrm{Tr}\ \mathrm{Op}\left(\partial_\tau^k a(t,\tau+i\gamma)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}\mathrm{tr}\,\partial_\tau^k a(t,\tau+i\gamma)\,d\tau\\ &= 0\end{aligned}$$

since

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{tr}\,\partial_\tau^k a(t,\tau+i\gamma)\,d\tau &= \partial_\tau^{k-1}\,\mathrm{tr}\,a(t,\tau+i\gamma)\,|_{-\infty}^{\infty}\\ &= 0.\end{aligned}$$

Using the notation $(a)_\gamma$ for the formal complex shift by $i\gamma$, we have

$$(a\circ b)_\gamma=(a)_\gamma\circ(b)_\gamma,$$

so, the formal complex shift is an automorphism or the formal Leibniz product $\circ$. With this notation the second lines in (3.38) and (3.39) may be written as

$$\mathrm{Op}\left((\rho_+(a_+^{-1})_{\gamma_+}\circ\rho_i(a_+)_{\gamma_+}-\rho_+r_i\circ\rho_i a_i)\,|_N\right)\tag{3.41}$$

and

$$\begin{aligned}&\mathrm{Op}\left((\rho_+(a_+)_{\gamma_+}\circ\rho_i(a_+^{-1})_{\gamma_+}-\rho_+a_i\circ\rho_i r_i)\,|_N\right)\\ =\ &\mathrm{Op}\left((\rho_i(a_+^{-1})_{\gamma_+}\circ\rho_+(a_+)_{\gamma_+}-\rho_i r_i\circ\rho_+a_i)\,|_N\right).\end{aligned}\tag{3.42}$$

Here we have changed the order of factors since $\rho_+(a_+)_{\gamma_+}$ is equivalent to ρ_+a_i by compatibility condition, and similarly $\rho_i(a_+^{-1})_{\gamma_+}$ is equivalent to $\rho_i r_i$. We move ρ_i in (3.41) through $(a_+)_{\gamma_+}$ and a_i as follows

$$\begin{aligned}\rho_i\,(a_+)_{\gamma_+} &= [\rho_i,(a_+)_{\gamma_+}]+(a_+)_{\gamma_+}\circ\rho_i\\ &= -[\rho_+,(a_+)_{\gamma_+}]+(a_+)_{\gamma_+}\circ\rho_i-[\rho_-,(a_+)_{\gamma_+}].\end{aligned}$$

Substituting this into (3.41) we may drop the last term since it vanishes on the support of ρ_+. Thus, for the trace of (3.41) we obtain

$$\begin{aligned}&\mathrm{Tr}\ \mathrm{Op}\left((\rho_+(a_+^{-1})_{\gamma_+}\circ[(a_+)_{\gamma_+},\rho_+]-\rho_+r_i\circ[a_i,\rho_+])\,|_N\right)\\ +\ &\mathrm{Tr}\ (\rho_+(1-r_i\circ a_i)\rho_i\,|_N).\end{aligned}$$

In (3.42) we move ρ_+ through $(a_+)_{\gamma_+}$ and a_i obtaining

$$\begin{aligned}-\ &\mathrm{Tr}\ \mathrm{Op}\left((\rho_i(a_+^{-1})_{\gamma_+}\circ[(a_+)_{\gamma_+},\rho_+]-\rho_i r_i\circ[a_i,\rho_+])\,|_N\right)\\ +\ &\mathrm{Tr}\ (\rho_i(1-r_i\circ a_i)\rho_+\,|_N).\end{aligned}$$

It remains to take the difference of these traces to obtain

$$\operatorname{Tr}\, \operatorname{Op}\left(((a_+^{-1})_{\gamma_+} \circ [(a_+)_{\gamma_+}, \rho_+] - r_i \circ [a_i, \rho_+]) \,|_N \right).$$

Now, since $(a_+^{-1})_{\gamma_+}$ and a_i do not depend on t on the support of ρ_+, we have

$$\begin{array}{rcl} [(a_+)_{\gamma_+}, \rho_+] & = & -i\partial_\tau (a_+)_{\gamma_+}\, \rho_+'(t), \\ [a_i, \rho_+] & = & -i\partial_\tau a_i\, \rho_+'(t). \end{array}$$

Using that

$$\int_{-\infty}^{\infty} \rho_+'(t)\, dt = 1,$$

we obtain finally

$$-\frac{i}{2\pi} \operatorname{tr}\, \left((a_+)_{\gamma_+}^{-1} \circ \partial_\tau (a_+)_{\gamma_+} - r_i \circ \partial_\tau a_i \right) |_{N-1},$$

which gives the second line in (3.28). In a similar way we obtain the third line completing the proof of Theorem 3.2.

□

4 The Index Formula

It remains to reduce the index formula (3.28) to (0.6). This may be done similarly to [FS96] using the scheme

analytical index $\mapsto$ algebraic index $\mapsto$ topological index.

Unfortunately, the methods of [FS96] can not be applied directly and we need some modifications.

Our basic observation consists in the fact that the homothety group

$$H_\lambda : \ \{a_+(\tau), a(t,\tau), a_-(\tau)\} \to \{a_+(\tau), a(\lambda t, \tau), a_-(\tau)\}$$

with $\lambda > 0$ acts on the set of elliptic corner operators. The index remains constant under this action. So, extracting the part of the expression (3.28) invariant under homotheties, we get the final formula (0.6).

4.1 The algebraic index

We start with the definition of the algebra of formal symbols where the algebraic index lives.

Definition 4.1 *A formal symbol is a formal power series*

$$a(t,\tau,h) = \sum_{k=0}^{\infty} h^k\, a_k(t,\tau) \tag{4.1}$$

where $a_k(t,\tau)$ are edge symbols from $\Sigma^{m-k}(\delta, \delta - l)$.

We assume further that $a_k \in \Sigma^{m-k}(\delta, \delta - l + \varepsilon)$ for $k > 0$, with some $\varepsilon > 0$. The leading term a_0 does not depend on t for large positive or negative t, while the higher coefficients $a_k(t, \tau)$, $k > 0$ vanish identically for large $|t|$.

The Leibniz product of two formal symbols is defined by

$$a \circ b = \sum_{k=0}^{\infty} \frac{(-ih)^k}{k!} \, \partial_\tau^k a \, \partial_t^k b. \tag{4.2}$$

We will use the notation

$$a \,|_N = \sum_{k=0}^{N-1} a_k$$

for the N-th partial sum of the series (4.1) at $h = 1$. This is a symbol from $\Sigma^m(\delta, \delta - l)$.

Define the trace of a formal symbol to be

$$\begin{aligned} \operatorname{Tr}\, a(t, \tau, h) &= \frac{1}{2\pi h} \int_{\mathbb{R}^2} \operatorname{tr} a(t, \tau, h)\, dt d\tau \\ &= \sum_{k=0}^{\infty} \frac{h^{k-1}}{2\pi} \int_{\mathbb{R}^2} \operatorname{tr} a_k(t, \tau)\, dt d\tau. \end{aligned} \tag{4.3}$$

Of course, all the symbols $a_k(t, \tau)$ must be trace class operators in $H^{s,\delta}(Y)$ and the integrals in (4.3) must converge. This will be the case, in particular, if $a_k(t, \tau) \in \Sigma_G^{-(n+2)}(\delta, \delta)$ for all k, and $\operatorname{tr} a_0 \equiv 0$ for $|t|$ large enough. Here and further on $\Sigma_G^m(\delta, \delta)$ stands for the Green symbols of order m.

