The conormal symbolic structure of corner boundary value problems

T. Krainer and B.-W. Schulze

Abstract. Ellipticity of operators on manifolds with conical singularities or parabolicity on space-time cylinders are known to be linked to parameter-dependent operators (conormal symbols) on a corresponding base manifold.

We introduce the conormal symbolic structure for the case of corner manifolds, where the base itself is a manifold with edges and boundary. The specific nature of parameter-dependence requires a systematic approach in terms of meromorphic functions with values in edge-boundary value problems. We develop here a corresponding calculus, and we construct inverses of elliptic elements.

Introduction

Boundary value problems for elliptic differential operators on a manifold with corners (or parabolic operators on a spatial configuration of that kind) can be studied in terms of symbolic structures that contain parameter-dependent operators on base manifolds of corresponding local cones (or cylinders). The analysis of such so called conormal symbols is the first essential step for establishing parametrices or regularity and asymptotics of solutions in weighted Sobolev spaces, as is done in Kontratyev's work [10] for the case of conical singularities with a smooth base.

The classical theory of parabolic boundary value problems in connection with parameter-dependent ellipticity is developed in Agranovich and Vishik [1]. New applications concern the construction of inverses of parabolic operators in infinite cylinders, cf. [11], the characterisation of resolvents of elliptic operators as meromorphic inverses with a specific dependence on parameters, cf. [27], or Maniccia and Schulze [15], and the evaluation of Shapiro-Lopatinskij elliptic edge conditions for elliptic operators on manifolds with edges, cf. [25], [29], or Nazaikinskij, Savin, Schulze, and Sternin [16].

Conormal symbols also play an important role in the recent develoment of the index theory on manifolds with geometric singularities, e.g., in analytic index formulas, cf. Schulze, Sternin, and Shatalov [32], or Fedosov, Schulze, and

Received by the editors December 1, 2001.
1991 *Mathematics Subject Classification.* Primary 99Z99; Secondary 00A00.
Key words and phrases. Class file, Birkart.

Tarkhanov [5], [6]. Another interesting aspect is the spectral flow of families of conormal symbols associated with edge singularities, cf. [17].

The main purpose of the present paper is to establish a new calculus of boundary value problems on a manifold W with edges with meromorphy in a complex parameter.

Such parameter-dependent operators should be contained in a future algebra of boundary value problems on manifolds with higher corner singularities as a component of a corresponding symbolic hierarchy.

Moreover, if the manifold W plays the role of a cross section of an infinite space-time cylinder our calculus (in a corresponding anisotropic form) can be related to iterated long-time asymptotics of solutions to parabolic equations, similarly as the author's joint paper [12] for the simpler case of conical singularities and without boundary.

Here we study isotropic parameter-dependent operators associated with the elliptic theory of boundary value problems on manifolds with corners and base W. The calculus as a whole contains interesting substructures, e.g., parameter-dependent operators that are flat in the cone and corner axial variables $r, t \in \mathbb{R}$, cf. [18], or smoothing operators with iterated asymptotic information for r and t tending to zero, cf. [3].

This paper is organised as follows.

Chapter 1 starts from manifolds with corner points where the base spaces are manifolds with edges and boundary. Configurations of that kind can be described by corresponding 'corner metrics'; the associated Laplacians are then corner degenerate. For such differential operators, we observe how corner conormal symbols appear as parameter-dependent edge degenerate differential operators. Moreover, we consider operators with the transmission property at the boundary for the case of manifolds with conical exits to infinity. Then we formulate the cone algebra on an infinite cone with discrete asymptotics at the tip of the cone.

Chapter 2 develops the calculus of boundary value problems on a manifold with edges in parameter-dependent form. We introduce edge amplitude functions taking values in boundary value problems on the infinite model cone. Discrete a symptotic data are formulated in terms of the meromorphic structure of subordinate cone conormal symbols and of the mapping properties of Green symbols. The edge algebra itself will be formulated with continuous asymptotic types, based on vector- and operator-valued analytic functionals in the complex plane of the Mellin covariable belonging to the axial variable of the model cone. We investigate parameter-dependent ellipticity of edge boundary value problems and obtain invertibility of operators in weighted edge Sobolev spaces for large absolute values of the parameter (Theorem 2.16).

An essential technical tool is the kernel cut-off procedure which generates elements that are holomorphic in the parameter (Theorem 2.6), cf. also [24], [28], [26], and [13].

In Chapter 3 we investigate 'corner conormal symbols', i.e., families of edge boundary value problems meromorphically depending on a parameter. Kernel cut-off produces a rich space of such families (Theorem 3.3), and parameter-dependent ellipticity is preserved in this process.

We finally show that the space of corner conormal symbols is closed under inversion of elliptic elements (Theorem 3.10).

1. Parameter-dependent boundary value problems

1.1. Differential operators on manifolds with edges

By a manifold W with edge Y and boundary we understand a topological space such that $W \setminus Y$ is a C^∞ manifold with boundary, Y a C^∞ manifold, and every $y \in Y$ has a neighbourhood U in W that is modelled on a wedge $X^\Delta \times \Omega$ with $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ for a compact C^∞ manifold X with boundary, and an open set $\Omega \subseteq \mathbb{R}^q$, $q = \dim Y$. In addition we require a specific behaviour of transition maps belonging to different 'singular charts' $U \to X^\Delta \times \Omega$.

It will be convenient to first pass to the double $\tilde{W} = 2W$ which is a manifold with edge Y (without boundary). The space $\tilde{W}$ may be described by a stretched manifold $\tilde{\mathbb{W}}$ associated with $\tilde{W}$ which is a C^∞ manifold with boundary $\partial\tilde{\mathbb{W}}$, and $\partial\tilde{\mathbb{W}}$ is an $\tilde{X}$-bundle over Y for $\tilde{X} = 2X$, the double of X. Then $\tilde{W}$ itself follows by squeezing down the fibres $(\partial\tilde{\mathbb{W}})_y$ for every $y \in Y$ to the single point y. This gives us the local structure of $\tilde{W}$ near a point of Y as $\tilde{X}^\Delta \times \Omega$, and transition maps are induced by the ones for a collar neighbourhood of $\partial\tilde{\mathbb{W}}$ in $\tilde{\mathbb{W}}$, represented by diffeomorphisms $\overline{\mathbb{R}}_+ \times \tilde{X} \times \Omega \to \overline{\mathbb{R}}_+ \times \tilde{X} \times \Omega$ that restrict to corresponding transition maps of the $\tilde{X}$-bundle $\partial\tilde{\mathbb{W}}$ on the boundary.

By construction, there is then a projection

$$\tilde{\pi} : \tilde{\mathbb{W}} \to \tilde{W},$$

defined as the map that restricts to the bundle projection $\partial\tilde{\mathbb{W}} \to Y$ on the boundary and to the identity map on $\operatorname{int}\tilde{\mathbb{W}}$. Let us write

$$\tilde{\mathbb{W}}_{\text{sing}} := \partial\tilde{\mathbb{W}}, \ \tilde{\mathbb{W}}_{\text{reg}} := \operatorname{int}\tilde{W}.$$

In the following, for convenience, we assume $\partial\tilde{\mathbb{W}}$ to be a trivial $\tilde{X}$-bundle on Y, i.e., $\partial\tilde{\mathbb{W}} = \tilde{X} \times Y$. The general case requires more comment on invariance properties of our operators below which is not the main intention of the present paper.

For references below, we identify neighbourhoods $\tilde{\mathbb{U}}$ near $\tilde{\mathbb{W}}_{\text{sing}}$ with $[0,1) \times \tilde{X} \times G$, where G is a coordinate neighbourhood on Y, and we fix diffeomorphisms

$$\chi : \tilde{\mathbb{U}} \to \overline{\mathbb{R}}_+ \times \tilde{X} \times \Omega, \quad \kappa : G \to \Omega \tag{1.1}$$

for an open set $\Omega \subseteq \mathbb{R}^q$, such that $\chi(r, x, y) = (r, x, \kappa(g))$ for $0 \leq r \leq \frac{1}{2}$. Now $\tilde{\mathbb{W}}$ can be regarded as the double of another stretched space $\mathbb{W}$ associated with our

W, with a projection

$$\pi := \tilde{\pi}|_{\mathbb{W}} : \mathbb{W} \to W.$$

To make the idea more transparent we consider an example, namely

$$\tilde{\mathbb{W}} := \overline{\mathbb{R}}_+ \times \tilde{X} \times \Omega,\ \mathbb{W} := \overline{\mathbb{R}}_+ \times X \times \Omega.$$

This shows in what sense $\tilde{\mathbb{W}}$ is equal to $2\mathbb{W}$ for $\tilde{X} = 2X$. Here $\pi : \overline{\mathbb{R}}_+ \times X \times \Omega \to X^\Delta \times \Omega = W$ is induced by $\tilde{\pi} : \overline{\mathbb{R}}_+ \times \tilde{X} \times \Omega \to \tilde{X}^\Delta \times \Omega = \tilde{W}$. The invariance of the latter construction under transition maps for a general stretched manifold $\tilde{\mathbb{W}}$ gives us correct global definitions also for $\mathbb{W}$. Let us set

$$\mathbb{W}_{\mathrm{sing}} := \pi^{-1} Y \quad \text{and} \quad \mathbb{W}_{\mathrm{reg}} := \mathbb{W} \setminus \mathbb{W}_{\mathrm{sing}}.$$

Then (under our assumption on $\partial\tilde{\mathbb{W}}$) we have $\mathbb{W}_{\mathrm{sing}} = X \times Y$, and $\mathbb{W}_{\mathrm{reg}}$ is a C^∞ manifold with boundary.

Let us now pass to a corner $W^\Delta = (\overline{\mathbb{R}}_+ \times W)/(\{0\} \times W)$ with base W, and consider the associated stretched corner $\mathbb{R}_+ \times \mathbb{W}$. We also look at the doubles $\tilde{W}^\Delta$ and $\mathbb{R}_+ \times \tilde{\mathbb{W}}$, respectively. Differential operators on $\mathbb{R}_+ \times \mathbb{W}$ may be defined in terms of restrictions of corresponding operators on $\mathbb{R}_+ \times \tilde{\mathbb{W}}$ to $\mathbb{R}_+ \times \mathbb{W}$. Typical examples are Laplace-Beltrami operators to corner metrics of the form

$$dt^2 + t^2 g_{\tilde{\mathbb{W}}}(t), \tag{1.2}$$

where $t \in \mathbb{R}_+$ is the corner axis variable and $g_{\tilde{\mathbb{W}}}(t)$ a family of edge-metrics on $\tilde{\mathbb{W}}$, smoothly dependent on t up to 0. By an edge metric we understand a Riemannian metric on $\tilde{\mathbb{W}}_{\mathrm{reg}}$ that is close to $\tilde{\mathbb{W}}_{\mathrm{sing}}$ in the splitting of the variables $(r, x, y) \in \mathbb{R}_+ \times \tilde{X} \times \Omega$ of the form

$$dr^2 + r^2 g_{\tilde{X}}(r) + g_\Omega(r), \tag{1.3}$$

where $g_{\tilde{X}}(r)$ and $g_\Omega(r)$ are families of Riemannian metrics on $\tilde{X}$ and Ω, respectively, smoothly dependent on r up to 0. In other words, (1.2) takes the form

$$dt^2 + t^2(dr^2 + r^2 g_{\tilde{X}}(t, r) + g_\Omega(t, r)) \tag{1.4}$$

with $g_{\tilde{X}}$ and g_Ω being smooth in $t, r \in \mathbb{R}_+ \times \mathbb{R}_+$ up to $(0, 0)$.

Let $\mathrm{Diff}^\mu(M)$ for a C^∞ manifold M denote the space of all differential operators on M (with smooth coefficients) of order μ; this is a Fréchet space in a natural way (all manifolds in this paper are assumed to be paracompact and locally compact). Moreover, let $\mathrm{Diff}^\mu_{\mathrm{edge}}(\tilde{\mathbb{W}})$ be the space of all elements of $\mathrm{Diff}^\mu(\tilde{\mathbb{W}}_{\mathrm{reg}})$ that have close to $\tilde{\mathbb{W}}_{\mathrm{sing}}$ in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times \tilde{X} \times \Omega$ the form

$$r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y)(-r\frac{\partial}{\partial r})^j (rD_y)^\alpha \tag{1.5}$$

with coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(\tilde{X}))$. Also $\mathrm{Diff}^\mu_{\mathrm{edge}}(\tilde{\mathbb{W}})$ is a Fréchet space in a canonical way. Then the Laplace-Beltrami operator associated with an edge metric on $\tilde{\mathbb{W}}_{\mathrm{reg}}$ belongs to $\mathrm{Diff}^2_{\mathrm{edge}}(\tilde{\mathbb{W}})$.

Moreover, the Laplace-Beltrami operator associated with a corner metric (1.2) has the form

$$\tilde{A} = t^{-\mu} \sum_{k=0}^{\mu} b_k(t)(-t\frac{\partial}{\partial t})^k \tag{1.6}$$

(for $\mu = 2$) with coefficients $b_k \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-k}_{\mathrm{edge}}(\tilde{\mathbb{W}}))$. For r near 0 it follows that

$$\tilde{A} = t^{-\mu} r^{-\mu} \sum_{k+j+|\alpha| \le \mu} c_{kj\alpha}(t,r,y)(-rt\frac{\partial}{\partial t})^k (-r\frac{\partial}{\partial r})^j (rD_y)^\alpha \tag{1.7}$$

with coefficients $c_{kj\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(k+j+|\alpha|)}(\tilde{X}))$.

The corner conormal symbol for the operator $\tilde{A}$ in our terminology is the operator function

$$\sigma_{\mathrm{c}}(\tilde{A})(w) = \sum_{k=0}^{\mu} b_k(0) w^k \tag{1.8}$$

with a complex variable w varying in $\mathbb{C}$ or on a weight line

$$\Gamma_\beta := \{w \in \mathbb{C} : \mathrm{Re}\, w = \beta\}$$

for some real β.

In other words, the Fuchs type derivative $-t\frac{\partial}{\partial t}$ is replaced by the covariable w in the Mellin transform $Mu(w) = \int_0^\infty t^{w-1}u(t)dt$. We also employ the weighted Mellin transform $(M_\gamma u)(w) = M(t^{-\gamma}u)(w+\gamma)$, $\gamma \in \mathbb{R}$ (such that $M = M_0$), with the inverse

$$M_\gamma^{-1} g(t) = (2\pi i)^{-1} \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-w} g(w) dw.$$

Setting

$$\mathrm{op}_M^\gamma(f)u(t) = (2\pi)^{-1} \iint \Big(\frac{t}{t'}\Big)^{-(\frac{1}{2}-\gamma+i\tau)} f(t,t',w)u(t')dt'd\tau,$$

$w = \frac{1}{2} - \gamma + i\tau$, which has the meaning of a Mellin pseudo-differential operator $M_\gamma^{-1} f M_\gamma$ with amplitude function $f(t,t',w)$ (to be specified below), we have $M_\gamma^{-1} w M_\gamma = -t\frac{\partial}{\partial t}$. More generally, for $f(t,w) = \sum_{k=0}^{\mu} b_k(t) w^k$ it follows that

$$\tilde{A} = t^{-\mu}\, \mathrm{op}_M^\gamma(f).$$

Here $\gamma \in \mathbb{R}$ is arbitrary when $\tilde{A}$ is applied to argument functions with compact support in $t \in \mathbb{R}$, otherwise, for extensions of the operator to weighted Sobolev spaces we have to specify the number γ, cf. [30].

Let $\tilde{\mathbb{W}}$ be compact. Writing $f(\tau) := \sigma_{\mathrm{c}}(\tilde{A})(\beta + i\tau)$ for any fixed $\beta \in \mathbb{R}$ we obtain a τ-dependent family of operators in the edge algebra on the space $\tilde{\mathbb{W}}$, acting in weighted edge Sobolev spaces $\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}})$, cf. [29], and Definition 1.1 below.

Operators in the edge algebra are characterised by a principal symbolic hierarchy which is particularly simple for the subalgebra generated by $\mathrm{Diff}^\mu_{\mathrm{edge}}(\tilde{\mathbb{W}})$, the edge-degenerate differential operators. We shall formulate symbols at once in parameter-dependent form with parameter $\tau \in \mathbb{R}$.

First, we have the standard homogeneous principal symbol of order μ

$$\sigma_\psi(f)(\tilde{x},\tilde{\xi},\tau) \in C^\infty((T^*\tilde{\mathbb{W}}_{\mathrm{reg}} \times \mathbb{R}) \setminus 0)$$

which is independent of β. For points $\tilde{x} \in \tilde{\mathbb{W}}_{\mathrm{reg}}$ close to $\tilde{\mathbb{W}}_{\mathrm{sing}}$ in the splitting of variables $\tilde{x} = (r,x,y) \in \mathbb{R}_+ \times \Sigma \times \Omega$, $\Sigma \subseteq \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^q$ open, $n = \dim \tilde{X}$, $q = \dim Y$, with covariables $(\varrho,\xi,\eta) \in \mathbb{R}^{1+n+q}$, we can write

$$\sigma_\psi(f)(r,x,y,\varrho,\xi,\eta,\tau) = r^{-\mu}\tilde{\sigma}_\psi(f)(r,x,y,r\varrho,\xi,r\eta,r\tau)$$

for a function

$$\tilde{\sigma}_\psi(f)(r,x,y,\tilde{\varrho},\xi,\tilde{\eta},\tilde{\tau}),$$

homogeneous in $(\tilde{\varrho},\xi,\tilde{\eta},\tilde{\tau}) \neq 0$ of order μ and smooth in r up to $r = 0$. This is immediate from the definition.

Furthermore, we have a so called homogeneous principal edge symbol

$$\sigma_\wedge(f)(y,\eta,\tau)$$

which is an operator valued function on $(T^*Y \times \mathbb{R}) \setminus 0$, acting between weighted Sobolev spaces on the infinite (stretched) model cone $\tilde{X}^\wedge$ of the local wedges that characterise $\tilde{\mathbb{W}}$ near $\tilde{\mathbb{W}}_{\mathrm{sing}}$. Representing $f(\tau)$ locally in the form

$$f(\tau) = r^{-\mu} \sum_{k+j+|\alpha| \leq \mu} c_{kj\alpha}(0,r,y)(-r(\beta+i\tau))^k(-r\frac{\partial}{\partial r})^j(rD_y)^\alpha$$

we have

$$\begin{aligned} \sigma_\wedge(f)(y,\eta,\tau) &= r^{-\mu} \sum_{k+j+|\alpha| \leq \mu} c_{kj\alpha}(0,0,y)(-ir\tau)^k(-r\frac{\partial}{\partial r})^j(r\eta)^\alpha \\ &: \mathcal{K}^{s,\gamma}(\widetilde{X}^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\widetilde{X}^\wedge). \end{aligned}$$

Let us briefly recall the definition of the involved spaces. Given a C^∞ manifold N, by $L^\mu_{\mathrm{cl}}(N;\mathbb{R}^l)$ we denote the space of all classical parameter-dependent pseudo-differential operators of order μ on N, with parameter $\lambda \in \mathbb{R}^l$. Recall that local (left-) amplitude functions $a(x,\xi,\lambda)$ are classical symbols in the covariables $(\xi,\lambda) \in \mathbb{R}^{n+l}$, $n = \dim N$, while $L^{-\infty}(N;\mathbb{R}^l)$ is identified with $A(\lambda) \in \mathcal{S}(\mathbb{R}^l, L^{-\infty}(N))$. Let N be compact. Then it is known that for every $s \in \mathbb{R}$ there exists an element $R^s(\lambda) \in L^s_{\mathrm{cl}}(N;\mathbb{R}^l)$ that induces isomorphisms $H^r(N) \to H^{r-s}(N)$ for all $r \in \mathbb{R}$. Let $\mathcal{H}^{s,\gamma}(N^\wedge)$ for $s,\gamma \in \mathbb{R}$, $N^\wedge = \mathbb{R}_+ \times N$, denote the completion of the space $C_0^\infty(N^\wedge)$ with respect to the norm

$$\big\{(2\pi i)^{-1}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\operatorname{Im} w)(Mu)(w)\|^2_{L^2(N)}dw\big\}^{1/2},$$

where $R^s(\tau) \in L^s(N;\mathbb{R})$ is any choice of an order reducing family.

In this paper, by a cut-off function on the half-axis we understand any real-valued $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ that is equal to 1 in a neighbourhood of 0. We then define the space

$$\mathcal{K}^{s,\gamma}(N^\wedge) = \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(N^\wedge),\ v \in H^s_{\mathrm{cone}}(N^\wedge)\}$$

for any cut-off function ω. Here $H^s_{\mathrm{cone}}(N^\wedge)$ for the unit sphere $N := S^n$ in $\mathbb{R}^{n+1}$ is defined as the subspace of all $v \in H^s_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \{0\})$ such that $(1-\omega)v \in H^s(\mathbb{R}^{n+1})$, otherwise, for general N we define $H^s_{\mathrm{cone}}(N^\wedge)$ by a simple reduction to the previous case by a localisation on sets $\mathbb{R}_+ \times U$ for coordinate neighbourhoods U on N, cf. [29].

Both $\mathcal{H}^{s,\gamma}(N^\wedge)$ and $\mathcal{K}^{s,\gamma}(N^\wedge)$ are Hilbert spaces with suitable scalar products, where $\mathcal{K}^{0,0}(N^\wedge) = \mathcal{H}^{0,0}(N^\wedge) = t^{-\frac{n}{2}}L^2(\mathbb{R}_+ \times N)$ with L^2 referring to the measure $dtdx$.

If N is equal to the double $\tilde{X} = 2X$ of a manifold X with boundary, consisting of two copies $X_\pm$ of X (and X identified with X_+) we set $\mathcal{K}^{s,\gamma}_0(X^\wedge_\pm) := \{\tilde{u} \in \mathcal{K}^{s,\gamma}(\tilde{X}^\wedge) : \operatorname{supp}\tilde{u} \subset X^\wedge_\pm\}$ which is a closed subspace of $\mathcal{K}^{s,\gamma}(\tilde{X}^\wedge)$, and

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\tilde{u}|_{\mathrm{int}\,X^\wedge} : \tilde{u} \in \mathcal{K}^{s,\gamma}(\tilde{X}^\wedge)\}$$

isomorphic to $\mathcal{K}^{s,\gamma}(X^\wedge) \cong \mathcal{K}^{s,\gamma}(\tilde{X}^\wedge)/\mathcal{K}^{s,\gamma}_0(X^\wedge_-)$ and endowed with the corresponding Hilbert space structure.

For references below we set

$$\mathcal{S}^\gamma(X^\wedge) = \{\omega u + (1-\omega)v : u \in \mathcal{K}^{\infty,\gamma}(X^\wedge),\ v \in \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))\}. \tag{1.9}$$

Given a Hilbert space E together with a strongly continuous group of isomorphisms $\kappa_\delta : E \to E, \delta \in \mathbb{R}_+$, such that $\kappa_\delta\kappa_{\delta'} = \kappa_{\delta\delta'}$ for all $\delta, \delta' \in \mathbb{R}_+$, we say that E is equipped with the group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$. More generally, if a Fréchet space E is written as a projective limit of Hilbert spaces $(E^j)_{j\in\mathbb{N}}$ with continuous embeddings $E^{j+1} \hookrightarrow E^j \hookrightarrow \ldots \hookrightarrow E^0$ for all j, and if κ is a group action on E^0 that restricts to group actions on E^j for all j, we say that E is endowed with the group action κ.

We apply this terminology to $E = \mathcal{K}^{s,\gamma}(N^\wedge)$ for $(\kappa_\delta u)(t,x) = \delta^{\frac{n+1}{2}}u(\delta t, x), \delta \in \mathbb{R}_+, n = \dim N$, and later on to a variety of Fréchet subspaces.

