Mellin-Edge Representations of Elliptic Operators

N. Dines and B.–W. Schulze

Abstract

We construct a class of elliptic operators in the edge algebra on a manifold M with an embedded submanifold Y interpreted as an edge. The ellipticity refers to a principal symbolic structure consisting of the standard interior symbol and an operator-valued edge symbol. Given a differential operator A on M for every (sufficiently large) s we construct an associated operator $\mathcal{A}_s$ in the edge calculus. We show that ellipticity of A in the usual sense entails ellipticity of $\mathcal{A}_s$ as an edge operator (up to a discrete set of reals s). Parametrices P of A then correspond to parametrices $\mathcal{P}_s$ of $\mathcal{A}_s$, interpreted as Mellin-edge representations of P.

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Contents

Introduction

Ellipticity of (pseudo-) differential operators $\mathcal{A}$ on a manifold M with edges Y is a bijectivity condition for the components of a principal symbolic hierarchy $\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\wedge(\mathcal{A}))$, where $\sigma_\psi(\mathcal{A})$ is the ('scalar') interior and $\sigma_\wedge(\mathcal{A})$ the (operator-valued) edge symbol. The edge symbol $\sigma_\wedge(\mathcal{A})$ has a 2×2 block matrix structure, with an upper left corner acting in weighted Sobolev spaces on infinite cones, while the other entries are of finite rank. The bijectivity of $\sigma_\wedge(\mathcal{A})$ is an analogue of the Shapiro-Lopatinskij condition for additional data of trace and potential type on the edge. In concrete cases it may be very difficult to control this condition explicitly. In particular, we need information on the position of 'non-linear' eigenvalues of a subordinate so called conormal symbol in the complex plane. It may happen that there are no edge conditions at all for a given elliptic (edge-degenerate) operator A on M; the existence is guaranteed if and only if a certain topological obstruction vanishes, cf. [17] or [21] (which is an analogue of a corresponding condition of Atiyah and Bott [1] for the case of boundary value

problems). If the obstruction vanishes, the number of trace and potential conditions depends on the dimension of kernels and cokernels of $\sigma_\wedge(A)$ as well as on the chosen weights in the Sobolev spaces.
The main result of the present paper is the explicit construction of a class of elliptic edge operators that are interesting for a number of applications. We start from elliptic operators $A : H^s(M) \longrightarrow H^{s-\mu}(M)$ on a C^∞ manifold M with an embedded C^∞ submanifold Y regarded as a 'fictitious' edge of codimension $\geqq 1$. We then show the ellipticity of associated edge operators $\mathcal{A}_s$ with trace and potential operators at Y depending on the choice of s. In the edge algebra we then have parametrices $\mathcal{P}_s$ of $\mathcal{A}_s$ which can be interpreted as 'Mellin-edge' representations of pseudo-differential parametrices P of the original operator A.
This result has relations to the asymptotic properties of potentials of densities supported by a hypersurface in M with edges, when the densities themselves have 'edge asymptotics', cf. Kapanadze and Schulze [9]. There are also connections with the solvability of the Zaremba problem in edge Sobolev spaces, see the author's joint paper with Harutjunian [4] and to crack problems with singular interfaces, cf. [20].
Our approach is based on ideas of the edge pseudo-differential calculus, cf. [17], which is a step in an iteratively organised hierarchy of calculi for configurations with higher geometric singularities, cf. [19]. In such cases the edges themselves may have (conical, etc.) singularities, as is typical in models of mechanics.
Our constructions are also related to problems of Sobolev type, where solutions of $Au = f$ on $M \setminus Y$ for an elliptic operator A are considered under conditions of 'Shapiro-Lopatinskij-type' on Y, cf. Sobolev [23], Sternin [24].
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1 Edge-representations of differential operators

1.1 Edge Sobolev spaces and operator-valued symbols

1.1.1 Mellin transform and Fuchs type operators

In this section we introduce some background on the Mellin transform on $\mathbb{R}_+$, $Mu(z) = \int_0^\infty r^{z-1}u(r)dr$ and weighted spaces on infinite cones.
$Mu(z)$ for $u \in C_0^\infty(\mathbb{R}_+)$ is an entire function in $z \in \mathbb{C}$; otherwise, for more general (weighted) distributions, z will vary on

$$\Gamma_\beta := \{z \in \mathbb{C} : \operatorname{Re} z = \beta\},$$

for a suitable real β. We will also apply the weighted Mellin transform, $(M_\gamma u)(z) := M(r^{-\gamma}u)(z+\gamma)$, with $\gamma \in \mathbb{R}$ interpreted as a power weight at

$r = 0$. Given a pseudo-differential symbol $f(r, r', z) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, S^\mu(\Gamma_{\frac{1}{2}-\gamma}))$ with $S^\mu(\mathbb{R})$ being Hörmander's space of symbols of order μ (with 'constant coefficients') and $S^\mu(\Gamma_\beta)$ the corresponding space on Γ_β under the identification $\Gamma_\beta \to \mathbb{R}$, $z \to \operatorname{Im} z$, we can form a corresponding Mellin pseudo-differential operator

$$\operatorname{op}_M^\gamma(f)u(r) = \int\int_0^\infty \left(\frac{r}{r'}\right)^{-(\frac{1}{2}-\gamma+i\rho)} f(r, r', z)u(r')\frac{dr'}{r'}đ\rho,$$

first on $C_0^\infty(\mathbb{R}_+)$ and later on extended to suitable distribution spaces. Analogous notation will be used for vector-valued u and operator-valued amplitude functions $f(r, r', z)$. Concerning more details on Mellin operators, see [17] or Dorschfeldt [5].
Our next goal is to define weighted Sobolev spaces on a stretched cone $X^\wedge := \mathbb{R}_+ \times X$ with a closed compact C^∞ manifold X as base, $n = \dim X$. Let $L_{\mathrm{cl}}^\mu(X; \mathbb{R}^l)$ denote the space of all classical parameter-dependent pseudo-differential operators on X. Homogeneous principal symbols and ellipticity in this context then refer to $(\xi, \lambda) \neq 0$ where ξ is the covariable on X and $\lambda \in \mathbb{R}^l$ the parameter. We use the fact that for every $\mu \in \mathbb{R}$ there exists an elliptic element $R^\mu(\lambda)$ in $L_{\mathrm{cl}}^\mu(X; \mathbb{R}^l)$ such that $R^\mu(\lambda)$ induces isomorphisms $H^s(X) \longrightarrow H^{s-\mu}(X)$ for all $\lambda \in \mathbb{R}^l$, $s \in \mathbb{R}$. Here $H^s(X)$ are the standard Sobolev spaces of smoothness $s \in \mathbb{R}$ on X.
We now choose such a family $R^\mu(\rho)$ with parameter $\rho \in \mathbb{R}$ and define $\mathcal{H}^{s,\gamma}(X^\wedge)$ as the completion of $C_0^\infty(X^\wedge)$ with respect to the norm $\left\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} ||R^\mu(\operatorname{Im} z)Mu(z)||^2_{L^2(X)}dz\right\}^{\frac{1}{2}}$. The space $L^2(X)$ is equipped with a scalar product, defined in terms of a fixed Riemannian metric on X.
The spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ have the meaning of Sobolev spaces based on the Fuchs type derivative in $r \in \mathbb{R}_+$ and (local) usual derivatives on X. More precisely, for $s \in \mathbb{N}$ we have

$$\begin{aligned}\mathcal{H}^{s,\gamma}(X^\wedge) = \Big\{u(r,x) \in r^{\gamma-\frac{n}{2}}L^2(X^\wedge) : \left(r\frac{\partial}{\partial r}\right)^k Du(r,x) \in r^{\gamma-\frac{n}{2}}L^2(X^\wedge),\\ 0 \leqq k \leqq s,\ D \in \operatorname{Diff}^{s-k}(X)\Big\}.\end{aligned} \tag{1.1}$$

Here $\operatorname{Diff}^m(X)$ denotes the space of all differential operators of order m on X (with smooth coefficients). It can easily be proved that (1.1) is an equivalent definition of $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \in \mathbb{N}$, and the full scale could be defined by duality and interpolation. Notice that $\mathcal{H}^{0,0}(X^\wedge) = r^{-\frac{n}{2}}L^2(\mathbb{R}_+ \times X)$ (with L^2 being taken with the measure $drdx$).
By a cut-off function on the half-axis we understand any real-valued $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1 in a neighbourhood of $r = 0$.
In the considerations below we will also use a modified scale of weighted Sobolev spaces, namely $\mathcal{K}^{s,\gamma}(X^\wedge)$, defined by

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega f + (1-\omega)g :\ f \in \mathcal{H}^{s,\gamma}(X^\wedge),\ g \in H^s_{\mathrm{cone}}(X^\wedge)\}$$

for any cut-off function ω. Here $H^s_{\mathrm{cone}}(X^\wedge)$ denotes the subspace of all $g = \tilde{g}|_{X^\wedge}$, $\tilde{g} \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)$ such that for every coordinate neighbourhood U on X, every diffeomorphism $\chi : U \longrightarrow V$ to an open set $V \subset S^n$, $\chi(x) = v$, and every $\varphi \in C_0^\infty(U)$ the function $\varphi(\chi^{-1}(v))(1-\omega(r))g(r, \chi^{-1}(v))$ belongs to the space $H^s(\mathbb{R}^{n+1})$ (where (r, v) has the meaning of polar coordinates in $\mathbb{R}^{n+1} \setminus \{0\} \cong (S^n)^\wedge$).
The spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ are independent of the specific choice of ω. They are Hilbert spaces with the scalar product

$$(f_1, f_2) = (\omega f_1, \omega f_2)_{\mathcal{H}^{s,\gamma}(X^\wedge)} + ((1-\omega) f_1, (1-\omega) f_2)_{H^s_{\mathrm{cone}}(X^\wedge)}$$

for any fixed ω. The space $C_0^\infty(\mathbb{R}_+)$ is dense in $\mathcal{K}^{s,\gamma}(X^\wedge)$ for every s, $\gamma \in \mathbb{R}$.

Remark 1.1.1. *Let $\chi : X \longrightarrow X$ be a diffeomorphism. Then the function pull back with respect to $\mathrm{id}_{\mathbb{R}_+} \times \chi : \mathbb{R}_+ \times X \longrightarrow \mathbb{R}_+ \times X$, first on $C_0^\infty(X^\wedge)$, extends to an isomorphism $\mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s,\gamma}(X^\wedge)$ for every s, $\gamma \in \mathbb{R}$.*

1.1.2 Edge spaces and symbols with twisted homogeneity

As noted in the beginning, operators in $\mathbb{R}^m$ as well as Sobolev spaces will be reformulated in an anisotropic manner with respect to a splitting $\mathbb{R}^m = \mathbb{R}^{1+n} \times \mathbb{R}^q$, where $\mathbb{R}^q$ is regarded as an edge and $\mathbb{R}^{1+n}$ as a model cone $(S^n)^\triangle := (\overline{\mathbb{R}}_+ \times S^n)/(\{0\} \times S^n)$ of the 'wedge' $\mathbb{R}^m \cong (S^n)^\triangle \times \mathbb{R}^q$.
Let E be a Hilbert space, and let $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ be a strongly continuous group of isomorphisms $\kappa_\delta : E \longrightarrow E$, such that $\kappa_\delta \kappa_{\delta'} = \kappa_{\delta\delta'}$, for all δ, $\delta' \in \mathbb{R}_+$ (strongly continuous means $\kappa_\delta e \in C(\mathbb{R}_+, E)$ for every $e \in E$). In that case we will say that E is endowed with a group action. In particular, for $E = H^s(\mathbb{R}^{1+n})$ we take $(\kappa_\delta u)(\tilde{x}) = \delta^{\frac{1+n}{2}} u(\delta \tilde{x})$, $\delta \in \mathbb{R}_+$. The anisotropic reformulation of the standard Sobolev space $H^s(\mathbb{R}^{1+n+q})$ with respect to the edge $\mathbb{R}^q$ is now formulated in terms of so called ('abstract') edge Sobolev spaces.

Definition 1.1.2. *Let E be a Hilbert space with group action $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$. Then $\mathcal{W}^s(\mathbb{R}^q, E)$, $s \in \mathbb{R}$, is defined as the completion of the space $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm*

$$||u||_{\mathcal{W}^s(\mathbb{R}^q, E)} = \left\{ \int \langle \eta \rangle^{2s} ||\kappa^{-1}_{\langle \eta \rangle} \hat{u}(\eta)||^2_E \, d\eta \right\}^{\frac{1}{2}},$$

where $\langle \eta \rangle = (1 + |\eta|^2)^{\frac{1}{2}}$, $\hat{u}(\eta) = Fu(\eta) = \int e^{-iy\eta} u(y)\, dy$.

More details on the functional analytic properties of this category of spaces may be found in [17], see also Hirschmann [8]. Note that we obtain an equivalent norm if we replace $\langle \eta \rangle$ by any other strictly positive function $p(\eta)$ with the property $c_1 \langle \eta \rangle \leqq p(\eta) \leqq c_2 \langle \eta \rangle$ for certain constants c_1, $c_2 > 0$. In particular, if $[\eta]$ denotes a strictly positive C^∞ function in $\mathbb{R}^q$ that is equal to $|\eta|$ for $|\eta| \geqq c$ for some $c > 0$, we may set $p(\eta) = [\eta]$.

If $\Omega \subseteq \mathbb{R}^q$ is an open set we introduce $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$, $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ similarly as in the scalar case: $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ is the space of all $u \in \mathcal{W}^s(\mathbb{R}^q, E)$ with compact $\operatorname{supp} u \subset \Omega$; moreover, $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ is the space of all $u \in \mathcal{D}'(\Omega, E)$ with $\varphi u \in \mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ for all $\varphi \in C_0^\infty(\Omega)$.

Remark 1.1.3. *For $E = H^s(\mathbb{R}^{1+n})$ with the abovementioned group action we have $H^s(\mathbb{R}^{1+n+q}) = \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^{1+n}))$ for every $s \in \mathbb{R}$.*

Example 1.1.4. *For the weighted cone Sobolev spaces $E = \mathcal{K}^{s,\gamma}(X^\wedge)$ with the group action $(\kappa_\delta u)(r,x) = \delta^{\frac{n+1}{2}} u(\delta r, x)$, $\delta > 0$, $n = \dim X$, we have so called weighted edge Sobolev spaces*

$$\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)).$$

They have the property $H^s_{\mathrm{comp}}(X^\wedge \times \mathbb{R}^q) \subset \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \subset H^s_{\mathrm{loc}}(X^\wedge \times \mathbb{R}^q)$ for all s, $\gamma \in \mathbb{R}$.

For references below we define for any open set $\Omega \subseteq \mathbb{R}^q$ the spaces

$$\mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(X^\wedge \times \Omega) := \mathcal{W}^s_{\mathrm{comp}}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge)),$$

and, analogously, $\mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}(X^\wedge \times \Omega)$.
Let us now consider a differential operator

$$\tilde{A} = \sum_{|\alpha|+|\beta| \leqq \mu} \tilde{a}_{\alpha,\beta}(\tilde{x}, y) D^\alpha_{\tilde{x}} D^\beta_y \tag{1.2}$$

with coefficients $\tilde{a}_{\alpha,\beta}(\tilde{x}, y) \in C^\infty(\mathbb{R}^{1+n+q})$, and write $A := \tilde{A}\big|_{(\mathbb{R}^{1+n}\setminus\{0\})\times\mathbb{R}^q}$ in polar coordinates with respect to the $\tilde{x}$- variables. Setting $r = |\tilde{x}|$ and $X := S^n$ we then obtain

$$A = r^{-\mu} \sum_{j+|\beta| \leqq \mu} a_{j\beta}(r, y) \Big(-r \frac{\partial}{\partial r} \Big)^j (r D_y)^\beta, \tag{1.3}$$

with coefficients $a_{j\beta}(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathrm{Diff}^{\mu-(j+|\beta|)}(X))$.
Differential operators of the form (1.3) will also be called edge-degenerate; clearly, not every such operator admits a reformulation (1.2).
With (1.3) we can associate the symbolic structure of the calculus of edge differential (or pseudo-differential) operators. First we have the standard homogeneous principal symbol of order μ which is in the variables $(r, x, y) \in \mathbb{R}_+ \times \Sigma \times \mathbb{R}^q$ for open sets $\Sigma \subseteq \mathbb{R}^n$, and covariables (ρ, ξ, η) of the form

$$\sigma_\psi(A)(r, x, y, \rho, \xi, \eta) := r^{-\mu} p_{(\mu)}(r, x, y, \rho, \xi, \eta),$$

with $p_{(\mu)}(r, x, y, \rho, \xi, \eta) = \tilde{p}_{(\mu)}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta})\big|_{\tilde{\rho}=r\rho,\, \tilde{\eta}=r\eta}$ for a polynomial $\tilde{p}_{(\mu)}$ in $(\tilde{\rho}, \xi, \tilde{\eta})$ of order μ with coefficients in $C^\infty(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}^q)$; $x \in \Sigma$ are local coordinates

on X.
Operators of the form (1.3) will play the role of local representatives of global operators on a manifold; therefore the behaviour of coefficients for large r and $|y|$ will be unessential, and we assume, for convenience, that the coefficients are independent of r and y for $r+|y|>C$ for some constant $C>0$.
Let us set

$$a(y,\eta):=r^{-\mu}\sum_{j+|\beta|\leqq\mu}a_{j\beta}(r,y)\Big(-r\frac{\partial}{\partial r}\Big)^{j}(r\eta)^{\beta},$$

regarded as a family of continuous operators

$$a(y,\eta):\mathcal{K}^{s,\gamma}(X^\wedge)\longrightarrow\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge), \tag{1.4}$$

for every $s,\gamma\in\mathbb{R}$. We want to interpret (1.4) as an operator-valued symbol in the variables y and covariables η. To this end we briefly recall the corresponding general definition.

Definition 1.1.5. *Let E and $\tilde{E}$ be Hilbert spaces, endowed with group actions $\kappa:=\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ and $\tilde{\kappa}:=\{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}$, respectively. Then $S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ for $\mu\in\mathbb{R}$ and an open set $U\subseteqq\mathbb{R}^p$ is defined to be the set of all $a(y,\eta)\in C^\infty(U\times\mathbb{R}^q,\mathcal{L}(E,\tilde{E}))$ such that*

$$\sup_{(y,\eta)\in K\times\mathbb{R}^q}\langle\eta\rangle^{-\mu+|\beta|}||\tilde{\kappa}^{-1}_{\langle\eta\rangle}\left\{D_y^\alpha D_\eta^\beta a(y,\eta)\right\}\kappa_{\langle\eta\rangle}||_{\mathcal{L}(E,\tilde{E})} \tag{1.5}$$

is finite for every $K\subset\subset U$, $\alpha\in\mathbb{N}^p$, $\beta\in\mathbb{N}^q$. The elements of $S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ are called operator-valued symbols on $U\times\mathbb{R}^q$.

The space $S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ is Fréchet with the semi-norm system (1.5); this allows us asymptotic summation, similarly as for standard symbols.