For an elliptic corner operator defined by a triple

$$A = \{a_+(\tau), a(t, \tau), a_-(\tau)\} \tag{4.4}$$

let us consider the function $a(t, \tau)$ as a formal symbol consisting of the leading term only. Then we have a fiberwise parametrix $r_0(t, \tau)$ such that

$$\begin{aligned} 1 - r_0 a &\in \Sigma_G^{-1}(\delta, \delta), \\ 1 - a r_0 &\in \Sigma_G^{-1}(\delta - l, \delta - l). \end{aligned}$$

Treating r_0 as a formal symbol consisting of the leading term only let us construct a formal symbol

$$\begin{aligned} r(t, \tau, h) &= r_0 \circ \sum_{k=0}^{N-1} (1 - a \circ r_0)^{\circ k} \\ &= \sum_{k=0}^{N-1} (1 - r_0 \circ a)^{\circ k} \circ r_0 \end{aligned}$$

where $\circ k$ means the k-th power with respect to the Leibniz product (4.2). Then,

$$\begin{array}{rcl} 1 - r \circ a & = & (1 - r_0 \circ a)^{\circ N} \\ & \in & \Sigma^{-N}(\delta, \delta) \end{array}$$

and

$$\begin{array}{rcl} 1 - a \circ r & = & (1 - a \circ r_0)^{\circ N} \\ & \in & \Sigma^{-N}(\delta - l, \delta - l). \end{array}$$

Taking $N > n + 1$ we see that the fiberwise trace exists, moreover, the constant term of the difference

$$\mathrm{tr}\,(1 - r \circ a) - \mathrm{tr}\,(1 - a \circ r)$$

is equal to zero. Indeed, it may be written as

$$\mathrm{tr}\,(1 - r_0 a)^N - \mathrm{tr}\,(1 - a r_0)^N = \mathrm{ind}\, a(t, \tau).$$

This index is independent of t, τ because of stability of the index. Thus,

$$\begin{array}{rcl} \mathrm{ind}\, a(t, \tau) & = & \mathrm{ind}\, a(+\infty, \tau) \\ & = & \mathrm{ind}\, a_+(\tau + i\gamma_+) \\ & \equiv & 0 \end{array}$$

since $a_+(\tau + i\gamma)$ is an invertible operator. The higher-order terms in $1 - r \circ a$ and $1 - a \circ r$ vanish for large $|t|$ since they contain derivatives in t of functions stabilizing for large $|t|$. Thus, the expression

$$\mathrm{Tr}\ (1 - r \circ a) - \mathrm{Tr}\ (1 - a \circ r)$$

is meaningful being understood as

$$\frac{1}{2\pi h} \int_{\mathbb{R}^2} (\mathrm{tr}\,(1 - r \circ a) - \mathrm{tr}\,(1 - a \circ r))\, dt d\tau. \qquad (4.5)$$

It is a formal power series in *positive* powers of h because the coefficient at h^{-1} vanishes as it has been explained above. The series (4.5) will be called an *algebraic index.*

Proposition 4.2 *The constant term is the only non-zero term in the algebraic index.*

Proof. Consider a homothety operator defined on formal symbols

$$H_\lambda : \ a(t, \tau, h) \mapsto a(\lambda t, \tau, \lambda h),$$

where λ is a positive number. It is straightforward to verify that H_λ is an automorphism of the Leibniz algebra, that is

$$(H_\lambda a) \circ (H_\lambda b) = H_\lambda (a \circ b).$$

Thus, if we replace the symbol $a(t,\tau)$ in the triple (4.4) by the symbol $H_\lambda a = a(\lambda t, \tau)$ we again obtain a corner elliptic operator. The corresponding parametrix (as a formal symbol) is $H_\lambda r = r(\lambda t, \tau, \lambda h)$. Further, denoting

$$\operatorname{tr}(1 - r \circ a) - \operatorname{tr}(1 - a \circ r) = \sum_{k=1}^{\infty} h^k\, c_k(t,\tau)$$

we obtain

$$\begin{aligned}\operatorname{tr}(1 - H_\lambda r \circ H_\lambda a) - \operatorname{tr}(1 - H_\lambda a \circ H_\lambda r) &= H_\lambda\{\operatorname{tr}(1 - r \circ a) - \operatorname{tr}(1 - a \circ r)\} \\ &= \sum_{k=1}^{\infty} \lambda^k h^k\, c_k(\lambda t, \tau).\end{aligned}$$

Changing the variables $\lambda t = t'$ and integrating we come to the following series

$$\sum_{k=1}^{\infty} \lambda^{k-1} h^{k-1} \frac{1}{2\pi} \int_{\mathbb{R}^2} c_k(t', \tau)\, dt'd\tau.$$

Thus, the part of the functional (4.5) invariant under homotheties is given by the constant term.

On the other hand, we can expect the stability property for the algebraic index resulting in

$$\operatorname{ind} H_\lambda a \equiv \operatorname{ind} a.$$

Unfortunately, the trace functional Tr is not a "genuine" trace, so we may not assert that the stability property for the functional (4.5) holds. We even can not define $\operatorname{Tr}(1 - r \circ a)$ and $\operatorname{Tr}(1 - a \circ r)$ separately, but only in a combination (4.5). However, the stability property for the functional (4.5) does hold. The matter is that our symbols are independent of t for large $|t|$. Thus, the homothety $a(\lambda t, \tau)$ does not change the leading terms for sufficiently large $|t|$. In other words, the symbol

$$\dot{a}(\lambda) = \frac{d}{d\lambda} a(\lambda)$$

has a compact support in t. It means that we may make use of the notion of *trace ideal* $\mathcal{J}$ in the algebra of formal symbols. Let $\mathcal{J}$ consist of formal symbols belonging to $\Sigma_G^{-(n+2)}$ and vanishing identically for $|t|$ large enough. Although $(1 - r \circ a)$ and $(1 - a \circ r)$ do not belong to $\mathcal{J}$ (since they do not vanish for large t) their derivatives in λ do belong and moreover,

$$\frac{d}{d\lambda} \operatorname{ind} a(\lambda) = \operatorname{Tr} \frac{d}{d\lambda}(1 - r \circ a) - \operatorname{Tr} \frac{d}{d\lambda}(1 - a \circ r)$$

where $\operatorname{ind} a(\lambda)$ is given by the functional (4.5). We proceed further in a standard way transforming the above expression to

$$\begin{aligned} &- \ \mathrm{Tr}\ (\dot r + r \circ \dot a \circ r) \circ a - \mathrm{Tr}\ r \circ \dot a \circ (1 - r \circ a) \\ &+ \ \mathrm{Tr}\ a \circ (\dot r + r \circ \dot a \circ r) + \mathrm{Tr}\ (1 - a \circ r) \circ \dot a \circ r \end{aligned}$$

where all the symbols under the Tr sign belong to $\mathcal{J}$. For such symbols cyclic permutations of factors under Tr sign are possible implying that the first and the second lines cancel, proving the proposition.

□

Thus, the algebraic index may be viewed as a number: the constant term of the formal series

$$\begin{aligned} &\frac{1}{2\pi h} \int_{\mathbb{R}^2} \{\mathrm{tr}\,(1 - r \circ a) - \mathrm{tr}\,(1 - a \circ r)\}\, dt d\tau \\ &\quad + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \mathrm{tr}\, \left\{ \sum_{k=0}^{\infty} (a_+^{-1} a_+')^{(k)} (\tau + i\gamma_+) \frac{(-i\gamma_+)^k}{k!} - (ra')(+\infty, \tau) \right\} d\tau \\ &\quad - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \mathrm{tr}\, \left\{ \sum_{k=0}^{\infty} (a_-^{-1} a_-')^{(k)} (\tau - i\gamma_-) \frac{(i\gamma_-)^k}{k!} - (ra')(-\infty, \tau) \right\} d\tau. \end{aligned} \tag{4.6}$$

Observe that for k large enough the integral

$$\int \mathrm{tr}\ \left(a_+^{-1} a_+'\right)^{(k)} (\tau + i\gamma_+)\, d\tau$$

converges and equals zero, so the formal Taylor series in (4.6) are meaningful since their terms with large k vanish after integration, so, they do not contribute the algebraic index. Observe also that the corner contributions at $t = \pm\infty$ are independent of h, that is they contribute the constant term only.

