Eta Invariant and the Spectral Flow[1]

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Chapter 3

Eta Invariant and the Spectral Flow

3.1. Introduction

In this chapter, we consider two types of invariants of elliptic operators, which arise as one attempts to generalize the following well-known finite-dimensional invariants to the infinite-dimensional case.

For a quadratic form defined on a finite-dimensional real space, there is a well-defined notion of *signature*, which is the difference between the numbers of positive and negative squares in the canonical representation of the form. Note that the signature can be treated as a spectral invariant of the symmetric operator defining the quadratic form. Moreover, the signature is homotopy invariant in the class of invertible operators (nondegenerate forms).

In the infinite-dimensional case, we replace symmetric matrices defining quadratic forms by elliptic self-adjoint operators. The number of eigenvalues is now infinite, and the signature in the usual sense is meaningless for a general operator. The *Atiyah–Patodi–Singer spectral η-invariant* is defined as a regularization (of the type of ζ-function) of the divergent series formed by the signs of the eigenvalues of the operator.[1] For elliptic operators on manifolds, this procedure gives a spectral invariant that inherits some properties of the signature. However, some new properties also arise. For example, the η-invariant can be an arbitrary real number, say, $-\sqrt{2}$. Moreover, the spectral η-invariant is not homotopy invariant even in the class of invertible operators.

There is yet another finite-dimensional invariant, the *spectral flow* of a homotopy of Hermitian matrices, which is the (integer) signature variation under the homotopy. One can obtain a sound definition of the spectral flow in the infinite-dimensional case by treating it as the net number of eigenvalues of the family passing through zero.

The η-invariant and the spectral flow have quite a few applications. A majority of these is based on index formulas, which often involve these two invariants. Index formulas will be discussed in subsequent chapters. In particular, index formulas in the theory of boundary value problems are discussed in Chap. 3.3, and index formulas on manifolds with singularities are treated in Part III.

Surprisingly, one can arrive at the η-invariant in a completely different way starting from another finite-dimensional invariant, the winding number of an invertible function on the circle. In the infinite-dimensional case, one deals with invertible families of elliptic operators with parameter instead of invertible functions. The corresponding η-invariant is known as the *Melrose η-invariant*. The coincidence of the Melrose and Atiyah–Patodi–Singer η-invariants is essentially based on Bott periodicity.

We start our exposition from the more elementary notion of spectral flow.

[1] In the finite-dimensional case, this sum is just the signature.

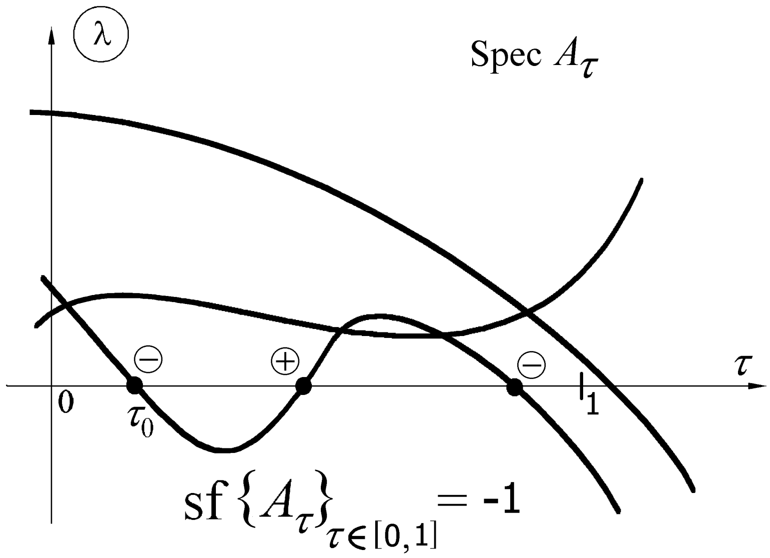

Figure 3.1. The spectral flow

3.2. The Classical Spectral Flow

3.2.1. Definition and main properties

The finite-dimensional case. Consider a continuous family A_t, $t \in [0,1]$, of Hermitian matrices. Informally, the *spectral flow* of this family is the net number of eigenvalues passing through zero as t varies from 0 to 1. Needless to say, one can easily give a rigorous definition in the finite-dimensional case: the spectral flow is just half the difference of signatures of the corresponding quadratic forms at the beginning and end of the homotopy.