Definition 1.1.
1. Let E be a Hilbert space with group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$. Then $\mathcal{W}^s(\mathbb{R}^q, E)$, $s = \mathbb{R}$, denotes the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm
$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,E)} = \left\{\int \langle\eta\rangle^{2s}\|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_E d\eta\right\}^{1/2},$$
where $\hat{u}(\eta) = Fu(\eta)$ is the Fourier transform in $\mathbb{R}^q$ and $\langle\eta\rangle = (1+|\eta|^2)^{1/2}$.
2. If $E = \varprojlim_{j\in\mathbb{N}} E^j$ is a Fréchet space with group action, we denote by $\mathcal{W}^s(\mathbb{R}^q, E)$, $s \in \mathbb{R}$, the projective limit of the spaces $\mathcal{W}^s(\mathbb{R}^q, E^j)$, $j \in \mathbb{N}$.

The space $\mathcal{W}^s(\mathbb{R}^q, E)$ in the case (ii) is Fréchet with $\|\cdot\|_{\mathcal{W}^s(\mathbb{R}^q,E^j)}$, $j \in \mathbb{N}$, as a system of norms.

There is also an analogue of standard 'comp' and 'loc' spaces in the present vector-valued case

$$W^s_{\text{comp}}(\Omega, E) \quad \text{and} \quad W^s_{\text{loc}}(\Omega, E)$$

for any open set $\Omega \subseteq \mathbb{R}^q$.

Applying Definition 1.1 to $E = \mathcal{K}^{s,\gamma}(N^\wedge)$ we obtain the spaces

$$\mathcal{W}^{s,\gamma}(N^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(N^\wedge))$$

for every $s, \gamma \in \mathbb{R}$. For $N = \tilde{X}$ the space $\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}})$ is defined as the subspace of all $u \in H^s_{\text{loc}}(\tilde{\mathbb{W}}_{\text{reg}})$ such that $\omega\varphi u = \chi^* v$ for some $v \in \mathcal{W}^{s,\gamma}(\tilde{X}^\wedge \times \mathbb{R}^q)$ for every χ of the kind (1.1) and any cut-off function $\omega(r)$ vanishing for $r > \frac{1}{2}$, and $\varphi \in C_0^\infty(G)$, cf. also Section 2.2 below.

We set

$$\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}) := \big\{\omega u + (1-\omega)v : u \in \mathcal{W}^{s,\gamma}(\tilde{X} \times \mathbb{R}^q),\ v \in H^s_{\text{loc}}(\tilde{\mathbb{W}}_{\text{reg}})\big\}$$

for any cut-off function $\omega(r)$, with r referring to the local splitting of variables (r, x, y) near $\tilde{\mathbb{W}}_{\text{sing}}$. For simplicity, pull backs under charts are suppressed in the latter notation; also below we shall identify a neighbourhood of $\tilde{\mathbb{W}}_{\text{sing}}$ with $\overline{\mathbb{R}}_+ \times \tilde{X} \times \mathbb{R}^q$. Now let us set

$$\mathcal{W}^{s,\gamma}_0(\mathbb{W}_\pm) = \{\tilde{u} \in \mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}) : \operatorname{supp} \tilde{u} \subseteq \mathbb{W}_\pm\}$$

with $\mathbb{W}_\pm$, where the two copies of $\mathbb{W}$ constitute the double $\tilde{\mathbb{W}}$ (with $\mathbb{W}$ being identified with $\mathbb{W}_+$), and

$$\mathcal{W}^{s,\gamma}(\mathbb{W}) = \big\{\tilde{u}|_{\operatorname{int}\mathbb{W}_{\text{reg}}} : \tilde{u} \in \mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}})\big\}.$$

The conormal symbol (1.8) can be written in the form

$$\sigma_{\text{c}}(\tilde{A})(w) = r^{-\mu}\omega(r) \sum_{k+j+|\alpha|\le\mu} c_{kj\alpha}(0, r, y)(rw)^k \big(-r\frac{\partial}{\partial r}\big)^j (rD_y)^\alpha \\ + (1-\omega)r^{-\mu}\sum_{k=0}^{\mu} b_k(0) w^j.$$

Proposition 1.2. (1.8) *represents a holomorphic family of continuous operators*

$$\sigma_{\text{c}}(\tilde{A})(w) : \mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}) \to \mathcal{W}^{s-\mu,\gamma-\mu}(\tilde{\mathbb{W}}). \tag{1.10}$$

.

Corollary 1.3. *The restriction of* (1.10) *to* $\operatorname{int}\mathbb{W}_{\text{reg}}$ *gives us a holomorphic family of continuous operators*

$$\sigma_{\text{c}}(A)(w) : \mathcal{W}^{s,\gamma}(\mathbb{W}) \to \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}). \tag{1.11}$$

for all $s, \gamma \in \mathbb{R}$.

The continuity of (1.10) for every fixed $w \in \mathbb{C}$ is a consequence of the fact that (1.10) near $\tilde{\mathbb{W}}_{\mathrm{sing}}$ can be regarded as a pseudo-differential operator with an operator-valued symbol of a specific kind. The general definition is as follows.

Let E and $\tilde{E}$ be Hilbert spaces with group actions $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ and $\tilde{\kappa} = \{\tilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+}$, respectively. Then

$$S^\mu(U \times \mathbb{R}^q; E, \tilde{E}) \tag{1.12}$$

for $\mu \in \mathbb{R}, U \subseteq \mathbb{R}^p$ open, is defined as the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$ such that

$$\sup_{y \in K, \eta \in \mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,\tilde{E})} < \infty \tag{1.13}$$

for every $K \Subset U, \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$. The expressions (1.13) form a semi-norm system which makes (1.12) a Fréchet space.

Elements of (1.12) are called operator-valued symbols of order μ.

We also have operator-valued symbols in the case when E or $\tilde{E}$ are Fréchet spaces, written as projective limits of Hilbert spaces, where the respective group actions extend to group actions in the former sense in all Hilbert spaces of the projective limits, cf. [29].

Let $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$ denote the space of classical symbols, that is, for $a(y,\eta)$ there exist homogeneous components $a_{(\mu-j)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E, \tilde{E})), j \in \mathbb{N}$, i.e.,

$$a_{(\mu-j)}(y, \delta\eta)) = \delta^{\mu-j}\tilde{\kappa}_\delta a_{(\mu-j)}(y,\eta)\kappa^{-1}_\delta$$

for all $\delta \in \mathbb{R}_+$, such that

$$a(y,\eta) - \chi(\eta)\sum_{j=0}^N a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; E, \tilde{E})$$

for all $N \in \mathbb{N}$ and any excision function χ. Also $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$ is a Fréchet space in a natural way.

If a relation holds both for classical or general symbols, we write '(cl)' as subscript.

The spaces $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$ of y-independent symbols are closed in $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})$.

Theorem 1.4. *Let $U = \Omega \times \Omega$ for $\Omega \subseteq \mathbb{R}^q$ open, and let $a(y,y',\eta) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^q; E, \widetilde{E})$.*

Then, setting $\mathrm{Op}(a)u(y) = \iint e^{i(y-y')\eta}a(y,y',\eta)u(y')dy'đ\eta$, $đ\eta = (2\pi)^{-q}d\eta$, the operator

$$\mathrm{Op}(a) : C^\infty_0(\Omega, E) \to C^\infty(\Omega, \widetilde{E})$$

extends to a continuous operator

$$\mathrm{Op}(a) : W^s_{\mathrm{comp}}(\Omega, E) \to W^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{E})$$

for every $s \in \mathbb{R}$.

1.2. Boundary value problems with the transmission property

Let X be a smooth manifold with boundary ∂X. In this section we discuss a few basic constructions on parameter-dependent pseudo-differential operators on X with the transmission property at the boundary. For simplicity, we assume X to be compact. The non-compact case will also be of interest in a variety of cases. We will tacitly use the corresponding generalisations, unless special precautions are necessary; those will be separately described.

Let $\mathrm{Vect}(\cdot)$ denote the set of all smooth complex vector bundles on the manifold in brackets. The manifolds in question are assumed to be equipped with Riemannian metrics and the vector bundles with Hermitian metrics. Operators will refer to Sobolev spaces $H^s(X,E)$ of distributional sections in vector bundles $E \in \mathrm{Vect}(X)$.

Parameter-dependent boundary value problems in our set-up will be families of continuous operators

$$\mathcal{A}(\lambda) : \begin{array}{c} H^s(X,E) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial X, J_-) \end{array} \to \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(\partial X, J_+) \end{array} \tag{1.14}$$

for $E, F \in \mathrm{Vect}(X)$, $J_-, J_+ \in \mathrm{Vect}(\partial X)$, $\mu \in \mathbb{Z}$, given in the form

$$\mathcal{A}(\lambda) = \begin{pmatrix} \mathrm{r}^+ A(\lambda) \mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G}(\lambda), \tag{1.15}$$

with the following ingredients.

We assume that

$$A(\lambda) \in L^\mu_{\mathrm{cl}}(\tilde{X}; \tilde{E}, \tilde{F}; \mathbb{R}^l) \tag{1.16}$$

for vector bundles $\tilde{E}, \tilde{F} \in \mathrm{Vect}(\tilde{X})$ on the double $\tilde{X}$ such that $\tilde{E}|_X = E, \tilde{F}|_X = F$. In other words, (1.16) is a family of classical parameter-dependent pseudo-differential operators on $\tilde{X}$ ($\tilde{E}, \tilde{F}$ in (1.16) means that the operators act between distributional sections of corresponding vector bundles). In addition we require the operators $A(\lambda)$ to have the transmission property at the interface ∂X, cf. [19], or [29]. Let $L^\mu_{\mathrm{cl}}(\tilde{X}; \tilde{E}, \tilde{F}; \mathbb{R}^l)_{\mathrm{tr}}$ denote the subspace of all elements of (1.16) with the transmission property. Moreover,

$$\mathrm{e}^+ : H^s(X,E) \to H^s(\tilde{X}, \tilde{E})$$

is the operator of extension by zero from $\mathrm{int}\ X$ to the double $\tilde{X}$ of X, $s > -\frac{1}{2}$, and

$$\mathrm{r}^+ : H^s(\tilde{X}, \tilde{E}) \to H^s(X,E)$$

the operator of restriction, i.e., $\mathrm{r}^+ u := u|_{\mathrm{int}\, X}$.

To explain $\mathcal{G}(\lambda)$ on the right hand side of (1.15) we first introduce some other notation. Choose any differential operator $T \in \mathrm{Diff}^1(X; E, E)$ on X with smooth coefficients up to the boundary (acting between sections in E) which is of the form $T = \frac{\partial}{\partial x_n} \otimes \mathrm{id}_E$ in the splitting of variables (x', x_n) in a collar neighbourhood $\cong \partial X \times [0,1)$ of ∂X.

Moreover, let $\mathcal{B}^{-\infty,0}(X;\boldsymbol{v})$ for $\boldsymbol{v}=(E,F;J_-,J_+)$ denote the space of all operators

$$\mathcal{C}\in\bigcap_{\substack{s,s'\in\mathbb{R}\\ s>-\frac{1}{2}}}\mathcal{L}(H^s(X,E)\oplus H^{s'}(\partial X,J_-),\ C^\infty(X,F)\oplus C^\infty(\partial X,J_+))$$

such that the formal adjoint $\mathcal{C}^*$ with respect to the respective L^2-scalar products represents an operator

$$\mathcal{C}^*\in\bigcap_{\substack{s,s'\in\mathbb{R}\\ s>-\frac{1}{2}}}\mathcal{L}(H^s(X,F)\oplus H^{s'}(\partial X,J_+),\ C^\infty(X,E)\oplus C^\infty(\partial X,J_-)).$$

The space $\mathcal{B}^{-\infty,0}(X;\boldsymbol{v})$ is Fréchet in a canonical way, and we set $\mathcal{B}^{-\infty,0}(X;\boldsymbol{v};\mathbb{R}^l)=\mathcal{S}(\mathbb{R}^l,\mathcal{B}^{-\infty,0}(X;\boldsymbol{v};\mathbb{R}^l))$. We then form the space $\mathcal{B}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^l)$ of all operator families

$$\mathcal{C}(\lambda)=\mathcal{C}_0(\lambda)+\sum_{j=1}^{d}\mathcal{C}_j(\lambda)\,\mathrm{diag}(T^j,0) \tag{1.17}$$

for arbitrary $\mathcal{C}_j(\lambda)\in\mathcal{B}^{-\infty,0}(X;\boldsymbol{v};\mathbb{R}^l)$.

Let us endow the spaces $L^2(\mathbb{R}_+,\mathbb{C}^N)\oplus\mathbb{C}^M$ and $\mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^N)\oplus\mathbb{C}^M$ for $\mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^N):=\varprojlim_{j\in\mathbb{N}}\langle x_n\rangle^{-j}H^j(\mathbb{R}_+,\mathbb{C}^N)$ with the group action $\varphi(x_n)\oplus c\to(\delta^{\frac{1}{2}}\varphi(\delta x_n))\oplus c$, $\delta\in\mathbb{R}_+$. Then we can form operator-valued symbols

$$g(x',\xi',\lambda)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n-1};L^2(\mathbb{R}_+,\mathbb{C}^e)\oplus\mathbb{C}^{j_-},\mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^f)\oplus\mathbb{C}^{j_+})$$

with e,f,j_- and j_+ in the meaning of fibre dimensions of the bundles E,F,J_- and J_+, respectively, $\Omega\subseteq\mathbb{R}^{n-1}$ open, such that the pointwise adjoints $g^*(x',\xi',\lambda)$ in the sense of $(g(x',\xi',\lambda)u,v)_{L^2(\mathbb{R}_+,\mathbb{C}^f)\oplus\mathbb{C}^{j_+}}=(u,g^*(x',\xi',\lambda)v)_{L^2(\mathbb{R}_+,\mathbb{C}^e)\oplus\mathbb{C}^{j_-}}$ have the property

$$g^*(x',\xi',\lambda)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n-1+l};\ L^2(\mathbb{R}_+,\mathbb{C}^f)\oplus\mathbb{C}^{j_+},\ \mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^e)\oplus\mathbb{C}^{j_-}).$$

Note that then

$$b(x',\xi',\lambda):=g_0(x',\xi',\lambda)+\sum_{j=1}^{d}g_j(x',\xi',\lambda)\,\mathrm{diag}(\frac{\partial^j}{\partial x_n^j},0)$$

for arbitrary $g_j(x',\xi',\lambda)$ of the abovementioned structure, of order $\mu-j$, represents a symbol

$$b(x',\xi',\lambda)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n-1+l};H^s(\mathbb{R}_+,\mathbb{C}^e)\oplus\mathbb{C}^{j_-},\mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^f)\oplus\mathbb{C}^{j_+}) \tag{1.18}$$

for all $s>d-\frac{1}{2}$. Also

$$\mathrm{diag}(1,\langle\xi',\lambda\rangle^{\frac{1}{2}})b(x',\xi',\lambda)\,\mathrm{diag}(1,\langle\xi',\lambda\rangle^{-\frac{1}{2}})=:h(x',\xi',\lambda) \tag{1.19}$$

is a classical operator-valued symbol, although with slightly modified group actions in the involved spaces, namely $\varphi(x_n)\oplus c\to\delta^{\frac{1}{2}}\varphi(\delta x_n)\oplus\delta^{\frac{1}{2}}c,\ \delta\in\mathbb{R}_+$, instead of the previous ones.

Choose an open covering of the collar neighbourhood $\partial X\times[0,1)$ of the boundary by charts $\chi_k : U'_k \times [0,1) \to \Omega \times \overline{\mathbb{R}}_+$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, $k = 1,\ldots,N$, such that $\{U'_1,\ldots,U'_N\}$ is an open covering of ∂X by induced charts $\chi'_k : U'_k \to \Omega$. Let $\{\varphi_1,\ldots,\varphi_N\}$, $\{\psi_1,\ldots,\psi_N\}$ be systems of functions $\varphi_k,\psi_k \in C_0^\infty(U'_l \times [0,1))$ such that $\sum \varphi_k = 1$ in a neighbourhood of ∂X and $\psi_k \equiv 1$ on $\operatorname{supp}\varphi_k$. Set $\varphi'_k = \varphi_k|_{\partial X}$, $\psi'_k = \psi_k|_{\partial X}$. Then, using symbols $h_k(x',\xi',\lambda)$ of the kind (1.19), we can pass to operator pull backs $\mathcal{H}_k(\lambda)$ of pseudo-differential operators $\mathrm{Op}_{x'}(h_k)(\lambda)$ to $U'_k \times (0,1)$ (that also take into account the transition functions of involved bundles) and form

$$\mathcal{H}(\lambda) := \sum_{k=1}^N \operatorname{diag}(\varphi_k,\varphi'_k)\mathcal{H}_k(\lambda)\operatorname{diag}(\psi_k,\psi'_k). \tag{1.20}$$

The operator $\mathcal{G}(\lambda)$ in (1.15) is assumed to be of the form

$$\mathcal{G}(\lambda) = \mathcal{H}(\lambda) + \mathcal{C}(\lambda) \tag{1.21}$$

for arbitrary $\mathcal{H}(\lambda)$ as in (1.20) and $\mathcal{C}(\lambda) \in \mathcal{B}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}^l)$.

Definition 1.5. Let $\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^l)$ for $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, $\boldsymbol{v} = (E,F;J_-,J_+)$ denote the space of all operator families of the form (1.15) for arbitrary $A(\lambda) \in L^\mu_{\mathrm{cl}}(\widetilde{X};\widetilde{E},\widetilde{F};\mathbb{R}^l)_{\mathrm{tr}}$ and $\mathcal{G}(\lambda) \in \mathcal{B}^{\mu,d}_G(X;\boldsymbol{v};\mathbb{R}^l)$, the subspace of all elements of the kind (1.21). Given any $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^l)$ we also write $d = d_{\mathcal{A}}$, called the type of $\mathcal{A}$. Moreover, let

$$\mathcal{B}^\mu(X;\boldsymbol{v};\mathbb{R}^l) = \bigcup_{d\in\mathbb{N}} \mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^l),$$

and, similarly, $\mathcal{B}^\mu_G(X;\boldsymbol{v};\mathbb{R}^l) = \bigcup_{d\in\mathbb{N}} \mathcal{B}^{\mu,d}_G(X;\boldsymbol{v};\mathbb{R}^l)$.

The operator families $\mathcal{A}(\lambda) \in \mathcal{B}^\mu(X;\boldsymbol{v};\mathbb{R}^l)$ have a parameter-dependent principal symbolic structure

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}),\sigma_\partial(\mathcal{A}))$$

with the homogeneous principal interior and boundary symbols $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$, respectively. The interior symbol of $\mathcal{A}(\lambda)$ is the restriction of the parameter-dependent interior symbol of (1.16) to $(T^*X\times\mathbb{R}^l)\setminus 0$, where 0 stands for $(\xi,\lambda) = 0$. The boundary symbol, expressed in a collar neighbourhood of ∂X in the variables (x',x_n), contains an ingredient from $\mathrm{r}^+A(\lambda)\mathrm{e}^+$, namely

$$\sigma_\partial(\mathrm{r}^+A\mathrm{e}^+))(x',\xi',\lambda) := \mathrm{r}^+\operatorname{op}_{x_n}(\sigma_\psi(\mathcal{A})\big|_{x_n=0})(x',\xi',\lambda)\mathrm{e}^+,$$

regarded as a family of maps

$$\sigma_\partial(\mathrm{r}^+A\mathrm{e}^+)(x',\xi',\lambda) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$$

for $s > -\frac{1}{2}$, invariantly defined as an operator family parametrised by $(T^*(\partial X)\times\mathbb{R}^l)\setminus 0$, with 0 being interpreted as $(\xi',\lambda) = 0$.

Another ingredient is defined by the summand $\mathcal{G}(\lambda)$ in (1.21), more precisely by $\mathcal{H}(\lambda)$ in (1.21), locally given by

$$\sigma_\partial(\mathcal{G})(x',\xi',\lambda) = \operatorname{diag}(1,|\xi',\lambda|^{\frac{1}{2}})b_{(\mu)}(x',\xi',\lambda)\operatorname{diag}(1,|\xi',\lambda|^{-\frac{1}{2}})$$

for $(\xi', \lambda) \neq 0$, cf. (1.18), (1.19), and

$$b_{(\mu)}(x', \xi', \lambda) = g_{0,(\mu)}(x', \xi', \lambda) + \sum_{j=1}^{d} g_{j,(\mu-j)}(x', \xi', \lambda)\, \mathrm{diag}\big(\frac{\partial^j}{\partial x_n^j}, 0\big),$$

with $g_{j,(\mu-j)}(x', \xi', \lambda)$ as the homogeneous principal components of the classical operator-valued symbols $g_j(x', \xi', \lambda)$ of order $\mu - j$, $j = 0, \ldots, d$. Then

$$\sigma_\partial(\mathcal{A})(x', \xi', \lambda) := \mathrm{diag}\big(\sigma_\partial(\mathrm{r}^+ A \mathrm{e}^+)(x', \xi', \lambda), 0\big) + \sigma_\partial(\mathcal{G})(x', \xi', \lambda)$$

represents an invariantly defined family of operators

$$\sigma_\partial(\mathcal{A})(x', \xi', \lambda) : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \to \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{array} \tag{1.22}$$

parametrised by $(T^*(\partial X) \times \mathbb{R}^l) \setminus 0$, with homogeneity

$$\sigma_\partial(\mathcal{A})(x', \delta\xi', \delta\lambda) = \delta^\mu \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \delta^{\frac{1}{2}} \end{pmatrix} \sigma_\partial(\mathcal{A}(x', \xi', \lambda) \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \delta^{\frac{1}{2}} \end{pmatrix}^{-1}$$

for all $\delta \in \mathbb{R}_+$.

Given a Fréchet space F and an open set $U \subseteq \mathbb{C}$ we denote by $\mathcal{A}(U, F)$ the space of all holomorphic functions in U with values in E. By

$$\mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{C} \times \mathbb{R}^q) \tag{1.23}$$

we will denote the space of all operator families

$$f(z, \eta) \in \mathcal{A}(\mathbb{C}, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{R}^q))$$

such that

$$f(\beta + i\varrho, \eta) \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{R}^{1+q}_{\varrho,\eta})$$

for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for every $c \leq c'$. For $q = 0$ we simply write $\mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{C})$. Moreover, we set

$$\mathcal{B}^\mu(X; \boldsymbol{v}; \mathbb{C} \times \mathbb{R}^q) = \bigcup_{d \in \mathbb{N}} \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{C} \times \mathbb{R}^q). \tag{1.24}$$

The space (1.23) Fréchet in a natural way.

Remark 1.6. As is well known, cf. [22], [23], or [9], the space $\mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{C} \times \mathbb{R}^q)$ is rich in the sense that for every operator family $p \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{R}^{1+q}_{\varrho,\eta})$ there exists an $f(z, \eta) \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{C} \times \mathbb{R}^q)$ such that

$$p(\varrho, \eta) = f(i\varrho, \eta) \bmod \mathcal{B}^{-\infty,d}(X; \boldsymbol{v}; \mathbb{R}^{1+q}).$$

The construction of f is based on a kernel cut-off construction, cf. Section 2.1 below.

The spaces $\mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^l)$ as well as (1.23) have a straightforward generalisation to the case of a non-compact C^∞ manifold X with boundary. This will tacitly be used below. Instead of (1.14) we then talk about continuous maps between Sobolev spaces with subscript 'comp' or 'loc'.