Remark 1.1.6. *The point-wise composition of operator functions gives us an inclusion $S^\mu(U\times\mathbb{R}^q;E_0,\tilde{E})\cdot S^\nu(U\times\mathbb{R}^q;E,E_0)\subseteq S^{\mu+\nu}(U\times\mathbb{R}^q;E,\tilde{E})$ for every $\mu,\nu\in\mathbb{R}$.*

By $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E})$ we denote the subspace of so called classical symbols $a(y,\eta)$ that can be written as an asymptotic sum $\sum_{j=0}^\infty\chi(\eta)a_{(\mu-j)}(y,\eta)$ for any excision function $\chi(\eta)$ in $\mathbb{R}^q$ and a suitable sequence of functions

$$a_{(\mu-j)}(y,\eta)\in C^\infty(U\times(\mathbb{R}^q\setminus\{0\}),\mathcal{L}(E,\tilde{E})),\ j\in\mathbb{N},$$

satisfying

$$a_{(\mu-j)}(y,\delta\eta)=\delta^{\mu-j}\tilde{\kappa}_\delta a_{(\mu-j)}(y,\eta)\kappa_\delta^{-1}, \tag{1.6}$$

for all $\delta\in\mathbb{R}_+$. The homogeneous components $a_{(\mu-j)}$ are uniquely determined by $a(y,\eta)$.
The space $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E})$ is also Fréchet in a canonical way (in a corresponding

stronger topology than that induced by $S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$).
The relation (1.6) will also be referred to as twisted homogeneity of the corresponding order. We also set

$$\sigma_\wedge(a)(y,\eta) := a_{(\mu)}(y,\eta)$$

for the homogeneous principal component of order μ (in our cases μ will be known from the context).
If a relation is valid in the classical or non-classical case, we often write as subscript '(cl)'. Let $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$ denote the subspace of all $a(\eta) \in S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})$ which are independent of y (i.e., symbols with 'constant coefficients'). The spaces $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$ are closed in $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})$, and we have

$$S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E}) = C^\infty(U, S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})).$$

Note that for the case $E = \tilde{E} = \mathbb{C}$ and $\kappa_\delta = \mathrm{id}_E$, $\tilde{\kappa}_\delta = \mathrm{id}_{\tilde{E}}$ for all $\delta \in \mathbb{R}_+$, we just recover the standard 'scalar' spaces of symbols, also used below.

Example 1.1.7. *Let $f(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$ be a function that is homogeneous of order μ for large $|\eta|$, i.e., $f(y, \delta\eta) = \delta^\mu \tilde{\kappa}_\delta f(y,\eta)\kappa_\delta^{-1}$ for all $\delta \geqq 1$, $(y,\eta) \in U \times \mathbb{R}^q$, $|\eta| \geqq c$ for some $c > 0$; then we have $f(y,\eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$.*

Example 1.1.8. *Let $\beta(\tilde{x}, y) \in C_0^\infty(\mathbb{R}^{d+q})$ and consider the y-dependent family of continuous operators $b(y) : u(\tilde{x}) \to \beta(\tilde{x}, y)u(\tilde{x})$, $b(y) \in C_0^\infty(\mathbb{R}^q, \mathcal{L}(H^s(\mathbb{R}^d), H^s(\mathbb{R}^d)))$. Then we have $b(y) \in S^0(\mathbb{R}^q_y \times \mathbb{R}^q_\eta; H^s(\mathbb{R}^d), H^s(\mathbb{R}^d))$ for every $s \in \mathbb{R}$ (although b is independent of the covariable η).*

Remark 1.1.9. *Let A be given by (1.3) and assume the coefficients $a_{j\alpha}$ to be independent of r for large r. Then the operator family $a(y,\eta)$ from (1.4) represents an element*

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

for all s, $\gamma \in \mathbb{R}$. If the coefficients $a_{j\alpha}$ are independent of r then $a(y,\eta)$ is classical.

There is a calculus of pseudo-differential operators with operator-valued symbols, similar to the scalar case (i.e., when $E = \tilde{E} = \mathbb{C}$ and the groups κ and $\tilde{\kappa}$ consist of the identity). Details may be found in [17]. The corresponding tools will be systematically employed here.
In particular, there is an analogue of the standard continuity of pseudo-differential operators in Sobolev spaces, here for the case of edge Sobolev spaces.

Theorem 1.1.10. *Let $a(y, y', \eta) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^q; E, \tilde{E})$, $\Omega \subseteqq \mathbb{R}^q$ open, and $\mathrm{Op}(a)u(y) = \iint e^{i(y-y')\eta} a(y, y', \eta) u(y')\, dy'\, đ\eta$, $đ\eta = (2\pi)^{-q} d\eta$. Then $\mathrm{Op}(a)$ is continuous as an operator $\mathrm{Op}(a) : C_0^\infty(\Omega, E) \longrightarrow C^\infty(\Omega, \tilde{E})$, and extends to continuous operators $\mathrm{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E})$ for every $s \in \mathbb{R}$. For the* *case* $\Omega = \mathbb{R}^q$ *and* $a(y,\eta)$ *independent of* y *for large* $|y|$, *we also have continuous operators*

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, E) \longrightarrow \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{E}), \tag{1.7}$$

for all $s \in \mathbb{R}$.

1.1.3 Trace and potential symbols

As noted before, an admitted choice for the spaces E or $\tilde{E}$ is also $\mathbb{C}^N$ for any $N \in \mathbb{N}$, with the trivial group action, i.e., the identity for all $\delta \in \mathbb{R}_+$.
Elements of the spaces $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \mathbb{C}^N)$ and $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \mathbb{C}^M, \tilde{E})$ will be called (abstract, classical) trace and potential symbols, respectively.

Example 1.1.11. (i) *Let* $E := H^s(\mathbb{R}^d)$, *endowed with the group action* $(\kappa_\delta u)(\tilde{x}) = \delta^{\frac{d}{2}} u(\delta \tilde{x})$, $\delta \in \mathbb{R}_+$, *and let* $s \in \mathbb{R}$, $\alpha \in \mathbb{N}^d$, $s > \frac{d}{2} + |\alpha|$. *Then the operator* $\gamma^\alpha : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}$, $\gamma^\alpha u := D^\alpha_{\tilde{x}} u(0)$ *extends to a classical trace symbol* $\gamma^\alpha \in S^{\frac{d}{2}+|\alpha|}_{\mathrm{cl}}(\mathbb{R}^q; H^s(\mathbb{R}^d), \mathbb{C})$ (*although it is independent of* $\eta \in \mathbb{R}^q$). *In fact, we have smoothness in* η *and homogeneity* $\gamma^\alpha u = \delta^{\frac{d}{2}+|\alpha|} \gamma^\alpha \kappa_\delta^{-1} u$, for all $\delta \in \mathbb{R}_+$. *Then, for*

$$t^\alpha(\eta) := [\eta]^{-\frac{d}{2}-|\alpha|} \gamma^\alpha$$

we have

$$t^\alpha(\eta) \in S^0_{\mathrm{cl}}(\mathbb{R}^q; H^s(\mathbb{R}^d), \mathbb{C}). \tag{1.8}$$

(ii) *Let* $\omega(\tilde{x}) \in C_0^\infty(\mathbb{R}^d)$, $\omega(\tilde{x}) \equiv 1$ *in a neighbourhood of* $\tilde{x} = 0$. *Then*

$$k^\alpha(\eta) c := [\eta]^{\frac{d}{2}} \frac{1}{\alpha!} ([\eta]\tilde{x})^\alpha \omega([\eta]\tilde{x}) c$$

for $c \in \mathbb{C}$ *defines a potential symbol*

$$k^\alpha(\eta) \in S^0_{\mathrm{cl}}(\mathbb{R}^q; \mathbb{C}, H^s(\mathbb{R}^d)), \tag{1.9}$$

for arbitrary $s \in \mathbb{R}$. *In fact, we have* $k^\alpha(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}(\mathbb{C}, H^s(\mathbb{R}^d))$ *and*

$$k^\alpha(\delta\eta) = \kappa_\delta k^\alpha(\eta) \text{ for all } \delta \geqq 1,\ |\eta| \geqq c,$$

for some $c > 0$. *This gives us the relation* (1.9), *cf. also Example* 1.1.7.

Remark 1.1.12. *If we define* $t^\alpha(\eta)$ *and* $k^\alpha(\eta)$ *in terms of* $\langle\eta\rangle$ *instead of* $[\eta]$ *we also obtain the relations* (1.8) *and* (1.9), *respectively. In both versions the homogeneous principal symbols of order* 0 *have the form*

$$\sigma_\wedge(t^\alpha)(\eta) = |\eta|^{-\frac{d}{2}-|\alpha|} \gamma^\alpha \text{ and } \sigma_\wedge(k^\alpha)(\eta) = |\eta|^{\frac{d}{2}} \frac{1}{\alpha!} (|\eta|\tilde{x})^\alpha \omega(|\eta|\tilde{x}). \tag{1.10}$$

1.2 Edge-representations

1.2.1 Decompositions of Sobolev spaces

Let M be a closed oriented C^∞ manifold, $m = \dim M$, and let $Y \subset M$ be a closed C^∞ submanifold with the induced orientation, $q = \dim Y < m - 1$. On M and Y we fix Riemannian metrics and assume that the metric on Y is induced by the one on M. We will interpret M as a manifold with edge Y; let us first assume that Y has a trivial normal bundle in M.
In this section we want to derive certain anisotropic decompositions of the standard Sobolev spaces $H^s(M)$, $s > m - q$, with respect to the edge Y.
Let us fix an atlas of charts on M

$$\chi_j : U_j \longrightarrow \mathbb{R}^m,\ j = 1, ..., N, \tag{1.11}$$

with coordinate neighbourhoods U_j on M. Assume (without loss of generality) that with U_i and U_j also $U_i \cup U_j$ is contained in a coordinate neighbourhood for each i, j and that $U_j \cap Y \neq \emptyset$ for $1 \leqq j \leqq L$ and $U_j \cap Y = \emptyset$ for $L+1 \leqq j \leqq N$. The charts χ_j for $1 \leqq j \leqq L$ can (and will) be chosen in such a way that the restrictions $\chi'_j := \chi_j\big|_{U'_j}$ to $U'_j = U_j \cap Y$ form an atlas $\chi'_j : U'_j \longrightarrow \mathbb{R}^q$ on Y. Then we have a splitting $\mathbb{R}^m = \mathbb{R}^q \times \mathbb{R}^d$, $d := m - q$, and we write $(y, \tilde{x})$ for the local coordinates near Y. We then assume that the transition maps belonging to the charts χ_j for $1 \leqq j \leqq L$ are independent of $\tilde{x}$ for $\tilde{x} < \varepsilon$ for some $\varepsilon > 0$. In other words the local coordinates $(y, \tilde{x})$ near Y are chosen in such a way that $\tilde{x}$ is an invariant coordinate in $\mathbb{R}^d$, i.e., remains unchanged under transition diffeomorphisms.
We first establish certain edge-decompositions of Sobolev spaces on $\mathbb{R}^q \times \mathbb{R}^d$ and then pass to a corresponding global construction.
Let us define the space

$$H^s_0(\mathbb{R}^d) := \left\{ u \in H^s(\mathbb{R}^d) : D^\alpha_{\tilde{x}} u(0) = 0 \text{ for all } |\alpha| < s - \frac{d}{2} \right\}$$

for any $s \geq 0$, $s - \frac{d}{2} \notin \mathbb{N}$. The space $H^s_0(\mathbb{R}^d)$ is closed in $H^s(\mathbb{R}^d)$, and $(\kappa_\delta u)(\tilde{x}) = \delta^{\frac{d}{2}} u(\delta \tilde{x})$, $\delta \in \mathbb{R}_+$, represents a group action in $H^s_0(\mathbb{R}^d)$.

Remark 1.2.1. *For every $s \in \mathbb{R}$, $s \geq 0$, we have*

$$\mathcal{K}^{s,s}(\mathbb{R}^d \setminus \{0\}) = H^s_0(\mathbb{R}^d),$$

cf. Kondratyev [12] *or* Dauge [3]*, see also* [10]*. The associated edge space*

$$\mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^d)) \subset H^s(\mathbb{R}^{d+q}), \tag{1.12}$$

(*cf. Definition* 1.1.2 *and Remark* 1.1.3) *can be characterised as follows:*

$$\mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^d)) = \left\{ u(\tilde{x}, y) \in H^s(\mathbb{R}^{d+q}) : D^\alpha_{\tilde{x}} u(0, y) = 0 \text{ for all } |\alpha| < s - \frac{d}{2} \right\}$$

for any $s \geq 0$, $s - \frac{d}{2} \notin \mathbb{N}$.

Let us now construct complementary spaces in $H^s(\mathbb{R}^{d+q})$ in terms of families of edge-potential operators.
First, we form the vectors of symbols

$$t(\eta,\lambda) := {}^{\mathrm{t}}(t^{\alpha}(\eta,\lambda) : |\alpha| < s - \frac{d}{2}),\ k(\eta,\lambda) := (k^{\alpha}(\eta,\lambda) : |\alpha| < s - \frac{d}{2}), \tag{1.13}$$

cf. (1.8), (1.9), here, with covariables $(\eta,\lambda) \in \mathbb{R}^{q+l}$ instead of $\eta \in \mathbb{R}^q$. We then have

$$t(\eta,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; H^s(\mathbb{R}^d), \mathbb{C}^{\sigma(s)}),\ k(\eta,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; \mathbb{C}^{\sigma(s)}, H^s(\mathbb{R}^d)),$$

for $\mathbb{C}^{\sigma(s)} = \oplus_{|\alpha|<s-\frac{d}{2}}\mathbb{C}$, and

$$t(\eta,\lambda)k(\eta,\lambda) = \mathrm{id}_{\mathbb{C}^{\sigma(s)}} \quad \text{for all } (\eta,\lambda) \in \mathbb{R}^{q+l}. \tag{1.14}$$

From the relation (1.7) and Remark 1.1.3 we see that the associated pseudo-differential operators with respect to y induce families of continuous operators

$$K(\lambda) := \mathrm{Op}(k)(\lambda) : H^s(\mathbb{R}^q, \mathbb{C}^{\sigma(s)}) \longrightarrow \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^d)) = H^s(\mathbb{R}^{d+q}), \tag{1.15}$$

$$T(\lambda) := \mathrm{Op}(t)(\lambda) : \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^d)) \longrightarrow H^s(\mathbb{R}^q, \mathbb{C}^{\sigma(s)}).$$

As a consequence of (1.14) we obtain that

$$T(\lambda)K(\lambda) : H^s(\mathbb{R}^q, \mathbb{C}^{\sigma(s)}) \longrightarrow H^s(\mathbb{R}^q, \mathbb{C}^{\sigma(s)}),$$

is the identity map for all $\lambda \in \mathbb{R}^l$. The operator (1.15) is surjective, and we have $\ker T(\lambda) = \mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^d))$ for all $\lambda \in \mathbb{R}^l$. Moreover,

$$K(\lambda)T(\lambda) : \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^d)) \longrightarrow \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^d))$$

is a family of continuous projections to the spaces

$$\mathcal{V}^s(\mathbb{R}^{d+q}; \lambda) := \left\{ K(\lambda)w : w \in H^s(\mathbb{R}^q, \mathbb{C}^{\sigma(s)}) \right\},$$

for all $\lambda \in \mathbb{R}^l$. Thus for each $s \in \mathbb{R}$, $s \geq 0$, $s - \frac{d}{2} \notin \mathbb{N}$, it follows that

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^d)) = \mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^d)) \oplus \mathcal{V}^s(\mathbb{R}^{d+q}; \lambda),\ \lambda \in \mathbb{R}^l.$$

The latter relation is referred to as an edge-decomposition of the space $H^s(\mathbb{R}^{d+q})$ with $\mathbb{R}^q$ being regarded as an edge, embedded in $\mathbb{R}^{d+q}$ and $\mathbb{R}^d$ as the model cone of the corresponding 'wedge' $\mathbb{R}^d \times \mathbb{R}^q$.
The operators

$$T(\lambda) : \mathcal{V}^s(\mathbb{R}^{d+q}; \lambda) \longrightarrow H^s(\mathbb{R}^q, \mathbb{C}^{\sigma(s)}) \tag{1.16}$$

are isomorphisms, and $K(\lambda)$ is the inverse of (1.16) for every $\lambda \in \mathbb{R}^l$.
Summing up, we obtain the following assertions:

Proposition 1.2.2. *Let* $E : \mathcal{W}^s(\mathbb{R}^q, H_0^s(\mathbb{R}^d)) \longrightarrow \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^d))$ *be the canonical embedding, cf. the formula* (1.12)*; then the operators*

$$(E \quad K(\lambda)) : \begin{array}{c} \mathcal{W}^s(\mathbb{R}^q, H_0^s(\mathbb{R}^d)) \\ \oplus \\ H^s(\mathbb{R}^q, \mathbb{C}^{\sigma(s)}) \end{array} \longrightarrow H^s(\mathbb{R}^{d+q}) \tag{1.17}$$

are isomorphisms, and

$$\begin{pmatrix} P(\lambda) \\ T(\lambda) \end{pmatrix} : H^s(\mathbb{R}^{d+q}) \longrightarrow \begin{array}{c} \mathcal{W}^s(\mathbb{R}^q, H_0^s(\mathbb{R}^d)) \\ \oplus \\ H^s(\mathbb{R}^q, \mathbb{C}^{\sigma(s)}) \end{array} \tag{1.18}$$

for $P(\lambda) := 1 - K(\lambda)T(\lambda)$ *define the inverses of* (1.17)*, for all* $\lambda \in \mathbb{R}^l$.

Lemma 1.2.3. *Let* $\varphi \in C_0^\infty(\mathbb{R}^{d+q})$ *be an arbitrary function. Then for every* $\alpha \in \mathbb{N}^d$ *we have*

$$\tilde{x}^\alpha \varphi(\tilde{x}, y) k(\eta, \lambda) \in S_{\mathrm{cl}}^{-|\alpha|}(\mathbb{R}_y^q \times \mathbb{R}^{q+l}; \mathbb{C}^{\sigma(s)}, H^s(\mathbb{R}^d)),$$

$$t(\eta, \lambda) \tilde{x}^\alpha \varphi(\tilde{x}, y) \in S_{\mathrm{cl}}^{-|\alpha|}(\mathbb{R}_y^q \times \mathbb{R}^{q+l}; H^s(\mathbb{R}^d), \mathbb{C}^{\sigma(s)}).$$

Moreover,

$$\tilde{x}^\alpha k(\eta, \lambda) \in S_{\mathrm{cl}}^{-|\alpha|}(\mathbb{R}^{q+l}; \mathbb{C}^{\sigma(s)}, H^s(\mathbb{R}^d)),\ t(\eta, \lambda) \tilde{x}^\alpha \in S_{\mathrm{cl}}^{-|\alpha|}(\mathbb{R}^{q+l}; H^s(\mathbb{R}^d), \mathbb{C}^{\sigma(s)}).$$

The potential symbols exist for all $s \in \mathbb{R}$*, the trace symbols for all* $|\alpha| < s - \frac{d}{2}$.