Proposition 4.3 *The analytical index defined by* (3.28) *and the algebraic index defined by* (4.6) *coincide.*

Proof. Take an algebraic parametrix

$$r = r_0 \circ \sum_{k=0}^{N_1 - 1} (1 - a \circ r_0)^{\circ k}$$

with N_1 sufficiently large, and define an analytic parametrix taking

$$R = \mathrm{Op}\,(r\,|_{N_2})\,.$$

Denoting $r\,|_{N_2} = b$, write the analytical index (3.28) in the form

$$
\begin{aligned}
\operatorname{ind} A &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \left(\operatorname{tr}(1-b\circ a) - \operatorname{tr}(1-a\circ b)\right)|_N\, dt d\tau \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \operatorname{tr}\left\{\sum_{k=0}^{N-1} (a_+^{-1}a_+')^{(k)}(\tau+i\gamma_+)\frac{(-i\gamma_+)^k}{k!} - (ba')(+\infty,\tau)\right\} d\tau \\
&- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \operatorname{tr}\left\{\sum_{k=0}^{N-1} (a_-^{-1}a_-')^{(k)}(\tau-i\gamma_-)\frac{(i\gamma_-)^k}{k!} - (ba')(-\infty,\tau)\right\} d\tau.
\end{aligned}
$$

In this formula we may replace the symbol b by the formal symbol r since the difference

$$\Delta b = r\,|_{N_2'} - r\,|_{N_2''}$$

for N_2', N_2'' sufficiently large does not affect the index. Indeed, the corner contributions remain unchanged since $\Delta b(\pm\infty,\tau) \equiv 0$. As for the interior term its change is proportional to

$$\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} (\operatorname{tr}\Delta b\circ a - \operatorname{tr} a\circ\Delta b)\,|_N\, dt.$$

This expression is the sum of the terms

$$\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} \operatorname{tr}\partial_\tau^k (\Delta b)_l\; \partial_t^k a\, dt - \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} \operatorname{tr}\partial_\tau^k a\, \partial_t^k (\Delta b)_l\; dt,$$

where Δb means a summand of Δb. Integration by parts shows that these terms are equal to zero. Next, the number N may be enlarged arbitrarily. We may also take $N=\infty$ in the corner contributions as was explained earlier. The result is

$$
\begin{aligned}
\operatorname{ind} A &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \left(\operatorname{tr}(1-r\circ a) - \operatorname{tr}(1-a\circ r)\right)|_N\, dt d\tau \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \operatorname{tr}\left\{\sum_{k=0}^{\infty} (a_+^{-1}a_+')^{(k)}(\tau+i\gamma_+)\frac{(-i\gamma_+)^k}{k!} - (ra')(+\infty,\tau)\right\} d\tau \\
&- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \operatorname{tr}\left\{\sum_{k=0}^{\infty} (a_-^{-1}a_-')^{(k)}(\tau-i\gamma_-)\frac{(i\gamma_-)^k}{k!} - (ra')(-\infty,\tau)\right\} d\tau.
\end{aligned}
$$

We also can independently take $N=\infty$ in the interior term since only the constant term of the formal series

$$\frac{1}{2\pi h}\int_{\mathbb{R}^2} \left(\operatorname{tr}(1-r\circ a) - \operatorname{tr}(1-a\circ r)\right) dt d\tau$$

(cf. (4.5)) is not equal to zero. This non-zero term gives precisely the algebraic index.

□

4.2 Explicit expression

It remains to extract explicitly the constant term in (4.5). Here we follow exactly [FS96, Section 4]. Introduce an algebra $\widehat{\mathcal{A}}$ consisting of differential forms of even degree on the plane $\mathbb{R}^2$,

$$a(t,\tau) = a_0(t,\tau) + a_1(t,\tau)\, d\tau \wedge dt \tag{4.7}$$

where the coefficients are operator-valued symbols belonging to the classes $\Sigma^m(\delta, \delta - l)$, $l \geq m$, and stabilizing for large t. A product $\widehat{\circ}$ of two elements $a, b \in \widehat{\mathcal{A}}$ is defined by

$$\begin{aligned} a \widehat{\circ} b &= a \wedge b - \frac{i}{2}\, da \wedge db \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)\, d\tau \wedge dt - \frac{i}{2} \left(\frac{\partial a_0}{\partial \tau} \frac{\partial b_0}{\partial t} - \frac{\partial a_0}{\partial t} \frac{\partial b_0}{\partial \tau} \right) d\tau \wedge dt. \end{aligned}$$

One checks immediately that this product is associative.

Any function $a(t,\tau)$ may be considered as an element of $\widehat{\mathcal{A}}$ consisting of the 0-component only. So, for functions a, b, we have three products:

- ab, the usual point-wise product of operator-valued functions;
- $$a \circ b = ab - ih \partial_\tau a \partial_t b + \ldots, \tag{4.8}$$
  the Leibniz product of formal symbols consisting of leading terms only;
- $$a \widehat{\circ} b = ab - \frac{i}{2}\, da \wedge db, \tag{4.9}$$
  the product in the algebra $\widehat{\mathcal{A}}$.

Comparing (4.8) and (4.9) one can deduce a simple rule to pass from the product $a \circ b$ to the product $a \widehat{\circ} b$ of functions: keep the terms linear in h, alternate the derivatives $\partial/\partial\tau$ and $\partial/\partial t$, and then replace h by $d\tau \wedge dt$. This rule may be extended by induction to any number of factors: $a_1 \circ a_2 \circ \ldots \circ a_k$ and $a_1 \widehat{\circ} a_2 \widehat{\circ} \ldots \widehat{\circ} a_k$.

Next, we introduce a trace functional on the algebra $\widehat{\mathcal{A}}$ by

$$\begin{aligned} \operatorname{Tr} a &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \operatorname{tr} a \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \operatorname{tr} a_1(t,\tau)\, dt \wedge d\tau \end{aligned}$$

for an element a given by (4.7). We assume that $\mathbb{R}^2$ is oriented by the form $dt \wedge d\tau$. Of course, this functional is defined if $\operatorname{tr} a$ exists and the integral converges. In particular, this is the case if a belongs to an ideal $\widehat{\mathcal{J}} \subset \widehat{\mathcal{A}}$ consisting of Green symbols $\Sigma_G^{-\infty}(\delta, \delta)$ vanishing for large t.

Now, starting with the algebraic index formula (4.6) and applying the rule to pass from the $\circ$-product to the $\widehat{\circ}$-product, we come to the following index formula.

Proposition 4.4 *For N large enough,*

$$\begin{aligned}
\operatorname{ind} A &= \operatorname{Tr}\,(1-r_0\widehat{\circ}a)^{\widehat{\circ}N} - \operatorname{Tr}\,(1-a\widehat{\circ}r_0)^{\widehat{\circ}N} \\
&+\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\operatorname{tr}\left\{\sum_{k=0}^{\infty}(a_+^{-1}a_+')^{(k)}(\tau+i\gamma_+)\frac{(-i\gamma_+)^k}{k!}-r_0\sum_{k=0}^{N-1}(1-ar_0)^k a'|_{t=+\infty}\right\}d\tau \\
&-\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\operatorname{tr}\left\{\sum_{k=0}^{\infty}(a_-^{-1}a_-')^{(k)}(\tau-i\gamma_-)\frac{(i\gamma_-)^k}{k!}-r_0\sum_{k=0}^{N-1}(1-ar_0)^k a'|_{t=-\infty}\right\}d\tau.
\end{aligned} \tag{4.10}$$

where r_0 is a fiberwise parametrix of a.

Proof. Taking an algebraic parametrix in the form

$$r = r_0 \circ \sum_{k=0}^{N-1}(1-a\circ r_0)^{\circ k}$$

and substituting into (4.6), we obtain corner contributions as in (4.10) and the interior contribution in the form

$$\frac{1}{2\pi h}\int_{\mathbb{R}^2}\left(\operatorname{tr}(1-r_0\circ a)^{\circ N}-\operatorname{tr}(1-a\circ t_0)^{\circ N}\right)dt\wedge d\tau. \tag{4.11}$$

Let us extract the constant term in (4.11). So, we may calculate the integrand keeping only the terms linear in h. Thus,

$$1-r_0\circ a = 1-r_0a+ih\,\frac{\partial r_0}{\partial\tau}\,\frac{\partial a}{\partial t}+\dots$$

where the dots mean higher-degree terms in h. Further,

$$\begin{aligned}
(1-r_0\circ a)^{\circ N} &\sim (1-r_0a) \\
&+ih\sum_{k=0}^{N-1}(1-r_0a)^k\,\frac{\partial r_0}{\partial\tau}\,\frac{\partial a}{\partial t}\,(1-r_0a)^{N-1-k} \\
&-ih\sum_{k+p+q=N-2}(1-r_0a)^k\,\frac{\partial(1-r_0a)}{\partial\tau}\,(1-r_0a)^p\,\frac{\partial(1-r_0a)}{\partial t}\,(1-r_0a)^q
\end{aligned} \tag{4.12}$$

where $\sim$ means that the linear terms on both sides coincide. The second sum may be written as

$$\sum_{k=0}^{N-1}\frac{\partial(1-r_0a)^k}{\partial\tau}\,\frac{\partial(1-r_0a)}{\partial t}\,(1-r_0a)^{N-k-1} \tag{4.13}$$

or

$$\sum_{k=0}^{N-1} (1-r_0 a)^k \, \frac{\partial (1-r_0 a)}{\partial \tau} \, \frac{\partial (1-r_0 a)^{N-k-1}}{\partial t}. \tag{4.14}$$