An interesting situation with a specific control of the non-compactness at infinity is the case of an infinite stretched cone $X^\wedge = \mathbb{R}_+ \times X \ni (r,x)$ for a compact C^∞ manifold X with boundary. Let us formulate a few basic notions for this case. We first discuss operators far from $r=0$; operators close to $r=0$ will be studied in the following section.

The typical situation is the half-space $\overline{\mathbb{R}^{n+1}_+} = \{\tilde{x} = (\tilde{x}', \tilde{x}_{n+1}) \in \mathbb{R}^{n+1} : \tilde{x}_{n+1} \geq 0\}$, $\tilde{x}' = (\tilde{x}_1, \ldots, \tilde{x}_n)$; in this case X is equal to $S^n_+ = S^n \cap \overline{\mathbb{R}^{n+1}_+}$ the upper half of the unit sphere. A simple localisation on conical subsets Γ of $\overline{\mathbb{R}^{n+1}_+}$, using invariance of our constructions under transition maps $\kappa : \Gamma \to \tilde{\Gamma}$ that satisfy $\kappa(\lambda\tilde{x}) = \lambda\kappa(\tilde{x})$ for $|\tilde{x}| \geq C$ for some $C > 0$ and $\lambda \geq 1$, allows us to pass to operators on $X^\wedge$. In other words, we mainly look at $\overline{\mathbb{R}^{n+1}_+}$.

In this case we define a global calculus of operators of the class

$$\mathcal{B}^{\mu,d;\varrho}(\overline{\mathbb{R}^{n+1}_+};\boldsymbol{v}) \tag{1.25}$$

with a weight $\varrho \in \mathbb{R}$ at infinity. We do not need any parameter-dependent variant here.

Similarly to $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$, cf. Definition 1.5, the elements of (1.25) consist of 2×2 block matrices of continuous operators

$$\mathcal{A} : \begin{array}{c} H^{s;\delta}(\mathbb{R}^{n+1}_+, \mathbb{C}^e) \\ \oplus \\ H^{s-\frac{1}{2};\delta}(\mathbb{R}^n, \mathbb{C}^{j_-}) \end{array} \to \begin{array}{c} H^{s-\mu;\delta-\varrho}(\mathbb{R}^{n+1}_+, \mathbb{C}^f) \\ \oplus \\ H^{s-\mu-\frac{1}{2};\delta-\varrho}(\mathbb{R}^n, \mathbb{C}^{j_+}) \end{array}$$

for $s > d - \frac{1}{2}$. Here $H^{s;\delta}(\mathbb{R}^{n+1}_+) := \langle\tilde{x}\rangle^{-\delta} H^s(\mathbb{R}^{n+1}_+)$, $H^{s';\delta}(\mathbb{R}^n) := \langle\tilde{x}'\rangle^{-\delta} H^{s'}(\mathbb{R}^n)$. For simplicity, we consider the case $e = f = 1$, $j_- = j_+ = 0$. The constructions for the general case are straightforward and left to the reader.

First, on $\mathbb{R}^{n+1}$ we have the standard calculus of pseudo-differential operators with exit conditions. Let $S^{\mu;\varrho}(\mathbb{R}^{2(n+1)}_{\tilde{x},\tilde{\xi}})$ for $\mu,\varrho \in \mathbb{R}$ denote the set of all $a(\tilde{x},\tilde{\xi}) \in C^\infty(\mathbb{R}^{2(n+1)})$ such that

$$\sup_{(\tilde{x},\tilde{\xi})\in\mathbb{R}^{2(n+1)}} \langle\tilde{x}\rangle^{-\varrho+|\alpha|}\langle\tilde{\xi}\rangle^{-\mu+|\beta|}|D^\alpha_{\tilde{x}} D^\beta_{\tilde{\xi}} a(\tilde{x},\tilde{\xi})| < \infty \tag{1.26}$$

for all $\alpha,\beta \in \mathbb{N}^{n+1}$. Observe that $S^{\mu;\varrho}(\mathbb{R}^{2(n+1)})$ for $\varrho \geq 0$ and $\mu \geq 0$, respectively contains the subspaces $S^\mu_{(\mathrm{cl})}(\mathbb{R}^{n+1}_{\tilde{\xi}})$ and

$S^\varrho_{(\mathrm{cl})}(\mathbb{R}^{n+1}_{\tilde{x}})$ 'with constant coefficients' ('(cl)' means classical or non-classical in the respective variables, treated as covariables). Note that $S^\mu_{\mathrm{cl}}(\mathbb{R}^{n+1})$ is nuclear in the natural Fréchet topology. Let us define

$$S^{\mu;\varrho}_{\mathrm{cl}}(\mathbb{R}^{2(n+1)}) := \mathcal{S}^\mu_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{\xi}})\hat{\otimes}_\pi \mathcal{S}^\varrho_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{x}})$$

which is the set of all $a(\tilde{x},\tilde{\xi}) \in S^{\mu;\varrho}(\mathbb{R}^{2(n+1)})$ that are classical both in $\tilde{\xi}$ and $\tilde{x}$. Moreover, let $S^{\mu;\varrho}_{\mathrm{cl}}(\mathbb{R}^{2(n+1)})_{\mathrm{tr}}$ be the subspace of all elements with the transmission property at $\tilde{x}_{n+1} = 0$.

Let us set

$$L^{\mu;\varrho}_{\mathrm{cl}}(\mathbb{R}^{n+1})_{\mathrm{tr}} = \{\mathrm{Op}_{\tilde{x}}(a) : a(\tilde{x},\tilde{\xi}) \in S^{\mu;\varrho}_{\mathrm{cl}}(\mathbb{R}^{2(n+1)})_{\mathrm{tr}}\}. \tag{1.27}$$

Then $\mathrm{r}^+ A\mathrm{e}^+$ for $A \in L^{\mu;\varrho}_{\mathrm{cl}}(\mathbb{R}^{n+1})_{\mathrm{tr}}$ (with e^+ being the operator of extension by zero from $\mathbb{R}^{n+1}_+$ to $\mathbb{R}^{n+1}$ and r^+ the restriction from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}_+$) induces continuous operators

$$\mathrm{r}^+ A\mathrm{e}^+ : H^{s;\delta}(\mathbb{R}^{n+1}_+) \to H^{s-\mu;\delta-\varrho}(\mathbb{R}^{n+1}_+)$$

for all $s > -\frac{1}{2};\ \delta \in \mathbb{R}$.

Symbols $a(\tilde{x},\tilde{\xi})$ that are classical in $\tilde{\xi}$ of order μ and $\tilde{x}$ of order ϱ have a triple of principal components, consisting of

$$\sigma_\psi(a)(\tilde{x},\tilde{\xi}) \in C^\infty(\mathbb{R}^{n+1}_{\tilde{\xi}} \setminus \{0\}, S^{\varrho}_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{x}})),$$

homogeneous of order μ in $\tilde{\xi} \neq 0$,

$$\sigma_{\mathrm{e}}(a)(\tilde{x},\tilde{\xi}) \in C^\infty(\mathbb{R}^{n+1}_{\tilde{x}} \setminus \{0\}, S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{\xi}})),$$

homogeneous of order ϱ in $\tilde{x} \neq 0$, and

$$\sigma_{\psi,\mathrm{e}}(a)(\tilde{x},\tilde{\xi}) \in C^\infty((\mathbb{R}^{n+1}_{\tilde{x}} \setminus \{0\}) \times (\mathbb{R}^{n+1}_{\tilde{\xi}} \setminus \{0\})),$$

homogeneous in $\tilde{\xi} \neq 0$ of order μ and $\tilde{x} \neq 0$ of order ϱ. For $A = \mathrm{Op}(a)$ we then set

$$\sigma_\psi(\mathrm{r}^+ A\mathrm{e}^+) = \sigma_\psi(a),\ \sigma_{\mathrm{e}}(\mathrm{r}^+ A\mathrm{e}^+) = \sigma_{\mathrm{e}}(a),\ \sigma_{\psi,\mathrm{e}}(\mathrm{r}^+ A\mathrm{e}^+) = \sigma_{\psi,\mathrm{e}}(a).$$

There is also a variant of operator-valued symbols with exit conditions in $\mathbb{R}^n_{\tilde{x}'}$, acting between Hilbert spaces E and $\tilde{E}$ with group action, cf. (1.12) and (1.13). Let $S^{\mu;\varrho}(\mathbb{R}^{2n}; E, \tilde{E})$ be the set of all $a(\tilde{x}',\tilde{\xi}') \in C^\infty(\mathbb{R}^{2n}, \mathcal{L}(E,\tilde{E}))$ such that

$$\sup_{(\tilde{x}',\tilde{\xi}')\in\mathbb{R}^{2n}} \langle\tilde{x}'\rangle^{-\varrho+|\alpha|}\langle\tilde{\xi}'\rangle^{-\mu+|\beta|}\|\tilde{\kappa}^{-1}_{\langle\xi'\rangle}\{D^\alpha_{\tilde{x}'}D^\beta_{\tilde{\xi}'}a(\tilde{x}',\tilde{\xi}')\}\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(E,\tilde{E})} < \infty$$

for all $\alpha,\beta \in \mathbb{N}^n$.

Also in the operator-valued case there is a natural notion of classical symbols in both variables $\tilde{x}'$ and $\tilde{\xi}'$, for more details, cf. [9, Chapter 3]. Let $S^{\mu;\varrho}_{\mathrm{cl}}(\mathbb{R}^{2n}; E, \tilde{E})$ denote the corresponding subspace of $S^{\mu;\varrho}(\mathbb{R}^{2n}; E, \tilde{E})$. Finally, we can generalise such symbol spaces to the case of Fréchet spaces $E, \tilde{E}$ with group actions.

Classical operator-valued symbols $a(\tilde{x}',\tilde{\xi}')$ in $(\tilde{x}',\tilde{\xi}')$ of order μ in $\tilde{\xi}'$ and ϱ in $\tilde{x}'$ have also a triple of principal symbols, namely

$$\sigma_\partial(a)(\tilde{x}') \in C^\infty(\mathbb{R}^n_{\tilde{\xi}'} \setminus \{0\}, S^{\varrho}_{\mathrm{cl}}(\mathbb{R}^n_{\tilde{x}'}; E, \tilde{E})),$$

homogeneous of order μ in $\tilde{\xi}' \neq 0$ in the sense

$$\sigma_\partial(a)(\tilde{x}', \lambda\tilde{\xi}') = \lambda^\mu \tilde{\kappa}_\lambda \sigma_\partial(a)(\tilde{x}',\tilde{\xi}')\kappa^{-1}_\lambda \tag{1.28}$$

for all $\lambda \in \mathbb{R}_+$,

$$\sigma_{\mathrm{e}'}(a)(\tilde{x}', \tilde{\xi}') \in C^\infty(\mathbb{R}^n_{\tilde{x}'} \setminus \{0\}, S^\mu_{\mathrm{cl}}(\mathbb{R}^n_{\tilde{\xi}'}; E, \tilde{E})),$$

homogeneous of order ϱ in $\tilde{x}' \neq 0$ in the sense

$$\sigma_{\mathrm{e}'}(a)(\delta\tilde{x}', \tilde{\xi}') = \delta^\varrho \sigma_{\mathrm{e}'}(a)(\tilde{x}', \tilde{\xi}') \tag{1.29}$$

for all $\delta \in \mathbb{R}_+$, and a corresponding mixed term

$$\sigma_{\partial,\mathrm{e}'}(a)(\tilde{x}', \tilde{\xi}') \in C^\infty((\mathbb{R}^n_{\tilde{x}'} \setminus \{0\}) \times (\mathbb{R}^n_{\tilde{\xi}'} \setminus \{0\}), \mathcal{L}(E, \tilde{E})),$$

homogeneous in $\tilde{\xi}' \neq 0$ of order μ as (1.28) and in $\tilde{x}' \neq 0$ of order ϱ as (1.29).

In particular, we can talk about so called Green symbols $g(\tilde{x}', \tilde{\xi}')$ of type 0 in the half-space, defined by the conditions

$$g(\tilde{x}', \tilde{\xi}'), g^*(\tilde{x}', \tilde{\xi}') \in S^{\mu;\varrho}_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n; L^2(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$$

with g^* being the pointwise adjoint with respect to the $L^2(\mathbb{R}_+)$-scalar product. Moreover, Green symbols $g(\tilde{x}', \tilde{\xi}')$ of type $d \in \mathbb{N}$ are defined by

$$g(\tilde{x}', \tilde{\xi}') = \sum_{j=0}^{d} g_j(\tilde{x}', \tilde{\xi}') \frac{\partial^j}{\partial \tilde{x}^j_{n+1}}$$

with Green symbols $g_j(\tilde{x}', \tilde{\xi}')$ of type 0 and order $(\mu - j, \varrho)$, $j = 0, \ldots, d$.

Let $\mathcal{B}^{-\infty,d;-\infty}(\overline{\mathbb{R}^{n+1}_+})$ denote the set of all operators

$$C = \sum_{j=0}^{d} C_j \frac{\partial^j}{\partial \tilde{x}^j_{n+1}},$$

where C_j are integral operators with kernels in

$$\mathcal{S}(\overline{\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+}) = \mathcal{S}(\mathbb{R}^{2(n+1)})|_{\overline{\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+}}.$$

An element $\psi \in C^\infty(\overline{\mathbb{R}^{n+1}_+})$ is called an admissible cut-off function, if it has the following properties:

(i) There are constants $R < R'$ such that

$$\psi = 1 \quad \text{for} \quad \tilde{x} \in L_R \ , \ \psi = 0 \quad \text{for} \quad \tilde{x} \notin L_{R'}$$

for $L_c := \{\tilde{x} \in \mathbb{R}^{n+1} : |\tilde{x}_{n+1}| \leq c|\tilde{x}'|\} \cup \{\tilde{x} \in \mathbb{R}^{n+1} : |\tilde{x}_{n+1}| \leq c\}$, $c > 0$.

(ii) $\psi(\delta\tilde{x}) = \psi(\tilde{x})$ for all $\delta \geq 1$, $|\tilde{x}| \geq \tilde{c}$ for a sufficiently large $\tilde{c} > 0$.

Now $\mathcal{B}^{\mu,d;\varrho}(\overline{\mathbb{R}^{n+1}_+})$ is defined as the set of all operators

$$A := \mathrm{r}^+ \operatorname{Op}_{\tilde{x}}(a)\mathrm{e}^+ + \psi_0 \operatorname{Op}_{\tilde{x}'}(g)\psi_1 + C \tag{1.30}$$

for arbitrary $a(\tilde{x}, \tilde{\xi}) \in S^{\mu;\varrho}_{\mathrm{cl}}(\mathbb{R}^{2(n+1)})_{\mathrm{tr}}$, a Green symbol $g(\tilde{x}', \tilde{\xi}')$ of type d, admissible cut-off functions ψ_0, ψ_1, and a smoothing operator $C \in \mathcal{B}^{-\infty,d;-\infty}(\overline{\mathbb{R}^{n+1}_+})$.

Let N be a closed compact C^∞ manifold, $n = \dim N$, and form a C^∞ manifold $N_\asymp \cong \mathbb{R} \times N \ni (r, x)$ with conical exits for $r \to \pm\infty$ (i.e., on $N_\asymp$ we fix a Riemannian metric that has the form $dr^2 + r^2 g_N$ for $|r| > R$ for some $R > 0$ where

g_N is a Riemannian metric on N). Moreover, let $H^{s;\delta}(N_{\asymp})$ for $s, \delta \in \mathbb{R}$ denote the subspace of all $u \in H^s_{\mathrm{loc}}(N_{\asymp})$ such that $(1-\omega)u, (1-\omega)u^{\vee} \in \langle r\rangle^{-\delta} H^s_{\mathrm{cone}}(N^{\wedge})$ for $u^{\vee}(r,x) = u(-r,x)$ and any cut-off function $\omega(r)$ on the positive half-axis. On $N_{\asymp}$ we then have a calculus of pseudo-differential operators $L^{\mu;\varrho}(N_{\asymp}), \mu, \varrho \in \mathbb{R}$, the local symbols of which on subsets of $U_{\asymp} \cong \mathbb{R} \times U$ (for any coordinate neighbourhood U on N) satisfy the symbolic estimates of the form (1.26) in coordinates $\tilde{x} \in \mathbb{R}^{n+1}$ (the corresponding charts $\chi : U_{\asymp} \to \Gamma$ are assumed to be homogeneous in the sense $\chi(\lambda r, x) = \lambda\tilde{x}$ for $\lambda \geq 1, |\tilde{x}| > R$ for some $R > 0$). In this calculus the smoothing operators have kernels in $\mathcal{S}(N_{\asymp} \times N_{\asymp}) \cong \mathcal{S}(\mathbb{R} \times \mathbb{R}, C^{\infty}(N \times N))$.

Taking classical local symbols in $(\tilde{x}, \tilde{\xi})$ we get the subspace $L^{\mu;\varrho}_{\mathrm{cl}}(N_{\asymp})$ of classical operators with exit behaviour for $|r| \to \infty$.

Given vector bundles $\tilde{E}, \tilde{F} \in \mathrm{Vect}(N_{\asymp})$ we have similarly the spaces

$$L^{\mu;\varrho}_{(\mathrm{cl})}(N_{\asymp}; \tilde{E}, \tilde{F})$$

of operators $\tilde{A}$ acting between corresponding spaces of distributional sections

$$\tilde{A} : H^{s;\delta}(N_{\asymp}, \tilde{E}) \to H^{s-\mu,\delta-\varrho}(N_{\asymp}, \tilde{F}).$$

Applying this picture to $N := 2X$, the double of a compact C^{∞} manifold X with boundary, there is a subspace $L^{\mu;\varrho}_{\mathrm{cl}}((2X)_{\asymp}; \tilde{E}, \tilde{F})_{\mathrm{tr}}$ of operators which have the transmission property at $(\partial X)_{\asymp}$. Then, if $\mathrm{r}^+ : H^{s;\delta}((2X)_{\asymp}, \tilde{E}) \to H^{s;\delta}(X_{\asymp}, E)$ (for $E = \tilde{E}|_{X_{\asymp}}$) is the operator of restriction to $\mathrm{int}\, X_{\asymp}$, $\mathrm{e}^+ : H^{s;\delta}(\mathrm{int}\, X_{\asymp}, E) \to H^{s;\delta}((2X)_{\asymp}, \tilde{E})$ the extension by zero to the opposite side of $\mathrm{int}\, X_{\asymp}$ in $(2X)_{\asymp}$, every $\tilde{A} \in L^{\mu;\varrho}_{\mathrm{cl}}((2X)_{\asymp}; \tilde{E}, \tilde{F})_{\mathrm{tr}}$ gives rise to continuous operators

$$\mathrm{r}^+ \tilde{A} \mathrm{e}^+ : H^{s;\delta}(X_{\asymp}, E) \to H^{s-\mu;\delta-\varrho}(X_{\asymp}, F)$$

for every $s, \delta \in \mathbb{R}, s > -\frac{1}{2}$.

There is also a generalisation of Green and smoothing operators of the class $\mathcal{B}^{\mu,d;\varrho}(\overline{\mathbb{R}}^{n+1}_+)$ to the case of a smooth manifold $X_{\asymp}$ with boundary and conical exits for $|r| \to \infty$, including the aspect of operators between sections of vector bundles and additional trace and potential operators, cf. also [9, Chapter 3]. This gives us spaces of boundary value problems

$$\mathcal{B}^{\mu,d;\varrho}(X_{\asymp}; \boldsymbol{v}) \quad \text{for} \quad \boldsymbol{v} = (E, F; J_-, J_+), \tag{1.31}$$

$E, F \in \mathrm{Vect}(X_{\asymp}),\ J_{\pm} \in \mathrm{Vect}((\partial X)_{\asymp})$. The operators in (1.31) have the form

$$\mathcal{A} = \begin{pmatrix} \mathrm{r}^+ \tilde{A} \mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G},$$

where $\mathcal{G}$ is a 2×2 block matrix of Green, trace and potential operators with exit behaviour for $|r| \to \infty$, and $\tilde{A} \in L^{\mu;\varrho}_{\mathrm{cl}}((2X)_{\asymp}; E, F)_{\mathrm{tr}}$. The operators $\mathcal{A}$ in (1.31) are continuous in the sense

$$\begin{matrix} H^{s;\delta}(X_{\asymp}, E) \\ \oplus \\ H^{s-\frac{1}{2};\delta}((\partial X)_{\asymp}, J_-) \end{matrix} \to \begin{matrix} H^{s-\mu;\delta-\varrho}(X_{\asymp}, F) \\ \oplus \\ H^{s-\mu-\frac{1}{2};\delta-\varrho}((\partial X)_{\asymp}, J_+) \end{matrix} \tag{1.32}$$

for all $s > d - \frac{1}{2}$. Let us set

$$\mathcal{B}^{\mu,d;\varrho}(X^\wedge;\boldsymbol{v}) = \{\mathcal{A}|_{X^\wedge} : \mathcal{A} \in \mathcal{B}^{\mu,d;\varrho}(X_\asymp;\boldsymbol{v})\}; \tag{1.33}$$

in this notation $X^\wedge$ is regarded as a subset of $X_\asymp$ in a canonical way, and bundles on $X_\asymp$ $((\partial X)_\asymp)$ and their restrictions to $X^\wedge$ $((\partial X)^\wedge)$ are denoted by the same letters.

Operators $\mathcal{A} \in \mathcal{B}^{\mu,d;\varrho}(X_\asymp;\boldsymbol{v})$ with $A := \mathrm{r}^+\widetilde{A}\mathrm{e}^+$ for $\widetilde{A} \in L^{\mu;\varrho}_{\mathrm{cl}}((2X)_\asymp;E,F)_{\mathrm{tr}}$ in the upper left corner have a principal symbolic structure

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_{\mathrm{e}}(\mathcal{A}), \sigma_{\psi,\mathrm{e}}(\mathcal{A}); \sigma_\partial(\mathcal{A}), \sigma_{\mathrm{e}'}(\mathcal{A}), \sigma_{\partial,\mathrm{e}'}(\mathcal{A})).$$

Here $\sigma_\psi(\mathcal{A})$ is the standard homogeneous principal symbol of A of order μ, further $\sigma_{\mathrm{e}}(\mathcal{A})$ is the homogeneous principal exit symbol of A of order ϱ (by definition, this concerns homogeneity in the variable r for $r \to \pm\infty$) and $\sigma_{\psi,\mathrm{e}}(\mathcal{A})$ is the homogeneous principal part of $\sigma_{\mathrm{e}}(\mathcal{A})$ of order μ in the $X_\asymp$-covariables. Moreover, $\sigma_\partial(\mathcal{A})$ is the standard homogeneous principal symbol of $\mathcal{A}$ of order μ, further $\sigma_{\mathrm{e}'}(\mathcal{A})$ is the homogeneous principal exit symbol of $\mathcal{A}$ of order ϱ (which refers again to homogeneity of order ϱ in the variable r for $r \to \pm\infty$), and $\sigma_{\partial,\mathrm{e}'}(\mathcal{A})$ is the homogeneous principal part of $\sigma_{\mathrm{e}'}(\mathcal{A})$ of order μ in the $\partial X_\asymp$-covariables.

An operator $\mathcal{A} \in \mathcal{B}^{\mu,d;\varrho}(X_\asymp;\boldsymbol{v})$ is said to be elliptic, if all components of $\sigma(\mathcal{A})$ are bijective on the respective sets of variables and covariables.