Our next objective is to derive similar relations on the manifold M with respect to the edge Y. Let us choose functions φ_j, $\psi_j \in C_0^\infty(U_j)$, $1 \leqq j \leqq L$, such that $\sum_{j=1}^L \varphi_j = 1$ in a neighbourhood of Y and $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$. Then the functions $\varphi_j' := \varphi_j\big|_Y \in C_0^\infty(U_j')$, $1 \leqq j \leqq L$, form a partition of unity on Y, subordinate to the covering $\{U_1', ..., U_L'\}$, and $\psi_j' := \psi_j\big|_Y$ are functions in $C_0^\infty(U_j')$ such that $\psi_j' \equiv 1$ on $\operatorname{supp} \varphi_j'$, $1 \leqq j \leqq L$. After the above splitting of variables near Y into $y \in Y$ and a global normal variable $\tilde{x} \in \mathbb{R}^d$ without loss of generality we assume that the functions φ_j and ψ_j are of the form

$$\varphi_j = \varphi_j' \omega_0,\ \psi_j = \psi_j' \omega_1$$

for functions $\omega_0(\tilde{x})$, $\omega_1(\tilde{x}) \in C_0^\infty(\mathbb{R}^d)$ which are equal to 1 in a neighbourhood of $\tilde{x} = 0$ and have the property $\omega_1 \equiv 1$ on $\operatorname{supp} \omega_0$.
We now apply the operator push forwards of $\mathrm{Op}(t)(\lambda)$ and $\mathrm{Op}(k)(\lambda)$ to operators on the manifold Y with respect to the charts $\chi_j' : U_j' \longrightarrow \mathbb{R}^q$.
Let α_j, $\beta_j \in C_0^\infty(\mathbb{R}^{d+q})$ denote functions with the property $\chi_j^* \alpha_j = \varphi_j$, $\chi_j^* \beta_j = \psi_j$,

and set $\alpha'_j := \alpha_j\big|_{\mathbb{R}^q}$, $\beta'_j := \beta_j\big|_{\mathbb{R}^q}$, such that $\varphi'_j = \chi'^*_j \alpha'_j$, $\psi'_j = \chi'^*_j \beta'_j$.
We now form the operator-valued amplitude functions

$$b_j(y, y', \eta, \lambda) := \alpha'_j t(\eta, \lambda)\beta_j \in S^0(\mathbb{R}^q_y \times \mathbb{R}^q_{y'} \times \mathbb{R}^{q+l}; H^s(\mathbb{R}^d), \mathbb{C}^{\sigma(s)}), \tag{1.19}$$

$$g_j(y, y', \eta, \lambda) := \alpha_j k(\eta, \lambda)\beta'_j \in S^0(\mathbb{R}^q_y \times \mathbb{R}^q_{y'} \times \mathbb{R}^{q+l}; \mathbb{C}^{\sigma(s)}, H^s(\mathbb{R}^d)). \tag{1.20}$$

The factor $\beta_j = \beta_j(\tilde{x}, y')$ in the formula (1.19) is interpreted as an operator of multiplication by $\beta_j(\tilde{x}, y')$ in the space $H^s(\mathbb{R}^q)$ for every fixed y; this represents an element $\beta_j \in S^0(\mathbb{R}^q_{y'} \times \mathbb{R}^{q+l}; H^s(\mathbb{R}^d), H^s(\mathbb{R}^d))$, cf. Example 1.1.8. In a similar manner we interpret α_j in the expression (1.20) as an operator-valued symbol.
The operators of multiplication by $\alpha'_j(y)$ and $\beta'_j(y')$ in (1.19) and (1.20), respectively only contribute a dependence of amplitude functions on the respective variables y or y' and represent symbols of order zero, not depending on the covariables.
The associated pseudo-differential operators give us families of maps

$$\mathrm{Op}(b_j)(\lambda) : H^s_{\mathrm{comp}}(\mathbb{R}^{d+q}) \longrightarrow H^s_{\mathrm{comp}}(\mathbb{R}^q),\ \mathrm{Op}(g_j)(\lambda) : H^s_{\mathrm{comp}}(\mathbb{R}^q) \longrightarrow H^s_{\mathrm{comp}}(\mathbb{R}^{d+q}).$$

Remark 1.2.4. *Let $\tilde{\alpha}_j \in C_0^\infty(\mathbb{R}^{d+q})$ be any other function such that $\alpha_j - \tilde{\alpha}_j$ vanishes in some neighbourhood of $\tilde{x} = 0$. Then setting $\tilde{g}_j := \tilde{\alpha}_j k(\eta, \lambda)\beta'_j$, we have $\mathrm{Op}(\tilde{g}_j)(\lambda) = \mathrm{Op}(g_j)(\lambda)$ modulo a smoothing potential operator, depending on $\lambda \in \mathbb{R}^l$ as a Schwartz function. A similar remark holds for the trace operators when we replace β_j by $\tilde{\beta}_j$ such that $\beta_j - \tilde{\beta}_j$ vanishes near $\tilde{x} = 0$.*

The pull backs under χ_j and χ'_j induce isomorphisms $\chi^*_j : H^s_{\mathrm{comp}}(\mathbb{R}^{d+q}) \longrightarrow H^s_{\mathrm{comp}}(U_j)$, $\chi'^*_j : H^s_{\mathrm{comp}}(\mathbb{R}^q) \longrightarrow H^s_{\mathrm{comp}}(U'_j)$ which give us

$$B_j(\lambda) := \chi'^*_j \mathrm{Op}(b_j)(\lambda)(\chi^*_j)^{-1} : H^s_{\mathrm{comp}}(U_j) \longrightarrow H^s_{\mathrm{comp}}(U'_j),$$

$$G_j(\lambda) := \chi^*_j \mathrm{Op}(g_j)(\lambda)(\chi'^*_j)^{-1} : H^s_{\mathrm{comp}}(U'_j) \longrightarrow H^s_{\mathrm{comp}}(U_j).$$

We now form the operator families $B(\lambda) := \sum_{j=1}^L B_j(\lambda)$ and $G(\lambda) := \sum_{j=1}^L G_j(\lambda)$

$$B(\lambda) : H^s(M) \longrightarrow H^s(Y, \mathbb{C}^{\sigma(s)}),\ G(\lambda) : H^s(Y, \mathbb{C}^{\sigma(s)}) \longrightarrow H^s(M).$$

Definition 1.2.5. *Let $H^s_0(M, Y)$ denote the subspace of all $u \in H^s(M)$ such that for every $\varphi \in C_0^\infty(U_j)$ and every chart $\chi_j : U_j \longrightarrow \mathbb{R}^{d+q}$ that induces by restriction a chart $\chi'_j : U'_j \longrightarrow \mathbb{R}^q$, $1 \leqq j \leqq L$, we have*

$$\varphi u = \chi^*_j v \text{ for some } v \in \mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^d)), \tag{1.21}$$

$s - \frac{d}{2} \notin \mathbb{N}$.

This is an invariant definition with respect to transition maps associated with the system of charts $\{\chi_1, ..., \chi_L\}$ on M near Y.

Remark 1.2.6. *The space $H_0^s(M,Y)$ can equivalently be characterised as the subspace of all $u \in H^s(M)$ such that for every chart $\chi_j : U_j \longrightarrow \mathbb{R}^{d+q}$ as mentioned before we have $D^\alpha_{(y,\tilde{x})}(\chi_j^{-1})^* u|_{\tilde{x}=0} = 0$ for all $|\alpha| < s - \frac{d}{2}$ and all y. This is an invariant property under coordinate changes and does not employ the assumption that the normal bundle of Y is trivial.*

Let $E : H_0^s(M,Y) \longrightarrow H^s(M)$ denote the canonical embedding. Similarly as the operators in (1.17), (1.18) we now form operators

$$(E \;\; G(\lambda)) : \begin{array}{c} H_0^s(M,Y) \\ \oplus \\ H^s(Y, \mathbb{C}^{\sigma(s)}) \end{array} \longrightarrow H^s(M) \;, \tag{1.22}$$

and

$$\begin{pmatrix} P(\lambda) \\ B(\lambda) \end{pmatrix} : \; H^s(M) \longrightarrow \begin{array}{c} H_0^s(M,Y) \\ \oplus \\ H^s(Y, \mathbb{C}^{\sigma(s)}) \end{array} \tag{1.23}$$

for

$$P(\lambda) := \sum_{j=1}^{L} \chi_j^* \left\{\alpha_j P_j(\lambda) \beta_j\right\} (\chi_j^*)^{-1}. \tag{1.24}$$

Here $P_j(\lambda) : H^s(\mathbb{R}^{d+q}) \longrightarrow \mathcal{W}^s(\mathbb{R}^q, H_0^s(\mathbb{R}^d))$ is the first component of (1.18), interpreted as an operator in local coordinates $(\tilde{x}, y)$ belonging to the chart $\chi_j : U_j \longrightarrow \mathbb{R}^{d+q}$. The mapping property $P(\lambda) : H^s(M) \longrightarrow H_0^s(M,Y)$ follows from the corresponding property of $P_j(\lambda)$ together with the definition of the space $H_0^s(M,Y)$ and the fact that the multiplications by C^∞ functions respect the spaces $\mathcal{W}^s(\mathbb{R}^q, H_0^s(\mathbb{R}^d))$.

Lemma 1.2.7. *The composition $B(\lambda)G(\lambda)$ represents a parameter-dependent elliptic element of $L^0_{\mathrm{cl}}(Y;\mathbb{R}^l) \otimes \mathbb{C}^{\sigma(s)} \otimes \mathbb{C}^{\sigma(s)}$.*

Proof. By definition the operator $B(\lambda)G(\lambda)$ is a finite sum of compositions

$$\chi_j'^* \mathrm{Op}(b_j)(\lambda)(\chi_j^*)^{-1} \chi_i^* \mathrm{Op}(g_i)(\lambda)(\chi_i'^*)^{-1} \tag{1.25}$$

for i, $j = 1, ..., L$. For every fixed i, j we may transform (1.25) into a representation in local coordinates in $\mathbb{R}^q$, using the fact that U_i and U_j belong to a common coordinate neighbourhood on M. This gives us (1.25) in the form

$$\mathrm{Op}_y(\alpha_j' t(\eta,\lambda)\beta_j)\mathrm{Op}_y(\alpha_i k(\eta,\lambda)\beta_i') + R(\lambda) \tag{1.26}$$

where α_j', β_j and α_i, β_i' are the former functions in the chosen local coordinate system, and $R(\lambda)$ is an operator family in $L^{-1}_{\mathrm{cl}}(\mathbb{R}^q;\mathbb{R}^l)$.
We have

$$\mathrm{Op}(\alpha_j'(y)t(\eta,\lambda)\beta_j(\tilde{x}',y')) = \mathrm{Op}(t(\eta,\lambda)\alpha_j'(y')\beta_j(\tilde{x}',y')) + R_1(\lambda), \tag{1.27}$$

$$\mathrm{Op}(\alpha_i(\tilde{x},y)k(\eta,\lambda)\beta_i'(y')) = \mathrm{Op}(\alpha_i(\tilde{x},y)\beta_i'(y)k(\eta,\lambda)) + R_2(\lambda) \tag{1.28}$$

for trace and potential families $R_1(\lambda)$ and $R_2(\lambda)$, respectively, of order -1.
Note that the notation of y or y' in the latter formulas mean that y is regarded as a 'left' variable, y' as a 'right' variable in the sense of double symbols with variables (y,y') under the oscillatory integrals. Writing (by Taylor formula at $\tilde{x}=0$)

$$\alpha_j'(y)\beta_j(\tilde{x},y) - \alpha_j'(y) = \sum_{m=1}^{d} \tilde{x}_m\gamma_m(\tilde{x},y)$$

with C^∞ functions γ_m (depending on j) we obtain for the first summand on the right of (1.27) $\mathrm{Op}(\alpha_j'(y)t(\eta,\lambda)) + \sum_{m=1}^{d}\mathrm{Op}(t(\eta,\lambda)\tilde{x}_m\gamma_m(\tilde{x},y))$ which is equal to $\mathrm{Op}(\alpha_j'(y)t(\eta,\lambda))$ modulo a trace operator family of order -1 (where we use Lemma 1.2.3). In a similar manner we can argue for (1.28) and replace this modulo a lower order term by $\mathrm{Op}(k(\eta,\lambda)\alpha_i'(y'))$.
In sum it follows that (1.26) is equal to $\alpha_j'(y)\mathrm{Op}_y(t)\mathrm{Op}_y(k)\alpha_i'(y') = \alpha_j'(y)\alpha_i'(y')$, modulo contributions of order -1 in λ. Carrying out the summation over i,j from 1 to L we obtain the identity operator modulo a term of order -1. Concerning the nature of the latter contribution we can easily verify that this is indeed a classical parameter-dependent pseudo-differential operator of order -1 on Y, i.e., the operator in question is, in fact, parameter-dependent elliptic of order 0 (in a trivial sense, insofar its parameter-dependent principal symbol of order 0 is equal to 1). □

Let us write the composition of (1.22) and (1.23) in the form

$$(E \quad G(\lambda))\,{}^{\mathrm{t}}(P(\lambda) \quad B(\lambda)) = I - C(\lambda) \tag{1.29}$$

for a corresponding operator family $C(\lambda) = -I - EP(\lambda) - G(\lambda)B(\lambda)$ for $I = \mathrm{id}_{H^s(M)}$. By construction, for arbitrary $\varphi,\,\psi \in C^\infty(M)$ such that $\operatorname{supp}\varphi \cap Y = \operatorname{supp}\psi \cap Y = \emptyset$ the operators $\varphi C(\lambda)\psi$ define an element in $\mathcal{S}(\mathbb{R}^l, L^{-\infty}(M))$. Locally near Y the operator $C(\lambda)$ is a pseudo-differential operator with respect to $y \in \mathbb{R}^q$ with classical operator-valued symbol in (η,λ) and homogeneous principal symbol of order -1. In order to verify this we first consider $G(\lambda)B(\lambda)$ which is locally defined as a sum of compositions of pseudo-differential operators in y. Its homogeneous principal symbol in (η,λ) of order zero is equal to $k_{(0)}(\eta,\lambda)t_{(0)}(\eta,\lambda)$ (cf. the notation in the formula (1.6)). Similarly as (1.14) we have $t_{(0)}(\eta,\lambda)k_{(0)}(\eta,\lambda) = \mathrm{id}_{\mathbb{C}^{\sigma(s)}}$ and, $1 - k_{(0)}(\eta,\lambda)t_{(0)}(\eta,\lambda)$ is a family of projections to $H_0^s(\mathbb{R}^d)$, while $k_{(0)}(\eta,\lambda)t_{(0)}(\eta,\lambda)$ is a family of projections to a complement of $H_0^s(\mathbb{R}^d)$ in $H^s(\mathbb{R}^d)$. Thus the sum of the principal symbols of $EP(\lambda)+G(\lambda)B(\lambda)$ is equal to $\mathrm{id}_{H^s(\mathbb{R}^d)}$ for every $(\eta,\lambda) \neq 0$ which shows that the local operator-valued symbols of $C(\lambda)$ are of order -1.
Thus we proved that the operator (1.23) is an approximation of a right inverse

of (1.22) modulo $C(\lambda)$. In a similar manner we can show that (1.23) also approximates a left inverse modulo a remainder of order -1. We want to improve this by a formal Neumann series construction. In fact, if we form

$$ {}^{\mathrm{t}}(P_N(\lambda) \;\; B_N(\lambda)) := {}^{\mathrm{t}}(P(\lambda) \;\; B(\lambda)) \sum_{k=0}^{N} C^k(\lambda), \tag{1.30} $$

then we obtain a similar approximation of the (two-sided) inverse of (1.22) modulo a remainder which is parameter-dependent smoothing outside Y and locally near Y an operator with a symbol of order $-N$.

Remark 1.2.8. *The composition $B_N(\lambda)G(\lambda)$ for every $N \in \mathbb{N}$ is a parameter-dependent elliptic element of $L^0_{\mathrm{cl}}(Y;\mathbb{R}^l) \otimes \mathbb{C}^{\sigma(s)} \otimes \mathbb{C}^{\sigma(s)}$ with $\mathrm{id}_{\mathbb{C}^{\sigma(s)}}$ being the homogeneous principal symbol.*

The relation (1.30) makes sense for every finite N. Let us generalise it for $N = \infty$ by passing to an asymptotic sum on the right of (1.30). To this end we first observe that the operators $C^k(\lambda)$ can be written in the form

$$ C^k(\lambda) = \sum_{j=1}^{L} (\chi_j^{-1})_* (\alpha_j \mathrm{Op}(c_j^k)(\lambda)\beta_j) + \mathcal{D}_k(\lambda) \tag{1.31} $$

for (left-) amplitude functions $c_j^k(y,\eta,\lambda) \in S^{-k}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^{q+l}; H^s(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$ and some element $\mathcal{D}_k(\lambda) \in \mathcal{S}(\mathbb{R}^l, \mathcal{L}(H^s(M), C^\infty(M)))$. The meaning of $\mathrm{Op}(\cdot)$ on the right hand side of (1.31) is $\mathrm{Op}_y(\cdot)$; but the operators can be interpreted as operators on M, more precisely, as operators on Y with symbols having values in operators with kernels in $(\tilde{x},\tilde{x}') \in \mathbb{R}^d \times \mathbb{R}^d$, supported in a tubular neighbourhood of Y, because of the specific structure of the amplitude functions generated by the abovementioned local trace and potential symbols. In addition as for pseudo-differential operators on a closed manifold with scalar symbols the local amplitude functions $c_j^k(y,\eta,\lambda)$ can be arranged in such a way that they behave invariant with respect to symbol push forwards under transition diffeomorphisms with respect to y, modulo symbols of order $-\infty$. This is true for every fixed k. This allows us to form the asymptotic sums

$$ c_j(y,\eta,\lambda) \sim \sum_{k=1}^{\infty} c_j^k(y,\eta,\lambda) \text{ in } S^{-1}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^{q+l}; H^s(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)) $$

which are invariant under transition maps as before and also have the abovementioned property of the support of kernels in $(\tilde{x},\tilde{x}')$. Thus, if we form

$$ {}^{\mathrm{t}}(P_\infty(\lambda) \;\; B_\infty(\lambda)) := {}^{\mathrm{t}}(P(\lambda) \;\; B(\lambda)) \left\{ 1 + \sum_{j=1}^{L} \alpha_j (\chi_j^{-1})_* \mathrm{Op}(c_j)(\lambda)\beta_j \right\} $$

we obtain another version of the relation (1.29), namely

$$(E \quad G(\lambda))\,{}^{\mathrm{t}}(P_\infty(\lambda) \quad B_\infty(\lambda)) = I - C_r(\lambda) \tag{1.32}$$

where now $C_r(\lambda) \in \mathcal{S}(\mathbb{R}^l, \mathcal{L}(H^s(M), C^\infty(M)))$. A similar construction is possible for approximating the inverse from the left. This means that we also have

$${}^{\mathrm{t}}(P_\infty(\lambda) \quad B_\infty(\lambda))(E \quad G(\lambda)) = I - C_l(\lambda) \tag{1.33}$$

for some $C_l(\lambda) \in \mathcal{S}\left(\mathbb{R}^l; \mathcal{L}\left(\begin{array}{c} H_0^s(M,Y) \\ \oplus \\ H^s(Y, \mathbb{C}^{\sigma(s)}) \end{array}, \begin{array}{c} H_0^s(M,Y) \cap C^\infty(M) \\ \oplus \\ C^\infty(Y, \mathbb{C}^{\sigma(s)}). \end{array}\right)\right)$.