In what follows, we show that although the number of eigenvalues in the case of operators can be infinite (so that the above definition in terms of signatures becomes meaningless), one still can give a well-defined notion of spectral flow at least for *elliptic* self-adjoint operators.

The infinite-dimensional case. Let $\{A_\tau\}_{\tau\in[0,1]}$ be a continuous family of elliptic self-adjoint operators on a closed compact manifold M.

DEFINITION 3.1. The *spectral flow* of the family $\{A_\tau\}_{\tau\in[0,1]}$ is the integer equal to the net number of eigenvalues of A_τ that change their sign as τ varies, the change of sign from minus to plus being counted as $+1$ (see Fig. 3.1). The spectral flow is denoted by $\mathrm{sf}\,\{A_\tau\}_{\tau\in[0,1]}$.

We point out that this definition makes sense only in general position, where the graph of the spectrum of the family and the line $\lambda = 0$ intersect transversally. Otherwise, one obtains a well-defined formula for the spectral flow by bringing the intersection to general position. The simplest way to do this is to perturb the line $\lambda = 0$ slightly, namely, replace it by a broken line (see Fig. 3.2) with alternating horizontal and vertical segments such that the horizontal segments avoid the spectrum of the family. Such broken lines are said to be *admissible*.

EXERCISE 3.2. Prove the existence of an admissible broken line.

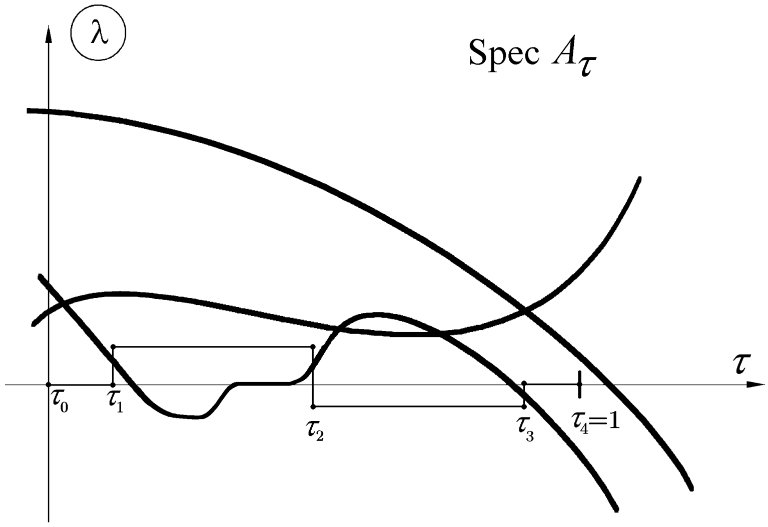

Figure 3.2. The spectral flow. Computation via a broken line

◀ *Hint.* Use the fact that the spectrum of an elliptic self-adjoint operator is discrete. ▶

Using an admissible broken line with vertices $\{(\tau_i, \lambda_i)\}_{i=0,N}$, one can compute[2] the spectral flow as the net sum over all vertices of the numbers of eigenvalues (counted according to their multiplicities) that leave the region under the broken line:

$$\mathrm{sf}\,\{A_\tau\}_{\tau\in[0,1]} = \sum_{i=1}^{N-1} (-1)^{sgn(\lambda_{i-1}-\lambda_i)} N(\lambda_i, \lambda_{i-1}), \tag{3.1}$$

where

$$N(\lambda_i, \lambda_{i-1}) = \#\left\{\lambda \in \mathrm{spec} A_{\tau_i} \middle| \lambda \text{ lies between } \lambda_i \text{ and } \lambda_{i-1}\right\}.$$

(Here $\#$ stands for the cardinality of a set.)