Theorem 1.7. *An operator $\mathcal{A} \in \mathcal{B}^{\mu,d;\varrho}(X_\asymp;\boldsymbol{v})$ is elliptic if and only if $\mathcal{A}$ defines a Fredholm operator* (1.32) *for any $s \in \mathbb{R}, s > \max(\mu,d) - \frac{1}{2}$.*

If $\mathcal{A} \in \mathcal{B}^{\mu,d;\varrho}(X_\asymp;\boldsymbol{v})$ is elliptic, there is a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,h;-\varrho}(X_\asymp;\boldsymbol{v}^{-1})$ for $\boldsymbol{v}^{-1} := (F,E;J_+,J_-)$ and $h = \max(d-\mu,0)$ such that

$$\mathcal{I} - \mathcal{P}\mathcal{A} \in \mathcal{B}^{-\infty,d_{\mathrm{l}};0}(X_\asymp;\boldsymbol{v}_{\mathrm{l}}), \quad \mathcal{I} - \mathcal{A}\mathcal{P} \in \mathcal{B}^{-\infty,d_{\mathrm{r}};0}(X_\asymp;\boldsymbol{v}_{\mathrm{r}})$$

for $d_{\mathrm{l}} = \max(\mu,d), d_{\mathrm{r}} = \max(d-\mu,0)$, and $\boldsymbol{v}_{\mathrm{l}} = (E,E;J_-,J_-), \boldsymbol{v}_{\mathrm{r}} = (F,F;J_+,J_+)$.

Boundary value problems with the transmission property on a manifold with exits to infinity have been studied systematically by Schrohe [21] where one can find, in particular, the necessity of the ellipticity for the Fredholm property of (1.32). In the present paper we refer to the corresponding calculus of [9, Chapter 3].

1.3. The cone algebra

We now turn to boundary value problems of the classes $\mathcal{B}^{\mu,d}$ globally on $X^\wedge = \mathbb{R}_+ \times X \ni (r,x)$. Close to $r = 0$ we impose the structure of the cone algebra. For $r \to \infty$ the calculus corresponds to operators on a manifold with conical exit to infinity as formulated in the preceding section.

Let $\omega_0(r), \omega_1(r), \omega_2(r)$ be cut-off functions on the half-axis, $\omega_1 \equiv 1$ on $\operatorname{supp}\omega_0$, and $\omega_0 \equiv 1$ on $\operatorname{supp}\omega_2$, and set

$$\mathcal{A} = \omega_0\mathcal{A}_{\mathrm{cone}}\omega_1 + (1-\omega_0)\mathcal{A}_{\mathrm{exit}}(1-\omega_2) + \mathcal{C}. \tag{1.34}$$

Here $\mathcal{A}_{\mathrm{exit}} \in \mathcal{B}^{\mu,d;0}(X^\wedge;\boldsymbol{v})$, and $\mathcal{A}_{\mathrm{cone}}$ is of the form

$$\mathcal{A}_{\mathrm{cone}} = r^{-\mu}\{\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) + \sum_{j=0}^{k} r^j\left[\mathrm{op}_M^{\delta_j-\frac{n}{2}}(f_j) + \mathrm{op}_M^{\tilde{\delta}_j-\frac{n}{2}}(\tilde{f}_j)\right]\} \tag{1.35}$$

for some $k \in \mathbb{N}$ and weights

$$\delta_0 = \tilde{\delta}_0 = \gamma \text{ and } \delta_j = \gamma - \frac{1}{3},\ \tilde{\delta}_j = \gamma - \frac{2}{3} \text{ for } 1 \le j \le k. \tag{1.36}$$

Note that in (1.36) we could fix any other weights $\delta_j, \tilde{\delta}_j$ such that $\gamma - j \le \delta_j,\ \tilde{\delta}_j \le \gamma$ and $\delta_j \ne \tilde{\delta}_j$ for $j \ge 1$.

We assume

$$h(r,z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{C})). \tag{1.37}$$

In order to define f_j and $\tilde{f}_j$ we first introduce so called discrete and continuous asymptotic types of Mellin symbols.

Given a set $A \subseteq \mathbb{C}$ we define $A^I := \{\lambda z_1 + (1-\lambda)z_2 : z_1, z_2 \in A, \operatorname{Re} z_1 = \operatorname{Re} z_2, 0 \le \lambda \le 1\}$. Let $\mathcal{V}$ denote the system of all closed subsets V of $\mathbb{C}$ such that $V \cap \{c \le \operatorname{Re} z \le c'\}$ is compact for every $c \le c'$, and $V^I = V$. A V-excision function is any $\chi \in C^\infty(\mathbb{C})$ such that $\chi(z) = 0$ for $\operatorname{dist}(z,V) < \varepsilon_0$, $(z) = 1$ for $\operatorname{dist}(z,V) > \varepsilon_1$ for certain $0 < \varepsilon_0 < \varepsilon_1$.

For every $V \in \mathcal{V}$ we form the space

$$\mathcal{B}_V^{-\infty,d}(X;\boldsymbol{v};\mathbb{C}) \tag{1.38}$$

of all $f(z) \in \mathcal{A}(\mathbb{C} \setminus V, \mathcal{B}^{-\infty,d}(X;\boldsymbol{v}))$ such that for every V-excision function $\chi(z)$, we have

$$\chi(z)f(z)|_{z=\beta+i\varrho} \in \mathcal{B}^{-\infty,d}(X;\boldsymbol{v};\mathbb{R}_\varrho)$$

for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$ for every $c \le c'$. The space (1.38) is Fréchet in a canonical way.

Consider, in particular, a compact set, and let $\mathcal{A}'(K,E)$ be the space of all analytic functionals, carried by K, with values in a Fréchet space E. Then, for $K \in \mathcal{V}$ we have

$$f(z) := M_{\delta, r\to z}\omega(r)\langle\zeta, r^{-z}\rangle \in \mathcal{B}_K^{-\infty,d}(X;\boldsymbol{v};\mathbb{C})$$

for every $\zeta \in \mathcal{A}'(K, \mathcal{B}^{-\infty,d}(X;\boldsymbol{v}))$ and a cut-off function ω. Here $\delta \in \mathbb{R}$ is any weight such that $K \subset \{z : \operatorname{Re} z < \frac{1}{2} - \delta\}$. Moreover,

$$\mathcal{B}_K^{-\infty,d}(X;\boldsymbol{v};\mathbb{C}) = \{M_\delta\omega\langle\zeta, r^{-z}\rangle + f_0 : \zeta \in \mathcal{A}'(K, \mathcal{B}^{-\infty,d}(X;\boldsymbol{v})),$$
$$f_0 \in \mathcal{B}^{-\infty,d}(X;\boldsymbol{v};\mathbb{C})\}.$$

For $V, \tilde{V} \in \mathcal{V}$, $V + \tilde{V} := (V \cup \tilde{V})^I$, we have the relation

$$\mathcal{B}_{V+\tilde{V}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{C}) = \mathcal{B}_V^{-\infty,d}(X;\boldsymbol{v};\mathbb{C}) + \mathcal{B}_{\tilde{V}}^{-\infty,d}(X;\boldsymbol{v};\mathbb{C}) \tag{1.39}$$

as a non-direct sum of Fréchet spaces. The sets $V \in \mathcal{V}$ will be called continuous asymptotic types for Mellin symbols of the cone algebra.

A useful property of decompositions like (1.39) is that for every $V_1 \in \mathcal{V}$ and $\delta, \tilde{\delta} \in \mathbb{R}$, $\delta \neq \tilde{\delta}$, there exist sets $V, \tilde{V} \in \mathcal{V}$ such that

$$V_1 = V + \tilde{V} \text{ and } \Gamma_\delta \cap V = \Gamma_{\tilde{\delta}} \cap \tilde{V} = \emptyset. \tag{1.40}$$

In other words, Mellin symbols l with continuous asymptotic types can be decomposed as sums $l = f + \tilde{f}$, where $f \in \mathcal{B}_V^{-\infty,d}(X;\boldsymbol{v};\mathbb{C})$ and $\tilde{f} \in \mathcal{B}_{\tilde{V}}^{-\infty}(X;\boldsymbol{v};\mathbb{C})$ are holomorphic in a neighbourhood of Γ_δ and $\Gamma_{\delta'}$, respectively.

Now the Mellin symbols f_j and $\tilde{f}_j$ in (1.35) are assumed as

$$f_j(z) \in \mathcal{B}_{V_j}^{-\infty,d}(X;\boldsymbol{v};\mathbb{C}) \quad \text{and} \quad \tilde{f}_j(z) \in \mathcal{B}_{\tilde{V}_j}^{-\infty,d}(X;\boldsymbol{v};\mathbb{C}), \tag{1.41}$$

respectively, with sets $V_j, \tilde{V}_j \in \mathcal{V}$ such that $\Gamma_{\frac{n+1}{2}-\delta_j} \cap V_j = \Gamma_{\frac{n+1}{2}-\tilde{\delta}_j} \cap \tilde{V}_j = \emptyset$ for all j.

It is interesting also to consider Mellin symbols, where the asymptotic information is supported by a discrete set $\{p_j\}_{j\in\mathbb{Z}} \subset \mathbb{C}$.

A sequence of triples

$$R = \{(p_j, m_j, L_j)\}_{j\in\mathbb{Z}}$$

for $p_j \in \mathbb{C}, m_j \in \mathbb{N}$, is called a discrete asymptotic type for Mellin symbols if

(i) $\pi_{\mathbb{C}}R := \{p_j\}_{j\in\mathbb{N}}$ has a finite intersection with $\{z : c \leq \operatorname{Re} z \leq c'\}$ for every $c \leq c'$,

(ii) $L_j \subset \mathcal{B}^{-\infty,d}(X;\boldsymbol{v})$ is a finite-dimensional subspace of operators of finite rank, for every $j \in \mathbb{N}$.

We denote by $\mathcal{B}_R^{-\infty,d}(X;\boldsymbol{v};\mathbb{C})$ the subspace of all $f(z) \in \mathcal{B}_{\pi_{\mathbb{C}}R}^{-\infty,d}(X;\boldsymbol{v};\mathbb{C})$ that are meromorphic with poles at the points p_j of multiplicity $m_j + 1$ and Laurent coefficients at $(z-p_j)^{-(k+1)}$ belonging to L_j for all $0 \leq k \leq m_j$, $j \in \mathbb{N}$.

It remains to explain the nature of operators $\mathcal{C}$ in the expression (1.34). To this end we first recall the notion of continuous and discrete asymptotics in the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$.

First, if $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$, is a weight interval, we set

$$\mathcal{K}_\Theta^{s,\gamma}(X^\wedge) := \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(X^\wedge)$$

which is a Fréchet space in a canonical way. Let Θ be finite, consider any $V \in \mathcal{V}$, $V \subset \{z : \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$, and set $K := V \cap \{z : \frac{n+1}{2} - \gamma - \vartheta \leq \operatorname{Re} z \leq \frac{n+1}{2} - \gamma\}$ which is a compact set. Then

$$\mathcal{E}_K(X^\wedge) := \{\omega(r)\langle\zeta, r^{-z}\rangle : \zeta \in \mathcal{A}'(K, C^\infty(X))\}$$

is a Fréchet subspace of $\mathcal{K}^{\infty,\gamma}(X^\wedge)$, and we define

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_K(X^\wedge) \tag{1.42}$$

in the Fréchet topology of the non-direct sum. Subscript 'P' has the meaning of a so called continuous asymptotic type associated with V and weight data $\boldsymbol{g} = (\gamma, \Theta)$. We interpret P as the quotient space $\mathcal{A}'(K, C^\infty(X))/\sim$, where equivalence $\zeta \sim \zeta'$ means that $\omega(r)\langle\zeta - \zeta',\ r^{-z}\rangle$ belongs to $\mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge)$. Let us set

$$\pi_{\mathbb{C}}P := K \cap \{z : \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} z\}. \tag{1.43}$$

For $\Theta = (-\infty, 0]$ we choose a sequence $\{\vartheta_j\}_{j\in\mathbb{N}}$, $\vartheta_{j+1} < \vartheta_j < 0$ for all j and $\vartheta_j \to -\infty$ for $j \to \infty$. Then, if P_j is the continuous asymptotic type, associated with V and weight data $\boldsymbol{g}_j = (\gamma, (\vartheta_j, 0])$, there are continuous embeddings

$$\mathcal{K}^{s,\gamma}_{P_{j+1}}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}_{P_j}(X^\wedge) \tag{1.44}$$

for all j. We then set $\mathcal{K}^{s,\gamma}_P(X^\wedge) = \varprojlim_{j\in\mathbb{N}} \mathcal{K}^{s,\gamma}_{P_j}(X^\wedge)$, where P is represented by the sequence of P_j, $j \in \mathbb{N}$, and is regarded as a continuous asymptotic type associated with V and $\boldsymbol{g} = (\gamma, (-\infty, 0])$. This construction is independent of the specific choice of the sequence $\{\vartheta_j\}_{j\in\mathbb{N}}$. Let $\operatorname{As}(X, \boldsymbol{g})$ denote the set of all continuous asymptotic types P, belonging to any $V \in \mathcal{V}$ and $\boldsymbol{g} = (\gamma, \Theta)$. Similarly to (1.43) we write

$$\pi_{\mathbb{C}}P = V,$$

if P belongs to $V \in \mathcal{V}$.

Let us write

$$P_1 \subseteq P_2 \quad \text{for} \quad \pi_{\mathbb{C}}P_1 \subseteq \pi_{\mathbb{C}}P_2. \tag{1.45}$$

A sequence

$$P = \{(p_j, m_j, L_j)\}_{0\le j\le N},$$

with $p_j \in \mathbb{C}, m_j \in \mathbb{N}$, $N = N(P) \le \infty$, is called a discrete asymptotic type for weighted distributions on $X^\wedge$ near $r = 0$, associated with weight data $\boldsymbol{g} = (\gamma, \Theta)$, $\Theta = (\vartheta, 0]$, if

(i) $\pi_{\mathbb{C}}P = \{p_j\}_{0\le j\le N} \subset \{z : \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$, $N(P) < \infty$ for finite Θ, $\pi_{\mathbb{C}}P \cap \{z : c \le \operatorname{Re} z \le c'\}$ finite for every $c \le c'$.

(ii) $m_j \in \mathbb{N}$ and $L_j \subset C^\infty(X, E)$ (for a given $E \in \operatorname{Vect}(X)$) is a finite-dimensional subspace, for every $j \in \mathbb{N}$.

Let $\operatorname{As}(X, \boldsymbol{g}^\bullet)$ denote the set of all discrete asymptotic types in that sense. For $P \in \operatorname{As}(X, \boldsymbol{g}^\bullet)$, $\boldsymbol{g} = (\gamma, \Theta)$, Θ finite, the space

$$\begin{aligned} \mathcal{E}_P(X^\wedge) :=\{\omega(r) \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r : c_{jk} \in L_j \\ \text{for } 0 \le k \le m_j,\ 0 \le j \le N\} \end{aligned}$$

(for any fixed cut-off function ω) is finite-dimensional, and we have $\mathcal{E}_P(X^\wedge) \subset \mathcal{K}^{\infty,\gamma}(X^\wedge)$ and $\mathcal{E}_P(X^\wedge) \cap \mathcal{K}^{\infty,\gamma}_\Theta(X^\wedge) = \{0\}$. We then set

$$\mathcal{K}^{s,\gamma}_P(X^\wedge) := \mathcal{K}^{s,\gamma}_\Theta(X^\wedge) + \mathcal{E}_P(X^\wedge) \tag{1.46}$$

in the Fréchet topology of the direct sum.

For $\Theta = (-\infty, 0]$ we choose a sequence $\{\vartheta_j\}_{j\in\mathbb{N}}$ as above, and form $P_j := \{(p,m,L) \in P : \mathrm{Re}\, p > \frac{n+1}{2} - \gamma + \vartheta_j\} \in \mathrm{As}(X, \boldsymbol{g}_j^\bullet)$ for $\boldsymbol{g}_j = (\gamma, \Theta_j)$, $\Theta_j = (\vartheta_j, 0]$. Then we have continuous embeddings (1.44) for all j, and we set again

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \varprojlim_{j\in\mathbb{N}} \mathcal{K}_{P_j}^{s,\gamma}(X^\wedge).$$

Moreover, let us set

$$\mathcal{S}_P^\gamma(X^\wedge) = \{\omega u + (1-\omega)v : u \in \mathcal{K}_P^{\infty,\gamma}(X^\wedge),\ v \in \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))\}. \tag{1.47}$$

Remark 1.8. The spaces $\mathcal{K}_P^{s,\gamma}(X^\wedge)$ and $\mathcal{S}_P^\gamma(X^\wedge)$ for $P \in \mathrm{As}(X, \boldsymbol{g})$ or $P \in \mathrm{As}(X, \boldsymbol{g}^\bullet)$ are Fréchet spaces with group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, $(\kappa_\delta u)(r,x) = \delta^{\frac{n+1}{2}} u(\delta r, x)$ (cf. the notation of Section 1.1).

An operator

$$\mathcal{C} : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, E) \\ \oplus \\ \mathcal{K}^{s',\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \end{array} \to \begin{array}{c} \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge, F) \\ \oplus \\ \mathcal{K}^{\infty,\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \end{array}$$

which is continuous for $s > -\frac{1}{2}$, $s' \in \mathbb{R}$, is called a Green operator of type 0 in the cone algebra on $X^\wedge$, if $\mathcal{C}$ and its formal adjoint $\mathcal{C}^*$ induce continuous operators

$$\mathcal{C} : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, E) \\ \oplus \\ \mathcal{K}^{s',\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \end{array} \to \begin{array}{c} \mathcal{S}_P^{\gamma-\mu}(X^\wedge, F) \\ \oplus \\ \mathcal{S}_{P'}^{\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \end{array} \tag{1.48}$$

and

$$\mathcal{C}^* : \begin{array}{c} \mathcal{K}^{s,-\gamma+\mu}(X^\wedge, F) \\ \oplus \\ \mathcal{K}^{s',-\gamma+\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \end{array} \to \begin{array}{c} \mathcal{S}_{Q'}^{-\gamma-\frac{1}{2}}((\partial X^\wedge, E) \\ \oplus \\ \mathcal{S}_{Q'}^{-\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \end{array}, \tag{1.49}$$

respectively, for all $s > -\frac{1}{2}$, $s' \in \mathbb{R}$, with asymptotic types

$$(P, P') \in \mathrm{As}(X, (\gamma-\mu, \Theta)) \times \mathrm{As}(\partial X, (\gamma-\mu-\frac{1}{2}, \Theta)) \tag{1.50}$$

and

$$(Q, Q') \in \mathrm{As}(X, (-\gamma, \Theta)) \times \mathrm{As}(\partial X, (-\gamma-\frac{1}{2}, \Theta)), \tag{1.51}$$

depending on $\mathcal{C}$ (not on s, s'). The formal adjoint is defined in terms of the respective $\mathcal{K}^{0,0}(X^\wedge, \cdot) \oplus \mathcal{K}^{0,-\frac{1}{2}}((\partial X)^\wedge, \cdot)$-scalar products (where dots stand for the corresponding bundles).

An operator of the form

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{d} \mathcal{C}_j \,\mathrm{diag}(D^j, 0)$$

for Green operators $\mathcal{C}_j$ of type 0, $0 \le j \le d$, and D^j as in (1.17), is called a Green operator of type d.

Definition 1.9. Let $\mathcal{C}^{\mu,d}(X^\wedge, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, $\Theta = (-(k+1), 0]$, $\boldsymbol{v} = (E, F; J_-, J_+)$, denote the space of all operators of the form (1.34), such that $\mathcal{A}_{\mathrm{exit}} \in \mathcal{B}^{\mu,d;0}(X^\wedge; \boldsymbol{v})$, moreover, $\mathcal{A}_{\mathrm{cone}}$ given by (1.34), (1.37), (1.41), and a Green operator $\mathcal{C}$ of type d. For $\Theta = (-\infty, 0]$ we set $\mathcal{C}^{\mu,d}(X, \boldsymbol{g}; \boldsymbol{v}) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^{\mu,d}(X; \boldsymbol{g}_k; \boldsymbol{v})$ for $\boldsymbol{g}_k := (\gamma, \gamma-\mu, (-(k+1), 0])$.

Theorem 1.10. *Every $\mathcal{A} \in \mathcal{C}^{\mu,d}(X^\wedge, \boldsymbol{g}; \boldsymbol{v})$ induces continuous operators*

$$\begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge, E) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \end{matrix} \to \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge, F) \\ \oplus \\ \mathcal{K}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \end{matrix} \tag{1.52}$$

as well as

$$\begin{matrix} \mathcal{K}_P^{s,\gamma}(X^\wedge, E) \\ \oplus \\ \mathcal{K}_{P'}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \end{matrix} \to \begin{matrix} \mathcal{K}_Q^{s-\mu,\gamma-\mu}(X^\wedge, F) \\ \oplus \\ \mathcal{K}_{Q'}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \end{matrix} \tag{1.53}$$

for every $s \in \mathbb{R}, s > d - \frac{1}{2}$, and every pair of asymptotic types (1.50) *with some resulting* (1.51).

This theorem may be found in [9, Section 2.1.7], cf. also [29, Theorem 2.3.55] for the case without boundary.

Operators $\mathcal{A} \in \mathcal{C}^{\mu,d}(X^\wedge, \boldsymbol{g}; \boldsymbol{v})$ have a principal symbolic hierarchy

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_{\mathrm{c}}(\mathcal{A}), \sigma_E(\mathcal{A})),$$

where $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ are the homogeneous interior and boundary symbols of $\mathcal{A}$, regarded as an element of $\mathcal{B}^{\mu,d}(X^\wedge; \boldsymbol{v})$. Moreover, $\sigma_{\mathrm{c}}(\mathcal{A})$ is the principal conormal symbol, defined as

$$\sigma_{\mathrm{c}}(\mathcal{A})(z) = h(0, z) + f_0(z), \tag{1.54}$$

cf. (1.35) (without loss of generality, we assume $\tilde{f}_0 = 0$, otherwise we have to replace the summand $f_0(z)$ in (1.54) by $f_0(z) + \tilde{f}_0(z)$). By definition,

$$\sigma_{\mathrm{c}}(\mathcal{A}) : \begin{matrix} H^s(X, E) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial X, J_-) \end{matrix} \to \begin{matrix} H^{s-\mu}(X, F) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(\partial X, J_+) \end{matrix} \tag{1.55}$$

is an element of $\mathcal{B}_V^{\mu,d}(X; \boldsymbol{v}; \mathbb{C})$ for some $V \in \mathcal{V}, V \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$.

Finally, $\sigma_E(\mathcal{A})$ is the tuple of exit symbolic components

$$\sigma_E(\mathcal{A}) = (\sigma_{\mathrm{e}}(\mathcal{A}), \sigma_{\psi,\mathrm{e}}(\mathcal{A}), \sigma_{\mathrm{e}'}(\mathcal{A}), \sigma_{\partial,\mathrm{e}'}(\mathcal{A}))$$

for $r \to \infty$, cf. the notation of the preceding section.

An operator $\mathcal{A} \in \mathcal{C}^{\mu,d}(X^\wedge, \boldsymbol{g}; \boldsymbol{v})$ is called elliptic (with respect to the weight $\gamma \in \mathbb{R}$), if all components of $\sigma(\mathcal{A})$ are bijective; for (1.54) this means that (1.55) is a family of isomorphisms for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ and any $s \in \mathbb{R}, s > \max(\mu, d) - \frac{1}{2}$ (this condition is independent of the choice of s).