Theorem 1.2.9. *Let us fix $s \in \mathbb{R}$, $s > \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{N}$. Then there is a constant $C > 0$ such that the operators* (1.22) *and* (1.23) *are isomorphisms for all $\lambda \in \mathbb{R}^l$, $|\lambda| \geqq C$.*

Proof. From the relation (1.32) we see that the operator (1.22) has a right inverse for large $|\lambda|$. Analogously the relation (1.33) shows us the existence of a left inverse for large $|\lambda|$. This gives us the invertibility of (1.22) for large $|\lambda|$. In a similar manner we can argue for (1.23). $\square$

1.2.2 Edge-representations of differential operators

As in the preceding section, let M be a manifold of dimension m with an embedded manifold Y of dimension q, and let

$$A : H^s(M) \longrightarrow H^{s-\mu}(M) \tag{1.34}$$

be a differential operator, $s - \mu > \frac{d}{2}$ for $d = m - q$. Let us fix $|\lambda| \geqq C$ in such a way that the operators (1.22), (1.23) are isomorphisms, and write the corresponding operators as $(E_s \quad G_s)$ and ${}^{\mathrm{t}}(P_s \quad B_s)$, respectively. The operator (1.34) is then equivalent to the block matrix

$$\mathcal{A}_s := \begin{pmatrix} A_0 & K \\ T & Q \end{pmatrix} := \begin{pmatrix} P_{s-\mu} \\ B_{s-\mu} \end{pmatrix} A(E_s \quad G_s), \tag{1.35}$$

for $A_0 = P_{s-\mu}AE_s$, $T = B_{s-\mu}AE_s$, $K = P_{s-\mu}AG_s$, $Q = B_{s-\mu}AG_s$ and $\mathcal{A}_s$ is continuous as an operator

$$\mathcal{A}_s : \begin{array}{c} H_0^s(M,Y) \\ \oplus \\ H^s(Y, \mathbb{C}^{\sigma(s)}) \end{array} \longrightarrow \begin{array}{c} H_0^{s-\mu}(M,Y) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{C}^{\sigma(s-\mu)}) \end{array}. \tag{1.36}$$

We want to identify $\mathcal{A}_s$ with an operator in the edge algebra on M with respect to the edge Y. To this end we associate with $\mathcal{A}_s$ a principal symbol

$$\sigma(\mathcal{A}_s) = (\sigma_\psi(\mathcal{A}_s), \sigma_\wedge(\mathcal{A}_s)),$$

where $\sigma_\psi(\mathcal{A}_s)$ is the standard homogeneous principal symbol of A_0 of order μ (or of A which is the same), and $\sigma_\wedge(\mathcal{A}_s)$ the so called principal edge symbol which is operator-valued (more details on the edge-calculus will be given below in Section 2.1).
Let us now calculate $\sigma_\wedge(\mathcal{A}_s)$. The function $\sigma_\psi(\mathcal{A}_s) \in C^\infty(T^*M \setminus 0)$ in local coordinates $(\tilde{x}, y) \in \mathbb{R}^d \times \mathbb{R}^q$ with covariables $(\tilde{\xi}, \eta)$ has the form

$$\sigma_\psi(A)(\tilde{x}, y, \tilde{\xi}, \eta) = \sum_{|\alpha|+|\beta|=\mu} a_{\alpha\beta}(\tilde{x}, y)\tilde{\xi}^\alpha \eta^\beta.$$

Freezing coefficients at $\tilde{x} = 0$ and replacing $\tilde{\xi}^\alpha$ by $D^\alpha_{\tilde{x}}$ yields a (y,η)-dependent family of continuous operators

$$\sigma_\wedge(A)(y,\eta) := \sum_{|\alpha|+|\beta|=\mu} a_{\alpha\beta}(0,y) D^\alpha_{\tilde{x}} \eta^\beta : H^s(\mathbb{R}^d) \longrightarrow H^{s-\mu}(\mathbb{R}^d). \tag{1.37}$$

Observe that for $(\kappa_\delta u)(\tilde{x}) := \delta^{\frac{d}{2}} u(\delta \tilde{x})$ we have

$$\sigma_\wedge(A)(y, \delta\eta) = \delta^\mu \kappa_\delta \sigma_\wedge(A)(y,\eta)\kappa_\delta^{-1}$$

for all $\delta \in \mathbb{R}_+$, cf. the formula 1.6.

Proposition 1.2.10. *Let A be elliptic on M in the standard sense. Then* (1.37) *is a family of isomorphisms for all $(y,\eta) \in T^*Y \setminus 0$ and $s \in \mathbb{R}$.*

Proof. Let $(y,\eta) \in \mathbb{R}^q \times \mathbb{R}^q$, $\eta \neq 0$. Then we can write $\sigma_\wedge(A)(y,\eta) = \mathrm{Op}_{\tilde{x}}(p)(y,\eta)$, where $p(y,\tilde{\xi},\eta) = \sum_{|\alpha|+|\beta|=\mu} \tilde{a}_{\alpha\beta}(0,y)\tilde{\xi}^\alpha\eta^\beta$ has constant coefficients with respect to $\tilde{x}$ and

$$p(y,\tilde{\xi},\eta) \neq 0, \text{ for all } \tilde{\xi} \in \mathbb{R}^d, \tag{1.38}$$

since $\sigma_\psi(A)(\tilde{x}, y, \tilde{\xi}, \eta) \neq 0$, for all $(\tilde{\xi},\eta) \neq 0$ and $\eta \neq 0$ in (1.38). It follows then that $p(y,\tilde{\xi},\eta)^{-1} \in S^{-\mu}_{\mathrm{cl}}(\mathbb{R}^d)$ is an elliptic symbol of order $-\mu$. It is now evident that $\sigma_\wedge(A)(y,\eta)^{-1} = \mathrm{Op}_{\tilde{x}}(p^{-1})(y,\eta)$ is the inverse of the map (1.37). $\square$

From the construction of the operators E_s, G_s, $P_{s-\mu}$, $B_{s-\mu}$ we have η-dependent families of isomorphisms $\sigma_\wedge \begin{pmatrix} P_{s-\mu} \\ B_{s-\mu} \end{pmatrix}(\eta)$ and $\sigma_\wedge(E_s \quad G_s)(\eta)$

$$H^{s-\mu}(\mathbb{R}^d) \longrightarrow \begin{matrix} H_0^{s-\mu}(\mathbb{R}^d) \\ \oplus \\ \mathbb{C}^{\sigma(s-\mu)} \end{matrix} \quad \text{and} \quad \begin{matrix} H_0^{s}(\mathbb{R}^d) \\ \oplus \\ \mathbb{C}^{\sigma(s)} \end{matrix} \longrightarrow H^s(\mathbb{R}^d)\ ,$$

respectively, given by

$$\sigma_\wedge \begin{pmatrix} P_{s-\mu} \\ B_{s-\mu} \end{pmatrix}(\eta) = \begin{pmatrix} 1 - \sigma_\wedge(G_{s-\mu})(\eta)\sigma_\wedge(B_{s-\mu})(\eta) \\ \sigma_\wedge(B_{s-\mu})(\eta) \end{pmatrix},$$

and

$$\sigma_\wedge(E_s \quad G_s)(\eta) = (\sigma_\wedge(E_s)(\eta) \quad \sigma_\wedge(G_s)(\eta)),$$

where

$$\sigma_\wedge(G_s)(\eta) = (\sigma_\wedge(k_s^\alpha)(\eta) : |\alpha| < s - \frac{d}{2})$$

and

$$\sigma_\wedge(B_{s-\mu})(\eta) = {}^{\mathrm{t}}(\sigma_\wedge(t_{s-\mu}^\alpha)(\eta) : |\alpha| < s - \mu - \frac{d}{2})$$

for the corresponding symbols

$$k_s^\alpha(\eta) \in S_{\mathrm{cl}}^0(\mathbb{R}^q; \mathbb{C}, H^s(\mathbb{R}^d)),\ t_{s-\mu}^\alpha(\eta) \in S_{\mathrm{cl}}^0(\mathbb{R}^q; H^{s-\mu}(\mathbb{R}^d), \mathbb{C}),$$

as in Section 1.1.3. Moreover, the canonical embedding

$$\sigma_\wedge(E_s)(\eta) : H_0^s(\mathbb{R}^d) \longrightarrow H^s(\mathbb{R}^d)$$

is also an operator-valued symbol of order 0 between the corresponding spaces. We then define

$$\sigma_\wedge(\mathcal{A}_s)(y,\eta) := \sigma_\wedge \begin{pmatrix} P_{s-\mu} \\ B_{s-\mu} \end{pmatrix} (\eta) \sigma_\wedge(A)(y,\eta) \sigma_\wedge(E_s \quad G_s)(\eta),$$

as a family of continuous operators

$$\sigma_\wedge(\mathcal{A}_s)(y,\eta) : \begin{array}{c} H_0^s(\mathbb{R}^d) \\ \oplus \\ \mathbb{C}^{\sigma(s)} \end{array} \longrightarrow \begin{array}{c} H_0^{s-\mu}(\mathbb{R}^d) \\ \oplus \\ \mathbb{C}^{\sigma(s-\mu)} \end{array}, \tag{1.39}$$

which is homogeneous of order μ in the sense

$$\sigma_\wedge(\mathcal{A}_s)(y,\delta\eta) = \delta^\mu \begin{pmatrix} \kappa_\delta & 0 \\ 0 & 1 \end{pmatrix} \sigma_\wedge(\mathcal{A}_s)(y,\eta) \begin{pmatrix} \kappa_\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

for all $\delta \in \mathbb{R}_+$.

Corollary 1.2.11. *Let A be elliptic on M. Then (1.39) is a family of isomorphisms for all $(y,\eta) \in T^*Y \setminus 0$ and $s \in \mathbb{R}$, $s - \mu > \frac{d}{2}$.*

2 Parametrices in the edge calculus

2.1 Edge calculus

2.1.1 Manifolds with edges and edge-degenerate symbols

By a manifold W with edge Y we understand a topological space such that $W \setminus Y$ and Y are C^∞ manifolds. In addition we assume that every $y \in Y$ has a neighbourhood V which is homeomorphic to a wedge $X^\triangle \times \Omega$, where $\Omega \subseteq \mathbb{R}^q$ is an open set, $q = \dim Y$, and $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$, the cone with a closed compact C^∞ manifold X as base, $n = \dim X$. We further assume that the homeomorphism $V \to X^\Delta \times \Omega$ restricts to diffeomorphisms $V \cap Y \to \Omega$ and $V \setminus Y \to X^\wedge \times \Omega$ for $X^\wedge := \mathbb{R}_+ \times X$, and the transition maps $X^\wedge \times \Omega \to X^\wedge \times \Omega$ for different (admitted) choices of diffeomorphisms of the latter kind are required to be restrictions of diffeomorphisms

$$\overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \Omega \tag{2.1}$$

to $X^\wedge \times \Omega$. The latter conditions give rise to a C^∞ manifold $\mathbb{W}$ with boundary $\partial\mathbb{W}$ such that $\mathbb{W} \setminus \partial\mathbb{W}$ is an X-bundle over Y, and diffeomorphisms $V \longrightarrow X^\Delta \times \Omega$ correspond to diffeomorphisms

$$\chi : \mathbb{V} \longrightarrow \overline{\mathbb{R}}_+ \times X \times \Omega \tag{2.2}$$

from corresponding open subsets $\mathbb{V} \subset \mathbb{W}$ which intersect $\partial\mathbb{W}$. A 'singular chart' (2.2) on $\mathbb{W}$ gives us a splitting of variables (r, x, y), and the restrictions of the transition maps for the system of charts (2.2) to $r = 0$ just define the transition maps

$$X \times \Omega \longrightarrow X \times \Omega \tag{2.3}$$

for the X-bundle $\partial\mathbb{W}$.
We call $\mathbb{W}$ the stretched manifold of W, and we set

$$\mathbb{W}_{\mathrm{reg}} := \mathbb{W} \setminus \partial\mathbb{W}, \quad \mathbb{W}_{\mathrm{sing}} := \partial\mathbb{W}. \tag{2.4}$$

Example 2.1.1. *As in the beginning of Section* 1.1.2 *we can interpret the Euclidean space* $\mathbb{R}^{1+n+q}$ *as a manifold* W *with edge* $\{0\} \times \mathbb{R}^q$ *and model cone* $\mathbb{R}^{1+n} = (S^n)^\Delta$. *Then the stretched manifold is equal to* $\overline{\mathbb{R}}_+ \times S^n \times \mathbb{R}^q$.

A manifold W with edge Y can be equivalently defined as the quotient space $\mathbb{W}/\sim$, starting from an arbitrary C^∞ manifold with boundary $\partial\mathbb{W}$ which is an X-bundle over Y for a compact closed C^∞ manifold X. The equivalence relation $\sim$ means the contraction of fibres over any $y \in Y$ to the point y. It is then possible to choose a collar neighbourhood $\cong [0,1) \times \partial\mathbb{W}$ of $\partial\mathbb{W}$ such that the transition diffeomorphisms in that neighbourhood are independent of r for $0 \leqq r < \varepsilon$ for some $\varepsilon > 0$. This gives us the following observation:

Remark 2.1.2. *We find representations of the maps* (2.2) *such that the functions*

$$(\tilde{r}(r,x,y),\ \tilde{x}(r,x,y),\ \tilde{y}(r,x,y)) \tag{2.5}$$

belonging to the transition maps (2.1) *are independent of* r *for* $0 \leqq r < \varepsilon$. *In particular* $x \to \tilde{x}(r,x,y)$ *gives us the* y*-dependent family of diffeomorphisms* $X \longrightarrow X$ *for* $0 \leqq r < \varepsilon$ *which belongs to the transition maps* (2.3) *for* $\partial\mathbb{W}$.

The connection of our considerations to manifolds with edges is that we interpret the C^∞ manifold M with the embedded manifold Y as a manifold W with edge

Y. In this case we have $X = S^n$ when $d = n+1$ is the codimension of Y in M, and $X^\triangle \cong \mathbb{R}^d$ can be interpreted as the fiber of the normal bundle of Y in M.
In local coordinates near a point of Y we have a splitting of variables into $(\tilde{x}, y) \in \mathbb{R}^d \times \Omega$, $\Omega \subseteqq \mathbb{R}^q$ open, and in $\mathbb{R}^d \setminus \{0\}$ we introduce polar coordinates (r, x). Then the operator (1.34) in local coordinates takes the form (1.3) for coefficients $a_{j\beta}(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\,\mu-(j+|\beta|)}(X))$. If we also introduce local coordinates on X, varying in an open set $\Sigma \subseteqq \mathbb{R}^n_x$, the operator (1.3) has an amplitude function of the form $r^{-\mu} p(r, x, y, \rho, \xi, \eta)$ for

$$p(r, x, y, \rho, \xi, \eta) = \tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta})\big|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta}, \tag{2.6}$$

where $\tilde{p}$ is a polynomial in $(\tilde{\rho}, \xi, \tilde{\eta})$ of order μ with coefficients smooth up to $r = 0$.

Definition 2.1.3. *A symbol $p(r, x, y, \rho, \xi, \eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$ is called edge-degenerate, if it can be written in the form* (2.6) *for some $\tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}_{\tilde{\rho},\xi,\tilde{\eta}})$.*

With $p(r, x, y, \rho, \xi, \eta)$ we associate pseudo-differential operators on $\mathbb{R}_+ \times \Sigma$, depending on the variables $(y, \eta) \in \Omega \times \mathbb{R}^q$. We set

$$\mathrm{op}_{r,x}(p)(y, \eta) u(r, x) = \int\!\!\int e^{i(r-r')\rho + i(x-x')\xi} p(r, x, y, \rho, \xi, \eta) u(r', x') dr' dx' đ\rho đ\xi,$$

which gives us a family of continuous maps

$$\mathrm{op}_{r,x}(p)(y, \eta) : C^\infty_0(\mathbb{R}_+ \times \Sigma) \longrightarrow C^\infty(\mathbb{R}_+ \times \Sigma).$$

To establish a calculus of operators in a suitable scale of weighted Sobolev spaces we now formulate the Mellin operator convention ('Mellin quantisation') with respect to the variable $r \in \mathbb{R}_+$. To this end we consider symbols that are holomorphic in the complex variable $z \in \mathbb{C}$. If F is a Fréchet space and $U \subseteqq \mathbb{C}$ an open set, by $\mathcal{A}(U, F)$ we denote the space of all holomorphic functions on U with values in F.

Definition 2.1.4. *Let $S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ denote the space of all*

$$h(r, x, y, z, \xi, \eta) \in \mathcal{A}(\mathbb{C}_z, C^\infty(\mathbb{R}_+ \times \Sigma \times \Omega \times \mathbb{R}^{n+q}))$$

such that $h(r, x, y, \beta + i\rho, \xi, \eta) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}_{\rho,\xi,\eta})$ for every $\beta \in \mathbb{R}$, uniformly in $c \leqq \beta \leqq c'$ for every $c \leqq c'$.

Theorem 2.1.5. *Let $p(r, x, y, \rho, \xi, \eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$ be an edge-degenerate symbol in the sense of Definition* 2.1.3. *Then there exists an $\tilde{h}(r, x, y, z, \xi, \tilde{\eta}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q}_{\xi,\tilde{\eta}})$ such that for*

$$h(r, x, y, z, \xi, \eta) := \tilde{h}(r, x, y, z, \xi, r\eta)$$

we have

$$\mathrm{op}_{r,x}(p)(y, \eta) = \mathrm{op}^\gamma_M \mathrm{op}_x(h)(y, \eta) \bmod C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+ \times \Sigma; \mathbb{R}^q)) \tag{2.7}$$

for every $\gamma \in \mathbb{R}$. The element $\tilde{h}$ is uniquely determined by p, modulo an element in $S^{-\infty}(\mathbb{R}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$.

The idea of the relation (2.7) is to replace the pseudo-differential action in $r \in \mathbb{R}_+$ based on the Fourier transform by a corresponding action based on the Mellin transform, under a corresponding change $p \to h$ of amplitude functions. Incidentally, this process will be referred to as a Mellin operator convention. Theorem 2.1.5 is to be understood first as a relation between families of pseudo-differential on $\mathbb{R}_+ \times \Sigma$ as continuous operators $C_0^\infty(\mathbb{R}_+ \times \Sigma) \longrightarrow C^\infty(\mathbb{R}_+ \times \Sigma)$; then (2.7) is valid for all $\gamma \in \mathbb{R}$. However, the expression in Mellin terms admits an extension to weighted Sobolev spaces; in this moment γ becomes fixed.

