The minimmum number of zeros of Lipschitz-Killing curvature

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Abstract. In the previous paper [Pi1] are given some sufficient topological conditions expressed in terms of the homotopy groups of the two given differentiable manifolds M, N in order that the φ-category of the pair (M, N) be infinite. As an application we observe in this paper that a lower bound for the minimmum number of points of zeros of the Lipschitz-Killing curvature of some m-dimensional manifold immersible in $\mathbf{R}^{m+k}$, with respect to all this kind of immersions, can be given in terms of φ-category.

1. Introduction

Recall that the φ-category of the pair (M, N) of closed manifolds is defined as

$$\varphi(M, N) = \min\{\#C(f) \mid f \in C^{\infty}(M, N)\}$$

where $C(f)$ is the critical set of f. A survey containing general results concerning φ-category is contained in the paper [An].

Let M^m be a differentiable manifold immersible in $\mathbf{R}^{m+k}$, $f : M \to \mathbf{R}^{m+k}$ be an immersion and $v \in (df)_p(T_p(M))^{\perp} \cap S^{m+k-1}$, where $p \in M$ is a given point. Consider the second fundamental form $\phi^{v}_{f,p}$ of f and the projections $N_f : \mathcal{N}^{f,1} \to S^{m+k-1}$, $\pi : \mathcal{N}^{f,1} \to M$, where

$$\begin{aligned}\mathcal{N}^{f,1} &= \{(p, v) \in M \times S^{m+k-1} \ : \ v \perp (df)_p(T_p(M))\} = \\ &= \bigcup_{p \in M} \Big(\{p\} \times (S^{m+k-1} \cap (df)_p(T_p(M))^{\perp})\Big).\end{aligned}$$

Observe that π is a fibration with fibre S^{k-1}. Recall that the *Lipschitz-Killing curvature* of f at the point $p \in M$ in the direction v is defined as

$$L_{f,v} = \frac{\det(\phi^{v}_{f,p})}{\det(g_f(p))}$$

and that $\phi^{v_0}_{f,p_0}$ is degenerated if and only if (p_0, v_0) is a critical point of N_f (for more details we refer to the book [Bu]). Therefore

$$C(N_f) = \{(p, v) \in \mathcal{N}^{f,1} \ : \ L_{f,v}(p) = 0\},$$

that is the φ-category of the pair $(\mathcal{N}^{f,1}, S^{m+k-1})$ is a lower bound for the minimmum number of points of zeros of the Lipschitz-Killing curvature with respect to all immersions of M in $\mathbf{R}^{m+k}$. This fact can be shortly written by means of the inequality

$$L_k(M) \geq \varphi(\mathcal{N}^{f,1}, S^{m+k-1}),$$

where

$$L_k(M) = \min\{\#C(N_f) \ : \ f \in Imm^k(M)\}$$

[1]This paper is supported by the European Research and Training Network "GEOMETRIC ANALYSIS". It was written in a period when the second author had a research appointment position at "Paul Sabatier" University, Toulouse, France (November 1, 2000-May 1, 2001)

is the so called L_k-*category* or the *immersiability category* of M, and $Imm^k(M)$ is the set of all immersions of M into $\mathbf{R}^{m+k}$. The immersiability category $L_k(M)$ is studied in the case of surfaces and $k=1$ in the recent second author's paper [Pi2]. To prove that the $L_k(M)$ is infinite in certain situations, we will use the following theorem.

Theorem 1.1. *(*[Pi1]*) Let M, N be compact connected differentiable manifolds having the same dimension m. In these conditions the following statements are true:*

(i) If $m \geq 3$ and $\pi_1(M)$ cannot be embedded as a subgroup in $\pi_1(N)$, then $\varphi(M,N) \geq \aleph_0$;

(ii) If $m \geq 4$ and $\pi_q(M) \not\simeq \pi_q(N)$ for some $q \in \{2, \dots, m-2\}$, then $\varphi(M,N) \geq \aleph_0$.

2. The implications of Theorem 1.1 on the L_k-category

Because the conditions of Theorem 1.1 are expressed in terms of homotopy groups of the given manifolds and our lower bound for the minimmum number of points of zeros of the Lipschitz-Killing curvature is $\varphi(\mathcal{N}^{f,1}, S^{m+k-1})$, we will try to relate in what follows the homotopy groups of $\mathcal{N}^{f,1}$ with those of M. Recall that $k(M)$ denotes the smallest natural number such that M is mmersable in $\mathbf{R}^{m+k(M)}$. Observe, by means of Whitney's theorem, that $k(M) \leq 2m+1$ and if M^m is a compact manifold, then $k(M) \leq 2m-1$.