Theorem 1.11. *An operator $\mathcal{A} \in \mathcal{C}^{\mu,d}(X^\wedge, \boldsymbol{g}; \boldsymbol{v})$ is elliptic if and only if* (1.52) *is a Fredholm operator for some $s > \max(\mu, d) - \frac{1}{2}$.*

Ellipticity entails the Fredholm property of (1.52) *for every $s > \max(\mu, d) - \frac{1}{2}$, and the operator $\mathcal{A}$ has a paramatrix $\mathcal{P} \in \mathcal{C}^{-\mu,h}(X^\wedge, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ for $h = \max(d - \mu, 0)$, $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, $\boldsymbol{v}^{-1} = (F, E; J_+, J_-)$, i.e.,*

$$\mathcal{I} - \mathcal{P}\mathcal{A} \in \mathcal{C}_G^{d_l}(X^\wedge, \boldsymbol{g}_{\mathrm{e}}; \boldsymbol{v}_{\mathrm{e}}), \quad \mathcal{I} - \mathcal{A}\mathcal{P} \in \mathcal{C}_G^{d_r}(X^\wedge, \boldsymbol{g}_{\mathrm{r}}; \boldsymbol{v}_{\mathrm{r}})$$

for $d_{\mathrm{l}} = \max(\mu, d)$, $\boldsymbol{g}_{\mathrm{l}} = (\gamma, \gamma; \Theta)$, $\boldsymbol{v}_{\mathrm{l}} = (E, E; J_-, J_-)$, and $d_{\mathrm{r}} = \max(d - \mu, 0)$, $\boldsymbol{g}_{\mathrm{r}} = (\gamma - \mu, \gamma - \mu, \Theta)$, $\boldsymbol{v}_{\mathrm{r}} = (F, F; J_+, J_+)$.

The proof of the Fredholm property of (1.52) follows from a parametrix construction, combining a local parametrix near the tip of the cone from [20] with a parametrix far from the tip up to infinity, using the second part of Theorem 1.7. The necessity of the ellipticity for the Fredholm property can be obtained by writing $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_{\mathrm{int}} + \mathcal{A}_\infty$ modulo a Green operator, where $\mathcal{A}_0$ is localised near the tip of the cone, $\mathcal{A}_{\mathrm{int}}$ far from the tip as well as far from ∞, and $\mathcal{A}_\infty$ localised near ∞. Now different variants of 'Gohberg's lemma' allows us to treat the principal symbolic components separately, cf. analogously [26, Section 2.2.1] for $\mathcal{A}_0$, [19, Section 3.1.1.1] for $\mathcal{A}_{\mathrm{int}}$, and [21] for $\mathcal{A}_\infty$.

2. The edge algebra with parameters

2.1. Edge-amplitude functions

Edge-amplitude functions $a(y, \eta)$ as they will be defined in this section are particular operator-valued symbols in the sense of (1.12), with E and $\tilde{E}$ being of the form

$$\mathcal{K}^{s,\gamma}(X^\wedge, E) \oplus \mathcal{K}^{s-\frac{1}{2}, \gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \oplus \mathbb{C}^{l_-}$$

and

$$\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge, F) \oplus \mathcal{K}^{s-\mu-\frac{1}{2}, \gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \oplus \mathbb{C}^{l_+},$$

respectively, cf. the notation of Section 1.3. The group actions in both spaces are

$$\kappa_\delta : u(r, x) \oplus u'(r, x') \oplus c \to \delta^{\frac{n+1}{2}}(u(\delta r, x) \oplus u'(\delta r, x') \oplus c), \tag{2.1}$$

$\delta \in \mathbb{R}_+$. The 2×2 upper left corners of the operator functions $a(y, \eta)$ take values in the cone algebra on $X^\wedge$, and there is a specific dependence on the parameter η.

These amplitude functions will constitute a space

$$\mathcal{R}^{\mu,d}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w}), \tag{2.2}$$

$d \in \mathbb{N}$, with weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, for a finite weight interval $\Theta = (-(k + 1), 0]$, and tuples $\boldsymbol{w} = (E, F; J_-, J_+; l_-, l_+)$, with vector bundles $E, F \in \mathrm{Vect}(X)$, $J_\pm \in \mathrm{Vect}(\partial X)$. The numbers $l_\pm$ are the fibre dimensions of bundles $L_\pm \in \mathrm{Vect}(Y)$ in the global calculus below.

Moreover, we will single out a subspace of so called Green edge-symbols

$$\mathcal{R}_G^{\mu,d}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w}). \tag{2.3}$$

Then the elements $a(y,\eta)$ of (2.2) will have the form

$$a(y,\eta) = \operatorname{diag}(b(y,\eta),0) + g(y,\eta) \tag{2.4}$$

for arbitrary $g(y,\eta) \in \mathcal{R}_G^{\mu,d}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ and $b(y,\eta) \in \mathcal{R}^{\mu,d}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{v} := (E,F;J_-,J_+)$ (the latter space corresponds to (2.2) for the case $l_- = l_+ = 0$).

Let us now turn to more details. The structure of $b(y,\eta)$ is as follows. Choose arbitrary cut-off functions $\sigma(r), \tilde{\sigma}(r)$ and $\omega(r), \tilde{\omega}(r), \tilde{\tilde{\omega}}(r)$, where we assume $\tilde{\omega} = 1$ on $\operatorname{supp}\omega$, and $\omega = 1$ on $\operatorname{supp}\tilde{\tilde{\omega}}$. Let us set $\omega_\eta(r) := \omega(r[\eta])$ for some strictly positive function $[\eta] \in C^\infty(\mathbb{R}^q)$ such that $[\eta] = |\eta|$ for $|\eta| > c$ for some constant $c > 0$.

(i) We choose an arbitrary element

$$\tilde{p}(r,y,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}})) \tag{2.5}$$

and form the family of operators $\operatorname{op}_r(p)(y,\eta)$ for

$$p(r,\varrho,\eta) := \tilde{p}(r,r\varrho,r\eta). \tag{2.6}$$

(ii) Let $\tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}^{\mu,d}(X;\boldsymbol{v};\mathbb{C} \times \mathbb{R}^q_{\tilde{\eta}}))$ be an element such that

$$\operatorname{op}_r(p)(y,\eta) = \operatorname{op}_M^\beta(h)(y,\eta) \bmod C^\infty(U, \mathcal{B}^{-\infty,d}(X^\wedge;\boldsymbol{v};\mathbb{R}^q_\eta))$$

for

$$h(r,y,z,\eta) := \tilde{h}(r,y,z,r\eta). \tag{2.7}$$

(iii) We set

$$m(y,\eta) = \omega_\eta r^{-\mu} \sum_{j+|\alpha| \le k} r^j \{\operatorname{op}_M^{\delta_j - \frac{n}{2}}(f_{j\alpha})(y) + \operatorname{op}_M^{\tilde{\delta}_j - \frac{n}{2}}(\tilde{f}_{j\alpha})(y)\}\eta^\alpha \tilde{\omega}_\eta \tag{2.8}$$

with weights $\delta_j, \tilde{\delta}_j$ as in (1.36) and functions

$$f_{j\alpha} \in C^\infty(U, \mathcal{B}^{-\infty,d}_{V_{j\alpha}}(X;\boldsymbol{v};\mathbb{C})), \quad \tilde{f}_{j\alpha} \in C^\infty(U, \mathcal{B}^{-\infty,d}_{\tilde{V}_{j\alpha}}(X;\boldsymbol{v};\mathbb{C})) \tag{2.9}$$

for certain $V_{j\alpha}, \tilde{V}_{j\alpha} \in \mathcal{V}$, satisfying the conditions

$$\Gamma_{\frac{n+1}{2}-\delta_j} \cap V_{j\alpha} = \emptyset, \quad \Gamma_{\frac{n+1}{2}-\tilde{\delta}_j} \cap \tilde{V}_{j\alpha} = \emptyset \tag{2.10}$$

for all j, α.

(iv) The operator function $b(y,\eta)$ in (2.4) has the form

$$\begin{aligned} b(y,\eta) = \sigma r^{-\mu}\{&\omega_\eta \operatorname{op}_M^{\gamma - \frac{n}{2}}(h)(y,\eta)\tilde{\omega}_\eta \\ &+ (1-\omega_\eta)\operatorname{op}_r(p)(y,\eta)(1-\tilde{\tilde{\omega}}_\eta)\}\tilde{\sigma} + m(y,\eta) \end{aligned} \tag{2.11}$$

for p, h and m as in (i), (ii) and (iii), respectively.

Let us set, for abbreviation,

$$\boldsymbol{E}^{0,\gamma} := \mathcal{K}^{0,\gamma}(X^\wedge, E) \oplus \mathcal{K}^{0,\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \oplus \mathbb{C}^{l_-},$$

$$\boldsymbol{\mathcal{S}}_{\mathcal{P}}^{\gamma-\mu} := \mathcal{S}_P^{\gamma-\mu}(X^\wedge, F) \oplus \mathcal{S}_{P'}^{\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \oplus \mathbb{C}^{l_+}$$

for asymptotic types $P \in \mathrm{As}(X, (\gamma, \Theta))$, $P' \in \mathrm{As}(X, (\gamma - \frac{1}{2}, \Theta))$, $\mathcal{P} = (P, P')$, and

$$\boldsymbol{F}^{0,-\gamma+\mu} := \mathcal{K}^{0,\gamma-\mu}(X^\wedge, F) \oplus \mathcal{K}^{0,\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \oplus \mathbb{C}^{l_+},$$

$$\boldsymbol{\mathcal{S}}_{\mathcal{Q}}^{-\gamma} := \mathcal{S}_Q^{-\gamma}(X^\wedge, E) \oplus \mathcal{S}_{Q'}^{-\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \oplus \mathbb{C}^{l_-}$$

for asymptotic types $Q \in \mathrm{As}(X, (-\gamma + \mu, \Theta))$, $Q' \in \mathrm{As}((\partial X)^\wedge, (-\gamma + \mu - \frac{1}{2}, \Theta))$, $\mathcal{Q} = (Q, Q')$.

A Green symbol $g(y, \eta)$ of type $d = 0$ in the sense of (2.3) is defined as a function

$$g(y, \eta) = \mathrm{diag}(1, \langle\eta\rangle^{\frac{1}{2}}, \langle\eta\rangle^{\frac{n+1}{2}}) g_0(y, \eta)\, \mathrm{diag}(1, \langle\eta\rangle^{\frac{1}{2}}, \langle\eta\rangle^{\frac{n+1}{2}})^{-1}$$

with an operator-valued symbol

$$g_0(y, \eta) \in S_{\mathrm{cl}}^{\mu}(U \times \mathbb{R}^q; \boldsymbol{E}^{0,\gamma}, \boldsymbol{\mathcal{S}}_{\mathcal{P}}^{\gamma-\mu})_{(\kappa^0, \kappa^0)}, \tag{2.12}$$

cf. notation (2.16) below, such that their pointwise formal adjoint $g_0^*(y, \eta)$ in the sense $(g_0 u, v)_{\boldsymbol{E}^{0,0}} = (u, g_0^* v)_{\boldsymbol{F}^{0,0}}$ for all $u \in C_0^\infty(X^\wedge, E) \oplus C_0^\infty((\partial X)^\wedge, J_-) \oplus \mathbb{C}^{l_-}$, $v \in C_0^\infty(X^\wedge, F) \oplus C_0^\infty((\partial X)^\wedge, J_+) \oplus \mathbb{C}^{l_+}$ represents an element

$$g_0^*(y, \eta) \in S_{\mathrm{cl}}^{\mu}(U \times \mathbb{R}^q; \boldsymbol{F}^{0,-\gamma+\mu}, \boldsymbol{\mathcal{S}}_{\mathcal{Q}}^{-\gamma})_{(\kappa^0, \kappa^0)}. \tag{2.13}$$

The asymptotic types

$$\mathcal{P} = (P, P') \in \mathrm{As}(X, (\gamma - \mu, \Theta)) \times \mathrm{As}(\partial X, (\gamma - \mu - \frac{1}{2}, \Theta)), \tag{2.14}$$

$$\mathcal{Q} = (Q, Q') \in \mathrm{As}(X, (-\gamma, \Theta)) \times \mathrm{As}(\partial X, (-\gamma - \frac{1}{2}, \Theta)) \tag{2.15}$$

depend on g_0. Subscripts '(κ^0, κ^0)' in the relations (2.12) and (2.13) mean that the spaces of symbols refer to the group actions

$$\kappa_\delta^0 : u(r, x) \oplus u'(r, x') \oplus c \to \delta^{\frac{n+1}{2}} u(\lambda r, x) \oplus \delta^{\frac{n}{2}} u'(\lambda r, x) \oplus c, \tag{2.16}$$

$\delta \in \mathbb{R}_+$, in contrast to (2.1). In this way we avoid distinguishing matrices of Douglis-Nirenberg orders for symbols and their adjoints. Clearly, the Green symbols $g(y, \eta)$ themselves are operator-valued symbols with respect to the group actions (2.1), cf. Remark 2.3 below.

Now a Green symbol $g(y, \eta)$ in (2.3) of type $d \in \mathbb{N}$ is defined as a linear combination

$$g(y, \eta) = h_0(y, \eta) + \sum_{j=1}^{d} h_j(y, \eta)\, \mathrm{diag}(T^j, 0, 0) \tag{2.17}$$

with T^j being as in (1.17) and $h_j \in \mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ of type zero, $j = 0, \ldots, d$.

Let

$$\mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w}) \tag{2.18}$$

be the subspace of all elements of (2.2) of the form

$$a(y,\eta) = m(y,\eta) + g(y,\eta) \tag{2.19}$$

for arbitrary $m(y,\eta)$ as in (iii) above, and a Green symbol $g(y,\eta)$ of type d.

Remark 2.1. The space $\mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ is equal to the subspace of all $a(y,\eta) \in \mathcal{R}^{\mu,d}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ such that $\tilde{p}(r, y, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}^{-\infty,d}(X; \boldsymbol{v}; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$.

For the aspect of holomorphic dependence of symbols in (2.18) on a further parameter it is interesting to single out Fréchet subspaces that are parametrised by the type $d \in \mathbb{N}$ and asymptotic types involved in (2.8) as well as $\mathcal{P}$ and $\mathcal{Q}$ in the Green summands $g(y,\eta)$. For this consideration it is not essential to fix the carrier sets $V_{j\alpha}, \tilde{V}_{j\alpha}$ particularly small. It suffices to choose tuples

$$(V_0, V_1, \mathcal{P}, \mathcal{Q}) =: R \tag{2.20}$$

where $V_0, V_1 \in \mathcal{V}$, $V_0 \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$, and to assume that the sets $V_{j\alpha}, \tilde{V}_{j\alpha}$ in (2.8) satisfy the condition

$$V_{j\alpha}, \tilde{V}_{j\alpha} \subseteq V_1$$

for all $0 < j + |\alpha| \le k$ (together with the relations (2.10)).

Lemma 2.2. *For every $V_1 \in \mathcal{V}$ there exist asymptotic types $\mathcal{P}_1, \mathcal{Q}_1$ of the kind* (2.14), (2.15) *such that for every sequence*

$$h_{j\alpha} \in C^\infty(U, \mathcal{B}^{-\infty,d}_{V_1}(X; \boldsymbol{v}; \mathbb{C})) \tag{2.21}$$

$0 < j + |\alpha| \le k$, *and decompositions*

$$h_{j\alpha} = f_{j\alpha} + \tilde{f}_{j\alpha}$$

into Mellin symbols (2.9) *for an arbitrary choice of $V_{j\alpha}, \tilde{V}_{j\alpha} \subseteq V_1$, the operator $m(y,\eta)$ given by* (2.8) *is uniquely determined by* (2.21), *modulo a Green symbol $g_1(y,\eta)$ with asymptotic types $\mathcal{P}_1$ and $\mathcal{Q}_1$.*

Let $\mathbf{As}_{M+G}(X, \boldsymbol{g})$ denote the set of all tuples (2.20) for arbitrary V_0, V_1 as mentioned before, and $\mathcal{P} \supseteq \mathcal{P}_1, \mathcal{Q} \supseteq \mathcal{Q}_1$ (the latter inclusions correspond to the inclusions of carrier sets, cf. the relation (1.45)).

Moreover, let

$$\mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_R \tag{2.22}$$

for $R \in \mathbf{As}_{M+G}(X, \boldsymbol{g})$ be the space of all $m(y,\eta) + g(y,\eta) \in \mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ of type d, with arbitrary Mellin symbols $f_{j\alpha}, \tilde{f}_{j\alpha}$ linked to certain $h_{j\alpha}$ as in (2.21), $f_0 \in C^\infty(U, \mathcal{B}^{-\infty,d}_{V_0}(X; \boldsymbol{v}; \mathbb{C}))$, and Green symbols $g(y,\eta)$ with asymptotic types $\mathcal{P}, \mathcal{Q}$. The space (2.22) is Fréchet in a natural way. It is clear that

$$\mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w}) = \bigcup_{R \in \mathbf{As}_{M+G}(X,\boldsymbol{g})} \mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q; \boldsymbol{g}; \boldsymbol{w})_R.$$

Remark 2.3. Let us set

$$\boldsymbol{E}^{s,\gamma} := \mathcal{K}^{s,\gamma}(X^\wedge, E) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-) \oplus \mathbb{C}^{l_-}, \tag{2.23}$$

$$\boldsymbol{E}^{s,\gamma}_{\mathcal{P}} := \mathcal{K}^{s,\gamma}_P(X^\wedge, E) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{P'}((\partial X)^\wedge, J_-) \oplus \mathbb{C}^{l_-}, \tag{2.24}$$

$$\boldsymbol{S}^{\gamma-\mu} := \mathcal{S}^{\gamma-\mu}(X^\wedge, F) \oplus \mathcal{S}^{\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \oplus \mathbb{C}^{l_+} \tag{2.25}$$

for

$$\mathcal{S}^{\gamma-\mu}(X^\wedge, F) := \{\omega u + (1-\omega)v : u \in \mathcal{K}^{\infty,\gamma-\mu}(X,F), v \in \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X,F))\},$$

cf. also the formula (1.9). Every $a(y,\eta) \in \mathcal{R}^{\mu,d}_{M+G}(U\times\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ represents operator-valued symbols

$$a(y,\eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \boldsymbol{E}^{s,\gamma}, \boldsymbol{\mathcal{S}}^{\gamma-\mu}) \tag{2.26}$$

and

$$a(y,\eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \boldsymbol{E}^{s,\gamma}_{\mathcal{P}}, \boldsymbol{\mathcal{S}}^{\gamma-\mu}_{\mathcal{Q}}) \tag{2.27}$$

of order μ with respect to (the first two components of) (2.1), $s > d - \frac{1}{2}$. The relation (2.27) is valid for every pair $\mathcal{P}$ of asymptotic types with some resulting $\mathcal{Q}$ depending on a and $\mathcal{P}$.

Note that in the second components of (2.23), (2.24) instead of $s - \frac{1}{2}$ we can insert any other $s' \in \mathbb{R}$.

Theorem 2.4. *The space* $\mathcal{R}^{\mu,d}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ *can equivalently be described as the space of all operator functions* (2.4) *where* $g(y,\eta)$ *is a Green symbol as before, and*

$$b(y,\eta) = \sigma r^{-\mu}\{\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\tilde{\omega} + (1-\omega)\operatorname{op}_r(p)(y,\eta)(1-\tilde{\tilde{\omega}}\}\tilde{\sigma} + m(y,\eta). \tag{2.28}$$

This result is a corollary of a corresponding theorem [9, Section 4.6.4] that the η-dependent cut-off functions in (2.4) may be replaced by η-independent ones, modulo a Green symbol with trivial asymptotic types; concerning the boundaryless case, see [7].

Let $\boldsymbol{E}^{s,\gamma}$ and $\boldsymbol{E}^{s,\gamma}_{\mathcal{P}}$ be as in Remark 2.3, and set

$$\boldsymbol{F}^{s-\mu} := \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge, F) \oplus \mathcal{K}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge, J_+) \oplus \mathbb{C}^{l_+},$$

$$\boldsymbol{F}^{s-\mu}_{\mathcal{Q}} := \mathcal{K}^{s-\mu,\gamma-\mu}_Q(X^\wedge, F) \oplus \mathcal{K}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}_{Q'}((\partial X)^\wedge, J_+) \oplus \mathbb{C}^{l_+},$$

$\mathcal{P} = (P, P')$, $\mathcal{Q} = (Q, Q')$, cf. the formulas (2.14), (2.15).

Theorem 2.5. *Every* $a(y,\eta) \in \mathcal{R}^{\mu,d}(U \times \mathbb{R}^q; \boldsymbol{g}; \boldsymbol{w})$ *represents operator-valued symbols*

$$a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; \boldsymbol{E}^{s,\gamma}, \boldsymbol{F}^{s-\mu,\gamma-\mu}) \tag{2.29}$$

and

$$a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; \boldsymbol{E}^{s,\gamma}_{\mathcal{P}}, \boldsymbol{F}^{s-\mu,\gamma-\mu}_{\mathcal{Q}}) \tag{2.30}$$

of order μ with respect to (2.1)*, for every $s > d - \frac{1}{2}$.* (2.30) *is valid for every pair $\mathcal{P}$ of asymptotic types with some resulting $\mathcal{Q}$ depending on a and $\mathcal{P}$ (not on s). We also get operator-valued symbols with respect to* (2.16) *with a scheme of Douglis-Nirenberg orders as in Remark* 2.3.

It is now essential for the applications below that the space (2.2) has a variant with a complex parameter $w \in \mathbb{C}$ as an additional covariable, such that the elements $a(y, \eta, w)$ are holomorphic in w.

Let

$$\mathcal{R}^{\mu,d}(U \times \mathbb{R}^q \times \mathbb{C}, \boldsymbol{g}; \boldsymbol{w}) \tag{2.31}$$

denote the corresponding space. Ingredients of (2.31) have been investigated in [18] and [3], namely

$$\sigma r^{-\mu}\{\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(y, \eta, w)\tilde{\omega} + (1-\omega)\operatorname{op}_r(p)(y, \eta, w)(1-\tilde{\tilde{\omega}})\}\tilde{\sigma}, \tag{2.32}$$

the holomorphic analogue of the first summand on the right hand side of (2.28), and the holomorphic analogues

$$g(y, \eta, w) \quad \text{and} \quad m(y, \eta, w) \tag{2.33}$$

of $g(y, \eta)$ in (2.4) and $m(y, \eta)$ in (2.11), respectively. In order to make the ingredients of the operator-valued symbols of the class (2.31) more transparent we now discuss the kernel cut-off constructions.

Kernel cut-off only concerns covariables. Therefore, to simplify considerations, we assume for a while that symbols have constant coefficients.

Let us first consider Green and smoothing Mellin symbols in the covariables $(\eta, \tau) \in \mathbb{R}^{q+1}$. These are operator-valued symbols

$$a(\eta, \tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{q+1}; E, \tilde{E}) \tag{2.34}$$

with E and $\tilde{E}$ running over specific scales of Hilbert spaces. The constructions may be performed for any fixed Hilbert spaces $E, \tilde{E}$; then they are valid also for the projective limits involved in the definition of Green symbols.

In the following we admit symbols to be classical or non-classical. Let $a(\eta, \tau) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+1}_{\eta,\tau}; E, \tilde{E})$, and set

$$k(a)(\eta, \varrho) := \int_{\mathbb{R}} e^{i\varrho\tau} a(\eta, \tau) đ\tau. \tag{2.35}$$

Then, for every $\psi(\varrho) \in C_0^{\infty}(\mathbb{R})$ the function

$$h(\psi)(a)(\eta, \tau) := \int_{\mathbb{R}} e^{-i\varrho\tau} \psi(\varrho) k(a)(\eta, \varrho) d\varrho \tag{2.36}$$

has a holomorphic extension $h(\psi)(a)(\eta, \zeta)$ into the complex plane of the variable $\zeta = \tau + i\delta$.