Using "integration by parts", transform (4.14) to the form

$$\begin{aligned} & \frac{\partial}{\partial t} \sum_{k=0}^{N-1} (1-r_0 a)^k \, \frac{\partial (1-r_0 a)}{\partial \tau} \, (1-r_0 a)^{N-k-1} \\ & \quad - \sum_{k=0}^{N-1} \frac{\partial (1-r_0 a)^k}{\partial t} \, \frac{\partial (1-r_0 a)}{\partial \tau} \, (1-r_0 a)^{N-k-1} \\ & \quad - \sum_{k=0}^{N-1} (1-r_0 a)^k \, \frac{\partial^2 (1-r_0 a)}{\partial \tau \partial t} \, (1-r_0 a)^{N-k-1}. \end{aligned} \tag{4.15}$$

We represent the second sum in (4.12) as a half sum of (4.13) and (4.15), then take a fiberwise trace tr, divide over $2\pi h$ and take a constant term. As a result we obtain the following expression

$$\begin{aligned} & -\frac{1}{2\pi i} \, N \operatorname{tr} (1-r_0 a)^{N-1} \left(\frac{\partial r_0}{\partial \tau} \frac{\partial a}{\partial t} - \frac{1}{2} \, \frac{\partial^2 (r_0 a)}{\partial \tau \partial t} \right) \\ & \quad + \frac{1}{4\pi i} \, \frac{\partial^2}{\partial \tau \partial t} \operatorname{tr} (1-r_0 a)^N \\ & \quad + \frac{1}{4\pi i} \sum_{k=0}^{N-1} \left(\frac{\partial (1-r_0 a)^k}{\partial \tau} \frac{\partial (1-r_0 a)}{\partial t} - \frac{\partial (1-r_0 a)^k}{\partial t} \frac{\partial (1-r_0 a)}{\partial \tau} \right) (1-r_0 a)^{N-1-k}. \end{aligned}$$

Integrating this expression over $t, \tau \in \mathbb{R}^2$, we see that

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\partial^2}{\partial t \partial \tau} \operatorname{tr} (1-r_0 a)^N \, dt d\tau &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \operatorname{tr} (1-r_0 a)^N \Big|_{\tau=-\infty}^{\tau=\infty} dt \\ &= 0, \end{aligned}$$

so, this term may be omitted. Next,

$$\frac{\partial r_0}{\partial \tau} \frac{\partial a}{\partial t} - \frac{1}{2} \, \frac{\partial^2 r_0 a}{\partial \tau \partial t} = \frac{1}{2} \, \frac{\partial r_0}{\partial \tau} \frac{\partial a}{\partial t} - \frac{1}{2} \, \frac{\partial r_0}{\partial t} \frac{\partial a}{\partial \tau} - \frac{1}{2} \, \frac{\partial^2 r_0}{\partial \tau \partial t} \, a - \frac{1}{2} \, r_0 \frac{\partial^2 a}{\partial \tau \partial t}$$

and

$$\begin{aligned} & \left(\frac{\partial r_0}{\partial \tau} \frac{\partial a}{\partial t} - \frac{\partial r_0}{\partial t} \frac{\partial a}{\partial \tau} \right) d\tau \wedge dt \\ & \quad = \; dr_0 \wedge da, \end{aligned}$$

$$\begin{aligned} & \left(\frac{\partial (1-r_0 a)^k}{\partial \tau} \, \frac{\partial (1-r_0 a)}{\partial t} - \frac{\partial (1-r_0 a)^k}{\partial t} \, \frac{\partial (1-r_0 a)}{\partial \tau} \right) d\tau \wedge dt \\ & \quad = \; d(1-r_0 a)^k \wedge d(1-r_0 a). \end{aligned}$$

Thus, taking into account the orientation, we get for the constant term of $\mathrm{Tr}\ (1 - r_0 \circ a)^{\circ N}$ an expression

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{R}^2}\Big(\frac{N}{2}\,\mathrm{tr}\,(1-r_0a)^{N-1}\,dr_0\wedge da \\ -\frac{1}{2}\sum_{k=1}^{N-1}\mathrm{tr}\,(1-r_0a)^{N-1-k}\,d(1-r_0a)^k\wedge d(1-r_0a)\Big) \\ +\frac{1}{2\pi i}\,\frac{N}{2}\int_{\mathbb{R}^2}\mathrm{tr}\,(1-r_0a)^{N-1}\left(\frac{\partial^2 r_0}{\partial\tau\partial t}a + r_0\frac{\partial^2 a}{\partial\tau\partial t}\right)d\tau\wedge dt.\end{aligned} \tag{4.16}$$

A similar expression may be obtained for the constant term of $\mathrm{Tr}\,(1-a\circ r_0)^{\circ N}$ simply by interchanging a and r_0 in (4.16).

Note that

$$\begin{aligned}\mathrm{tr}\,(1-ar_0)^{N-1}\,\frac{\partial^2 a}{\partial\tau\partial t}\,r_0 &= \mathrm{tr}\,r_0\,(1-ar_0)^{N-1}\,\frac{\partial^2 a}{\partial\tau\partial t} \\ &= \mathrm{tr}\,(1-r_0a)^{N-1}\,r_0\,\frac{\partial^2 a}{\partial\tau\partial t}.\end{aligned}$$

It implies that the last integral in (4.16) remains unchanged under a permutation of a and r_0. Thus, taking the difference of (4.16) and the corresponding expression obtained by a permutation of a and r_0, we find

$$\begin{aligned}&\mathrm{Tr}\ (1-r_0\circ a)^{\circ N} - \mathrm{Tr}\ (1-a\circ r_0)^{\circ N} \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}^2}\Big(\frac{N}{2}\,\mathrm{tr}\,(1-r_0a)^{N-1}\,dr_0\wedge da \\ &\quad -\frac{1}{2}\sum_{k=1}^{N-1}\mathrm{tr}\,(1-r_0a)^{N-1-k}\,d(1-r_0a)^k\wedge d(1-r_0a) \\ &\quad -\frac{N}{2}\,\mathrm{tr}\,(1-ar_0)^{N-1}\,da\wedge dr_0 \\ &\quad +\frac{1}{2}\sum_{k=1}^{N-1}\mathrm{tr}\,(1-ar_0)^{N-1-k}\,d(1-ar_0)^k\wedge d(1-ar_0)\Big).\end{aligned} \tag{4.17}$$

This is nothing but

$$\mathrm{Tr}\ (1-r_0\widehat{\circ}a)^{\widehat{\circ}N} - \mathrm{Tr}\ (1-a\widehat{\circ}r_0)^{\widehat{\circ}N}$$

which may be checked similarly to (4.12). This completes the proof of the proposition.

□

Finally, using the associativity of the $\widehat{\circ}$-product we reduce (4.10) to the index formula (0.6) of the introduction. To this end we first show that the number N in (4.10) may be reduced to $N=2$. Indeed, for any $N>2$, we have

$$\begin{aligned}
&\mathrm{Tr}\,(1-r_0\widehat{\circ}a)^{\widehat{\circ}N}-\mathrm{Tr}\,(1-a\widehat{\circ}r_0)^{\widehat{\circ}N}\\
&\quad=\mathrm{Tr}\,(1-r_0\widehat{\circ}a)^{\widehat{\circ}(N-1)}-\mathrm{Tr}\,(1-a\widehat{\circ}r_0)^{\widehat{\circ}(N-1)}\\
&\quad-\mathrm{Tr}\,(1-r_0\widehat{\circ}a)^{\widehat{\circ}(N-1)}\widehat{\circ}r_0\widehat{\circ}a+\mathrm{Tr}\,(1-a\widehat{\circ}r_0)^{\widehat{\circ}(N-1)}\widehat{\circ}a\widehat{\circ}r_0.
\end{aligned}$$