Theorem 2.1. *Let M^m be a connected closed manifold.*

(i) If $m \geq 1$ and $\pi_1(M)$ is not trivial, then $L_k(M) \geq \aleph_0$, $\forall k \geq \max\{3, k(M)\}$;

(ii) If $m \geq 2$ and $\pi_q(M)$ is not trivial for some for some $q \in \{2, \dots, k-1\}$, then $L_k(M) \geq \aleph_0$, $\forall k \geq \max\{3, k(M)\}$.

Proof. The homotopy sequence of the fibration $S^{k-1} \hookrightarrow \mathcal{N}^{f,1} \xrightarrow{\pi} M$ is

$$\cdots \to \pi_r(S^{k-1}) \to \pi_r(\mathcal{N}^{f,1}) \to \pi_r(M) \to \pi_{r-1}(S^{k-1}) \to \cdots .$$

Because the homotopy groups $\pi_q(S^{k-1}), \pi_{q-1}(S^{k-1})$ of the sphere are trivial for $q \in \{1, \dots, k-2\}$ and $\pi_{k-1}(S^{k-1}) \simeq \mathbf{Z}$ it follows that $\pi_q(\mathcal{N}^{f,1}) \simeq \pi_q(M)$ for all $q \in \{1, \dots, k-2\}$ and the morphism $\pi_{k-1}(\mathcal{N}^{f,1}) \to \pi_{k-1}(M)$, induced by the projection $\pi : \mathcal{N}^{f,1}) \to M$ is surjective. Therefore, to prove (i) apply Theorem 1.1 (i), taking into accout that $\pi_1(M) \not\simeq 0 \simeq \pi_1(S^{m+k-1})$. Annalogously, to prove (ii) apply theorem 1.1 (ii), taking into accout that for some for some $q \in \{2, \dots, k-1\}$ we have $\pi_q(M) \not\simeq 0 \simeq \pi_q(S^{m+k-1})$.

Theorem 2.2. *Let M^m be a connected closed manifold immersible in $\mathbf{R}^{m+2}$.*

(i) If $m \geq 2$ and $\pi_1(M)$ is not trivial, then $L_2(M) \geq \aleph_0$;

(ii) If $m \geq 3$ and $\pi_2(M) \not\simeq \mathbf{Z}$, then $L_2(M) \geq \aleph_0$;

(iii) If $m \geq 4$ and $\pi_q(M)$ is not trivial for some for some $q \in \{3, \dots, m-1\}$, then $L_2(M) \geq \aleph_0$.

Proof. The homotopy sequence of the fibration $S^1 \hookrightarrow \mathcal{N}^{f,1} \xrightarrow{\pi} M$ is

$$\cdots \to \pi_r(S^1) \to \pi_r(\mathcal{N}^{f,1}) \to \pi_r(M) \to \pi_{r-1}(S^1) \to \cdots .$$

Because the homotopy groups $\pi_q(S^1)$ of the circle are trivial for $q = 0$ and $q \geq 2$ and $\pi_1(S^1) \simeq \mathbf{Z}$, it follows that the morphism $\pi_1(\mathcal{N}^{f,1}) \to \pi_1(M)$ induced by the projection $\pi : \mathcal{N}^{f,1} \to M$ is surjective, meaning that $\pi_1(M) \not\simeq 0$ implies $\pi_1(\mathcal{N}^{f,1}) \not\simeq 0 \simeq \pi_1(S^{m+1})$. Therefore using Theorem 1.1 (i), it follows that $L_2(M) \geq \aleph_0$ and the statement (i) is proved. To prove the second statement, observe that the morphism

$$\pi_2(\mathcal{N}^{f,1}) \to \pi_2(M)$$

induced by the projection $\pi : \mathcal{N}^{f,1} \to M$ is injective and

$$\pi_2(M)/\pi_2(\mathcal{N}^{f,1}) \simeq \mathbf{Z}.$$

Combining this observation with the hypothesis of (ii) it follows that $\pi_2(\mathcal{N}^{f,1}) \neq 0 \simeq \pi_2(S^{m+1})$. Therefore using Theorem 1.1 (ii) conclude that $L_2(M) \geq \aleph_0$.