Theorem 2.6. *Let $a(\eta, \tau) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+1}; E, \tilde{E})$ and $\psi(\varrho) \in C_0^{\infty}(\mathbb{R})$.*

1. *We have*

$$h(\psi)(a)(\eta,\zeta) \in \mathcal{A}(\mathbb{C}_\zeta, C^\infty(\mathbb{R}^q, \mathcal{L}(E,\tilde{E})),$$

and

$$h(\psi)(a)(\eta,\tau+i\delta) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q+1}_{\eta,\tau}; E,\tilde{E}) \tag{2.37}$$

for every $\delta \in \mathbb{R}$, uniformly in $c \le \delta \le c'$ for arbitrary $c \le c'$.

2. *The map $\psi \to h(\psi)(a)(\tau+i\delta)$ given by (2.37) defines a continuous operator*

$$C_0^\infty(\mathbb{R}) \to S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q+1}; E,\tilde{E})$$

for every $\delta \in \mathbb{R}$, uniformly continuous in $c \le \delta \le c'$ for arbitrary $c \le c'$.

3. *If $\psi(\varrho)$ is equal to* 1 *in a neigbourhood of $\varrho = 0$, we have*

$$g(\eta,\tau) = h(\psi)(a)(\eta,\tau) \bmod S^{-\infty}(\mathbb{R}^{q+1}; E,\tilde{E}). \tag{2.38}$$

The map

$$h(\psi): S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q+1}; E,\tilde{E}) \to S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q+1}; E,\tilde{E})$$

will be called a kernel cut-off operator; the notation is motivated by the relation between (2.35) and the distributional kernel $k(a)(\eta, t-t')$ of the (η-dependent) pseudo-differential operator $\mathrm{Op}_t(a)(\eta)$.

Remark 2.7. Under the assumptions of Theorem 2.6 for every $\beta,\delta \in \mathbb{R}$ there are coefficients $c_k(\beta,\delta)$ (depending on ψ) such that

$$h(\psi)(a)(\eta,\tau+i\beta) \sim \sum_{k=0}^\infty c_k(\beta,\delta) D_\tau^k H(\psi)(a)(\eta,\tau+i\delta)$$

in the sense of an asymptotic sum in $S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q+1}; E,\tilde{E})$. If $\psi(0)=1$, it follows that $c_0(\beta,\delta) = 1$ for every $\beta,\delta \in \mathbb{R}$.

A particularly simple proof of Theorem 2.6 may be found in [13].

In our applications, the complex variable plays the role of a Mellin covariable with imaginary part τ. Therefore, we now slightly change the notation and pass from $\zeta = \tau + i\delta$ to $w = \delta + i\tau$. Instead of Theorem 2.6 we could consider an antiholomorphic variant as well by talking about $\tau - i\delta$ rather than $\tau + i\delta$; the change to w then gives us an analogue of Theorem 2.6 with interchanged real and imaginary parts.

Definition 2.8. Let $S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q\times\mathbb{C}; E,\tilde{E})$ denote the space of all $a(y,\eta,w) \in \mathcal{A}(\mathbb{C}_w, C^\infty(U\times\mathbb{R}^q, \mathcal{L}(E,\tilde{E}))$ such that

$$a(y,\eta,\delta+i\tau) \in S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^{q+1}_{\eta,\tau}; E,\tilde{E})$$

for every $\delta \in \mathbb{R}$, uniformly in $c \le \delta \le c'$ for every $c \le c'$.

The space $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}; E, \tilde{E})$ is Fréchet in a natural way. In the following we write

$$S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \Gamma_\delta; E, \tilde{E}) \tag{2.39}$$

for symbols of the form $a(y, \eta, \delta + i\tau)$ that belong with respect to (y, η, τ) to the space $S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^{q+1}; E, \tilde{E})$.

Corollary 2.9. *Setting*

$$H(\psi)(a)(y, \eta, \beta + i\tau) := h(\psi)(\tilde{a})(y, \eta, \tau - i\beta)$$

for $\tilde{a}(y, \eta, \tau) := a(y, \eta, i\tau)$, $\psi \in C_0^\infty(\mathbb{R})$, *where* $h(\psi)$ *refers to the cut-off operator with respect to covariable* τ, *the assertions of Theorem* 2.6 (*combined with a translation in direction of* $\mathrm{Re}\, w$) *can be interpreted in the following way:*

1.
$$H(\psi) : S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \Gamma_\delta; E, \tilde{E}) \to S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \mathbb{C}; E, \tilde{E}) \tag{2.40}$$
is a continuous operator.
2. *The map* $\psi \to H(\psi)a$ *for fixed* $a(y, \eta, w) \in S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \Gamma_\delta; E, \tilde{E})$ *defines a continuous operator*
$$C_0^\infty(\mathbb{R}) \to S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \mathbb{C}; E, \tilde{E}).$$
3. *For every* $a(y, \eta, w) \in S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \Gamma_\delta; E, \tilde{E})$ *there is an* $h(y, \eta, w) \in S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \mathbb{C}; E, \tilde{E})$ *such that*
$$a(y, \eta, w) = h(y, \eta, w)\big|_{\mathrm{Re}\, w = \delta} \in S^{-\infty}(U \times \mathbb{R}^q \times \Gamma_\delta; E, \tilde{E}).$$

Remark 2.10. $a(y, \eta, w) \in S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \mathbb{C}; E, \tilde{E})$ and $a(y, \eta, w)\big|_{\mathrm{Re}\, w = \delta} \in S^{\mu-1}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \Gamma_\delta; E, \tilde{E})$ for some $\delta \in \mathbb{R}$ implies $a(y, \eta, w) \in S^{\mu-1}_{(\mathrm{cl})}(U \times \mathbb{R}^q \times \mathbb{C}; E, \tilde{E})$.

We now apply the kernel cut-off operator $H(\psi)$ with respect to the covariable $\tau = \mathrm{Im}\, w, w \in \Gamma_\delta$, to symbols of the space

$$\mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q \times \Gamma_\delta, \boldsymbol{g}; \boldsymbol{w})_R \tag{2.41}$$

for $R = (V_0, V_1, \mathcal{P}, \mathcal{Q}) \in \mathbf{As}_{M+G}(X, \boldsymbol{g})$; the meaning of Γ_δ in (2.41) is analogous to (2.39).

There is a specialisation of Definition 2.8 for the symbol space (2.41) which yields a corresponding space of holomorphic symbols in $w \in \mathbb{C}$:

$$\mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q \times \mathbb{C}, \boldsymbol{g}; \boldsymbol{w})_R. \tag{2.42}$$

This is studied in detail in De Donno and Schulze [3]. The definition of the subclass $R^{\mu,d}_G(U \times \mathbb{R}^q \times \mathbb{C}, \boldsymbol{g}; \boldsymbol{w})_{\mathcal{P},\mathcal{Q}}$ is straightforward, and we can set

$$\mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q \times \mathbb{C}, \boldsymbol{g}; \boldsymbol{w})_R := \{H(\psi)a + g : a(y, \eta, w) \in \mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q \times \Gamma_\delta, \boldsymbol{g}; \boldsymbol{w})_R,\ g(y, \eta, w) \in \mathcal{R}^{\mu,d}_G(U \times \mathbb{R}^q \times \mathbb{C}, \boldsymbol{g}; \boldsymbol{w})_{\mathcal{P},\mathcal{Q}}\}$$

for any $\psi \in C_0^\infty(\mathbb{R})$ which is equal to 1 in a neigbourhood of 0.

More generally, we have a map

$$H(\psi) : \mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q \times \Gamma_\delta, \boldsymbol{g}; \boldsymbol{w})_R \to \mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q \times \mathbb{C}, \boldsymbol{g}; \boldsymbol{w})_R \tag{2.43}$$

for every $\psi \in C_0^\infty(\mathbb{R})$, that may be obtained in analogy of the preceding assertions on the abstract context.

Theorem 2.11. *For $\psi \in C_0^\infty(\mathbb{R})$ and every element $a(y,\eta,w)$ of* (2.41) *we have*

$$h(y,\eta,w) := H(\psi)a(y,\eta,w) \in \mathcal{R}^{\mu,d}_{M+G}(U \times \mathbb{R}^q \times \mathbb{C}, \boldsymbol{g}; \boldsymbol{w})_R,$$

where

$$a(y,\eta,w) = h(y,\eta,w)\big|_{\mathrm{Re}\,w=\delta} \quad \mathrm{mod}\ \mathcal{R}^{-\infty,d}_G(U \times \mathbb{R}^q \times \Gamma_\delta, \boldsymbol{g}, \boldsymbol{w})_{\mathcal{P},\mathcal{Q}}$$

when $\psi \equiv 1$ in a neighbourhood of 0.

Note that, although the map (2.43) is formally analogous to (2.40), it may appear surprising that operator-valued symbols which contain the τ-variable in the form $\omega_{\eta,w}(r) = \omega(r[\eta, \delta + i\tau])$, for a cut-off function ω, cf. the expression (2.8), can be transformed into holomorphic ones, modulo smoothing symbols, cf. also [3].

To complete the structure of (2.31) it remains to explain (2.32). Starting from arbitrary elements

$$\tilde{p}(r,y,\tilde{\varrho},\tilde{\eta},\tilde{w}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{R}^{1+q} \times \Gamma_\delta)),$$
$$\tilde{h}(r,y,z,\tilde{\eta},\tilde{w}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}^{\mu,d}(X; \boldsymbol{v}; \mathbb{C} \times \mathbb{R}^q \times \Gamma_\delta))$$

for any fixed $\delta \in \mathbb{R}$ and setting

$$p(r,y,\varrho,\eta,\delta + i\tau) := \tilde{p}(r,y,r\varrho, r(\delta + i\tau)), \tag{2.44}$$

$$h(r,y,z,\eta,\delta + i\tau) := \tilde{h}(r,y,z,r\eta, r(\delta + i\tau)), \tag{2.45}$$

it suffices to apply a kernel cut-off operator (2.43) with respect to the variable τ, in order to generate the required holomorphic dependence in w. The new element here, compared with the analogous procedure of Remark 1.6, is the extra degeneracy in the variable $r \in \mathbb{R}_+$. Details are elaborated in [18], see also [8]. Concerning the present situation, to generate the space (2.31), it suffices to insert

$$(H(\psi)h)(r,y,z,\eta,w) \quad \text{and} \quad (H(\psi)p)(r,y,\varrho,\eta,w)$$

in (2.32) in place of h and p there, for arbitrary families (2.44) and (2.45), respectively.

This definition is correct; the choice of the cut-off function ψ only affects (2.32) modulo a remainder of the kind (2.33).

2.2. The edge algebra

Let W be a compact manifold with edge Y and boundary and $\mathbb{W}$ its stretched manifold. Concerning notation, in particular, for the double $2\mathbb{W} = \tilde{\mathbb{W}}$, we refer to Section 1.1. In the present section we study parameter-dependent edge-boundary value problems. These will be families $\mathcal{A}(\lambda)$ of operators, with parameter $\lambda \in \mathbb{R}^l$, constituting a vector space

$$\mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l), \tag{2.46}$$

with $\mu \in \mathbb{Z}$ as order and $d \in \mathbb{N}$ as type, and weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\gamma \in \mathbb{R}$, with a (finite or infinite) weight interval Θ, and a tuple $\boldsymbol{w} = (E, F; J_-, J_+; L_-, L_+)$ of vector bundles $E, F \in \operatorname{Vect}(\mathbb{W})$, $J_\pm \in \operatorname{Vect}(\mathbb{W}')$, $L_\pm \in \operatorname{Vect}(Y)$. Here $\mathbb{W}'$ is the stretched manifold associated with the boundary W' of W (in the sense $W' = \partial(W \setminus Y) \cup Y$). Also W' is a manifold with edge Y, now without boundary; the base of the model cone near Y is ∂X.

In local considerations vector bundles $E \in \operatorname{Vect}(\mathbb{W})$ will be restricted to $\mathbb{W}_{\mathrm{reg}}$ or to neighbourhoods of $\mathbb{W}_{\mathrm{sing}}$. To simplify notation we will denote them again by E. Moreover, if

$$\chi : [0,1) \times X \times U'' \to \overline{\mathbb{R}}_+ \times X \times \Omega \tag{2.47}$$

is a 'singular chart', we assume χ to be of the form

$$\chi(r, x, y) = (\chi_0(r), x, \chi''(y))$$

for a diffeomorphism $\chi_0 : [0,1) \to \overline{\mathbb{R}}_+$, $\chi_0(r) = r$ for $0 \le r \le \varepsilon$ for some $\varepsilon > 0$, and a chart $\chi'' : U'' \to \Omega$ on Y. The pull back of E under χ^{-1} will also be denoted by E; also for the restriction of E to $X^\wedge$ for any fixed $y \in \Omega$ we use the same letter. Similar notation will be applied for bundles over $\mathbb{W}'$. Clearly, in pull backs of operators on $\mathbb{R}_+ \times X \times \Omega$ to $(0,1) \times X \times U''$, we take into account transition maps from the bundles, using corresponding invariance properties of our constructions.

Choose an open covering of a neighbourhood $\mathbb{V}$ of $\mathbb{W}_{\mathrm{sing}}$ in $\mathbb{W}$ by charts

$$\chi_k : [0,1) \times X \times U_k'' \to \overline{\mathbb{R}}_+ \times X \times \Omega, \tag{2.48}$$

$k = 1, \ldots, N$, $\Omega \subseteq \mathbb{R}^q$ open, such that $\{U_1'', \ldots, U_N''\}$ is an open covering of Y by induced charts $\chi_k'' : U_k'' \to \Omega$. Let $\{\varphi_1, \ldots, \varphi_N\}$, $\{\psi_1, \ldots, \psi_N\}$ be systems of functions $\varphi_k, \psi_k \in C_0^\infty([0,1) \times X \times U_k'')$ such that $\sum \varphi_k = 1$ in a neighbourhood of $\mathbb{W}_{\mathrm{sing}}$, and $\psi_k \equiv 1$ on $\operatorname{supp} \varphi_k$.

Set $\varphi_k' = \varphi_k|_{[0,1)\times(\partial X)\times U_k''}$, $\psi_k' = \psi_k|_{[0,1)\times(\partial X)\times U_k''}$, $\varphi_k'' = \varphi_k|_Y$, and $\psi_k'' = \psi_k|_Y$. Then, using edge-amplitude functions

$$a_k(y, \eta, \lambda) \in \mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^{q+l}_{\eta,\lambda}, \boldsymbol{g}; \boldsymbol{w}_k),$$

$\boldsymbol{w}_k = (E, F; J_-, J_+; l_-, l_+)$, we can pass to operator pull backs $\mathcal{A}_k(\lambda)$ of pseudo-differential operators $\operatorname{Op}_y(a_k)(\lambda)$ to $[0,1) \times X \times U_k''$ with respect to (2.46) (which also take into account the transition functions of the involved bundles) and form

$$\mathcal{A}_{\mathrm{edge}}(\lambda) := \sum_{k=1}^{N} \operatorname{diag}(\varphi_k, \varphi_k', \varphi_k'')\mathcal{A}_k(\lambda) \operatorname{diag}(\psi_k, \psi_k', \psi_k''). \tag{2.49}$$

By definition, the local amplitude functions a_k contain certain fixed cut-off factors $\sigma(r)$ and $\tilde{\sigma}(r)$, cf. the formula (2.11). For the global construction we assume them to be independent of k. Without loss of generality, let $\tilde{\sigma} \equiv 1$ on $\operatorname{supp}\sigma$. Moreover, choose another cut-off function $\tilde{\tilde{\sigma}}$ such that $\sigma \equiv 1$ on $\operatorname{supp}\tilde{\tilde{\sigma}}$.

The space (2.46) is then defined as the set of all operator families of the form

$$\mathcal{A}(\lambda) = \mathcal{A}_{\mathrm{edge}}(\lambda) + \operatorname{diag}((1-\sigma)\mathcal{B}_{\mathrm{reg}}(\lambda)(1-\tilde{\tilde{\sigma}}), 0) + \mathcal{C}(\lambda) \tag{2.50}$$

where $\mathcal{A}_{\mathrm{edge}}(\lambda)$ has the form (2.49), further $\mathcal{B}_{\mathrm{reg}}(\lambda)$ is an arbitrary element of $\mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}}; \boldsymbol{v}; \mathbb{R}^l)$ for $\boldsymbol{v} = (E, F; J_-, J_+)$, and $\mathcal{C}(\lambda)$ belongs to

$$\mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l). \tag{2.51}$$

The latter space of smoothing operators is defined as follows:

First recall that in Section 1.1 we introduced the spaces $\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}})$. For every $\tilde{E} \in \operatorname{Vect}(\tilde{\mathbb{W}})$ there is a straightforward generalisation to spaces $\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}, \tilde{E})$ of corresponding distributional sections. In $\tilde{E}$ we fix a Hermitian metric and on $\tilde{\mathbb{W}}$ a Riemannian metric, where $\tilde{\mathbb{W}}$ is treated as a compact C^∞ manifold with boundary. We then have a natural scalar product in the space $L^2(\tilde{\mathbb{W}}, \tilde{E})$.

Let h^γ denote any strictly positive function in $C^\infty(\tilde{\mathbb{W}}_{\mathrm{reg}})$ that is equal to r^γ in a small neighbourhood of $\tilde{\mathbb{W}}_{\mathrm{sing}}$. The multiplication by $h^{-\frac{n}{2}}$ induces a bijection

$$h^{-\frac{n}{2}} : L^2(\tilde{\mathbb{W}}, \tilde{E}) \to \mathcal{W}^{0,0}(\tilde{\mathbb{W}}, \tilde{E})$$

which gives us a scalar product also in $\mathcal{W}^{0,0}(\tilde{\mathbb{W}}, \tilde{E})$. This gives rise to non-degenerate sesquilinear pairings

$$\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}, \tilde{E}) \times \mathcal{W}^{-s,-\gamma}(\tilde{\mathbb{W}}, \tilde{E}) \to \mathbb{C}$$

for all $s, \gamma \in \mathbb{R}$.

Also for arbitrary $s, \gamma \in \mathbb{R}$ the space $\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}, \tilde{E})$ is a Hilbert space. To define a scalar product, we consider the double $2\tilde{\mathbb{W}}$ of $\tilde{\mathbb{W}}$ which is a closed compact C^∞ manifold. Let $2\tilde{E}$ denote any vector bundle on $2\tilde{\mathbb{W}}$ such that $2\tilde{E}|_{\tilde{\mathbb{W}}} = \tilde{E}$. We then have the standard Sobolev space $H^s(2\tilde{\mathbb{W}}, 2\tilde{E})$ of sections of smoothness $s \in \mathbb{R}$.

Let $\sigma \in C^\infty(\tilde{\mathbb{W}})$ be any function that is equal to 1 on $[0, \frac{1}{3}) \times X \times Y$ and vanishes outside $[0, \frac{1}{2}) \times X \times Y$. Let us identify $(1-\sigma)u$ for $u \in H^s_{\mathrm{loc}}(\tilde{\mathbb{W}}_{\mathrm{reg}}, \tilde{E})$ with an element of $H^s(2\tilde{\mathbb{W}}, 2\tilde{E})$ (vanishing on $2\tilde{\mathbb{W}} \setminus \tilde{\mathbb{W}}_{\mathrm{reg}}$). Then a scalar product in $\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}, \tilde{E})$ is

$$(u,v)_{\mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}},\tilde{E})} = \sum_{k=1}^{N} \int \langle\eta\rangle^{2s} (\kappa_{\langle\eta\rangle}^{-1} F(((\chi_k'')^{-1})^* \varphi_k'' \sigma u)(\eta), \\ \kappa_{\langle\eta\rangle}^{-1} F(((\chi_k'')^{-1})^* \varphi_k'' \sigma v)(\eta))_{\mathcal{K}^{s,\gamma}(\tilde{X}^\wedge,\tilde{E})} d\eta \\ + ((1-\sigma)u, (1-\sigma)v)_{H^s(2\tilde{\mathbb{W}},2\tilde{E})}.$$

Analogously to the constructions of Section 1.1 we have the spaces

$$\mathcal{W}_0^{s,\gamma}(\mathbb{W}_\pm, \tilde{E}) = \{\tilde{u} \in \mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}, \tilde{E}) : \operatorname{supp} \tilde{u} \subseteq \mathbb{W}_\pm\}$$

and

$$\mathcal{W}^{s,\gamma}(\mathbb{W}, E) = \{\tilde{u}\big|_{\operatorname{int} \mathbb{W}_{\mathrm{reg}}} : \tilde{u} \in \mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}, \tilde{E})\} \tag{2.52}$$

for $\tilde{E}|_{\mathbb{W}} = E$, with a natural identification

$$\mathcal{W}^{s,\gamma}(\mathbb{W}, E) = \mathcal{W}^{s,\gamma}(\tilde{\mathbb{W}}, \tilde{E}) / \mathcal{W}_0^{s,\gamma}(\mathbb{W}_-, \tilde{E}); \tag{2.53}$$

this gives us a Hilbert space structure also in $\mathcal{W}^{s,\gamma}(\mathbb{W}, E)$.

Let us now pass to subspaces with asymptotics. First, on $\tilde{X}^\wedge \times \mathbb{R}^q$ we can form the spaces

$$\mathcal{W}_{\tilde{P}}^{s,\gamma}(\tilde{X}^\wedge \times \mathbb{R}^q, \tilde{E}) := \mathcal{W}^{s,\gamma}(\mathbb{R}^q, \mathcal{K}_{\tilde{P}}^{s,\gamma}(\tilde{X}^\wedge, \tilde{E}))$$

for every $\tilde{P} \in \mathrm{As}(\tilde{X}, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \Theta)$. Then $\mathcal{W}_{\tilde{P}}^{s,\gamma}(\tilde{\mathbb{W}}, \tilde{E})$ is defined as the subspace of all $u \in H_{\mathrm{loc}}^s(\tilde{\mathbb{W}}_{\mathrm{reg}}, \tilde{E})$ such that $\omega\varphi u = \chi^* v$ for some $v \in \mathcal{W}_{\tilde{P}}^{s,\gamma}(\tilde{X}^\wedge \times \mathbb{R}^q, \tilde{E})$ for every χ of the kind (1.1). Then, by the same scheme as (2.52), (2.53) we can form Fréchet spaces

$$\mathcal{W}_P^{s,\gamma}(\mathbb{W}, E) \tag{2.54}$$

with asymptotics of type $P \in \mathrm{As}(X, \boldsymbol{g})$, where we use the fact that every P can be obtained from some $\tilde{P} = \{(p_j, m_j, \tilde{L}_j)\}$ by restricting the spaces $\tilde{L}_j$ from $\tilde{X}$ to X.

Analogous constructions apply to $\mathbb{W}'$, vector bundles $J \in \mathrm{Vect}(\mathbb{W}')$ and asymptotic types $P' \in \mathrm{As}(\partial X, \boldsymbol{g}')$ for weight data $\boldsymbol{g}' = (\gamma', \Theta)$, i.e., we have the spaces

$$\mathcal{W}^{s,\gamma'}(\mathbb{W}', J) \quad \text{and} \quad \mathcal{W}_{P'}^{s,\gamma'}(\mathbb{W}', J), \tag{2.55}$$

respectively, $s, \gamma' \in \mathbb{R}$.

For every $\gamma, \varrho \in \mathbb{R}$ we have non-degenerate sesquilinear pairings

$$\mathcal{W}^{s,\gamma-\varrho}(\mathbb{W}', J) \times \mathcal{W}^{-s,-\gamma-\varrho}(\mathbb{W}', J) \to \mathbb{C},$$

$s \in \mathbb{R}$, induced by the $\mathcal{W}^{0,-\varrho}(\mathbb{W}', J)$-scalar product. We will apply this to $\varrho = \frac{1}{2}$. In the following E, F, J_-, J_+ are bundles as before, and we assume $L_-, L_+ \in \mathrm{Vect}(Y)$.