The last line gives the trace of the commutator

$$-\mathrm{Tr}\,[(1-r_0\widehat{\circ}a)^{\widehat{\circ}(N-1)}\widehat{\circ}r_0,a]$$

understood with respect to the $\widehat{\circ}$-product. This is equal to

$$\begin{aligned}
&-\frac{1}{2\pi i}\int_{\mathbb{R}^2}\mathrm{tr}\,d\left((1-r_0\widehat{\circ}a)^{\widehat{\circ}(N-1)}\widehat{\circ}r_0\right)\wedge da\\
&\quad=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{tr}\,(1-r_0\widehat{\circ}a)^{\widehat{\circ}(N-1)}\widehat{\circ}r_0\,\partial_\tau a\Big|_{t=-\infty}^{t=\infty}d\tau,
\end{aligned}$$

which coincides with the last summand (with $k=N-1$) in the corner contributions

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\mathrm{tr}\left\{\sum_{k=0}^{\infty}(a_+^{-1}a_+')^{(k)}(\tau+i\gamma_+)\frac{(-i\gamma_+)^k}{k!}-\sum_{k=0}^{N-1}(1-r_0a)^kr_0a'|_{t=+\infty}\right\}d\tau\\
&\quad-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\mathrm{tr}\left\{\sum_{k=0}^{\infty}(a_-^{-1}a_-')^{(k)}(\tau-i\gamma_-)\frac{(i\gamma_-)^k}{k!}-\sum_{k=0}^{N-1}(1-r_0a)^kr_0a'|_{t=-\infty}\right\}d\tau.
\end{aligned}$$

For the case $N=2$ we use directly (4.17). This gives

$$\frac{1}{2\pi i}\int_{\mathbb{R}^2}\mathrm{tr}\,(1-r_0a)\,dr_0\wedge da-\mathrm{tr}\,(1-ar_0)\,da\wedge dr_0$$

since

$$\begin{aligned}
-\mathrm{tr}\,d(1-r_0a)\wedge d(1-r_0a)&=\mathrm{tr}\,d(1-ar_0)\wedge d(1-ar_0)\\
&=0.
\end{aligned}$$

Integrating by parts in the first summand, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathrm{tr}\,(1-r_0a)\,dr_0\wedge da\\
&\quad=\int_{\mathbb{R}^2}d\,(\mathrm{tr}\,(1-r_0a)r_0da)-\int_{\mathbb{R}^2}\mathrm{tr}\,d(1-r_0a)\wedge r_0da\\
&\quad=\int_{-\infty}^{\infty}\mathrm{tr}\,(1-r_0a)r_0\partial_\tau a\Big|_{t=-\infty}^{t=\infty}d\tau+\int_{\mathbb{R}^2}\mathrm{tr}\,(dr_0ar_0da+r_0dar_0da).
\end{aligned}$$

The first summand gives a part of the corner contributions at $N = 2$, that is the very last term in the expression

$$\int_{-\infty}^{\infty} \operatorname{tr} \left\{ \sum_{k=0}^{\infty} \partial_\tau^k (a_+^{-1}(\tau + i\gamma_+)\partial_\tau a_+(\tau + i\gamma)) - (r_0 + (1 - r_0 a) r_0)\, \partial_\tau a \,|_{t=\infty} \right\} d\tau.$$

Further,

$$\int_{\mathbb{R}^2} \operatorname{tr} (1 - r_0 a)\, da \wedge dr_0 = -\int_{\mathbb{R}^2} \operatorname{tr} dr_0 \wedge (1 - a r_0)\, da,$$

so, gathering all the terms, we come to the index formula (0.6).

5 Trace Properties of Edge and Corner Pseudodifferential Operators

This appendix contains an auxiliary material related to the trace of edge and corner pseudodifferential operators. Most of this material is scattered in our previous publications [FS96, FST98, FST99]. For the reader's convenience we repeat it here more or less systematically in a proper form.

We will consider here the edge symbols introduced in Section 3. A symbol class $\Sigma^m(\delta, \delta - l)$, $l \geq m$, (of order m with respect to weight data $(\delta, \delta - l)$) consists of operator-valued functions

$$a(t, \tau) = a_{\mathrm{int}}(t, \tau) + a_{\mathrm{edge}}(t, \tau) + a_G(t, \tau). \tag{5.1}$$

The first two summands act in the spaces

$$H^{s,\delta}(Y) \to H^{s-m,\delta-m}(Y)$$

while their derivatives

$$\partial_t^\alpha \partial_\tau^\beta \left(a_{\mathrm{int}} + a_{\mathrm{edge}}\right)$$

act in

$$H^{s,\delta}(Y) \to H^{s-m+\beta,\delta-m+\beta}(Y).$$

All these spaces are considered with the family of norms $\|\cdot\|_\tau$ (see Subsection 1.2), and all these operators are bounded in these norms uniformly with respect to τ.

The third item called a Green symbol and its derivatives act boundedly in the spaces

$$\partial_t^\alpha \partial_\tau^\beta a_G(t, \tau) : \begin{array}{c} H^{s_1,\delta}(Y) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} H^{s_2,\delta-l+\varepsilon}(Y) \\ \oplus \\ \mathbb{C}^{N_+} \end{array} \tag{5.2}$$

for some $\varepsilon > 0$, with the norms $\|\cdot\|_\tau$ defined by (1.13). These operators are bounded for all s_1, s_2, that is they are infinitely smoothing on the fibers Y, but *the norm bounds are not uniform in* τ. However, for any $s_1 = s$ and $s_2 = s - l + \beta$, the operators (5.2) *are bounded uniformly in* τ.

The whole sum (5.1) acts in the spaces

$$a(t,\tau) : \begin{array}{c} H^{s,\delta}(Y) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} H^{s-m,\delta-l}(Y) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}, \tag{5.3}$$

the summands a_{int} and a_{edge} are viewed as (2×2)-matrices with the only non-zero entry in the left upper corner. Since $\delta - m \geq \delta - l$ the operator (5.3) is bounded uniformly in τ. All the summands are supposed to stabilize for $t \to \pm\infty$.

The following simple observation is crucial in the sequel.

Proposition 5.1 *If* $\beta > 0$ *then for some* $\varepsilon > 0$ *the operator*

$$\partial_t^\alpha \partial_\tau^\beta a(t,\tau) : \begin{array}{c} H^{s,\delta}(Y) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} H^{s-m+\beta,\delta-l+\varepsilon}(Y) \\ \oplus \\ \mathbb{C}^{N_+} \end{array},$$

is bounded uniformly in τ *with respect to the norms* $\|\cdot\|_\tau$.

Proof. Assuming ε in (5.2) less than 1 we have

$$\delta - m + \beta > \delta - l + \varepsilon,$$

for $\beta > 0$ and $l \geq m$. Thus,

$$H^{s-m+\beta,\delta-m+\beta}(Y) \hookrightarrow H^{s-m+\beta,\delta-l+\varepsilon}(Y).$$

This proves the proposition since the norm of the embedding is uniformly bounded by Lemma 2.1.

□

Thus, for given cone weight data $(\delta, \delta-l)$ with $l \geq m$ (usually one starts with weight data $(\delta, \delta - m)$), the Green summand gives a gain in weight by ε. The whole symbol (5.3) looses this gain in general but after derivation in τ we regain it. Or, more briefly, the inclusion $a \in \Sigma^m(\delta, \delta - l)$, $l \geq m$, implies for $\beta > 0$

$$\partial_t^\alpha \partial_\tau^\beta a \in \Sigma^{m-\beta}(\delta, \delta - l + \varepsilon). \tag{5.4}$$

5.1 Compatibility

A corner operator $A \in \Psi^m(M)$ is defined via (3.23) by a triple

$$\{a_-(\tau), a(t,\tau), a_+(\tau)\}$$

of edge symbols satisfying compatibility conditions. The symbol $a(t,\tau)$ belongs to $\Sigma^m(\delta, \delta - l)$ and, for large t, stabilizes to $a_0(\pm\infty, \tau)$. The latter may be viewed as edge symbols from $\Sigma^m(\delta, \delta - l)$ independent of t. The symbols $a_\pm(\tau)$ are defined on horizontal lines $\Im\tau = \pm i\gamma_\pm$ and as functions in $\Re\tau$ belong to the symbol class $\Sigma^m(\delta, \delta - l)$, that is

$$a_\pm(\tau \pm i\gamma_\pm) \in \Sigma^m(\delta, \delta - m) \tag{5.5}$$

with a real variable τ.

Definition 5.2 *The symbol $a(t, \tau)$ is compatible with the symbol* (5.5) *if for any N and some $\varepsilon > 0$*

$$\begin{aligned} T_N(\tau) \;&:=\; a(\pm\infty, \tau) - \sum_{k<N} a_\pm^{(k)}(\tau \pm i\gamma_\pm)\, \frac{(\mp i\gamma_\pm)^k}{k!} \\ &\in\; \Sigma^{m-N}(\delta, \delta - l + \varepsilon). \end{aligned} \tag{5.6}$$

Were $a(t, \tau)$ an entire function in τ as in the case of differential operators, the symbols $a_\pm(\tau)$ might be taken as restrictions of $a(\pm\infty, \tau)$ to the corner weight lines $\Im\tau = \pm\gamma_\pm$, the condition (5.6) would be fulfilled automatically in virtue of (5.4) since $T_N(\tau)$ in this case would be a remainder in the Taylor formula. In general, $T_N(\tau)$ is the difference between $a(\pm\infty, \tau)$ and a formal complex shift of $a_\pm(\tau \pm i\gamma_\pm)$ by $\mp i\gamma_\pm$, it should be "small" in the sense of (5.6).