The boundary points $\tau = 0$ and $\tau = 1$ deserve special attention. Here the choice of the horizontal segment is ambiguous. We adopt the following convention: if the family is invertible at these points, then the first (and accordingly, the last) segment lies on the abscissa axis $\lambda = 0$. In the general case, these segments are taken below the abscissa axis so close to it that no eigenvalues of A_0 and A_1 lie between the abscissa axis and the corresponding horizontal segment.

The main properties of the spectral flow are given by the following theorem.

THEOREM 3.3. 1) (*Well-definedness*) *The spectral flow* (3.1) *is well defined, i.e., independent of the choice of an admissible broken line.*

2) (*Additivity*) *The spectral flow of the concatenation of two homotopies is the sum of the spectral flows of the two parts*:

$$\mathrm{sf}\{A_t\}_{t\in[0,2]} = \mathrm{sf}\{A_t\}_{t\in[0,1]} + \mathrm{sf}\{A_t\}_{t\in[1,2]}.$$

[2]Or, more precisely, define

3) (*Homotopy invariance*) *The spectral flow is invariant under deformations of the family* $\{A_t\}_{t\in[0,1]}$ *such that the operators* A_0 *and* A_1 *either do not change at all or change in such a way that the multiplicity of the zero eigenvalue remains constant.*

Proof. 1) To prove that the notion of spectral flow is well defined, it suffices to verify that the expression (3.1) is invariant under the following transformations of the admissible broken line:

- subdivision of segments or taking the union of neighboring adjacent horizontal segments;
- a vertical displacement of a horizontal segment.

Indeed, two arbitrary admissible broken lines can be transformed into each other by a sequence of such transformations via a sequence of admissible broken lines.

The invariance of the spectral flow with respect to the first transformation is fairly obvious. As to the second transformation, one should exhaust all possible mutual arrangements of the segment to be moved and its nearest neighbors. This is an easy but somewhat painstaking task, for there are eight distinct possibilities. However, one can avoid the exhaustion by rewriting the expression for the spectral flow in terms of the index of Fredholm operators as follows:

$$\mathrm{sf}\,\{A_\tau\}_{\tau\in[0,1]} = \sum_{i=1}^{N-1} \mathrm{ind}(\Pi_{\lambda_i}(A_{\tau_i}), \Pi_{\lambda_{i-1}}(A_{\tau_i})). \tag{3.2}$$

Here $\Pi_\lambda(\cdot)$ is the *spectral projection* of a self-adjoint operator on the subspace spanned by eigenvectors with eigenvalues $\geq \lambda$, and $\mathrm{ind}(P,Q)$ is the *relative index* of projections P and Q with compact difference, defined as the index of the following Fredholm operator:

$$\mathrm{ind}(P,Q) \stackrel{\mathrm{def}}{=} \mathrm{ind}(Q : \operatorname{Im} P \longrightarrow \operatorname{Im} Q).$$

EXERCISE 3.4. Check that the expressions (3.1) and (3.2) for the spectral flow are equivalent.

Now if one broken line is obtained from another by the replacement $\lambda_i \mapsto \lambda_i'$ in a single segment, then the difference of the spectral flows computed via these broken lines is equal to the difference

$$\mathrm{ind}(\Pi_{\lambda_i}(A_{\tau_i}), \Pi_{\lambda_i'}(A_{\tau_i})) - \mathrm{ind}(\Pi_{\lambda_i}(A_{\tau_{i+1}}), \Pi_{\lambda_i'}(A_{\tau_{i+1}})) \tag{3.3}$$

of relative indices. To obtain this relation, it is convenient to use the cyclic property

$$\mathrm{ind}(P,Q) + \mathrm{ind}(Q,R) = \mathrm{ind}(P,R)$$

of the relative index of projections.