Given an operator

$$(2.56)\qquad \mathcal{A}:\begin{array}{c}\mathcal{W}^{s,\gamma}(\mathbb{W},E)\\ \oplus\\ \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{W}',J_-)\\ \oplus\\ H^{s-\frac{n+1}{2}}(Y,L_-)\end{array}\to\begin{array}{c}\mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W},F)\\ \oplus\\ \mathcal{W}^{s-\frac{1}{2}-\mu,\gamma-\mu-\frac{1}{2}}(\mathbb{W}',J_+)\\ \oplus\\ H^{s-\frac{n+1}{2}-\mu}(Y,L_-)\end{array}$$

that is continuous for every $s>-\frac{1}{2}$ we define its formal adjoint $\mathcal{A}^*$ in the sense

$$(\mathcal{A}u,v)_{\mathfrak{W}^0_+}=(u,\mathcal{A}^*v)_{\mathfrak{W}^0_-}$$

for $\mathfrak{W}^0_+=\mathcal{W}^{0,0}(\mathbb{W},F)\oplus\mathcal{W}^{0,-\frac{1}{2}}(\mathbb{W}',J_+)\oplus H^0(Y,L_+)$, $\mathfrak{W}^0_-=\mathcal{W}^{0,0}(\mathbb{W},E)\oplus\mathcal{W}^{0,-\frac{1}{2}}(\mathbb{W}',J_-)\oplus H^0(Y,L_-)$, for all $u\in C_0^\infty(\operatorname{int}\mathbb{W}_{\mathrm{reg}},E)\oplus C_0^\infty(\mathbb{W}'_{\mathrm{reg}},J_-)\oplus C^\infty(Y,L_-)$, $v\in C_0^\infty(\operatorname{int}\mathbb{W}_{\mathrm{reg}},F)\oplus C_0^\infty(\mathbb{W}'_{\mathrm{reg}},J_+)\oplus C^\infty(Y,L_+)$. An operator (2.50) is said to belong $\mathcal{Y}^{-\infty,0}(\mathbb{W},\boldsymbol{g};\boldsymbol{w})$ for $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$, $\boldsymbol{w}=(E,F;J_-,J_+;L_-,L_+)$, if there are asymptotic types $(P,P')\in\operatorname{As}(X,(\gamma-\mu,\Theta))\times\operatorname{As}(\partial X,(\gamma-\mu-\frac{1}{2},\Theta))$ and $(Q,Q')\in\operatorname{As}(X,(-\gamma,\Theta))\times\operatorname{As}(\partial X,(-\gamma-\frac{1}{2},\Theta))$ such that $\mathcal{A}$ and $\mathcal{A}^*$ induce continuous operators

$$(2.57)\qquad \mathcal{A}:\begin{array}{c}\mathcal{W}^{s,\gamma}(\mathbb{W},E)\\ \oplus\\ \mathcal{W}^{s',\gamma-\frac{1}{2}}(\mathbb{W}',J_-)\\ \oplus\\ H^{s''}(Y,L_-)\end{array}\to\begin{array}{c}\mathcal{W}^{\infty,\gamma-\mu}_P(\mathbb{W},F)\\ \oplus\\ \mathcal{W}^{\infty,\gamma-\mu-\frac{1}{2}}_{P'}(\mathbb{W}',J_+)\\ \oplus\\ C^\infty(Y,L_+)\end{array}$$

and

$$(2.58)\qquad \mathcal{A}^*:\begin{array}{c}\mathcal{W}^{s,-\gamma+\mu}(\mathbb{W},F)\\ \oplus\\ \mathcal{W}^{s',-\gamma+\mu-\frac{1}{2}}(\mathbb{W}',J_+)\\ \oplus\\ H^{s''}(Y,L_+)\end{array}\to\begin{array}{c}\mathcal{W}^{\infty,-\gamma}_Q(\mathbb{W},E)\\ \oplus\\ \mathcal{W}^{\infty,-\gamma-\frac{1}{2}}_{Q'}(\mathbb{W}',J_-),\\ \oplus\\ C^\infty(Y,L_-)\end{array}$$

respectively, for all $s>-\frac{1}{2}$, $s',s''\in\mathbb{R}$. Moreover, $\mathcal{Y}^{-\infty,d}(\mathbb{W},\boldsymbol{g};\boldsymbol{w})$ denotes the space of all operators

$$(2.59)\qquad \mathcal{C}=\mathcal{C}_0+\sum_{j=1}^{d}\mathcal{C}_j\operatorname{diag}(T^j,0,0)$$

for arbitrary $\mathcal{C}_j\in\mathcal{Y}^{-\infty,0}(\mathbb{W},\boldsymbol{g};\boldsymbol{w})$ and differential operators T^j of order j on $\mathbb{W}$, $T^j=\tilde{T}^j|_{\mathbb{W}}$ for differential operators $\tilde{T}^j$ on $\tilde{\mathbb{W}}$ with smooth coefficients up to $\partial\tilde{\mathbb{W}}=\tilde{\mathbb{W}}_{\mathrm{sing}}$ which are in the splitting of variables (r,x,y) near $\tilde{\mathbb{W}}_{\mathrm{sing}}$ of the form $\frac{\partial^j}{\partial x_n^j}\otimes\mathrm{id}_{\tilde{E}}$ for all j. Elements of $\mathcal{Y}^{-\infty,d}(\mathbb{W},\boldsymbol{g};\boldsymbol{w})$ are said to be of type d.

Let $\mathcal{Y}^{-\infty,d}(\mathbb{W},\boldsymbol{g};\boldsymbol{w})_{\mathcal{P},\mathcal{Q}}$ denote the set of all $\mathcal{C}\in\mathcal{Y}^{-\infty,d}(\mathbb{W},\boldsymbol{g};\boldsymbol{w})$ such that the summands $\mathcal{C}_j$ in (2.59) satisfy the continuity properties (2.57) and (2.58) for fixed pairs of asymptotic types $\mathcal{P}, \mathcal{Q}$, $j = 0, \ldots, d$. This space is Fréchet in a natural way, and we set

$$\mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}, \mathbb{R}^l)_{\mathcal{P},\mathcal{Q}} = \mathcal{S}(\mathbb{R}^l, \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})_{\mathcal{P},\mathcal{Q}}). \tag{2.60}$$

Moreover, let $\mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)$ be the union of all spaces (2.60) over $\mathcal{P}, \mathcal{Q}$, and set

$$\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l) = \bigcup_{d \in \mathbb{N}} \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l).$$

Thus the class of parameter-dependent operators (2.46) is completely defined.

Parameter-dependent operators will also be employed in the variant

$$\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \Gamma_\delta \times \mathbb{R}^l) \tag{2.61}$$

which is defined as the space of all operator families $\mathcal{A}(w, \lambda)$, parametrised by $(w, \lambda) \in \Gamma_\delta \times \mathbb{R}^l$, such that $\mathcal{A}(\delta + i\tau, \lambda)$ belongs to $\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^{1+l}_{\tau,\lambda})$.

Given a pair of vector bundles $\mathfrak{E} := (E, J)$ for $E \in \mathrm{Vect}(\mathbb{W})$, $J \in \mathrm{Vect}(\mathbb{W}')$, we set

$$\mathcal{W}^{s,\gamma}(\mathbb{W}, \mathfrak{E}) := \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{W}', J), \tag{2.62}$$

$$\mathcal{W}^{s,\gamma}_{\mathcal{P}}(\mathbb{W}, \mathfrak{E}) := \mathcal{W}^{s,\gamma}_{P}(\mathbb{W}, E) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{P'}(\mathbb{W}', J) \tag{2.63}$$

for asymptotic types $\mathcal{P} = (P, P') \in \mathrm{As}(X, (\gamma, \Theta)) \times \mathrm{As}(\partial X, (\gamma - \frac{1}{2}, \Theta))$.

In the following theorem we set $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta = (-(k+1), 0]$, $k \in \mathbb{N} \cup \{\infty\}$, and $\boldsymbol{w} = (E, F; J_-, J_+; L_-, l_+)$, $\mathfrak{E}_- = (E, J_-)$, $\mathfrak{E}_+ = (F, J_+)$.

Theorem 2.12. *Every $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)$ induces families of continuous operators*

$$\mathcal{A}(\lambda) : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, \mathfrak{E}_-) \\ \oplus \\ H^{s-\frac{n+1}{2}}(Y, L_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, \mathfrak{E}_+) \\ \oplus \\ H^{s-\mu-\frac{n+1}{2}}(Y, L_+) \end{array} \tag{2.64}$$

and

$$\mathcal{A}(\lambda) : \begin{array}{c} \mathcal{W}^{s,\gamma}_{\mathcal{P}}(\mathbb{W}, \mathfrak{E}_-) \\ \oplus \\ H^{s-\frac{n+1}{2}}(Y, L_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathcal{Q}}(\mathbb{W}, \mathfrak{E}_+) \\ \oplus \\ H^{s-\frac{n+1}{2}}(Y, L_+) \end{array} \tag{2.65}$$

for $s > d - \frac{1}{2}$, $d = d_{\mathcal{A}}$ (the type of $\mathcal{A}$) and every pair $\mathcal{P}$ of asymptotic types, with some resulting pair $\mathcal{Q}$, depending on $\mathcal{A}$ and $\mathcal{P}$ (not on s).

The proof of this result can be obtained by considering the summands of (2.50) separately. $\mathcal{C}(\lambda)$ is clear by definition, and the second summand is essentially a variant of (1.14). The first term on the right of (2.50) is a finite sum of local operators with amplitude functions as in Theorem 2.5. It is now sufficient to apply the Theorem 1.4 to our specific situation, taking into account the definition of the edge Sobolev spaces and their subspaces with asymptotics.

Parameter-dependent edge boundary value problems $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)$ have a parameter-dependent principal symbolic structure

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A})). \tag{2.66}$$

To give a definition we write $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2,3}$ and observe that

$$\mathcal{A}_{11} \in L^{\mu}_{\mathrm{cl}}(\operatorname{int} \mathbb{W}_{\mathrm{reg}}; E, F; \mathbb{R}^l), \tag{2.67}$$

$$\mathcal{A}' := (\mathcal{A}_{ij})_{i,j=1,2} \in \mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}}, \boldsymbol{v}; \mathbb{R}^l) \tag{2.68}$$

for $\boldsymbol{v} = (E, F; J_-, J_+)$. We then set $\sigma_\psi(\mathcal{A}) := \sigma_\psi(\mathcal{A}_{11})$ which is the standard homogeneous principal symbol of order μ with parameter $\lambda \in \mathbb{R}^l$; moreover $\sigma_\partial(\mathcal{A}) := \sigma_\partial(\mathcal{A}')$ is the principal boundary symbol of $\mathcal{A}'$ with parameter $\lambda \in \mathbb{R}^l$, cf. Section 1.2, in particular, the generalisation of (1.22) to the non-compact manifold $\mathbb{W}_{\mathrm{reg}}$ with boundary $\mathbb{W}'_{\mathrm{reg}}$ (and with the corresponding vector bundles).

Because of the edge-degeneracy in the splitting of variables $(r, x, y) \in \overline{\mathbb{R}}_+ \times X \times Y$ we can write

$$\sigma_\psi(\mathcal{A})(r, x, y, \varrho, \xi, \eta, \lambda) = r^{-\mu} \tilde{\sigma}_\psi(\mathcal{A})(r, x, y, r\varrho, \xi, r\eta, r\lambda)$$

where

$$\tilde{\sigma}_\psi(\mathcal{A})(r, x, y, \tilde{\varrho}, \xi, \tilde{\eta}, \tilde{\lambda}), \quad (\tilde{\varrho}, \xi, \tilde{\eta}, \tilde{\lambda}) \neq 0,$$

is smooth in r up to 0. Analogously for $(r, x', y) \in \overline{\mathbb{R}}_+ \times \partial X \times Y$ we have

$$\sigma_\partial(\mathcal{A})(r, x', y, \varrho, \xi', \eta, \lambda) = r^{-\mu} \tilde{\sigma}_\partial(\mathcal{A})(r, x', y, r\varrho, \xi', r\eta, r\lambda)$$

with

$$\tilde{\sigma}_\partial(\mathcal{A})(r, x', y, \tilde{\varrho}, \xi', \tilde{\eta}, \tilde{\lambda}), \quad (\tilde{\varrho}, \xi', \tilde{\eta}, \tilde{\lambda}) \neq 0$$

being smooth in r up to 0.

Finally, $\sigma_\wedge(\mathcal{A})(y, \eta, \lambda)$ denotes the parameter-dependent homogeneous principal edge symbol which is an operator family that can be expressed in terms of the local amplitude functions

$$a(y, \eta, \lambda) \in R^{\mu,d}(U \times \mathbb{R}^q \times \mathbb{R}^l, \boldsymbol{g}; \boldsymbol{w}),$$

cf. the notation in the formula (2.2). Here $U \subseteq \mathbb{R}^q$ is open and corresponds to a chart on the edge Y. Applying (2.4) in the variant with (η, λ) in place of η we set

$$\sigma_\wedge(\mathcal{A})(y, \eta, \lambda) := \sigma_\wedge(a)(y, \eta, \lambda)$$

for $\sigma_\wedge(a)(y, \eta, \lambda) := \operatorname{diag}(\sigma_\wedge(b)(y, \eta, \lambda), 0) + \sigma_\wedge(g)(y, \eta, \lambda)$. Here $\sigma_\wedge(g)$ is the homogeneous principal part of g as a classical operator-valued symbols, while

$$\begin{aligned} \sigma_\wedge(b)(y, \eta, \lambda) \;&:=\; r^{-\mu}\{\omega(r|\eta, \lambda|) \operatorname{op}_M^{\gamma - \frac{n}{2}}(h_0)(y, \eta, \lambda) \tilde{\omega}(r|\eta, \lambda|) \\ &\quad+\; (1 - \omega(r|\eta, \lambda|)) \operatorname{op}_r(p_0)(y, \eta, \lambda)(1 - \tilde{\tilde{\omega}}(r|\eta, \lambda|)) \\ &\quad+\; \sigma_\wedge(m)(y, \eta, \lambda) \end{aligned}$$

with $\sigma_\wedge(m)$ being the homogeneous principal part of m as a classical operator-valued symbol, and

$$h_0(r, y, z, \eta, \lambda) := \tilde{h}(0, y, z, r\eta, r\lambda), \quad p_0(r, \varrho, \eta, \lambda) := \tilde{p}(0, r\varrho, r\eta, r\lambda),$$

cf. the λ-dependent analogue of (2.6) and (2.7).

Theorem 2.13. *$\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ for*

$$\boldsymbol{g} = (\gamma - \nu, \gamma - (\mu + \nu), \Theta), \quad \boldsymbol{v} = (\widetilde{E}, F; \widetilde{J}, J_+; \widetilde{L}, L_+)$$

and $\mathcal{B} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{h}; \boldsymbol{w}; \mathbb{R}^l)$ for

$$\boldsymbol{h} = (\gamma, \gamma - \nu, \Theta), \quad \boldsymbol{w} = (E, \widetilde{E}; J_-, \widetilde{J}; L_-, \widetilde{L})$$

inplies

$$\mathcal{A}\mathcal{B} \in \mathcal{Y}^{\mu+\nu}(\mathbb{W}, \boldsymbol{g} \circ \boldsymbol{h}; \boldsymbol{v} \circ \boldsymbol{w}; \mathbb{R}^l)$$

for $\boldsymbol{g} \circ \boldsymbol{h} = (\gamma, \gamma - (\mu + \nu), \Theta)$, $\boldsymbol{v} \circ \boldsymbol{w} = (E, F; J_-, J_+; L_-, L_+)$, and $d_{\mathcal{A}\mathcal{B}} = \max(\nu + d_{\mathcal{A}}, d_{\mathcal{B}})$, and we have

$$\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

with componentwise composition (cf. the formula (2.66)).

Theorem 2.13 is a parameter-dependent analogue of a corresponding composition result on edge boundary values problems, cf. [9, Section 4.5.2]. The proof in the parameter-dependent case does not contain additional difficulties.

2.3. Ellipticity of edge-boundary value problems

This section studies parameter-dependent ellipticity of edge-degenerate boundary value problems on a (compact) manifold W with edge Y and boundary. As usual we formulate results on the associated stretched manifold $\mathbb{W}$.

Let $\boldsymbol{w} = (E, F; J_-, J_+; L_-, L_+)$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta = (-(k+1), 0]$, $k \in \mathbb{N} \cup \{\infty\}$.

Definition 2.14. An element $\mathcal{A}(\lambda) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)$ is called parameter-dependent elliptic of order μ, if

1.

$$\sigma_{\psi}(\mathcal{A}) : \pi^*_{\mathbb{W}_{\mathrm{reg}}} E \to \pi^*_{\mathbb{W}_{\mathrm{reg}}} F \tag{2.69}$$

for $\pi_{\mathbb{W}_{\mathrm{reg}}} : (T^*\mathbb{W}_{\mathrm{reg}} \times \mathbb{R}^l) \setminus 0 \to \mathbb{W}_{\mathrm{reg}}$ is an isomorphism and, near $\mathbb{W}_{\mathrm{sing}}$ in the splitting of variables $(r, x, y) \in \overline{\mathbb{R}}_+ \times X \times Y$,

$$\tilde{\sigma}_{\psi}(\mathcal{A})(r, x, y, \tilde{\varrho}, \xi, \tilde{\eta}, \tilde{\lambda})$$

are isomorphisms between the respective fibres of E and F over all (r, x, y), including $r = 0$, for all $(\tilde{\varrho}, \xi, \tilde{\eta}, \tilde{\lambda}) \neq 0$.

2.

$$\sigma_{\partial}(\mathcal{A}) : \pi^*_{\mathbb{W}'\,\mathrm{reg}} \begin{pmatrix} H^s(\mathbb{R}_+, E') \\ \oplus \\ J_- \end{pmatrix} \to \pi^*_{\mathbb{W}'\,\mathrm{reg}} \begin{pmatrix} H^{s-\mu}(\mathbb{R}_+, F') \\ \oplus \\ J_+ \end{pmatrix} \tag{2.70}$$

for $\pi_{\mathbb{W}'_{\mathrm{reg}}} : (T^*\mathbb{W}'_{\mathrm{reg}} \times \mathbb{R}^l) \setminus 0 \to \mathbb{W}'_{\mathrm{reg}}$ is an isomorphism for any $s > \max(\mu, d) - \frac{1}{2}$, $d = d_{\mathcal{A}}$, and near $\mathbb{W}'_{\mathrm{sing}}$, in the splitting of variables $(r, x', y) \in \overline{\mathbb{R}}_+ \times (\partial X) \times Y$,

$$\tilde{\sigma}_\partial(\mathcal{A})(r, x', y, \tilde{\varrho}, \xi', \tilde{\eta}, \tilde{\lambda})$$

are isomorphisms between the respective fibres of $H^s(\mathbb{R}_+, E') \oplus J_-$ and $H^{s-\mu}(\mathbb{R}_+, F') \oplus J_+$ over all (r, x', y), including $r = 0$, for all $(\tilde{\varrho}, \xi', \tilde{\eta}, \tilde{\lambda}) \neq 0$.

3.

$$\sigma_\wedge(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge, \mathfrak{E}_-) \\ \oplus \\ L_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge, \mathfrak{E}_+) \\ \oplus \\ L_+ \end{pmatrix} \tag{2.71}$$

for $\pi_Y : (T^*Y \times \mathbb{R}^l) \setminus 0 \to Y$ is an isomorphism for any $s > \max(\mu, d) - \frac{1}{2}$, $d = d_{\mathcal{A}}$.

Conditions (ii) and (iii) are independent of the choice of s. Note that in (ii) the spaces may equivalently be replaced by $\mathcal{S}(\overline{\mathbb{R}}_+, E) \oplus J_-$ and $\mathcal{S}(\overline{\mathbb{R}}_+, F) \oplus J_+$, and in (iii) by $\mathcal{S}^\gamma(X^\wedge, \mathfrak{E}_-) \oplus L_-$ and $\mathcal{S}^{\gamma-\mu}(X^\wedge, \mathfrak{E}_+) \oplus L_+$, respectively. Here, for instance, for $\mathfrak{E}_- = (E, J_-)$,

$$\mathcal{S}^\gamma(X^\wedge, \mathfrak{E}_-) := \mathcal{S}^\gamma(X^\wedge, E) \oplus \mathcal{S}^{\gamma-\frac{1}{2}}((\partial X), J_-),$$

cf. also the formula (2.25).

In the sequel we also talk about ellipticity rather than parameter-dependent ellipticity of order μ.

Theorem 2.15. *Let $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)$ be elliptic. Then $\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1}; \mathbb{R}^l)$ for $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, $\boldsymbol{w}^{-1} = (E, F; J_+, J_-; L_+, L_-)$ of type $d_{\mathcal{P}} = \max(d_{\mathcal{A}} - \mu, 0)$ where*

$$\mathcal{I} - \mathcal{P}\mathcal{A} = \mathcal{C}_{\mathrm{l}} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}_{\mathrm{l}}; \boldsymbol{w}_{\mathrm{l}}; \mathbb{R}^l),$$

for $\boldsymbol{g}_{\mathrm{l}} = (\gamma, \gamma, \Theta)$, $\boldsymbol{w}_{\mathrm{l}} = (E, E; J_-, J_-; L_-, L_-)$,

$$\mathcal{I} - \mathcal{A}\mathcal{P} = \mathcal{C}_{\mathrm{r}} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}_{\mathrm{r}}; \boldsymbol{w}_{\mathrm{r}}; \mathbb{R}^l),$$

and $\boldsymbol{g}_{\mathrm{r}} = (\gamma - \mu, \gamma - \mu, \Theta)$, $\boldsymbol{w}_{\mathrm{r}} = (F, F; J_+, J_+; L_+, L_+)$ and we have $d_{\mathcal{C}_{\mathrm{l}}} = \max(\mu, d_{\mathcal{A}})$, $d_{\mathcal{C}_{\mathrm{r}}} = \max(d_{\mathcal{A}} - \mu, 0)$.

Theorem 2.16. *Let $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)$ be elliptic. Then $\mathcal{A}$ induces a family of Fredholm operators (2.64), $s > \max(\mu, d_{\mathcal{A}}) - \frac{1}{2}$, of index zero. There exists a constant $c > 0$ such that (2.64) is invertible for all $|\lambda| \geq c$.*

Theorem 2.17. [14] *For every $\mu \in \mathbb{Z}$, $\gamma \in \mathbb{R}$, $E \in \mathrm{Vect}(\mathbb{W})$, there exists an elliptic operator*

$$R^\mu \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; (E, E); \mathbb{R}^l)$$

of type 0, $\boldsymbol{g} = (\gamma, \gamma - \mu, (-\infty, 0])$, such that

$$R^\mu(\lambda) : \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \to \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, E)$$

is a family of isomorphisms for all $s > \max(\mu, 0) - \frac{1}{2}$ *and all* $\lambda \in \mathbb{R}^l$*, and we have*

$$(R^\mu(\lambda))^{-1} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}; (E, E); \mathbb{R}^l),$$

$\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, (-\infty, 0])$, *also of being type* 0.

Remark 2.18. Theorems 2.15 and 2.16 generalise analogous results for a manifold W' with edge Y and without boundary, cf. [2] and [4]. Also Theorem 2.17 has a corresponding analogue for the boundaryless case. Moreover, in a refined version, order reducing results also hold for boundary value problems without the transmission property, see [31].