The statement (iii) follows immediately from Theorem 1.1 (ii) taking into account that for $q \in \{3, \ldots, m-1\}$ we have $\pi_q(\mathcal{N}^{f,1}) \simeq \pi_r(M) \not\simeq 0 \simeq \pi_q(S^{m+1})$.

3. Some manifolds with infinite L_k-category

In this section we are going to use the results of the previous section to obtain some manifolds with infinite L_k-category.

Proposition 3.1. *(i) For $m \geq 1$ we have that*

$$L_k(P^m(\mathbf{R})) \geq \aleph_0,\ \forall k \geq \max\{3, k(P^m(\mathbf{R}))\};$$

(ii) For $m \geq 1$ we have that

$$L_k(L^{2m-1}(q_1, \ldots, q_p)) \geq \aleph_0,\ \forall k \geq \max\{3, k(L^{2m-1}(q_1, \ldots, q_p))\};$$

(iii) For $m \geq 2$ we have that $L_k(SO_m) \geq \aleph_0,\ \forall k \geq \max\{3, k(SO_m)\}$.

Proof. Indeed, for $m \geq 3$ we have

$$\pi_1(P^2(\mathbf{R})) \simeq \pi_1(P^m(\mathbf{R})) \simeq \pi_1(SO_m) \simeq \mathbf{Z}_2 \neq 0,$$

while $P^1(\mathbf{R}) \sim_{Top} SO_2 \sim_{Top} S^1$ so that $\pi_1(P^1(\mathbf{R})) \simeq \pi_1(SO_2) \simeq \mathbf{Z} \neq 0$, and $\pi_1(L^{2m-1}(q_1, \ldots, q_p)) \simeq \mathbf{Z}_m \neq 0,\ \forall m \geq 1$. Therefore using Theorem 2.1 (i), proposition 3.1 is completely proved.

Proposition 3.2. *(i) $L_k(G_{p,l}) \geq \aleph_0,\ \forall k \geq \max\{3, k(G_{p,l})\}$;*
(ii) If $m \geq 3$, then $L_k(Spin_m) \geq \aleph_0,\ \forall k \geq \max\{4, k(Spin_m)\}$.

Proof. (i) Let us assume that $p \geq l$. In this case it follows, using [Whi, Theorem 10.16,pp. 204], that

$$\pi_q(G_{p,l}) \simeq \pi_q(V_{p+l,l}) \oplus \pi_{q-1}(O_l),\ \forall q.$$

On the other hand the inclusion $SO_l \hookrightarrow O_l$ induces isomorphism of homotopy groups $\pi_i(SO_l) \to \pi_i(O_l)$, so that for $l = 2$ we have $\pi_1(O_2) \simeq \pi_1(SO_2) \simeq \mathbf{Z}$ and for $l \geq 3$ we have $\pi_1(O_l) \simeq \pi_1(SO_l) \simeq \mathbf{Z}_2$. In any case $\pi_2(G_{p,l})$ has as a subgroup either $\mathbf{Z}$ or $\mathbf{Z}_2$, that is, $\pi_2(G_{p,l}) \not\simeq 0$, meaning by Theorem 2.1 (ii), that $L_k(G_{p,l}) \geq \aleph_0,\ \forall k \geq \max\{3, k(G_{p,l})\}$.

(*ii*) Indeed $\pi_q(Spin_m) \simeq \pi_q(SO_m)$, $\forall k \geq 2$, $Spin_m$ being the universal cover of SO_m. Hence

$$\pi_3(Spin_m) \simeq \pi_3(SO_m) \simeq \begin{cases} \mathbf{Z} & \text{if } m = 3 \text{ and } m \geq 5 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } m = 4. \end{cases}$$

For the above mentioned homotopy groups, see for instance [DNF, pp.224]. Therefore in the given conditions, $\pi_3(Spin_m) \not\simeq 0$, meaning by Theorem 2.1 (*ii*), that $L_k(Spin_m) \geq \aleph_0$, $\forall k \geq \max\{4, k(Spin_m)\}$.

Observation 3.3. For $k \geq \max\{3, p+1, k(G_{p,l})\}$ the first point of Proposition 3.2 can be also proved by means of [Whi, Theorem 10.13,pp. 203], from which it follows that $\pi_p(V_{p+l,l})$ is either infinit cyclic or cyclic of order two.

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