2.1.2 The global Mellin operator convention

Let X be a closed compact C^∞ manifold, $n = \dim X$, and choose an atlas of charts $\chi_j : U_j \longrightarrow \mathbb{R}^n$, $j = 1, ..., N$. Let $\{\varphi_j\}_{j=1,\dots,N}$ be a partition of unity on X subordinate to $\{U_1, ..., U_N\}$, and let $\{\psi_j\}_{j=1,\dots,N}$ be a system of functions $\psi_j \in C_0^\infty(U_j)$ such that $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$, for all j.
Consider the space $L^{-\infty}(X)$ of all smoothing operators on X, i.e., operators $Cu(x) = \int_X c(x, x')u(x')dx'$ for kernels $c \in C^\infty(X \times X)$, where dx refers to a Riemannian metric on X. The space $L^{-\infty}(X)$ is Fréchet via its bijection to $C^\infty(X \times X)$, and we set $L^{-\infty}(X; \mathbb{R}^q) := \mathcal{S}(\mathbb{R}^q, L^{-\infty}(X))$. Moreover, let $p_j(x, \xi, \eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^{n+q}_{\xi,\eta})$, $j = 1, ..., N$, be a system of local amplitude functions. We then define parameter-dependent pseudo-differential operators $\mathrm{op}_x(p_j)(\eta)$ in $\mathbb{R}^n$ and form the η-wise operator pull backs $P_j(\eta) := (\chi_j^{-1})_* \mathrm{op}_x(p_j)(\eta)$ to the manifold X. Then $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q)$ is defined as the set of all $P(\eta) = \sum_{j=1}^N \varphi_j P_j(\eta)\psi_j + C(\eta)$ for arbitrary $P_j(\eta)$ as mentioned before, and $C(\eta) \in L^{-\infty}(X; \mathbb{R}^q)$. Note that $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q)$ is a Fréchet space in a natural way.

Definition 2.1.6. *Let $\mathcal{M}^\mu_{\mathcal{O}}(X; \mathbb{R}^q)$ denote the subspace of all elements*

$$h(z, \eta) \in \mathcal{A}(\mathbb{C}_z, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q_\eta))$$

such that $h(\beta + i\rho, \eta) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}_\rho \times \mathbb{R}^q_\eta)$ for every $\beta \in \mathbb{R}$, uniformly in $c \leqq \beta \leqq c'$, for every $c \leqq c'$. For $q = 0$ we simply write $\mathcal{M}^\mu_{\mathcal{O}}(X)$.

As a consequence of the definition we have in $\mathcal{M}^\mu_{\mathcal{O}}(X; \mathbb{R}^q)$ a natural semi-norm system which makes it to a Fréchet space.

Theorem 2.1.7. *Let $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$, $\Omega \subseteq \mathbb{R}^q$ open, and set $p(r, y, \rho, \eta) := \tilde{p}(r, y, \tilde{\rho}, \tilde{\eta})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta}$. Then there exists an element $\tilde{h}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathcal{M}^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}))$ such that $h(r, y, z, \eta) := \tilde{h}(r, y, z, \tilde{\eta})|_{\tilde{\eta}=r\eta}$ satisfies the relation*

$$\mathrm{op}_r(p)(y, \eta) = \mathrm{op}^\gamma_M(h)(y, \eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta))$$

for all $\gamma \in \mathbb{R}$*, and* $\tilde{h}(r, y, z, \tilde{\eta})$ *is uniquely defined* $\bmod C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathcal{M}_{\mathcal{O}}^{-\infty}(X; \mathbb{R}^q_{\tilde{\eta}}))$ *by* $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta})$*. For* $p_0(r, y, \rho, \eta) := \tilde{p}(0, y, r\rho, r\eta)$ *and* $h_0(r, y, z, \eta) := \tilde{h}(0, y, z, r\eta)$ *it follows that* $\mathrm{op}_r(p_0)(y, \eta) = \mathrm{op}_M^\gamma(h_0)(y, \eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta))$ *for all* $\gamma \in \mathbb{R}$.

Proof. The entire proof is rather long. In order to make the specific nature of our Mellin-edge representations transparent, we sketch the main ideas of the proof; more details can be found in [6] (there are in fact different proofs which point out other aspects of what we call Mellin operator conventions, cf. [7]).
In the sequel we write $\sim$ if an equality holds $\bmod C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta))$. First we prove that for any $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$ there is an $\tilde{f}_0(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \Gamma_0 \times \mathbb{R}^q_{\tilde{\eta}}))$ such that

$$\mathrm{op}_r(p)(y, \eta) \sim \mathrm{op}_M^{\frac{1}{2}}(f_0)(y, \eta) + \mathrm{op}_r(p_1)(y, \eta)$$

where $f_0(r, y, z, \eta) := \tilde{f}_0(r, y, z, r\eta)$ and $p_1(r, y, \rho, \eta) = \tilde{p}_1(r, y, r\rho, r\eta)$ for an $\tilde{p}_1(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{\mu-1}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$. In fact, $\tilde{f}_0(r, y, z, \tilde{\eta})$ will be of the form $\tilde{f}_0(r, y, i\rho, \tilde{\eta}) = \tilde{p}(r, y, -\rho, \tilde{\eta})$.
By iterating this procedure we obtain a sequence $\tilde{f}_j(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{\mu-j}_{\mathrm{cl}}(X; \Gamma_0 \times \mathbb{R}^q_{\tilde{\eta}}))$ for all $j \in \mathbb{N}$ such that

$$\mathrm{op}_r(p)(y, \eta) \sim \sum_{j=0}^{L} \mathrm{op}_M^{\frac{1}{2}}(f_j)(y, \eta) + \mathrm{op}_r(p_{L+1})(y, \eta)$$

for $p_{L+1}(r, y, \rho, \eta) := \tilde{p}_{L+1}(r, y, r\rho, r\eta)$ with

$$\tilde{p}_{L+1}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{\mu-L-1}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})).$$

The asymptotic sum

$$\tilde{f}(r, y, z, \tilde{\eta}) := \sum_{j=0}^{\infty} \tilde{f}_j(r, y, z, \tilde{\eta}) \text{ in } C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \Gamma_0 \times \mathbb{R}^q_{\tilde{\eta}}))$$

then gives us an $f(r, y, z, \eta) = \tilde{f}(r, y, z, r\eta)$ such that

$$\mathrm{op}_r(p)(y, \eta) \sim \mathrm{op}_M^{\frac{1}{2}}(f)(y, \eta).$$

Now, by a kernel cut-off argument it follows that to every $\tilde{f}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \Gamma_0 \times \mathbb{R}^q_{\tilde{\eta}}))$ there is an $\tilde{h}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathcal{M}^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}))$ such that $h(r, y, z, \eta) := \tilde{h}(r, y, z, r\eta)$ satisfies

$$\mathrm{op}_M^{\frac{1}{2}}(h)(y, \eta) \sim \mathrm{op}_M^{\frac{1}{2}}(f)(y, \eta).$$

Finally, in view of the holomorphy of h in z, we obtain as a consequence of the Cauchy theorem that $\mathrm{op}_M^{\frac{1}{2}}(h)(y, \eta) = \mathrm{op}_M^\gamma(h)(y, \eta)$ on functions with compact support in $r \in \mathbb{R}_+$, for arbitrary $\gamma \in \mathbb{R}$. $\square$

2.1.3 Edge amplitude functions from the interior

Let

$$(p(r,y,\rho,\eta), h(r,y,z,\eta))$$

be a pair of operator families as in Theorem 2.1.7, related to each other by the Mellin operator convention. If $\omega(r)$, $\tilde{\omega}(r)$ are cut-off functions, we write $\omega \prec \tilde{\omega}$ if $\tilde{\omega} \equiv 1$ on $\operatorname{supp}\omega$. Given a cut-off function $\omega = \omega(r)$ we pass to η-dependent cut-off functions $\omega_\eta(r) := \omega(r[\eta])$, where $\eta \to [\eta]$ was defined in Section 1.1.2. Then $\omega \prec \tilde{\omega}$ implies $\omega_\eta \prec \tilde{\omega}_\eta$ for all $\eta \in \mathbb{R}^q$.
Let us now choose cut-off functions ω, $\tilde{\omega}$, $\tilde{\tilde{\omega}}$ such that

$$\omega \prec \tilde{\omega} \text{ and } \tilde{\tilde{\omega}} \prec \omega\,. \tag{2.8}$$

Then $\mathrm{op}_r(p)(y,\eta) \sim \mathrm{op}_M^\gamma(h)(y,\eta)$ implies that the following result holds:

Lemma 2.1.8. *Given (p,h) and cut-off functions ω, $\tilde{\omega}$, $\tilde{\tilde{\omega}}$ satisfying the relation* (2.8)*, we have*

$$\omega_\eta \mathrm{op}_M^\gamma(h)(y,\eta)\tilde{\omega}_\eta + (1-\omega_\eta)\mathrm{op}_r(p)(y,\eta)(1-\tilde{\tilde{\omega}}_\eta) \sim \mathrm{op}_r(p)(y,\eta)$$

for every $\gamma \in \mathbb{R}$.

Proposition 2.1.9. *For every $h(r,y,z,\eta) = \tilde{h}(r,y,z,\tilde{\eta})|_{\tilde{\eta}=r\eta}$ with $\tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathcal{M}^\mu_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}}))$ the operator family $\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\tilde{\omega}_\eta$ represents an element in $S^0(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma}(X^\wedge))$ for every s, $\gamma \in \mathbb{R}$.*

This result is an easy consequence of the smoothness in (y,η) as an $\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma}(X^\wedge))$-valued function together with the homogeneity of order 0 for large $|\eta|$ when $\tilde{h}$ is independent of r, otherwise combined with a simple tensor product argument.
The following Theorem 2.1.10 is crucial for the structure of the edge calculus. In the global calculus on $\mathbb{W}$ below we will refer to a generalisation to the case of non-trivial X-bundle $\partial\mathbb{W}$ (cf. the notation in connection with (2.3)). Therefore, we give the details of the proof; it employs ideas from [16]. One of the tools are pseudo-differential operators on $X^\wedge$ interpreted as a manifold with conical exit to infinity $r \to \infty$; the general background may be found in [18]. The calculus of such exit operators goes back to Parenti [14] and Cordes [2]. Other authors then developed (partly independently) the calculus in different generality, see Schrohe [15] or Chapter 3 in [10].
Let $L^{\mu;0}(X^\wedge)$ denote the space of all pseudo-differential operators of order μ on $X^\wedge$ in this exit calculus of order zero in $r \to \infty$ where r is treated as an additional covariable in the symbolic estimates. This space is Fréchet in a natural way. If E is a Fréchet space that is a (two-sided) module over an algebra A we denote by $[a]E$ ($E[b]$, $[a]E[b]$) the closure of the set of all ae (eb, aeb) for all $e \in E$ in the space E.

Theorem 2.1.10. *Let $\sigma(r)$, $\tilde{\sigma}(r)$ be arbitrary cut-off functions, and set*

$$b(y,\eta) := r^{-\mu}\sigma\left\{\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\tilde{\omega}_\eta + (1-\omega_\eta)\mathrm{op}_r(p)(y,\eta)(1-\tilde{\tilde{\omega}}_\eta)\right\}\tilde{\sigma}. \quad (2.9)$$

Then $b(y,\eta)$ as a C^∞ family (in $(y,\eta) \in \Omega \times \mathbb{R}^q$) of continuous operators $\mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$ defines an element

$$b(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)).$$

for every $s \in \mathbb{R}$.

Proof. First note that the aspect of dependence on $y \in \Omega$ will not affect the proof in an essential way. So we simply omit y and focus on the typical dependence of operators functions on η. Let us write

$$b_0(\eta) = r^{-\mu}\sigma\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}_\eta\tilde{\sigma}, \;\; b_1(\eta) = r^{-\mu}\sigma(1-\omega_\eta)\mathrm{op}_r(p)(\eta)(1-\tilde{\tilde{\omega}}_\eta)\tilde{\sigma}.$$

We then have $b(\eta) = b_0(\eta) + b_1(\eta)$, and it suffices to consider the summands separately. For $b_0(\eta)$ we apply Proposition 2.1.9 together with the fact that $b_0(\eta)$ can be written in the form $b_0(\eta) = \sigma\tilde{\omega}_\eta r^{-\mu}\left\{\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}_\eta\right\}\tilde{\sigma}$. We have

$$\sigma\tilde{\omega}_\eta r^{-\mu} \in S^\mu(\mathbb{R}^q; \mathcal{K}^{s-\mu,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)),\; \tilde{\sigma} \in S^0(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$$

for every $s, \gamma \in \mathbb{R}$, and then $b_0(\eta)$ has the required property, see Remark 1.1.6. Concerning $b_1(\eta)$ we decompose the result into different observations that will be checked afterwards in more detail. We have

$$b_1(\eta) \in C^\infty(\mathbb{R}^q_\eta, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)), \quad (2.10)$$

for every s, $\gamma \in \mathbb{R}$. In fact, $\mathrm{op}_r(p)(\eta)$ is a C^∞ family of continuous operators

$$\mathrm{op}_r(p)(\eta) : H^s_{\mathrm{comp}}(X^\wedge) \longrightarrow H^{s-\mu}_{\mathrm{loc}}(X^\wedge),$$

for all s. Since the operators of multiplication

$$\tilde{\sigma}(1-\tilde{\tilde{\omega}}_\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow H^s_{\mathrm{comp}}(X^\wedge),$$

and

$$\sigma r^{-\mu}(1-\omega_\eta) : H^{s-\mu}_{\mathrm{loc}}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge),$$

both represent C^∞ families of continuous operators between the respective spaces, for all s, $\gamma \in \mathbb{R}$, we immediately obtain (2.10).
Choose any excision function $\chi(\eta)$ and write $b_1(\eta) = \chi(\eta)b_1(\eta) + (1-\chi(\eta))b_1(\eta)$. Then, from (2.10) it follows that

$$(1-\chi(\eta))b_1(\eta) \in C_0^\infty(\mathbb{R}^q_\eta, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))),$$

and hence

$$(1-\chi(\eta))b_1(\eta) \in S^{-\infty}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)),$$

for all s, $\gamma \in \mathbb{R}$.
It remains to consider $\chi(\eta)b_1(\eta)$. As we know, the operators of multiplication by σ or $\tilde{\sigma}$ represent operator-valued symbols. Thus, it suffices to study the operator function

$$b_2(\eta) = \chi(\eta)r^{-\mu}(1-\omega_\eta)\mathrm{op}_r(p)(\eta)(1-\tilde{\tilde{\omega}}_\eta). \tag{2.11}$$

Because of the factors σ or $\tilde{\sigma}$ in the original operator function, without loss of generality, we take p as a double symbol in r, $r' \in \mathbb{R}_+$ vanishing for $r > c$ or $r' > c$. In other words, p may assumed to be of the form $p(r, r', \rho, \eta) = \tilde{p}(r, r', r\rho, r\eta)$, for $\tilde{p}(r, r', \tilde{\rho}, \tilde{\eta}) \in C_0^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$.
A standard tensor product argument tells us that there is a representation

$$\tilde{p}(r, r', \tilde{\rho}, \tilde{\eta}) = \sum_{j=0}^\infty \lambda_j \varphi_j(r)\tilde{p}_j(\tilde{\rho}, \tilde{\eta})\psi_j(r') \tag{2.12}$$

convergent in the Fréchet subspace of all elements in $C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$ that are supported by the set $\left\{(r, r') \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ : 0 \leqq r \leqq c,\, 0 \leqq r' \leqq c\right\}$, where $(\lambda_j)_{j\in\mathbb{N}} \subset \mathbb{C}$ is a sequence with $\sum_{j=0}^\infty |\lambda_j| < \infty$, φ_j and ψ_j are null-sequences in $C^\infty([0, c)_0)$ (the latter space denotes the subspace of all elements of $C^\infty([0, c])$ vanishing of infinite order at c), and $\tilde{p}_j(\tilde{\rho}, \tilde{\eta})$ is a null-sequence in the space $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$. The operator of multiplication by some $\varphi \in C^\infty([0, c)_0)$ represents an element in our symbol space, in fact, it defines continuous maps

$$C^\infty([0, c)_0) \longrightarrow S^0(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge)),$$

for all s, $\gamma \in \mathbb{R}$.
We now look at the symbol $\tilde{p}_j(\tilde{\rho}, \tilde{\eta})$ in the middle of the summands on the right hand side of (2.12). In order to show that

$$\chi(\eta)(1-\omega_\eta)r^{-\mu}\mathrm{op}_r(p)(\eta)(1-\tilde{\tilde{\omega}}_\eta) = \sum_{j=0}^\infty \lambda_j\varphi_j\chi(\eta)(1-\omega_\eta)r^{-\mu}\mathrm{op}_r(p_j)(\eta)(1-\tilde{\tilde{\omega}}_\eta)\psi_j,$$

for $p_j(r, \rho, \eta) = \tilde{p}_j(r\rho, r\eta)$ converges in the space $S^\mu(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ it suffices to show that

$$c_j(\eta) := \chi(\eta)(1-\omega_\eta)r^{-\mu}\mathrm{op}_r(p_j)(\eta)(1-\tilde{\tilde{\omega}}_\eta) \tag{2.13}$$

is a C^∞ family of elements in $\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ which tends to zero for $j \to \infty$ in the operator norm, uniformly for $|\eta| < R$ for some sufficiently large $R > 0$. In fact, because of $c_j(\lambda\eta) = \lambda^\mu\kappa_\lambda c_j(\eta)\kappa_\lambda^{-1}$ for all $\lambda \geqq 1$, $|\eta| \geqq R$ for a suitable choice of R it follows then that $c_j(\eta) \in S^\mu(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$

tends to zero for $j \to \infty$ in that symbol space.
To complete the proof we show the required properties of (2.13). For abbreviation, let us omit subscript j, and fix for the moment $\eta \neq 0$. We then have to observe, in particular, that

$$c(\eta) := \chi(\eta)(1-\omega_\eta)r^{-\mu}\mathrm{op}_r(p)(\eta)(1-\tilde{\tilde{\omega}}_\eta) \in \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)),$$

for $p(r,\rho,\eta) = \tilde{p}(r\rho, r\eta)$ and that

$$L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}) \ni \tilde{p} \to c \in \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

is continuous.
We use the fact that $\mathrm{supp}(1-\omega_\eta)$ and $\mathrm{supp}(1-\tilde{\tilde{\omega}}_\eta)$ are contained in a half-axis like $[\alpha,\infty)$ for some $\alpha > 0$ (independent of η), which allows us to forget about γ and to replace everywhere $\mathcal{K}^{s,\gamma}(X^\wedge)$ by $H^s_{\mathrm{cone}}(X^\wedge)$.
From the exit calculus we know that the map $[\vartheta]L^{\mu;0}(X^\wedge)[\tilde{\vartheta}] \longrightarrow \mathcal{L}([\vartheta]H^s_{\mathrm{cone}}(X^\wedge), [\vartheta]H^{s-\mu}_{\mathrm{cone}}(X^\wedge))$ is continuous for every pair of excision functions $\vartheta(r), \tilde{\vartheta}(r)$ (i.e., vanishing for $r \leqq \alpha_0$ and $\vartheta(r) = 1$ for $r \geqq \alpha_1$) such that $\tilde{\vartheta} \equiv 1$ on $\mathrm{supp}\,\vartheta$. Moreover, the map $\tilde{p}(\tilde{\rho},\tilde{\eta}) \longrightarrow c(\eta)$ defines a continuous operator $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}) \longrightarrow [\vartheta]L^{\mu;0}(X^\wedge)[\tilde{\vartheta}]$, for fixed η, where $\vartheta(r) := \chi(\eta)(1-\omega_\eta(r))$ and $\tilde{\vartheta}(r) := \tilde{\chi}(\eta)(1-\tilde{\omega}_\eta(r))$ for another excision function $\tilde{\chi}(\eta)$ that is equal to 1 on $\mathrm{supp}\,\chi$. The latter assertions are true uniformly in η varying on a compact set. This completes the proof. $\square$

Remark 2.1.11. *Let us set with the notation of Theorem* 2.1.7

$$\begin{aligned} \sigma^{\mu-j}_\wedge(b)(y,\eta) &= r^{-\mu+j}\omega(r|\eta|)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(y,\eta)\tilde{\omega}(r|\eta|) \\ &+ r^{-\mu+j}(1-\omega(r|\eta|))\mathrm{op}_r(p_0)(y,\eta)(1-\tilde{\tilde{\omega}}(r|\eta|)) \end{aligned}$$

for every $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$. *Then*

$$\sigma^{\mu-j}_\wedge(b)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu+j,\gamma-\mu+j}(X^\wedge)$$

is a family of continuous operators and

$$\sigma^{\mu-j}_\wedge(b)(y,\delta\eta) = \delta^{\mu-j}\kappa_\delta\sigma^{\mu-j}_\wedge(b)(y,\eta)\kappa^{-1}_\delta$$

for all $\delta \in \mathbb{R}_+$ *and* $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$, $s \in \mathbb{R}$.