3. Corner conormal symbols

3.1. Meromorphic families

We now introduce families $\mathcal{A}(w, \lambda)$ of elements in $\mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)$ holomorphically depending on a parameter $w \in \mathbb{C}$.

The definition of the corresponding space

$$\mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C}_w \times \mathbb{R}^l) \tag{3.1}$$

will be given along the lines of the expression (2.50). The space (3.1) is defined as the set of all operator families

$$\mathcal{A}(w, \lambda) = \mathcal{A}_{\mathrm{edge}}(w, \lambda) + \mathrm{diag}\big((1 - \sigma)\mathcal{B}_{\mathrm{reg}}(w, \lambda)(1 - \tilde{\tilde{\sigma}}), 0\big) + \mathcal{C}(w, \lambda), \tag{3.2}$$

where the summands are defined as follows: We begin with the smoothing part $\mathcal{C}(w, \lambda) \in \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)$. The latter space is defined to be the set of all

$$\mathcal{C}(w, \lambda) \in \mathcal{A}(\mathbb{C}, \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)_{\mathcal{P},\mathcal{Q}}) \tag{3.3}$$

for any $d \in \mathbb{N}$ and asymptotics $\mathcal{P}$ and $\mathcal{Q}$ as in (2.60), such that

$$\mathcal{C}(\delta + i\tau, \lambda) \in \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^{1+l}_{\tau,\lambda})_{\mathcal{P},\mathcal{Q}}$$

for every $\delta \in \mathbb{R}$, uniformly in $c \le \delta \le c'$ for arbitrary $c \le c'$. The space $\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)$ is then defined as the union of all spaces $\mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)_{\mathcal{P},\mathcal{Q}}$ over all $d \in \mathbb{N}$ and $\mathcal{P}, \mathcal{Q}$. The operator family $\mathcal{C}(w, \lambda)$ in (3.2) is assumed to belong to that space.

Furthermore, by (1.24) (in the version of a non-compact C^∞ manifold with boundary) we have the space

$$\mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}}; \boldsymbol{v}; \mathbb{C} \times \mathbb{R}^l). \tag{3.4}$$

In (3.2) we assume that $\mathcal{B}_{\mathrm{reg}}(w, \lambda)$ is an element of (3.4).

The operator family $\mathcal{A}_{\mathrm{edge}}(w, \lambda)$ is defined in terms of summands $\mathcal{A}_k(w, \lambda)$, similarly as in (2.49). Locally, these operator functions have the form $\mathrm{Op}_y(a_k)(w, \lambda)$ with $a_k(y, \eta, w, \lambda)$ belonging to the space of edge amplitude functions

$$\mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^q_\eta \times \mathbb{C}_w \times \mathbb{R}^l_\lambda, \boldsymbol{g}; \boldsymbol{w}) \tag{3.5}$$

for $\boldsymbol{w} = (E, F; J_-, J_+; l_-, l_+)$ and some type $d = d_{\mathcal{A}}$, cf. Section 2.1 (recall that the local amplitude functions a_k contain cut-off factors $\sigma(r)$ and $\tilde{\sigma}(r)$ with $\tilde{\sigma} \equiv 1$ on $\operatorname{supp} \sigma$, similarly as in (2.11)). This completes the definition of (3.1).

Remark 3.1. $\mathcal{A}(w, \lambda) \in \mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)$ and $\mathcal{A}(\delta + i\tau, \lambda) \in \mathcal{Y}^{\mu-1,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w};\, \mathbb{R}^{1+l}_{\tau,\lambda})$ for any fixed $\delta \in \mathbb{R}$ implies $\mathcal{A}(w, \lambda) \in \mathcal{Y}^{\mu-1,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)$.

Let us set

$$\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l) = \bigcup_{d \in \mathbb{N}} \mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l).$$

Let us fix an element $R \in \mathbf{As}_{M+G}(X, \boldsymbol{g})$, $R = (V_0, V_1, \mathcal{P}, Q)$. Then

$$\mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)_R$$

denotes the subspace of all elements $\mathcal{A}(w, \lambda) \in \mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)$ such that the smoothing family $\mathcal{C}(w, \lambda)$ is as in (3.3), and the local amplitude functions a_k belong to

$$\mathcal{R}^{\mu,d}_{M+G}(\Omega \times \mathbb{R}^q \times \mathbb{C} \times \mathbb{R}^l; \boldsymbol{g}; \boldsymbol{w})_R$$

for all k. A similar notation will be used for spaces of (w, λ)-dependent families with w varying on Γ_δ instead of $\mathbb{C}$.

Remark 3.2. Every $\mathcal{A}(w, \lambda) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)$ induces a family of continuous operators (2.64), $s > d_{\mathcal{A}} - \frac{1}{2}$, which is holomorphic in $w \in \mathbb{C}$.

Theorem 3.3. *For every $\mathcal{A}(w, \lambda) \in \mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \Gamma_\delta \times \mathbb{R}^l)_R$, $\delta \in \mathbb{R}$, $R \in \mathrm{As}_{M+G}(X, \boldsymbol{g})$, there exists an $\mathcal{H}(w, \lambda) \in \mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)_R$ such that*

$$\mathcal{H}(w, \lambda)\big|_{\Gamma_\delta \times \mathbb{R}^l} - \mathcal{A}(w, \lambda) \in \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \Gamma_\delta \times \mathbb{R}^l).$$

Proof. Comparing (2.50) in the version with parameters $(w, \lambda) \in \Gamma_\delta \times \mathbb{R}^l$ and the expression (3.2) we see that we may consider the summands separately. The smoothing summand can obviously be ignored. To treat $\mathcal{B}_{\mathrm{reg}}(w, \lambda)$ in the middle we can apply kernel cut-off to the involved local (interior and boundary) amplitude functions with respect to $w \in \Gamma_\delta$, cf. Corollary 2.9. The first summand can also be modifed by kernel cut-off that we apply to the amplitude functions, (cf. also Theorem 2.11) as well as the other amplitude functions coming from the non-smoothing contributions, cf. also [18] and [3]. ☐

Proposition 3.4. *$\mathcal{A} \in \mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)_R$ and $\mathcal{A}(\delta + i\tau, \lambda) \in \mathcal{Y}^{\mu-1,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \Gamma_\delta \times \mathbb{R}^l)_R$ for some fixed $\delta \in \mathbb{R}$ implies $\mathcal{A} \in \mathcal{Y}^{\mu-1,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C} \times \mathbb{R}^l)_R$.*

Proof. The proof follows by applying Remark 2.10 to the various local amplitude functions which played a role in the proof Theorem 3.3. ☐

Definition 3.5. 1. Let $R \in \mathbf{As}_{M+G}(X, \boldsymbol{g})$, and let $\mathbf{As}(\mathbb{W})^{\bullet}_R$ denote the system of all sequences

$$T := \{(p_j, m_j, L_j)\}_{j \in \mathbb{Z}}, \tag{3.6}$$

with $p_j \in \mathbb{C},\ m_j \in \mathbb{N}$, such that $\pi_{\mathbb{C}}T := \{p_j\}_{j\in\mathbb{N}}$ intersects the strip $\{w : c \le \operatorname{Re} w \le c'\}$ in a finite set for every $c \le c'$, and $L_j \subset \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})_R$ is a finite-dimensional subspace of operators of finite rank, for some $d \in \mathbb{N}$.

2. Let

$$\mathcal{M}^{\mu,d}_{R,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}) \tag{3.7}$$

for $R = (V_0, V_1, \mathcal{P}, \mathcal{Q}) \in \mathbf{As}_{M+G}(X, \boldsymbol{g})$, and $T = \{(p_j, m_j, L_j)\}_{j\in\mathbb{Z}} \in \mathbf{As}(\mathbb{W})^{\bullet}_{R}$ denote the space of all operator functions

$$\mathcal{A}(w) = \mathcal{H}(w) + \mathcal{C}(w) \tag{3.8}$$

for arbitrary $\mathcal{H}(w) \in \mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C})_R$ and

$$\mathcal{C}(w) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}T,\ \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})_{\mathcal{P},\mathcal{Q}})$$

such that $\mathcal{C}(w)$ is meromorphic with poles at p_j of multiplicity $m_j + 1$ and Laurent coefficients at $(w - p_j)^{-(k+1)}$ belonging to L_j for all $0 \le k \le m_j,\ j \in \mathbb{Z}$, and

$$\chi(\delta + i\tau)\mathcal{C}(\delta + i\tau) \in \mathcal{Y}^{-\infty,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}_\tau)_{\mathcal{P},\mathcal{Q}}$$

for every $\pi_{\mathbb{C}}T$-excision function $\chi(w)$, and every $\delta \in \mathbb{R}$, uniformly in $c \le \delta \le c'$ for every $c \le c'$.

Let us write $\mathcal{M}^{\mu,d}_{R,\mathcal{O}}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})$ for the space (3.7) when $\pi_{\mathbb{C}}T = \emptyset$ (this coincides with $\mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{C})_R$). Moreover, set

$$\mathcal{M}^{\mu}_{R,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}) = \bigcup \mathcal{M}^{\mu,d}_{R,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}), \tag{3.9}$$

where the union is taken over all $d \in \mathbb{N}$. We finally write $\mathcal{M}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})$ for the union of the latter spaces over all $R \in \mathbf{As}_{M+G}(X, \boldsymbol{g})$, $T \in \mathbf{As}(\mathbb{W})^{\bullet}_{R}$.

By definition there is a decomposition as a non-direct sum

$$\mathcal{M}^{\mu}_{R,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}) = \mathcal{M}^{\mu}_{R,\mathcal{O}}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}) + \mathcal{M}^{-\infty}_{R,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}), \tag{3.10}$$

cf. analogously, for the case without boundary [15, formula (2.5.2)].

Theorem 3.6. *Let $\mathcal{A} \in \mathcal{M}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for*

$$\boldsymbol{g} = (\gamma - \nu, \gamma - (\mu + \nu), \Theta),\ \boldsymbol{v} = (\tilde{E}, F; \tilde{J}, J_+; \tilde{L}, L_+),$$

and $\mathcal{B} \in \mathcal{M}^{\nu}(\mathbb{W}, \boldsymbol{h}; \boldsymbol{w})$ for

$$\boldsymbol{h} = (\gamma, \gamma - \nu, \Theta),\ \boldsymbol{w} = (E, \tilde{E}; J_-, \tilde{J}; L_-, \tilde{L}).$$

Then we have

$$\mathcal{A}\mathcal{B} \in \mathcal{M}^{\mu+\nu}(\mathbb{W}, \boldsymbol{g} \circ \boldsymbol{h}; \boldsymbol{v} \circ \boldsymbol{w})$$

for $\boldsymbol{g} \circ \boldsymbol{h} = (\gamma, \gamma - (\mu + \nu); \Theta))$, $\boldsymbol{v} \circ \boldsymbol{w} = (E, F; J_-, J_+; L_-, L_+)$, and $d_{\mathcal{A}\mathcal{B}} = \max(\nu + d_{\mathcal{A}}, d_{\mathcal{B}})$.

This theorem can be regarded as an application of Theorem 2.13 to meromorphic parameter-dependent families. If both families are holomorphic this is fairly straightforward. In the general case we can apply the decomposition (3.10) which leads to the case that one factor is meromorphic and of order $-\infty$ with finite rank Laurent coefficients. The desired composition behaviour also this case can easily be obtained.

3.2. Ellipticity and meromorphic inverses

Definition 3.7. An element $\mathcal{A} \in \mathcal{M}^{\mu}_{R,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ is called elliptic of order μ, if for some $\beta \in \mathbb{R}$ with $\pi_{\mathbb{C}} T \cap \Gamma_\beta = \emptyset$ the family $\mathcal{A}(\beta + i\tau) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}_\tau)_R$ is parameter-dependent elliptic in the sense of Definition 2.14.

Remark 3.8. Definition 3.7 is independent of the choice of β.

In fact, writing $\mathcal{A}$ in the form (3.8), the ellipticity of $\mathcal{A}(\beta + i\tau)$ only depends on $\mathcal{H}(\beta + i\tau)$, since $\mathcal{C}(\beta + i\tau)$ is of order $-\infty$ in τ. Proposition 3.4 then shows that the principal symbols which determine the ellipticity are independent of β.

Theorem 3.9. *Let $\mathcal{A} \in \mathcal{M}^{\mu}_{R,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})$ be elliptic. Then there exists a countable set $D \subset \mathbb{C}$, such that $D \cap \{w \in \mathbb{C} : c \leq \operatorname{Re} w \leq c'\}$ is finite for every $c \leq c'$, and*

$$\mathcal{A}(w) : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, \boldsymbol{E}) \\ \oplus \\ H^s(Y, L_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, \boldsymbol{F}) \\ \oplus \\ H^{s-\mu}(Y, L_+) \end{array} \tag{3.11}$$

is invertible for all $w \in \mathbb{C} \setminus D$ and for all $s > \max(\mu, d_{\mathcal{A}}) - \frac{1}{2}$.

Proof. Because of Remark 3.2 and Definition 3.5 the family of operators (3.11) is holomorphic in $U := \mathbb{C} \setminus \pi_{\mathbb{C}} T$. At the same time it is elliptic in the space $\mathcal{Y}^{\mu,d}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})$ for every $w \in U$. Thus, (3.11) is a holomorphic Fredholm family in U. Let us write $\mathcal{A}(w)$ in the form (3.8). Then, by virtue of the parameter-dependent ellipticity of $\mathcal{H}(w)\big|_{\Gamma_\beta}$ for every $\beta \in \mathbb{R}$, there is a $C > 0$ such that $\mathcal{H}(w)$ is invertible for $|\operatorname{Im} w| > C$ and $\operatorname{Re} w = \beta$ for any fixed $\beta \in \mathbb{R}$. Since β is involved as a parameter that only affects lower order terms and because of the continuity in β, for every $c < c'$ we can choose a constant $C = C(c, c')$ such that $\mathcal{A}(w)$ is invertible for all $|\operatorname{Im} w| > C(c, c')$ and $c < \beta < c'$. We are now in a well known situation of a holomorphic Fredholm family $\mathcal{A}(w)$ in an open set $U \subset \mathbb{C}$ which is invertible in at least one point $w_1 \in U$; it follows then that $\mathcal{A}(w)$ is invertible for all $w \in U \setminus D$, except for a countable subset of U that intersects every $K \Subset U$ in a finite set. It remains to note that the points of $\pi_{\mathbb{C}} T$ are not accumulation points of D. The technique is the same as in [22, Lemma 4.3.13]. $\square$

Theorem 3.10. *Every elliptic element $\mathcal{A} \in \mathcal{M}^{\mu}_{R,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})$ has an inverse*

$$\mathcal{A}^{-1} \in \mathcal{M}^{-\mu}_{P,Q}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1}), d_{\mathcal{A}^{-1}} = \max(d_{\mathcal{A}} - \mu, 0),$$

for suitable asymptotic types

$$P \in \mathbf{As}_{M+G}(X, \boldsymbol{g}^{-1}) \quad \textit{and} \quad Q \in \mathbf{As}(\mathbb{W})^{\bullet}_{P}.$$

Proof. We invert $\mathcal{A}$ with respect to the multiplication of Theorem 3.6. Let us first apply Theorem 2.15 to $\mathcal{A}\big|_{\Gamma_\beta} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \Gamma_\beta)_R$ for any β such that $\pi_{\mathbb{C}} T \cap \Gamma_\beta = \emptyset$. Then there is a parametrix $\mathcal{P}_1 \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1}; \Gamma_\beta)_{P_1}$ for some $P_1 \in \mathbf{As}_{M+G}(X, \boldsymbol{g}^{-1})$. Theorem 3.3 applied to $\mathcal{P}_1$ gives us an element $\mathcal{P}_0 \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1}; \mathbb{C})_{P_1}$ such that $\mathcal{P}_0\big|_{\Gamma_\beta}$ coincides with $\mathcal{P}_1$ modulo a family of order $-\infty$. It follows that $\mathcal{P}_0\big|_{\Gamma_\beta}$ is also a parametrix of $\mathcal{A}\big|_{\Gamma_\beta}$. We now apply Theorem 3.6 and obtain $\mathcal{P}_0\mathcal{A} \in \mathcal{M}^0(\mathbb{W}, \boldsymbol{g}_1; \boldsymbol{w}_1)$ (cf. the notation of Theorem 2.15). Moreover, we have $\mathcal{I} - \mathcal{P}_0\mathcal{A}\big|_{\Gamma_\beta} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}_1; \boldsymbol{w}_1; \Gamma_\beta)$. From Remark 3.1 we obtain $\mathcal{N} := \mathcal{I} - \mathcal{P}_0\mathcal{A} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}_1; \boldsymbol{w}_1; \mathbb{C})_{P_2} = \mathcal{M}^{-\infty}_{P_2,\mathcal{O}}(\mathbb{W}, \boldsymbol{g}_1; \boldsymbol{w}_1)$ for some $P_2 \in \mathbf{As}_{M+G}(X, \boldsymbol{g}_1)$. It remains to observe that there is an $\mathcal{L} \in \mathcal{M}^{-\infty}_{P_3,S}(\mathbb{W}, \boldsymbol{g}_1; \boldsymbol{w}_1)$ such that $(\mathcal{I} - \mathcal{N})^{-1} = \mathcal{I} - \mathcal{L}$. Then it follows that $\mathcal{A}^{-1} = (\mathcal{I} - \mathcal{L})\mathcal{P}_0$ which belongs to the space $\mathcal{M}^{-\mu}_{P,Q}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ by Theorem 3.6. $\square$

References

[1] M.S. Agranovich, and M.I.Vishik, *Elliptic problems with parameter and parabolic problems of general type* Uspekhi Mat. Nauk **19**, 3, (1964), 53-161.

[2] S. Behm, *Pseudo-differential operators with parameters on manifolds with edges* Ph.D. thesis, University of Potsdam, 1995.

[3] G. De Donno, and B.-W. Schulze, *Meromorphic symbolic structures for boundary value problems on manifolds with edges*, Preprint 2003/10, Institut für Mathematik, Potsdam, 2003.

[4] Ch. Dorschfeldt, *Algebras of pseudo-differential operators near edge and corner singularities* Math. Res., vol. 102, Akademie Verlag, Berlin, 1998.

[5] B.V. Fedosov, B.-W. Schulze, and N.N. Tarkhanov, *The index of elliptic operators on manifolds with conical points* Selecta mathematica, New ser., **5**, 1999, 467-506.

[6] B.V. Fedosov, B.-W. Schulze, and N.N. Tarkhanov, *Analytic index formulas for elliptic corner operators* Ann. Inst. Fourier, **52**, 3 (2002), 899-982.

[7] J.B. Gil, B.-W. Schulze, J. Seiler, *Cone pseudodifferential operators in the edge symbolic calculus*, Osaka J. Math., **37**, (2000), 219-258.

[8] G. Harutjunjan and B.-W Schulze, *Asymptotics and relative index on a cylinder with conical cross section*, Preprint 2002/27, Institut für Mathematik, Potsdam, 2002.

[9] D. Kapanadze and B.-W. Schulze, *Crack theory and edge singularities*, Kluwer Academic Publ., Dordrecht, 2003.

[10] V.A. Kondratyev, *Boundary value problems for elliptic equations in domains with conical points*, Trudy Mosk. Mat. Obshch., **16**, (1967), 209-292.

[11] T. Krainer, *On the inverse of parabolic boundary value problems for large times*, Preprint 2002/12, Institut für Mathematik, Potsdam, 2002, Japanese J. Math. (to appear).

[12] T. Krainer, B.-W. Schulze, *Long-time asymptotics with geometric singularities in the spatial variables*, Preprint 2000/17, Institut für Mathematik, Potsdam, 2000, Israel Math. Conference Proceedings, Vol. **16**, 2003.

[13] T. Krainer, *Volterra families of pseudodifferential operators*, in Advances in Partial Differential Equations (Parabolicity, Volterra Calculus, and Conical Singularities), Oper. Theory Adv. Appl., Birkhäuser Verlag, Basel, **138**, (2002), 1-45.

[14] X. Liu, B.-W. Schulze, *Ellipticity on manifolds with edges and boundary*, (in preparation).

[15] L. Maniccia, B.-W. Schulze, *An algebra of meromorphic corner symbols*, Bull. des Sciences Math., **127**, 1, (2003), 55-99.

[16] V. Nazaikinskij, A. Savin, B.-W. Schulze, and B. Ju. Sternin, *Elliptic theory on manifolds with nonisolated singularities: II. Products in elliptic theory on manifolds with edges*, Preprint 2002/15, Institut für Mathematik, Potsdam, 2002.

[17] V. Nazaikinskij, A. Savin, B.-W. Schulze, and B. Ju. Sternin, *Elliptic theory on manifolds with nonisolated singularities: III. The spectral flow of families of conormal symbols*, Preprint 2002/20, Institut für Mathematik , Potsdam, 2002.

[18] A. Oliaro, B.-W. Schulze, *Parameter-dependent boundary value problems on manifolds with edges*, Preprint 2002/25, Institut für Mathematik , Potsdam, 2002.

[19] S. Rempel, B.-W. Schulze, *Index theory of elliptic boundary problems*, Akademie-Verlag, Berlin, 1982.

[20] E. Schrohe, B.-W. Schulze, *Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities II*, Advances in Partial Differential Equations (Boundary Value Problems, Schrödinger Operators, Deformation Quantization), Akademie Verlag, Berlin (1995), 70 - 205.

[21] E. Schrohe, *Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance*, Math. Nachr., **199**, (1999), 145-185.

[22] E. Schrohe, B.-W. Schulze, *Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities I*, Advances in Partial Differential Equations (Pseudo-Differential Calculus and Mathematical Physics), Akademie Verlag, Berlin, 1994, pp. 97-209.

[23] E. Schrohe, J. Seiler, *Ellipticity and invertibility in the cone algebra on L_p-Sobolev spaces*, Integral Equations Operator Theory, **41**, (2001), 93-114.

[24] B.-W. Schulze, *Corner Mellin operators and reduction of orders with parameters*, Ann. Sc. Norm. Sup. Pisa Cl. Sci., **16**, 1 (1989), 1-81.

[25] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, Symposium "Partial Differential Equations", Holzhau 1988, Teubner-Texte zur Mathematik, vol. 112, Teubner, Leipzig, 1989, pp.259-287.

[26] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

[27] B.-W. Schulze, *The Mellin pseudo-differential calculus on manifolds with corners*, *Pseudo-differential operators on manifolds with edges*, Symposium "Analysis in Domains and on Manifolds with Singularities", Breitenbrunn 1990, Teubner-Texte zur Mathematik, vol. 131, Teubner, Leipzig, 1992, pp.208-289.

[28] B.-W. Schulze, *Mellin representations of pseudo-differential operators on manifolds with corners*, Ann. Glob. Anal. Geom., **8**, 3, (1990), 261-297.

[29] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, J. Wiley, Chichester, 1998.

[30] B.-W. Schulze, *Operators with symbol hierarchies and iterated asymptotics*, Publications of RIMS, Kyoto University, **38**, 4 (2002), 735-802.

[31] B.-W. Schulze, J. Seiler, *The edge algebra structure of boundary value problems*, Annals of Global Analysis and Geometry, **22** (2002), 197-265.

[32] B.-W. Schulze, B. Ju. Sternin, and V. Shatalov, *On the index of differential operators on manifolds with conical singularities*, Ann. Global Anal. Geom., **16** (1998), 141-172.

Institut für Mathematik, Universität Potsdam, An Neuen Palais 10, 14469, Potsdam, Deutschland
E-mail address: `krainer@math.uni-potsdam.de`

Institut für Mathematik, Universität Potsdam, An Neuen Palais 10, 14469, Potsdam, Deutschland
E-mail address: `schulze@math.uni-potsdam.de`