We will use the compatibility condition in the following situation.

Lemma 5.3 *For any functions $f_1(t)$, $f_2(t)$ of $C_0^\infty(\mathbb{R})$, the operator*

$$A_N = f_1(t)\, \mathrm{Op}\,(T_N)\, f_2(t) \tag{5.7}$$

in the corner Sobolev space $H^{s,\delta,\gamma_+,\gamma_-}(M)$ belongs to the trace class provided $N \geq m + n + 2$.

Proof. From (5.6) it follows that

$$T_N(\tau) \in \Sigma^{-N}(\delta, \delta - l + \varepsilon).$$

Thus, the operator (5.7) may be factorized through

$$A_N :\; H^{s,\delta,\gamma_+,\gamma_-}(M) \to H^{s-N,\delta+\varepsilon,\gamma_++\varepsilon,\gamma_-+\varepsilon}(M).$$

The corner weights γ_+, γ_- are improved because f_1, f_2 have compact supports. Now the lemma follows from Corollary 1.8.

$\square$

5.2 Pseudolocality

We will need a pseudolocality property for corner ΨDO's, especially near corner points. Roughly speaking, it consists in the fact that a ΨDO

$$f_1 \operatorname{Op}^\gamma(a)\, f_2$$

belongs to the trace class if the supports of f_1 and f_2 have empty intersection. Here f_1 and f_2 are smooth functions either having compact supports or equal to 1 in a neighborhood of $\pm\infty$ and vanishing outside a larger neighborhood. The weight γ may be equal to 0, γ_+, γ_-, and the symbol $a(t,\tau) \in \Sigma^m(\delta, \delta - l)$ coincides with one of the symbols $a_i(t,\tau)$, $a_+(\tau)$, $a_-(\tau)$ entering the corner operator (3.22).

Below we consider one of such statements leaving to the reader similar statements for other situations. So, let

$$\begin{aligned} \operatorname{Op}^\gamma(a)u &= \frac{1}{2\pi}\int_{\Im\tau=\gamma} e^{i\tau t} a(\tau)\widehat{u}(\tau)\, d\tau \\ &= \frac{1}{2\pi}\int_{\Im\tau=\gamma} d\tau \int_{-\infty}^{\infty} e^{i\tau(t-t')} a(\tau) u(t')\, dt'. \end{aligned} \tag{5.8}$$

The function $a(\tau)$ is holomorphic in τ in some strip $|\Im\tau - \gamma| < \varepsilon$ and belongs to the symbol class $\Sigma^m(\delta, \delta - l)$ on any horizontal line in the strip, uniformly in $\Im\tau \in [\gamma - \varepsilon_0, \gamma + \varepsilon_0]$ with any $\varepsilon_0 < \varepsilon$. The operator (5.8) is considered in the space $H^{s,\delta,\gamma}(M)$ consisting of functions $u(t)$ with values in fiber spaces $H^{s,\delta}(Y)$ equipped with the family of norms $\|\cdot\|_\tau$, cf. (1.12). The norm $\|u(t)\|$ in $H^{s,\delta,\gamma}(M)$ coincides with the L^2-norm of the function $\|\widehat{u}(\tau)\|_\tau$, that is

$$\|u\|^2 = \int_{\Im\tau=\gamma} \|\widehat{u}(\tau)\|_\tau^2\, d\tau.$$

Lemma 5.4 *Let $f_1(t) \equiv 0$ for large negative t, and $f_1(t) \equiv 1$ for large positive t. Let the function f_2 be equal to 0 for large negative t, and $f_1(t)f_2(t) \equiv 0$. Then the operator*

$$A = f_1 \operatorname{Op}^\gamma(a)\, f_2 : \ H^{s,\delta,\gamma}(M) \to H^{s-m,\delta-l,\gamma}(M)$$

belongs to the trace class.

Proof. Our operator may be rewritten in the form

$$Au(t) = \frac{1}{2\pi}\int_{\Im\tau=\gamma} d\tau \int_{-\infty}^{\infty} e^{i\tau(t-t')}\, i^N\, \partial_\tau^N a(\tau)\, \frac{f_1(t)f_2(t')}{(t-t')^N}\, u(t')\, dt'$$

since $t - t' \neq 0$ on the support of $f_1(t)f_2(t')$. We will prove that A is bounded in the spaces

$$H^{s,\delta,\gamma}(M) \to H^{s-m+N,\delta+\varepsilon,\gamma+\varepsilon}(M),$$

for a large positive integer N and some $\varepsilon > 0$. Taking a suitable cut-off function $\omega_+(t)$ equal identically to 0 for large negative t and covering the function $f_1(t)$, that is $\omega_+(t) f_1(t) \equiv f_1(t)$, we see that the operator $A = \omega_+(t)\, A$ belongs to the trace class since multiplication by $\omega_+(t)$ followed by the embedding

$$H^{s-m+N,\delta+\varepsilon,\gamma+\varepsilon}(M) \to H^{s-m,\delta,\gamma}(M)$$

is a trace class operator by Corollary 1.8.

Passing to the Fourier transforms

$$\begin{aligned} (Au)(t) &= \frac{1}{2\pi} \int_{\Im\tau=\gamma} e^{i\eta t} \widehat{v}(\eta)\, d\eta, \\ u(t') &= \frac{1}{2\pi} \int_{\Im\xi=\gamma} e^{i\xi t'} \widehat{u}(\xi)\, d\xi, \end{aligned}$$

we find

$$\widehat{v}(\eta) = \int_{\Im\tau=\gamma} d\tau \int_{\Im\xi=\gamma} \partial_\tau^N a(\tau)\, b(\eta-\tau, \xi-\tau)\, \widehat{u}(\xi)\, d\xi$$

where

$$b(\eta,\xi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\eta t + i\xi t'}\; i^N \frac{f_1(t) f_2(t')}{(t-t')^N}\, dt dt'.$$

Note that $b(\eta,\xi)$ is holomorphic in η and τ in the lower half-planes $\Im\eta < 0$, $\Im\tau < 0$ and is rapidly decreasing on any horizontal lines

$$\begin{aligned} \Im\eta &= c_1 < 0; \\ \Im\tau &= c_2 < 0. \end{aligned}$$

Of course, N is supposed to be sufficiently large. We will use the notation

$$b(\eta,\xi) = O(\langle\xi\rangle^{-\infty})\, O(\langle\eta\rangle^{-\infty})$$

which means that the function

$$\langle\xi\rangle^p \langle\eta\rangle^q\, b(\eta,\xi)$$

considered on the lines $\Im\eta = c_1 < 0$ and $\Im\tau = c_2 < 0$ is bounded for any p, q, c_1, c_2.

The function $\partial_\tau^N a(\tau)$ is also holomorphic in a strip, so shifting the integration lines we get

$$\widehat{v}(\tau) = \int_{\Im\tau=\gamma+\varepsilon_0} d\tau \int_{\Im\xi=\gamma} \partial_\tau^N a(\tau)\, b(\eta-\tau, \xi-\tau)\, \widehat{u}(\xi)\, d\xi$$

where we take $\Im\eta = \gamma + \varepsilon_0/2$.