2.1.4 Edge amplitude functions of Green type

This section develops some necessary material on another category of amplitude functions that encodes (in analogy to boundary value problems) Green's functions as well as operators of trace and potential type with respect to the edge.
Let us form the Fréchet spaces $\mathcal{K}^{\infty,\gamma}(X^\wedge) := \varprojlim_{m\in\mathbb{N}} \mathcal{K}^{m,\gamma}(X^\wedge)$ and

$$S^\gamma_\varepsilon(X^\wedge) := \varprojlim_{m\in\mathbb{N}} \langle r\rangle^{-m}\mathcal{K}^{m,\gamma+\varepsilon-(1+m)^{-1}}(X^\wedge).$$

Remark 2.1.12. *The Fréchet spaces $\mathcal{K}^{\infty,\gamma}(X^\wedge)$, $\mathcal{S}^\gamma_\varepsilon(X^\wedge)$ can be considered with the group action $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, $(\kappa_\delta u)(r,x) = \delta^{\frac{n+1}{2}}u(\delta r, x)$, $\delta \in \mathbb{R}_+$, for $n = \dim X$. More precisely, if E is one of these spaces, we find a sequence of Hilbert spaces E^j, $j \in \mathbb{N}$, with continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all j and $E = \varprojlim_{j\in\mathbb{N}} E^j$, such that $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ induces a strongly continuous group of isomorphisms on E^j for all j.*

Definition 2.1.13. *An operator function*

$$g(y,\eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}, \mathcal{K}^{\infty,\beta}(X^\wedge) \oplus \mathbb{C}^{j_+})),$$

for some $s \in \mathbb{R}$ and $j_\pm \in \mathbb{N}$ is called a Green symbol of order $\mu \in \mathbb{R}$ (in the local edge calculus, with weights γ, $\beta \in \mathbb{R}$) if there is an $\varepsilon = \varepsilon(g) > 0$ such that

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}, \mathcal{S}^\beta_\varepsilon(X^\wedge) \oplus \mathbb{C}^{j_+}),$$

and

$$g^*(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\beta}(X^\wedge) \oplus \mathbb{C}^{j_+}, \mathcal{S}^{-\gamma}_\varepsilon(X^\wedge) \oplus \mathbb{C}^{j_-}),$$

for every $s \in \mathbb{R}$. The pointwise adjoint '$*$' refers to $(u, g^*v)_{\mathcal{K}^{0,0}(X^\wedge)\oplus\mathbb{C}^{j_-}} = (gu, v)_{\mathcal{K}^{0,0}(X^\wedge)\oplus\mathbb{C}^{j_+}}$ for all $u \in C_0^\infty(X^\wedge) \oplus \mathbb{C}^{j_-}$, $v \in C_0^\infty(X^\wedge) \oplus \mathbb{C}^{j_+}$.

Observe that there are useful (pointwise) kernel characterisations of Green symbols, cf. Seiler [22] which refer here to the individual weight intervals of width $\varepsilon > 0$.
Set $\boldsymbol{g} = (\gamma, \beta)$, and let $\mathcal{R}^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+)$ denote the space of all Green symbols. As classical symbols, the Green symbols $g(y,\eta)$ have a unique sequence $g_{(\mu-j)}(y,\eta)$ of components of homogeneity $\mu - j$ for all $j \in \mathbb{N}$. Let us set

$$\sigma^{\mu-j}_\wedge(g)(y,\eta) = g_{(\mu-j)}(y,\eta). \tag{2.14}$$

In particular, as introduced before in the abstract context, for $j = 0$ we have the homogeneous principal symbol of the Green symbol $g(y,\eta)$,

$$\sigma^\mu_\wedge(g)(y,\eta) \in C^\infty(T^*\Omega \setminus 0; \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}, \mathcal{S}^\beta_\varepsilon(X^\wedge) \oplus \mathbb{C}^{j_+})),$$

satisfying the relation $\sigma^\mu_\wedge(g)(y,\delta\eta) = \delta^\mu \mathrm{diag}(\kappa_\delta, 1)\sigma^\mu_\wedge(g)(y,\eta)\mathrm{diag}(\kappa_\delta^{-1}, 1)$ for all $\delta \in \mathbb{R}_+$.

2.1.5 Smoothing symbols of Mellin type

Let us fix $\beta \in \mathbb{R}$, set $S_\varepsilon(\beta) := \{z \in \mathbb{C} : \beta - \varepsilon < \mathrm{Re}(z) < \beta + \varepsilon\}$, and let $M^{-\infty}_\varepsilon(X)_\beta$ for any $\varepsilon > 0$ denote the space of all $f \in \mathcal{A}(S_\varepsilon(\beta), L^{-\infty}(X))$ such that $f(\alpha + i\rho) \in$

$L^{-\infty}(X;\mathbb{R}_\rho)$ for every $\alpha \in \mathbb{R}$, $\alpha \in (\beta-\varepsilon, \beta+\varepsilon)$, uniformly in compact subintervals of $(\beta-\varepsilon, \beta+\varepsilon)$. Let us set

$$M^{-\infty}(X)_\beta := \bigcup_{\varepsilon>0} M_\varepsilon^{-\infty}(X)_\beta.$$

We use this, in particular, for $\beta = \frac{n+1}{2} - \gamma$ for some weight $\gamma \in \mathbb{R}$, assume $f(y,z) \in C^\infty(\Omega, M^{-\infty}(X)_{\frac{n+1}{2}-\gamma})$, and formulate smoothing Mellin pseudo-differential operators $\omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(y)\tilde{\omega} : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{\infty,\gamma}(X^\wedge)$ which are continuous for all $s \in \mathbb{R}$, for any choice of cut-off functions ω, $\tilde{\omega}$.

Remark 2.1.14. *Let $f(y,z) \in C^\infty(\Omega, M^{-\infty}(X)_{\frac{n+1}{2}-\gamma})$, and let ω, $\tilde{\omega}$ be arbitrary cut-off functions. Then the operator family*

$$m(y,\eta) := r^{-\mu+j}\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(y)\eta^\alpha\tilde{\omega}_\eta \tag{2.15}$$

for every $\mu \in \mathbb{R}$, $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^q$, $|\alpha| \leqq j$, represents an element

$$m(y,\eta) \in S_{\mathrm{cl}}^{\mu-(j-|\alpha|)}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)),$$

for all $s \in \mathbb{R}$, and for $j > 0$ we have $m(y,\eta) \in \mathcal{R}_G^{\mu-(j-|\alpha|)}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$, for $\boldsymbol{g} = (\gamma, \gamma-\mu)$.

Note that the homogeneous principal symbol of (2.15) of order $\mu - (j - |\alpha|)$ is equal to

$$\sigma_\wedge^{\mu-(j-|\alpha|)}(m)(y,\eta) = r^{-\mu+j}\omega(r|\eta|)\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(y)\eta^\alpha\tilde{\omega}(r|\eta|). \tag{2.16}$$

Let $\mathcal{R}_{M+G}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+)$ denote the set of all operator-valued symbols of the form

$$c(y,\eta) = \begin{pmatrix} m(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + g(y,\eta),$$

for arbitrary $m(y,\eta)$ of the form (2.15) for $j=0$ and $g(y,\eta) \in \mathcal{R}_G^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+)$. For $j \in \mathbb{N} \setminus \{0\}$ we set $\mathcal{R}_{M+G}^{\mu-j}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+) := \mathcal{R}_G^{\mu-j}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+)$.

2.1.6 The edge algebra

We now pass to the space $\mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+)$ of so called edge amplitude functions of order $\mu \in \mathbb{R}$, with weight data $\boldsymbol{g} = (\gamma, \gamma-\mu)$ which is defined to be the set of all families of operators

$$a(y,\eta) = \begin{pmatrix} b(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + c(y,\eta), \tag{2.17}$$

for arbitrary symbols $b(y,\eta)$ of the type (2.9) and $c(y,\eta) \in \mathcal{R}^{\mu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+)$. More generally, let

$$\mathcal{R}^{\mu-j}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+) \tag{2.18}$$

for $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $j \in \mathbb{N} \setminus \{0\}$, denote the space of all $a(y,\eta)$ of the form (2.17) such that $c(y,\eta) \in \mathcal{R}^{\mu-j}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+)$ and $b(y,\eta)$ replaced by an operator family of the same structure as (2.9), but with $\mu - j$ in place of μ (which concerns the order of symbols as well as the weight factor $r^{-\mu+j}$).
We now turn to the global operators on a (stretched) manifold $\mathbb{W}$ with edge Y. First we need global versions of the weighted edge Sobolev spaces. Assume, for simplicity, that $\mathbb{W}$ is compact and choose finitely many 'singular' charts of the kind (2.2), namely,

$$\chi_j : \mathbb{V}_j \longrightarrow \overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q \text{ for } j = 1, ..., L$$

and 'regular' charts $\chi_j : U_j \longrightarrow \mathbb{R}^{1+n+q}$, for $j = L+1, ..., N$, such that the wedge neighbourhoods $\{\mathbb{V}_j\}_{j=1,...,L}$ together with the ordinary coordinate neighbourhoods $\{U_j\}_{j=L+1,...,N}$ form an open covering of $\mathbb{W}$. Choose a subordinate partition of unity $\{\varphi_j\}_{j=1,...,N}$, consisting of functions in $C^\infty(\mathbb{W})$, supported in $\mathbb{V}_j$ and U_j, respectively.

Definition 2.1.15. *The weighted edge Sobolev space $\mathcal{W}^{s,\gamma}(\mathbb{W})$ is defined to be the completion of $C_0^\infty(\mathbb{W}_{\text{reg}})$ with respect to the norm*

$$\left\{ \sum_{j=1}^{L} ||(\chi_j^{-1})^* \varphi_j u||^2_{\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)} + \sum_{j=L+1}^{N} ||(\chi_j^{-1})^* \varphi_j u||^2_{H^s(\mathbb{R}^{1+n+q})} \right\}^{\frac{1}{2}}.$$

Remark 2.1.16. *Definition* 2.1.15 *makes sense, i.e., it does not depend on the charts and the chosen partition of unity. This is a consequence of Proposition* 3.1.2 *below. Note, in particular, that Definition* 2.1.15 *does not require that $\partial\mathbb{W}$ is a trivial X-bundle over Y.*

Observe that

$$H^s_{\text{comp}}(\mathbb{W}_{\text{reg}}) \subset \mathcal{W}^{s,\gamma}(\mathbb{W}) \subset H^s_{\text{loc}}(\mathbb{W}_{\text{reg}})$$

for every $s, \gamma \in \mathbb{R}$, cf. Example 1.1.4.
A similar construction gives us global weighted edge Sobolev spaces of the kind $\mathcal{W}^{s,\gamma}_{\text{comp}}(\mathbb{W})$ and $\mathcal{W}^{s,\gamma}_{\text{loc}}(\mathbb{W})$ when $\mathbb{W}$ is not necessarily compact, using locally finite coverings by $\mathbb{V}_j$ and U_j, respectively. Let us point out that for $u \in \mathcal{W}^{s,\gamma}_{\text{comp}}(\mathbb{W})$ the support $\operatorname{supp} u$ may intersect $\mathbb{W}_{\text{sing}}$ in a non-empty compact set; similarly, for $u \in \mathcal{W}^{s,\gamma}_{\text{loc}}(\mathbb{W})$ we may have $\operatorname{supp} u \cap \mathbb{W}_{\text{sing}} \neq \emptyset$ (cf. also the meaning of 'comp' and 'loc' spaces for abstract edge Sobolev spaces on an open set $\Omega \ni y$ with values in E, where 'comp' and 'loc' only refers to y). From the identification $\mathcal{W}^{0,0}_{\text{loc}}(\mathbb{W}) = L^2_{\text{loc}}(\mathbb{W}_{\text{reg}})$ we have a corresponding sesquilinear paring $(\cdot,\cdot)_{\mathcal{W}^{0,0}_{\text{loc}}(\mathbb{W})}$ in $\mathcal{W}^{0,0}_{\text{loc}}(\mathbb{W})$

when one of the factors belongs to $\mathcal{W}^{0,0}_{\text{comp}}(\mathbb{W})$. Similarly, in $H^0_{\text{loc}}(Y,J)$ (for any $J \in \text{Vect}(Y)$ with a fixed Hermitian metric) we have a corresponding sesquilinear paring $(\cdot,\cdot)_{H^0_{\text{loc}}(Y,J)}$. For operators $\mathcal{C} : C_0^\infty(\mathbb{W}_{\text{reg}}) \oplus C_0^\infty(Y, J_-) \longrightarrow C^\infty(\mathbb{W}_{\text{reg}}) \oplus C^\infty(Y, J_+)$, $J_\pm \in \text{Vect}(Y)$, we define formal adjoints by $(\mathcal{C}u, v)_{\mathcal{W}^{0,0}_{\text{loc}}(\mathbb{W}) \oplus H^0_{\text{loc}}(Y,J_+)} = (u, \mathcal{C}^* v)_{\mathcal{W}^{0,0}_{\text{loc}}(\mathbb{W}) \oplus H^0_{\text{loc}}(Y,J_-)}$ for all $u \in C_0^\infty(\mathbb{W}_{\text{reg}}) \oplus C_0^\infty(Y, J_-)$, $v \in C_0^\infty(\mathbb{W}_{\text{reg}}) \oplus C_0^\infty(Y, J_+)$.

For the following notation we choose a cut-off function $\sigma \in C^\infty(\mathbb{W})$ which is equal to 1 in a collar neighbourhood of $\partial\mathbb{W}$ and vanishes outside another neighbourhood of $\partial\mathbb{W}$. Let

$$\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; J_-, J_+) \text{ for } \boldsymbol{g} = (\gamma, \gamma - \mu),$$

γ, $\mu \in \mathbb{R}$, denote the space of all smoothing operators $\mathcal{C} = (\mathcal{C}_{ij})_{i,j=1,2}$ in the sense that $\mathcal{C}$ and $\mathcal{C}^*$ induce continuous operators

$$\begin{matrix} \mathcal{W}^{s,\gamma}_{\text{comp}}(\mathbb{W}) \\ \oplus \\ H^s_{\text{comp}}(Y, J_-) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{\infty,\gamma-\mu}_{\text{loc}}(\mathbb{W}) \\ \oplus \\ H^\infty_{\text{loc}}(Y, J_+) \end{matrix} \quad \text{and} \quad \begin{matrix} \mathcal{W}^{s,-\gamma+\mu}_{\text{comp}}(\mathbb{W}) \\ \oplus \\ H^s_{\text{comp}}(Y, J_+) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{\infty,-\gamma}_{\text{loc}}(\mathbb{W}) \\ \oplus \\ H^\infty_{\text{loc}}(Y, J_-) \end{matrix}$$

for all $s \in \mathbb{R}$ such that for every $\theta \in C_0^\infty(Y)$ we have

$$\text{im}(\theta\sigma\mathcal{C}_{11} \;\; \theta\mathcal{C}_{12}) \subset \mathcal{W}^{\infty,\gamma-\mu+\varepsilon}_{\text{comp}}(\mathbb{W}), \;\; \text{im}(\theta\sigma\mathcal{C}^*_{11} \;\; \theta\mathcal{C}^*_{21}) \subset \mathcal{W}^{\infty,-\gamma+\varepsilon}_{\text{comp}}(\mathbb{W})$$

for some $\varepsilon = \varepsilon(\theta, \mathcal{C}) > 0$.

Definition 2.1.17. *The space*

$$\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$$

is defined to be the set of all operator functions

$$\mathcal{A} = \sigma\mathcal{A}_{\text{edge}}\tilde{\sigma} + \begin{pmatrix} (1-\sigma)A_{\text{int}}(1-\tilde{\tilde{\sigma}}) & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{C},$$

with cut-off functions $\sigma(r)$, $\tilde{\sigma}(r)$, $\tilde{\tilde{\sigma}}(r)$ satisfying $\sigma \prec \tilde{\sigma}$, $\tilde{\tilde{\sigma}} \prec \sigma$ such that

(i) $\mathcal{A}_{\text{edge}}$ *is a locally finite sum of operators of the form $(\chi^{-1})_*\text{Op}_y(a)$ for arbitrary singular charts of the form $\chi : \mathbb{V} \longrightarrow \overline{\mathbb{R}}_+ \times X \times \Omega$ (cf. the notation in the beginning of Section* 2.1.1) *and arbitrary edge amplitude functions $a(y,\eta) \in \mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; j_-, j_+)$ (with $j_\pm$ being the fiber dimension of $J_\pm$),*

(ii) $A_{\text{int}} \in L^\mu_{\text{cl}}(\mathbb{W}_{\text{reg}})$,

(iii) $\mathcal{C} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$.

Similarly as (2.18) we can define operator spaces

$$\mathcal{Y}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; J_-, J_+) \tag{2.19}$$

for $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $j \in \mathbb{N}$, by requiring the amplitude functions in (i) to belong to the symbol class (2.18) and in (ii) the interior pseudo-differential operator A_{int} to be of order $\mu - j$.

Theorem 2.1.18. *Every $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$ induces continuous operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}_{\text{comp}}(\mathbb{W}) \\ \oplus \\ H^{s}_{\text{comp}}(Y, J_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}_{\text{loc}}(\mathbb{W}) \\ \oplus \\ H^{s-\mu}_{\text{loc}}(Y, J_+) \end{array} \tag{2.20}$$

for every $s \in \mathbb{R}$.

Given $\varphi \in C^{\infty}(\mathbb{W})$, $\varphi' \in C^{\infty}(Y)$, by $\mathcal{M}_{\varphi,\,\varphi'}$ we denote the operator of multiplication by $\text{diag}(\varphi,\, \varphi' \cdot \text{id}_J)$ for some $J \in \text{Vect}(Y)$. The bundle J will be clear from the context, so it is omitted in the notation. Note that when $\psi \in C^{\infty}(\mathbb{W})$ and $\psi' \in C^{\infty}(Y)$ are functions such that $\text{supp}\,\varphi \cap \text{supp}\,\psi = \text{supp}\,\varphi' \cap \text{supp}\,\psi' = \emptyset$, we have $\mathcal{M}_{\varphi,\,\varphi'}\mathcal{A}\mathcal{M}_{\psi,\,\psi'} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$ for every $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$. Let us now formulate the principal symbolic structure of operators in $\mathcal{Y}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$, first for the local situation $\mathbb{W} = \overline{\mathbb{R}}_+ \times X \times \Omega$ and $J_{\pm}$ replaced by $\Omega \times \mathbb{C}^{j_{\pm}}$. In this case we have the homogeneous principal interior symbol in local coordinates x on X with x varying in an open set $\Sigma \subset \mathbb{R}^n$, namely

$$\sigma^{\mu-j}_{\psi}(\mathcal{A})(r, x, y, \rho, \xi, \eta) \tag{2.21}$$

which is of the form $r^{-\mu+j}\tilde{p}_{(\mu-j)}(r, x, y, r\rho, \xi, r\eta)$ for a C^{∞} function $\tilde{p}_{(\mu-j)}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta})$ in $\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times (\mathbb{R}^{1+n+q} \setminus \{0\})$, homogeneous in $(\tilde{\rho}, \xi, \tilde{\eta})$ of order $\mu - j$.
As in the usual pseudo-differential calculus, $\sigma^{\mu-j}_{\psi}(\mathcal{A})$ is invariantly defined, with respect to $x \in X$ and $y \in Y$.
Moreover, we have the homogeneous principal edge symbol

$$\sigma^{\mu-j}_{\wedge}(\mathcal{A})(y, \eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^{\wedge}) \\ \oplus \\ J_{-,\,y} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu+j,\gamma-\mu+j}(X^{\wedge}) \\ \oplus \\ J_{+,\,y} \end{array} \tag{2.22}$$

of order $\mu - j$, $s \in \mathbb{R}$, $(y, \eta) \in T^*\Omega \setminus 0$. The homogeneity means

$$\sigma^{\mu-j}_{\wedge}(\mathcal{A})(y, \delta\eta) = \delta^{\mu-j}\text{diag}(\kappa_{\delta},\, \text{id})\sigma^{\mu-j}_{\wedge}(\mathcal{A})(y, \eta)\text{diag}(\kappa^{-1}_{\delta},\, \text{id})$$

for all $\delta \in \mathbb{R}_+$, $(y, \eta) \in T^*\Omega \setminus 0$; the identities in the right lower corners refer to the fibers $J_{-,\,y}$ and $J_{+,\,y}$, respectively. The entries in (2.22) were defined in Remark 2.1.11, and (2.14), (2.16).
For $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$, $\boldsymbol{g} = (\gamma, \gamma - \mu)$, we simply write

$$\sigma(\mathcal{A}) = (\sigma_{\psi}(\mathcal{A}), \sigma_{\wedge}(\mathcal{A}))$$

with $\sigma_{\psi}(\mathcal{A}) := \sigma^{\mu}_{\psi}(\mathcal{A})$, $\sigma_{\wedge}(\mathcal{A}) := \sigma^{\mu}_{\wedge}(\mathcal{A})$.

Remark 2.1.19. *$\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$ and $\sigma(\mathcal{A}) = 0$ entails $\mathcal{A} \in \mathcal{Y}^{\mu-1}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$. In addition, if $\mathbb{W}$ is compact the operator* (2.20) *is compact for every $s \in \mathbb{R}$ (in this case the subscripts 'comp' and 'loc' are superfluous).*

Let us now discuss a class of examples in connection with the material of Section 1.2.2. Let M be a C^∞ manifold with an embedded manifold Y of dimension q (with a trivial normal bundle). Then there is a stretched manifold $\mathbb{W}$ with edge such that $\mathbb{W}_{\text{reg}} \cong M \setminus Y$ and $\mathbb{W}_{\text{sing}} = X \times Y$ for $X := S^{d-1}$. For $s > \frac{d}{2}$, we may write

$$H_0^s(\mathbb{R}^d) = \mathcal{K}^{s,s}(X^\wedge) \text{ and } H_0^s(M, Y) = \mathcal{W}^{s,s}(\mathbb{W}),$$

see Remark 1.2.1, and Definition 1.1.2 in connection with the formula (1.21).

Remark 2.1.20. *Let A be a differential operator on M of order μ (with smooth coefficients). Then, for $s > \mu + \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{N}$ the operator* (1.36) *is an element of $\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \mathbb{C}^{\sigma(s)}, \mathbb{C}^{\sigma(s-\mu)})$ for $\boldsymbol{g} = (s, s-\mu)$. The principal interior symbol $\sigma_\psi(\mathcal{A}_s)$ near Y can be obtained by inserting polar coordinates into the standard homogeneous principal symbol of A (according to the transformation of* (1.2) *to* (1.3)*). Moreover, the homogeneous principal edge symbol of* (1.36) *is nothing other than* (1.39).

Remark 2.1.21. *The principal edge symbol $\sigma_\wedge(\mathcal{A})$ of an operator $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$ determines a so called conormal symbol $\sigma_c(\mathcal{A})$, derived from the upper left corner of* (2.22) *in the framework of the cone algebra on $X^\wedge$ for every fixed $(y, \eta) \in T^*Y \setminus 0$. It has the form $\sigma_c(\mathcal{A})(y, z) = \tilde{h}(0, y, z, 0) + f(y, z)$ with the Mellin symbols $\tilde{h}(r, y, z, \tilde{\eta})$ contained in the non-smoothing part of the (local) amplitude function $b(y, \eta)$, cf. the formula* (2.9)*, and the smoothing Mellin symbol $f(y, z)$ in the $M + G$-part of* (2.17)*, cf. the formula* (2.15) *for $j = |\alpha| = 0$. The conormal symbol defines a family of continuous operators*

$$\sigma_c(\mathcal{A})(y, z) : H^r(X) \longrightarrow H^{r-\mu}(X), \tag{2.23}$$

$r \in \mathbb{R}$, parametrised by $(y, z) \in Y \times \Gamma_{\frac{n+1}{2}-\gamma}$.

2.2 Ellipticity

2.2.1 Elliptic operators in the edge algebra

Let us now pass to the ellipticity in the edge pseudo-differential calculus.

Definition 2.2.1. *An operator $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$, $\boldsymbol{g} = (\gamma, \gamma - \mu)$, is called elliptic if*

(i) $\sigma_\psi(\mathcal{A}) \neq 0$ *on* $T^*(\mathbb{W}_{\mathrm{reg}}) \setminus 0$, *and, near* $\mathbb{W}_{\mathrm{sing}}$ *in the splitting of variables* (r, x, y),

$$r^\mu \sigma_\psi(\mathcal{A})(r, x, y, r^{-1}\rho, \xi, r^{-1}\eta) \neq 0$$

up to $r = 0$, *for* $(\rho, \xi, \eta) \neq 0$,

(ii)

$$\sigma_\wedge(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ J_- \end{pmatrix} \longrightarrow \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ J_+ \end{pmatrix} \tag{2.24}$$

is an isomorphism (*with* $\pi_Y : T^*Y \setminus 0 \longrightarrow Y$ *being the canonical projection*).

Theorem 2.2.2. *Let* A *be an elliptic differential operator on* M, *and form the operator* $\mathcal{A}_s$, *according to* (1.36), *cf. also Remark* 2.1.20. *Then the operator* $\mathcal{A}_s$ *is elliptic in the sense of Definition* 2.2.1.

Proof. As noted in Remark 2.1.20 the operator $\mathcal{A}_s$ belongs to the edge calculus (s is fixed as before). The ellipticity of A on M in the standard sense entails the ellipticity also in the sense of Definition 2.2.1 (i) which can be easily verified by introducing polar coordinates in the local representations of A near Y. Moreover, Corollary 1.2.11 gives us the bijectivity of the edge symbol (2.24) when we identify the spaces $H_0^s(\mathbb{R}^d)$ with $\mathcal{K}^{s,s}((S^{d-1})^\wedge)$. □

Remark 2.2.3. *If an operator* $\mathcal{A}$ *in the edge calculus is elliptic, the conormal symbol* $\sigma_c(\mathcal{A})(y, z)$ *represents an element in* $C^\infty(Y, L^\mu_{\mathrm{cl}}(X; \Gamma_{\frac{n+1}{2}-\gamma}))$ *which is* (*for every* $y \in Y$) *parameter-dependent elliptic with parameter* $|\mathrm{Im}\, z|$ *for* $z \in \Gamma_{\frac{n+1}{2}-\gamma}$, *and* (2.23) *is a family of isomorphisms, parametrised by* $y \in Y$, $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. *In fact, the bijectivity of* (2.24) *for any fixed* $y \in Y$ *and* $\eta \neq 0$ *has the consequence that the operators in the upper left corner are Fredholm between the weighted Sobolev spaces on* $X^\wedge$. *Since these operators for every fixed* $\eta \neq 0$ *belong to the cone algebra on the* (*infinite stretched*)*cone* $X^\wedge$ *their principal symbols from the cone algebra are bijective, in particular, the principal conormal symbol* (2.23) *on the respective weight line. For the operators as in Example* 2.2.2 *we have* $n = d - 1$ *and* $\gamma = s$, *i.e.,*

$$\sigma_c(\mathcal{A}_s)(y, z) : H^r(X) \longrightarrow H^{r-\mu}(X)$$

is a family of isomorphisms for all $y \in Y$, $z \in \Gamma_{\frac{d}{2}-s}$, *and for all* $r \in \mathbb{R}$.

2.2.2 Invertibility and parametrices

Given an $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; J_-, J_+)$ for $\boldsymbol{g} = (\gamma, \gamma - \mu)$, an operator $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; J_+, J_-)$ for $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma)$ is said to be a parametrix of $\mathcal{A}$, if for every φ, $\psi \in C_0^\infty(\mathbb{W})$, φ', $\psi' \in C^\infty(Y)$ with $\psi \equiv 1$ on $\operatorname{supp}\varphi$ and $\psi' \equiv 1$ on $\operatorname{supp}\varphi'$ we have

$$\mathcal{M}_{\varphi,\varphi'} - \mathcal{M}_{\varphi,\varphi'}\mathcal{P}\mathcal{M}_{\psi,\psi'}\mathcal{A} \in \mathcal{Y}^{-\infty}(\mathbb{W},(\gamma,\gamma);J_-,J_-), \tag{2.25}$$

and

$$\mathcal{M}_{\varphi,\varphi'} - \mathcal{M}_{\varphi,\varphi'}\mathcal{A}\mathcal{M}_{\psi,\psi'}\mathcal{P} \in \mathcal{Y}^{-\infty}(\mathbb{W},(\gamma-\mu,\gamma-\mu);J_+,J_+). \tag{2.26}$$

Theorem 2.2.4. *Every elliptic operator $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W},\boldsymbol{g};J_-,J_+)$ has a parametrix $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W},\boldsymbol{g}^{-1};J_+,J_-)$.*

A proof of Theorem 2.2.4 in the edge calculus with asymptotics may be found in [18], see also [10]. Note that when $\mathbb{W}$ is compact, the relations (2.25), (2.26) simplify to

$$1-\mathcal{P}\mathcal{A} \in \mathcal{Y}^{-\infty}(\mathbb{W},(\gamma,\gamma);J_-,J_-),\ 1-\mathcal{A}\mathcal{P} \in \mathcal{Y}^{-\infty}(\mathbb{W},(\gamma-\mu,\gamma-\mu);J_+,J_+).$$

These remainders are compact in the respective spaces and we can conclude elliptic regularity of solutions in our spaces. Another consequence is the following result:

Theorem 2.2.5. *Let $\mathbb{W}$ be compact and let $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W},\boldsymbol{g};J_-,J_+)$ be elliptic. Then $\mathcal{A}$ induces Fredholm operators*

$$\mathcal{A}: \begin{array}{c}\mathcal{W}^{s,\gamma}(\mathbb{W})\\ \oplus\\ H^s(Y,J_-)\end{array} \longrightarrow \begin{array}{c}\mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W})\\ \oplus\\ H^{s-\mu}(Y,J_+)\end{array} \tag{2.27}$$

for all $s \in \mathbb{R}$, and the dimensions of $\ker\mathcal{A}$ *and* $\operatorname{coker}\mathcal{A}$ *are independent of s. Moreover, if* (2.27) *is an invertible operator* (*for some s which entails the invertibility for all s*), *we have $\mathcal{A}^{-1} \in \mathcal{Y}^{-\mu}(\mathbb{W},\boldsymbol{g}^{-1};J_+,J_-)$.*

We can make now a few remarks on relations between pseudo-differential operators based on the Fourier transform and operators based on the Mellin transform. We want to do this locally in $\mathbb{R}^{d+q} \ni (\tilde{x},y)$. Let (1.2) be a differential operator in $\mathbb{R}^{d+q}$ with coefficients in $C^\infty(\mathbb{R}^{d+q})$. Then, by introducing polar coordinates $\mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}_+ \times S^{d-1}$, $\tilde{x} \longrightarrow (r,x)$, we can transform A into edge-degenerate form (1.3). Let us form an operator-valued Mellin symbol

$$h(r,y,z,\eta) := \tilde{h}(r,y,z,r\eta)$$

for $\tilde{h}(r,y,z,\tilde{\eta}) := \sum_{j+|\beta|\leqq\mu} a_{j\beta}(r,y)z^j\tilde{\eta}^\beta$, cf. the formula (1.3). Then the operator A can be written as

$$A = r^{-\mu}\mathrm{Op}_y\mathrm{op}_M^\gamma(h)$$

for every weight $\gamma \in \mathbb{R}$, if we interpret A as an operator $C_0^\infty(X^\wedge \times \mathbb{R}^q) \longrightarrow C_0^\infty(X^\wedge \times \mathbb{R}^q)$ for $X := S^{d-1}$. In that sense we can talk about a Mellin-edge representation of the operator $\tilde{A}$, originally regarded as an operator in terms of the Fourier transform in $\mathbb{R}^d \times \mathbb{R}^q$. Mellin reformulations of a similar kind for pseudo-differential operators are usually connected with smoothing remainders outside the 'edge' $\{0\} \times \mathbb{R}^q$; however, those remainders have singularities at the edge. Such remainders may be very undesirable in certain applications to the asymptotics of potentials, cf. the article [9]. However, Theorem 2.2.2 combined with Theorem 2.2.4 gives us also Mellin-edge representations of parametrices of elliptic differential operators with remainders in the respective $\mathcal{Y}^{-\infty}$-classes. The mapping properties of remainders in $\mathcal{Y}^{-\infty}$ are characterised in Section 2.1.6.

3 Invariance properties of the edge algebra

3.1 Edge Sobolev spaces

We now consider the case of embedded manifolds $Y \subset M$ without the assumption of a trivial normal bundle $N(Y)$ in M. As is known, there is a tubular neighbourhood of Y in M which can be identified with the ball bundle induced by $N(Y)$, equipped with some Euclidean metric. We assume in this connection that $N(Y)$ is represented by a system of trivialisations

$$U_j' \times \mathbb{R}^d,\, j = 1, ..., L$$

and transition maps which are isomorphisms of $\mathbb{R}^d$ of determinant 1. Then we obtain an induced sphere bundle on Y with fibre S^{d-1}. If we define global weighted edge Sobolev spaces on M we have to take into account the non-trivial transition diffeomorphisms of the base of the cone.
In the following considerations we assume X to be an arbitrary closed compact C^∞ manifold of dimension n.

Lemma 3.1.1. *Let $\tau_0 : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ be a diffeomorphism such that $\tau_0(y) = y$ for all $y \in \mathbb{R}^q$, $|y| \geqq c$, for a constant $c > 0$. Then the pull back under τ_0 induces isomorphisms $\tau_0^* : \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \longrightarrow \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ and the same for the 'comp', 'loc' spaces for all s, $\gamma \in \mathbb{R}$.*

A proof may be found in [18].

Proposition 3.1.2. *Let $\tau_1 : X \times \mathbb{R}^q \longrightarrow X \times \mathbb{R}^q$ be an isomorphism in the sense of X-bundles on $\mathbb{R}^q$. Then the pull back under τ_1 induces an isomorphism*

$$\tau_1^* : \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \longrightarrow \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \tag{3.1}$$

for all $s, \gamma \in \mathbb{R}$.

Proof. Write $(\tilde{x}, \tilde{y}) = \tau_1(x, y) = (\tau(x, y), \tau_0(y))$, with $\tilde{x} = \tilde{x}(x, y)$ and $\tilde{y} = \tilde{y}(y)$. Here, $\tau_0 : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is a diffeomorphism with $\tau_0(y) = y$, for $|y| \geq c$ for some $c > 0$ and $\tau(\cdot, y) : X \longrightarrow X$ is also a C^∞-diffeomorphism for all $y \in \mathbb{R}^q$. Because of

$$(x, y) \mapsto (x, \tau_0^{-1}(y)) \mapsto (\tau(x, \tau_0^{-1}(y)), y)$$

without loss of generality, we can take $\tau_0(y) = y$ for each $y \in \mathbb{R}^q$. Moreover, by virtue of Remark 1.1.1 the pull back under $\tau_y := \tau(\cdot, y)$ induces an isomorphism

$$\tau_y^* : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s,\gamma}(X^\wedge) \tag{3.2}$$

for each $y \in \mathbb{R}^q$.

Setting $a(y) := \tau_y^*$ we have that $a(y) \in C^\infty(\mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge)))$. It can easily be checked that $\kappa_\lambda^{-1} a(y) \kappa_\lambda = a(y)$, $\lambda \in \mathbb{R}_+$, where $\kappa_\lambda : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s,\gamma}(X^\wedge)$, $\kappa_\lambda u(r, x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$ is the group action on $\mathcal{K}^{s,\gamma}(X^\wedge)$. This implies

$$a(y) \in S^0_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$$

for every s, $\gamma \in \mathbb{R}$. It follows that the associated pseudo-differential operator $\mathrm{Op}(a)$ is continuous as an operator

$$\mathrm{Op}(a) : \mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(X^\wedge \times \mathbb{R}^q) \longrightarrow \mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}(X^\wedge \times \mathbb{R}^q).$$

Because of

$$\mathrm{Op}(a) u(r, x, y) = (F^{-1} a(y) F) u(r, x, y) = a(y) u(r, x, y)$$

for every $u(r, x, y) \in \mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(X^\wedge \times \mathbb{R}^q)$ it follows that $\mathrm{Op}(a)$ is the same as the operator of multiplication by $a(y) = \tau_y^*$.

Since (3.2) is an isomorphism for each $y \in \mathbb{R}^q$, we see that

$$\mathrm{Op}(a) : \mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(X^\wedge \times \mathbb{R}^q) \longrightarrow \mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}(X^\wedge \times \mathbb{R}^q)$$

is an isomorphism for all $s, \gamma \in \mathbb{R}$. Finally, using Lemma 3.1.1 we obtain that the pull back under τ_1 induces the isomorphism (3.1) for all s, $\gamma \in \mathbb{R}$. □

We now extend our constructions to the case when Y has not necessarily a trivial normal bundle in M. Recall that then M corresponds to a manifold W with edge Y where in the notation of Section 2.1.1 the singular subset $\mathbb{W}_{\mathrm{sing}}$ of the stretched manifold $\mathbb{W}$ is an X-bundle on Y, not necessarily trivial. Recall from Section 2.1.6 that we have the scale of global weighted Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathbb{W})$ also in this general case. The constructions for Proposition 3.1.2 and Definition 2.1.15 can be generalised as follows.