The derivative $\partial^N a(\tau)$ of the symbol $a(\tau) \in \Sigma^m(\delta, \delta - l)$ belongs to the class $\Sigma^{m+N}(\delta, \delta - N + \varepsilon)$ (see (5.4)), and we are going to show that v belongs to the space $H^{s-m+N,\delta+\varepsilon,\gamma+\varepsilon/2}(M)$. Let us estimate the norm $\|\widehat{v}(\eta)\|_\eta$ in the space $H^{s-m+N,\delta+\varepsilon}(Y)$. We have

$$\|\widehat{v}(\eta)\|_\eta \leq \int_{\Im\tau=\gamma+\varepsilon_0} d\tau \int_{\Im\xi=\gamma} O(\langle\eta-\tau\rangle^{-\infty})\, O(\langle\xi-\tau\rangle^{-\infty})\, \|\partial_\tau^N a(\tau)\widehat{u}(\xi)\|_\eta\, d\xi.$$

But by Lemma 1.3

$$\begin{array}{rcl} \|\partial_\tau^N a(\tau)\, \widehat{u}(\xi)\|_\eta & \leq & C\, \|\partial_\tau^N a(\eta)\, \widehat{u}(\xi)\|_\tau \, \langle\eta-\tau\rangle^q \\ & \leq & C\, \|\partial_\tau^N a(\tau)\|_\tau \, \|\widehat{u}(\xi)\|_\tau \, \langle\eta-\tau\rangle^q \\ & \leq & C\, \langle\tau-\xi\rangle^q \, \langle\eta-\tau\rangle^q \, \|\widehat{u}(\xi)\|_\xi. \end{array}$$

Here $\|\widehat{u}(\xi)\|_\tau$ means the norm in $H^{s,\delta}(Y)$. By the definition of the classes $\Sigma^{m+N}(\delta,\delta-l+\varepsilon)$, the norm $\|\partial_\tau^N a(\tau)\|_\tau$ is bounded. So, we obtain finally

$$\|\widehat{v}(\eta)\|_\eta \leq C \int_{\Im\tau=\gamma+\varepsilon_0} d\tau \int_{\Im\xi=\gamma} O(\langle\eta-\tau\rangle^{-\infty})\, O(\langle\xi-\tau\rangle^{-\infty})\, \|\widehat{u}(\xi)\|_\xi\, d\xi.$$

But such an integral operator in L^2 is bounded, implying that A is bounded from $H^{s,\delta,\gamma}(M)$ to $H^{s-m+N,\delta+\varepsilon,\gamma+\varepsilon_0/2}(M)$. This completes the proof of the lemma.

□

5.3 Regularized trace of a product

We again consider a neighborhood of a corner point $t=+\infty$, so all functions $u(t)$ and symbols $a(t,\tau)$ are supposed to vanish for $t<0$. Thus, the corner weight γ_- at $t=-\infty$ is inessential and we drop it from our notation, writing $H^{s,\delta,\gamma_+}(M)$ instead of $H^{s,\delta,\gamma_+,\gamma_-}(M)$. We consider here a corner version of the theorem on the regularized trace of a product. In [FS96] and [FST98] the corresponding cone and edge versions were proved.

Consider two symbols $a(t,\tau)$, $b(t,\tau)$ vanishing for $t<0$ and stabilizing to $a(+\infty,\tau)$, $b(+\infty,\tau)$ for large positive t. We assume that they are holomorphic in the strip $|\Im\tau-\gamma_+|<\varepsilon$, and on each horizontal line within the strip we have

$$\begin{array}{rcl} a(t,\tau) & \in & \Sigma^m(\delta,\delta-l), \\ b(t,\tau) & \in & \Sigma^{-m}(\delta-l,\delta) \end{array}$$

uniformly with respect to $\Im\tau$ in each smaller strip. The corresponding corner ΨDO $A=\mathrm{Op}^{\gamma_+}(a)$ defined by (5.8) may be considered in any of the spaces $H^{s,\delta,\gamma}(M)$ with $\gamma\in(\gamma_+-\varepsilon,\gamma_++\varepsilon)$, since the integration line $\Im\tau=\gamma_+$ in (5.8) may be shifted to $\Im\tau=\gamma$. Thus, the operator A may be extended to a bounded operator

$$A=\mathrm{Op}^\gamma(a):\ H^{s,\delta,\gamma}(M)\to H^{s-m,\delta-l,\gamma}(M).$$

Consider the operator

$$\begin{array}{rcl} C_N & = & BA-\mathrm{Op}^\gamma(b\circ a\,|_N) \\ & = & \mathrm{Op}^\gamma(b)\,\mathrm{Op}^\gamma(a)-\mathrm{Op}^\gamma(b\circ a\,|_N) \end{array}$$

in the spaces

$$C_N:\ H^{s,\delta,\gamma}(M) \to H^{s,\delta,\gamma}(M).$$

Lemma 5.5 *For $N > n+2$, the operator C_N is of trace class.*

Proof. We pass to the Fourier transform of symbols. Denote

$$\begin{aligned} \widehat{a}(\xi,\tau) &= \int_{-\infty}^{\infty} e^{-i\xi t}\, a(t,\tau)\, dt \\ &= \frac{1}{i\xi}\int_{-\infty}^{\infty} e^{-i\xi t}\, \partial_t a(t,\tau)\, dt. \end{aligned} \tag{5.9}$$

The integral is defined for $\Im\xi < 0$, the last equality being obtained by integration by parts. The integrand in the last integral is a function with compact support because of the stabilization condition. Thus, $\widehat{a}(\xi,\tau)$ is a holomorphic function in $\xi \in \mathbb{C}$ except the only first-order pole $\xi = 0$. The operator A may be written as

$$\widehat{Au}(\xi) = \int_{\Im\tau=\gamma} \widehat{a}(\xi-\tau,\tau)\, \widehat{u}(\tau)\, d\tau.$$

Here we need to take $\Im\xi < \gamma$ since according to (5.9) $\widehat{a}(\xi-\tau,\tau)$ is defined for $\Im(\xi-\tau) < 0$.

Similarly, for the operator BA we obtain

$$\widehat{BAu}\,(\xi) = \int_{\Im\xi=\gamma_1}\int_{\Im\tau=\gamma} \widehat{b}(\zeta-\xi,\xi)\, \widehat{a}(\xi-\tau,\tau)\, \widehat{u}(\tau)\, d\tau$$

with $\gamma_1 < \gamma$ and $\Im\zeta < \gamma_1$. Expanding by Taylor's formula, we obtain

$$\widehat{b}(\zeta-\xi,\xi) = \sum_{k<N} \partial_\tau^k \widehat{b}(\zeta-\xi,\tau)\, (\xi-\tau)^k/k! + R_N(\zeta-\xi,\xi,\tau),$$

with the remainder

$$R_N(\zeta-\xi,\xi,\tau) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!}\, \widehat{b}^{(N)}(\zeta-\xi,\tau+\theta(\xi-\tau))\, (\xi-\tau)^N\, d\theta$$

where $\widehat{b}^{(N)}$ means the derivative with respect to the second argument. The regularized product C_N may be written in the form

$$\begin{aligned} \widehat{C_N u}\,(\zeta) &= \int\limits_{\Im\xi=\gamma_1} d\xi \int\limits_{\Im\xi=\gamma} T_N(\zeta-\xi,\xi,\tau)\, \widehat{a}(\xi-\tau,\tau)\, d\tau \\ &= \int\limits_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} d\theta \int\limits_{\Im\xi=\gamma_1} d\xi \int\limits_{\Im\tau=\gamma} \widehat{b}^{(N)}(\zeta-\xi,\tau+\theta(\xi-\tau))(\xi-\tau)^N \widehat{a}(\xi-\tau,\tau)\widehat{u}(\tau) d\tau. \end{aligned} \tag{5.10}$$

For $N > 0$ the function $(\xi - \tau)^N \widehat{a}(\xi - \tau, \tau)$ has no pole in $\xi - \tau$, so, the restriction $\gamma_1 < \gamma$ is no longer needed. The only restriction remains $\Im\zeta < \gamma_1$. Thus, we can choose $\Im\zeta = \gamma_2$ with $\gamma_1 > \gamma_2 \geq \gamma$.

Show that C_N is bounded in the spaces

$$C_N : \; H^{s,\delta,\gamma}(M) \to H^{s+N,\delta_2,\gamma_2}(M) \tag{5.11}$$

with $N \geq n+2$ and some $\delta_2 > \delta$, $\gamma_2 > \gamma$. The gain in corner weight follows since we can choose $\gamma_2 = \Im\zeta > \gamma$. The gain in the cone weight is a consequence of Proposition 5.1 because of the term $\widehat{b}^{(N)}$ with $N > 0$ in (5.10). This term gives also a gain in smoothness by N and this is sufficient to prove (5.11).

More detailed estimates look as follows. Denote

$$\begin{array}{rcl} \widehat{v}(\zeta) & = & \widehat{C_N u}\,(\zeta), \\ \widehat{w}(\xi,\tau) & = & (\xi-\tau)^N\,\widehat{a}(\xi-\tau,\tau)\,\widehat{u}(\tau), \\ \sigma = \sigma(\xi,\tau,\theta) & = & \tau + \theta(\xi-\tau). \end{array} \tag{5.12}$$

Then (5.10) implies

$$\|\widehat{v}(\zeta)\|_\zeta \leq \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} d\theta \int_{\Im\xi=\gamma_1} d\xi \int_{\Im\tau=\gamma} \|\widehat{b}^{(N)}(\zeta-\xi,\sigma)\,\widehat{w}(\xi,\tau)\|_\zeta \, d\tau$$

where $\|\cdot\|_\zeta$ means the norm in $H^{s+N,\delta_2}(Y)$ depending on a complex parameter ζ, cf. (1.12) (recall that $\|\cdot\|_\zeta := \|\cdot\|_{\Re\zeta}$).

We will apply Lemma 1.3 to estimate this norm. Thus,

$$\|\widehat{b}^{(N)}(\zeta-\xi,\sigma)\,\widehat{w}(\xi,\tau)\|_\zeta \leq C\,\langle\zeta-\sigma\rangle^q\,\|\widehat{b}^{(N)}(\zeta-\xi,\sigma)\,\widehat{w}(\xi,\tau)\|_\sigma. \tag{5.13}$$

To estimate the norm of $\widehat{b}^{(N)}(\zeta-\xi,\sigma)$, consider the symbol

$$\widehat{b}^{(N)}(t,\sigma) \in \Sigma^{-m-N}(\delta - l, \delta_2),$$

the gain $\delta_2 - \delta$ in the cone weight is due to the derivation ($N > 0$) by Proposition 5.1. So, the norm of the operator

$$b^{(N)}(t,\sigma) : \; H^{s-m,\delta-l}(Y) \to H^{s+N,\delta_2}(Y)$$

is uniformly bounded in σ, both spaces being equipped with the family of norms $\|\cdot\|_\sigma$. It is also uniformly bounded in t since this symbol vanishes for $t < 0$ and stabilizes for large positive t. Its Fourier transform

$$\widehat{b}(\lambda,\sigma) = \int_{-\infty}^{\infty} e^{-i\lambda t} b^{(N)}(t,\sigma)\,dt$$

being holomorphic in λ, $\Im\lambda < 0$, is rapidly decreasing on any fixed horizontal line in the lower half-plane, that is

$$\|\widehat{b}^{(N)}(\lambda,\sigma)\|_\sigma = O(\langle\lambda\rangle^{-\infty})$$

uniformly in $\Re\sigma$ if $\Im\lambda < 0$ and $\Im\sigma$ are fixed. Thus, the estimate (5.13) may be continued to give

$$C\,\langle\zeta-\sigma\rangle^q\,O(\langle\zeta-\xi\rangle^{-\infty})\,\|\widehat{w}(\xi,\tau)\|_\sigma.$$

Similar arguments applied to (5.12) give

$$\begin{aligned}\|\widehat{w}(\xi,\tau)\|_\sigma &\le C\,\langle\sigma-\tau\rangle^q\,\|\widehat{w}(\xi,\tau)\|_\tau\\ &= C\,\langle\sigma-\tau\rangle^q\,O(\langle\xi-\tau\rangle^{-\infty})\,\|\widehat{u}(\tau)\|_\tau\end{aligned}$$

where the norm $\|\widehat{u}(\tau)\|_\tau$ is taken in the space $H^{s,\delta}(Y)$. Note that

$$(\xi-\tau)^N\,\widehat{a}(\xi-\tau,\tau)$$

is an entire function in $\xi-\tau$ rapidly decreasing on any horizontal line. Now, since

$$\begin{aligned}\langle\sigma-\tau\rangle^q &= \langle\theta(\xi-\tau)\rangle^q\\ &\le \langle\xi-\tau\rangle^q\end{aligned}$$

for $q>0$, and

$$\begin{aligned}\langle\zeta-\sigma\rangle^q &= \langle\zeta-\xi+(1-\theta)(\xi-\tau)\rangle^q\\ &\le C\,\langle\zeta-\xi\rangle^q\,\langle\xi-\tau\rangle^q\end{aligned}$$

by Peetre's inequality, we come to the final estimate in (5.13) of the form

$$O(\langle\zeta-\xi\rangle^{-\infty})\,O(\langle\xi-\tau\rangle^{-\infty})\,\|\widehat{u}(\tau)\|_\tau,$$

because

$$\langle\zeta-\xi\rangle^q\,O(\langle\zeta-\xi\rangle^{-\infty}) = O(\langle\zeta-\xi\rangle^{-\infty})$$

for finite q. Integrating over $\theta\in[0,1]$ and over the lines $\Im\xi=\gamma_1$, $\Im\tau=\gamma$, we obtain

$$\begin{aligned}\|\widetilde{v}(\zeta)\|_\zeta &\le \int_{\Im\xi=\gamma_1} d\xi \int_{\Im\tau=\gamma} O(\langle\zeta-\xi\rangle^{-\infty})\,O(\langle\xi-\tau\rangle^{-\infty})\,\|\widehat{u}(\tau)\|_\tau\,d\tau\\ &\le \int_{\Im\tau=\gamma} O(\langle\zeta-\tau\rangle^{-\infty})\,\|\widehat{u}(\tau)\|_\tau\,d\tau \qquad (5.14)\end{aligned}$$

for $\Im\zeta=\gamma_2>\gamma$.

It remains to note that the norm of $C_N u$ in $H^{s+N,\delta_2,\gamma_2}(M)$ is equal to the L^2-norm of $\|\widetilde{v}(\zeta)\|_\zeta$, and the norm of u in $H^{s,\delta,\gamma}(M)$ is equal to the L^2-norm of $\|\widehat{u}(\tau)\|_\tau$. So, the boundedness of (5.11) follows from the boundedness of the integral operator (5.14) with the kernel $O(\langle\zeta-\tau\rangle^{-\infty})$ in L^2.

□

Under the previous assumptions on A and B define a regularized trace of a product as

$$\mathrm{Tr}_N BA := \mathrm{Tr}\;C_N$$

for $N\ge n+2$. Similarly we define

$$\mathrm{Tr}_N AB = \mathrm{Tr}\;(AB-\mathrm{Op}(a\circ b\,|_N))\,.$$

Theorem 5.6 *The regularized trace of a product does not depend on the order of factors, that is*

$$\mathrm{Tr}_N BA = \mathrm{Tr}_N AB.$$

Proof. We now put $\gamma_1 > \gamma_2 = \gamma$ in (5.10). To calculate the trace of the integral operator (5.10) (which belongs to the trace class as we know) we integrate its kernel over the diagonal obtaining

$$\begin{aligned}
&\mathrm{Tr}_N BA \\
&\quad = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} d\theta \int_{\Im\xi=\gamma_1} d\xi \int_{\Im\tau=\gamma} \mathrm{tr}\, \widehat{b}^{(N)}(\tau-\xi, \tau+\theta(\xi-\tau))(\xi-\tau)^N \widehat{a}(\xi-\tau,\tau) d\tau \\
&\quad = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} d\theta \int_{\Im s=\gamma_1-\gamma} ds \int_{\Im\tau=\gamma} \mathrm{tr}\, \widehat{b}^{(N)}(-s, \tau+\theta s)\, s^N\, \widehat{a}(s,\tau) d\tau.
\end{aligned}$$

Here tr means the operator trace in $H^{s,\delta}(Y)$ (that is, the fiberwise trace of the operator-valued symbol). In the last integral we have changed variables $s = \xi - \tau$. Note that for $N \geq 2$ there are no poles in s in the integrand. Indeed, $\widehat{a}(s,\tau)$ and $\widehat{b}^{(N)}(-s, \tau+\theta s)$ have first-order poles at $s = 0$, which disappear because of the factor s^N, $N \geq 2$.

Therefore, we can shift the integration line $\Im s = \gamma_1 - \gamma$ to the real axis $\Im s = 0$ obtaining the integral

$$\int_{-\infty}^{\infty} ds \int_{\Im\tau=\gamma} \mathrm{tr}\, \widehat{b}_\tau^{(N)}(-s, \tau+\theta s)\, s^N\, \widehat{a}(s,\tau)\, d\tau.$$

Integrating by parts in τ and changing variables

$$\begin{aligned} s' &= -s, \\ \tau' &= \tau + \theta s, \end{aligned}$$

we get

$$\int_{-\infty}^{\infty} ds' \int_{\Im\tau=\gamma} \mathrm{tr}\, \widehat{b}(s',\tau')\, (s')^N\, \widehat{a}_{\tau'}^{(N)}(-s', \tau'+\theta\tau')\, d\tau'$$

which coincides with the corresponding expression for $\mathrm{Tr}_N AB$.

□

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