Let E be a Hilbert space with group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, consider a family of isomorphisms $E \longrightarrow E$ given by an element $\beta \in C^\infty(\Omega, \mathcal{L}(E, E))$, $\Omega \subseteqq \mathbb{R}^q$ open. Assume that

$$\beta(y) = \kappa_\lambda \beta(y) \kappa_\lambda^{-1}$$

for all $y \in \Omega$, $\lambda \in \mathbb{R}_+$. Then we have $\beta \in S^0_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E, E)$, and

$$\mathrm{Op}(\beta) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \longrightarrow \mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$$

is an isomorphism (the same is true for the subspaces with subscripts 'loc').
Let us interpret the isomorphisms β as transition maps $\Omega \times E \longrightarrow \Omega \times E$ of a vector bundle $\mathcal{E}$ on a C^∞ manifold Y. Then, as an easy generalisation of Definition 2.1.15 we can form 'abstract' edge Sobolev spaces $\mathcal{W}^s_{\mathrm{loc}}(Y, \mathcal{E})$ of distributional sections in the bundle $\mathcal{E}$; moreover, let $\mathcal{W}^s_{\mathrm{comp}}(Y, \mathcal{E})$ denote the subspace of elements with compact support.
Let $\tilde{E}$ be another Hilbert space with group action $\{\tilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+}$ and $\tilde{\beta} \in C^\infty(\Omega, \mathcal{L}(\tilde{E}, \tilde{E}))$ a family of isomorphisms, also with the homogeneity property $\tilde{\beta}(y) = \tilde{\kappa}_\lambda \tilde{\beta}(y) \tilde{\kappa}^{-1}_\lambda$ for all $y \in \Omega$, $\lambda \in \mathbb{R}_+$. Then for every

$$a(y, \eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E, \tilde{E}) \tag{3.3}$$

we can form the symbol

$$\tilde{\beta}(y) a(y, \eta) \beta^{-1}(y) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E, \tilde{E}). \tag{3.4}$$

A pseudo-differential operator

$$\mathrm{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E}) \tag{3.5}$$

can be interpreted as an operator between sections in the trivial bundles $\Omega \times E$ and $\Omega \times \tilde{E}$ respectively. Then, if we pass to other trivialisations via

$$\beta : \Omega \times E \longrightarrow \Omega \times E,\ \tilde{\beta} : \Omega \times \tilde{E} \longrightarrow \Omega \times \tilde{E} \tag{3.6}$$

with the induced isomorphisms

$$\beta_* : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \longrightarrow \mathcal{W}^s_{\mathrm{comp}}(\Omega, E),\ \tilde{\beta}_* : \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E}) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E}) \tag{3.7}$$

we obtain a corresponding push forward

$$\mathrm{Op}(\tilde{a}) = \tilde{\beta}_* \mathrm{Op}(a) (\beta_*)^{-1} : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E}) \tag{3.8}$$

with a symbol $\tilde{a}(y, \eta)$ which is equal to (3.4) modulo lower order terms. Combining (3.6) with a coordinate diffeomorphism $\chi : \Omega \longrightarrow \tilde{\Omega}$, i.e., replace (3.6) by corresponding bundle isomorphisms

$$(\chi, \beta) : \Omega \times E \longrightarrow \tilde{\Omega} \times E,\ (\chi, \tilde{\beta}) : \Omega \times \tilde{E} \longrightarrow \tilde{\Omega} \times \tilde{E} \tag{3.9}$$

we obtain an operator push forward

$$\mathrm{Op}(\tilde{a}) = (\chi, \tilde{\beta})_* \mathrm{Op}(a) (\chi, \beta)^{-1}_* : \mathcal{W}^s_{\mathrm{comp}}(\tilde{\Omega}, E) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\tilde{\Omega}, \tilde{E}).$$

Instead of (3.4) the transformation rule $a(y,\eta) \longrightarrow \tilde{a}(\tilde{y},\tilde{\eta})$ of associated symbols now has the form

$$\tilde{a}(\tilde{y},\tilde{\eta})\Big|_{\substack{\tilde{y}=\chi(y)\\ \tilde{\eta}=({}^t d\chi(y))^{-1}\eta}} = \tilde{\beta}(y)a(y,\eta)\beta^{-1}(y)$$

modulo lower order terms in $\tilde{\eta}$. This invariance shows that similarly as in the standard calculus of pseudo-differential operators on C^∞ manifolds we have global pseudo-differential operators between distributional sections in our vector bundles

$$A : \mathcal{W}^s_{\mathrm{comp}}(Y,\mathcal{E}) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(Y,\tilde{\mathcal{E}})$$

locally described by (3.5), modulo global (on Y) smoothing operators (characterised by $C : \mathcal{W}^s_{\mathrm{comp}}(Y,\mathcal{E}) \longrightarrow \mathcal{W}^t_{\mathrm{loc}}(Y,\tilde{\mathcal{E}})$ for all $s,\, t \in \mathbb{R}$). Let $L^\mu_{(\mathrm{cl})}(Y;\mathcal{E},\tilde{\mathcal{E}})$ denote the space of these operators. In the case of dependence on extra parameters $\lambda \in \mathbb{R}^l$ we write $L^\mu_{(\mathrm{cl})}(Y;\mathcal{E},\tilde{\mathcal{E}};\mathbb{R}^l)$.
Note that pseudo-differential operators on Hilbert bundles in another context have been studied by other authors before, cf. Luke [13]. The new aspect here is that our spaces of symbols are connected with group actions in the fibres of the Hilbert bundles.

3.2 Trace and potential operators

We now discuss the transformation behaviour of trace and potential symbols under the symbol push forward belonging to push forwards of associated pseudo-differential operators along Y. Consider the local representative of a potential symbol (also dependent on a parameter $\lambda \in \mathbb{R}^l$ as an additional component of the covariables) $k_{(j)}(y_{(j)},\eta_{(j)},\lambda)$ in the variables $y_{(j)} \in \mathbb{R}^q$ under the j th chart $\chi_j : U'_j \longrightarrow \mathbb{R}^q$, with the covariables $(\eta_{(j)},\lambda)$ (for brevity, we will omit subscripts '(j)' below). These local symbols are generated by the potential symbols (1.13) plus lower order terms coming from the global construction, see (1.20). The vectors in $E := \mathbb{C}^{\sigma(s)}$ have the interpretation of coefficients $(c_\alpha)_{|\alpha|<s-\frac{d}{2}}$ in the Taylor expansion of a function $u(\tilde{x}) \in H^s(\mathbb{R}^d)$ at $\tilde{x}=0$, i.e.,

$$u(\tilde{x}) = \sum_{|\alpha|<s-\frac{d}{2}} \frac{1}{\alpha!}c_\alpha \tilde{x}^\alpha \text{ modulo a flat remainder}$$

(i.e., belonging to $H^s_0(\mathbb{R}^d)$). These vectors will be regarded as elements in the fibre of a vector bundle $\mathcal{E} := \mathcal{J}^{\sigma(s)}$ over Y. The transition maps

$$\mathfrak{c} \longrightarrow \mathfrak{d} \text{ for } \mathfrak{c} := (c_\alpha)_{|\alpha|<s-\frac{d}{2}},\ \mathfrak{d} := (d_\gamma)_{|\gamma|<s-\frac{d}{2}}$$

for different trivialisations $\mathbb{R}^q \times \mathbb{C}^{\sigma(s)}$ of the bundle $\mathcal{J}^{\sigma(s)}$ are generated by

$$u(\tilde{x}) \longrightarrow v(\tilde{z}) = \sum_{|\gamma|<s-\frac{d}{2}} \frac{1}{\gamma!}d_\gamma \tilde{z}^\gamma \text{ modulo a flat remainder}$$

when we insert in the coordinates in $u(\tilde{x})$ the linear transition map $\tilde{x} \longrightarrow \nu(y)\tilde{x} = \tilde{z}$, $\mathbb{R}^d \longrightarrow \mathbb{R}^d$, with $\nu(y) \in C^\infty(\mathbb{R}^q, \mathrm{Gl}(\mathbb{R}, d))$ being the cocycle of the normal bundle of Y in M. These transformations just constitute the transitions $\beta : E \longrightarrow E$ in the formula (3.9) for $E = \mathbb{C}^{\sigma(s)}$ (up to the substitution of the coordinate diffeomorphism).
Moreover, we set $\tilde{E} = H^s(\mathbb{R}^d)$ and obtain by substituting the linear maps $\nu(y)$ in the functions $u \in H^s(\mathbb{R}^d)$ the cocycle $\tilde{\beta}$ for the bundle $\tilde{\mathcal{E}}$. The basic observation for the invariance of our potential symbols $k(y, \eta, \lambda)$ and $\tilde{k}(\tilde{y}, \tilde{\eta}, \lambda)$ is that the diagram

$$\begin{array}{ccc} E & \xrightarrow{k} & \tilde{E} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{\beta}} \\ E & \xrightarrow{\tilde{k}} & \tilde{E} \end{array}$$

is commutative modulo remainders which vanish in a neighbourhood of $\tilde{x} = 0$ and modulo lower order terms in the covariables (plus parameters λ). In fact, on the level of highest order terms in the variables (y, η, λ) we have

$$k(\eta, \lambda)\mathfrak{c} = [\eta, \lambda]^{\frac{d}{2}} \sum_{|\alpha| < s - \frac{d}{2}} \frac{1}{\alpha!}([\eta, \lambda]\tilde{x})^\alpha \omega([\eta, \lambda]\tilde{x}) c_\alpha. \tag{3.10}$$

Then the linear substitution $\tilde{x} \longrightarrow \nu(y)\tilde{x} = \tilde{z}$ which represents $\tilde{\beta}$ at the point y and $\mathfrak{c} \longrightarrow \mathfrak{d}$ which represents β at y gives us

$$\tilde{k}(\eta, \lambda)\mathfrak{d} = \tilde{\beta} k(\eta, \lambda) \beta^{-1} \mathfrak{d}$$

modulo the abovementioned remainders, because $\omega([\eta, \lambda]\tilde{x}) = \omega([\eta, \lambda]\tilde{z}(\tilde{x}))$ in a neighbourhood of $\tilde{x} = 0$. On the level of pseudo-differential operators on Y the latter remainders cause Schwartz functions in $\lambda \in \mathbb{R}^l$ with values in smoothing operators on Y.
Similarly as in Section 1.2.1 (using local representations $\mathrm{Op}_y(k)(\lambda)$ combined with partitions of unity and pull backs to the manifold) we now form global potential operators $G(\lambda)$

$$G(\lambda) : H^s(Y, \mathcal{J}^{\sigma(s)}) \longrightarrow H^s(M)$$

(the image consists of functions in $C^\infty(M)$ supported by a tubular neighbourhood of Y). $G(\lambda)$ is a parameter-dependent pseudo-differential operator of the class $L^0_{\mathrm{cl}}(Y; \mathcal{J}^{\sigma(s)}, \tilde{\mathcal{E}}; \mathbb{R}^l)$ on Y with operator-valued symbols $\boldsymbol{k}(y, y', \eta, \lambda)$ locally belonging to $S^0_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q_\eta \times \mathbb{R}^l_\lambda; \mathbb{C}^{\sigma(s)}, H^s(\mathbb{R}^d))$ and the homogeneous principal symbol

$$\sigma_\wedge G : \pi_Y^* \mathcal{J}^{\sigma(s)} \longrightarrow \pi_Y^* \tilde{\mathcal{E}}, \tag{3.11}$$

$\pi_Y : (T^*Y \times \mathbb{R}^l) \setminus 0 \longrightarrow 0$, is locally given by

$$\sigma_\wedge G(y, \eta, \lambda) : \mathfrak{c} \longrightarrow |\eta, \lambda|^{\frac{d}{2}} \sum_{|\alpha| < s - \frac{d}{2}} \frac{1}{\alpha!}(|\eta, \lambda|\tilde{x})^\alpha \omega(|\eta, \lambda|\tilde{x}) c_\alpha.$$

In a similar manner we form the trace operators

$$B(\lambda) : H^s(M) \longrightarrow H^s(Y, \mathcal{J}^{\sigma(s)}),$$

$B(\lambda) \in L^0_{\mathrm{cl}}(Y; \tilde{\mathcal{E}}, \mathcal{J}^{\sigma(s)}; \mathbb{R}^l)$, with corresponding operator-valued local amplitude functions

$$\boldsymbol{t}(y, y', \eta, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q_\eta \times \mathbb{R}^l_\lambda; H^s(\mathbb{R}^d), \mathbb{C}^{\sigma(s)})$$

that have left symbols (modulo lower order terms in (η, λ)) of the form

$$t(\eta, \lambda) := (t^\alpha(\eta, \lambda)u)_{|\alpha|<s-\frac{d}{2}},\ t^\alpha(\eta, \lambda)u = [\eta, \lambda]^{-\frac{d}{2}-|\alpha|} D^\alpha_{\tilde{x}} u(0). \tag{3.12}$$

The homogeneous principal symbol is a map

$$\sigma_\wedge B : \pi^*_Y \tilde{\mathcal{E}} \longrightarrow \pi^*_Y \mathcal{J}^{\sigma(s)}$$

with the projection π_Y as before.
Analogously as (1.24) we now form an operator family $P(\lambda) : H^s(M) \longrightarrow H^s_0(M, Y)$, using local operator families $P_j(\lambda)$ of the form $1 - \mathrm{Op}_y(k(\eta, \lambda)t(\eta, \lambda))$ with $k(\eta, \lambda)$ and $t(\eta, \lambda)$ as in (3.10) and (3.12), respectively.
The following result is an analogue of Theorem 1.2.9 for the case of an arbitrary embedded submanifold $Y \subset M$.

Theorem 3.2.1. *Let $s \in \mathbb{R}$, $s > \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{N}$. Then there is a constant $C > 0$ such that the operators $(E \ \ G(\lambda))$ and ${}^{\mathrm{t}}(P(\lambda) \ \ B(\lambda))$ induce isomorphisms*

$$\begin{matrix} H^s_0(M, Y) \\ \oplus \\ H^s(Y, \mathcal{J}^{\sigma(s)}) \end{matrix} \longrightarrow H^s(M) \quad \text{and} \quad H^s(M) \longrightarrow \begin{matrix} H^s_0(M, Y) \\ \oplus \\ H^s(Y, \mathcal{J}^{\sigma(s)}) \end{matrix},$$

respectively, for all $\lambda \in \mathbb{R}^l$, $|\lambda| \geqq C$.

Proof. The proof is similar to that of Theorem 1.2.9. The properties which are essential for an analogoue of the relations (1.32) and (1.33) are satisfied in the present situation, too, namely that

$$\sigma_\wedge(B)\sigma_\wedge(G) = \mathrm{id}_{\pi^*_Y \mathcal{J}^{\sigma(s)}}$$

and $1 - \sigma_\wedge(G)\sigma_\wedge(B)$ is a projection in the bundle $\tilde{\mathcal{E}}$ with fibres $H^s(\mathbb{R}^d)$ to the subbundle with fibres $H^s_0(\mathbb{R}^d)$. Thus, for similar reasons as in the proof of Theorem 1.2.9 we have

$$(E \ \ G(\lambda))\,{}^{\mathrm{t}}(P(\lambda) \ \ B(\lambda)) = I - C_{0,r}(\lambda),\ \ {}^{\mathrm{t}}(P(\lambda) \ \ B(\lambda))(E \ \ G(\lambda)) = I - C_{0,l}(\lambda)$$

where $C_{0,r}$ and $C_{0,l}$ have local amplitude functions of order -1 in (η, λ). Hence, we can apply a formal Neumann series argument to obtain remainders $C_r(\lambda)$ and $C_l(\lambda)$ which are strongly decreasing for $\lambda \longrightarrow \infty$, where $C_r(\lambda)$ is as before while

$$C_l(\lambda) \in \mathcal{S}\left(\mathbb{R}^l; \mathcal{L}\left(\begin{matrix} H^s_0(M, Y) \\ \oplus \\ H^s(Y, \mathcal{J}^{\sigma(s)}) \end{matrix}, \begin{matrix} H^s_0(M, Y) \cap C^\infty(M) \\ \oplus \\ C^\infty(Y, \mathcal{J}^{\sigma(s)}). \end{matrix}\right)\right).$$

□

Note that invariance properties of cone operators have been studied in another context in [11]. In our case here the situation (for cone operator-valued edge symbols) is simpler because we could assume r-independence of the transition maps for small r).

3.3 Ellipticity of edge operators in the general case

Let $\mathbb{W}$ be the stretched manifold of a manifold W with edge Y where the X-bundle over Y is not necessarily trivial. Then there is a straightforward extension of the definition of the spaces of edge operators $\mathcal{Y}^{\mu-j}(\mathbb{W},\boldsymbol{g};J_-,J_+)$ as well as of Definition 2.2.1. The Theorems 2.2.4 and 2.2.5 remain true in the corresponding modified form. We now formulate an analogue of Theorem 2.2.2 for the case of an arbitrary $Y\subset M$.

Theorem 3.3.1. *Let A be an elliptic differential operator on M and form* (*the analogue of*) *the operator* (1.36) *for any fixed $s-\mu>\frac{d}{2}$, $s-\frac{d}{2}\notin\mathbb{N}$, namely,*

$$\mathcal{A}_s:\quad \begin{array}{c} H_0^s(M,Y)\\ \oplus\\ H^s(Y,\mathcal{J}^{\sigma(s)}) \end{array} \longrightarrow \begin{array}{c} H_0^{s-\mu}(M,Y)\\ \oplus\\ H^{s-\mu}(Y,\mathcal{J}^{\sigma(s-\mu)}) \end{array}, \tag{3.13}$$

(*where the spaces $H_0^s(M,Y)$ can equivalently be replaced by $\mathcal{W}^{s,s}(\mathbb{W})$*)*. Then $\mathcal{A}_s$ is elliptic as an element of $\mathcal{Y}^\mu(\mathbb{W},\boldsymbol{g};\mathcal{J}^{\sigma(s)},\mathcal{J}^{\sigma(s-\mu)})$ for $\boldsymbol{g}=(s,s-\mu)$.*

Proof. The ellipticity is a local property for the interior symbol $\sigma_\psi(\mathcal{A}_s)$ in coordinate neighbourhoods on $\mathbb{W}$ and for the edge symbol $\sigma_\wedge(\mathcal{A}_s)$ in coordinate neighbourhoods on Y. Thus the bijectivities of the components can be obtained in the same way as for Theorem 2.2.2. □

Remark 3.3.2. *There is a straightforward extension of Theorem* 3.3.1 *for the case of differential operators*

$$A:H^s(M,E)\longrightarrow H^{s-\mu}(M,F)$$

acting between distributional sections of vector bundles E, F on M. In this case we have to replace (3.13) *by*

$$\mathcal{A}_s:\quad \begin{array}{c} \mathcal{W}^{s,s}(\mathbb{W},E)\\ \oplus\\ H^s(Y,E'\otimes\mathcal{J}^{\sigma(s)}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,s-\mu}(\mathbb{W},F)\\ \oplus\\ H^{s-\mu}(Y,F'\otimes\mathcal{J}^{\sigma(s-\mu)}) \end{array}$$

where $E':=E|_Y$, $F':=F|_Y$, and the interpretation of E in the $\mathcal{W}^{s,s}$-space is the pull back of the former E under the canonical projection $\mathbb{W}_{\mathrm{reg}}\longrightarrow M\setminus Y$.

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