Now the difference (3.3) is zero by virtue of the following fact of functional analysis: if a continuous family of self-adjoint operators does not have eigenvalues λ and μ, then the number of eigenvalues (counted according to their multiplicities) between λ and μ is constant. Thus we have proved that the spectral flow is well defined.

2) Additivity now follows by construction.

3) Homotopy invariance. Formula (3.2) shows that the spectral flow is invariant under small deformations of the family $\{A_t\}_{t\in[0,1]}$ as long as the broken line remains admissible. Now the invariance of the spectral flow under large deformations follows with regard to the fact that the definition is independent of the choice of an admissible broken line. □

Remark 3.5. The reader has possibly noticed that the definition of the spectral flow essentially uses only the self-adjointness of operators and the discreteness of the spectrum in the vicinity of zero. Thus the definition is actually given in the framework of functional analysis. A detailed discussion of the spectral flow from the viewpoint of functional analysis can be found in the literature cited at the end of the chapter.

3.2.2. The spectral flow formula for periodic families

First, note that the knowledge of the principal symbol is not sufficient for computing the spectral flow; this is already seen in the matrix case, where the symbol is zero identically. However, the spectral flow is completely determined by the principal symbol in the class of *periodic* families ($A_0 = A_1$). In this subsection, we compute the spectral flow in terms of the principal symbol.[3]

In fact, the spectral flow formula is written out in terms of some vector bundle. To construct this bundle, recall that the principal symbol of an elliptic self-adjoint operator A on a manifold M determines the vector bundle

$$\operatorname{Im}\Pi_+\sigma(A) \in \operatorname{Vect}(S^*M)$$

over the bundle S^*M of unit spheres (with respect to some Riemannian metric) in the cotangent bundle T^*M. The vector bundle is constructed as follows. The symbol of an elliptic self-adjoint operator is Hermitian and invertible, and so its positive spectral projection $\Pi_+\sigma(A)$ is a smooth function on the spheres. The desired vector bundle is just the range of this projection.

THEOREM 3.6. *The spectral flow of a periodic family $\mathcal{A} = \{A_t\}_{t\in\mathbb{S}^1}$ of elliptic self-adjoint operators is given by the formula*

$$\operatorname{sf}\{A_t\}_{t\in\mathbb{S}^1} = \left\langle \operatorname{ch}\left[\operatorname{Im}\Pi_+\sigma(\mathcal{A})\right]\operatorname{Td}\left(T^*M\otimes\mathbb{C}\right), \left[S^*M\times\mathbb{S}^1\right]\right\rangle. \tag{3.4}$$

Here $\operatorname{ch}\left[\operatorname{Im}\Pi_+\sigma(\mathcal{A})\right] \in H^{ev}\left(S^*M\times\mathbb{S}^1\right)$ *is the Chern character of the bundle*

$$\operatorname{Im}\Pi_+\sigma(\mathcal{A}) \in \operatorname{Vect}\left(S^*M\times\mathbb{S}^1\right)$$

determined by the principal symbol of the family, Td *is the Todd class, and* $\langle\cdot,\left[S^*M\times\mathbb{S}^1\right]\rangle$ *is the value of a homology class of the top degree on the fundamental cycle.*

The idea of the derivation of formula (3.4) is to express the spectral flow as the index of some elliptic operator on a closed manifold and then apply the Atiyah–Singer formula. We state these two parts (analytical and topological) of the proof in the form of the following two propositions.

PROPOSITION 3.7. *The spectral flow of a periodic family $\{A_t\}$ is equal to the index of the elliptic operator*

$$D = \frac{\partial}{\partial t} + A_t, \quad t\in\mathbb{S}^1, \tag{3.5}$$

on the torus $M\times\mathbb{S}^1$.

Sketch of proof. The operator (3.5) is elliptic, since the family A_t is elliptic and self-adjoint.

The equality of the spectral flow to the index can be proved by the technique of spectral boundary value problems (see Sec. 1.3). Namely, we cut the torus, thus obtaining the cylinder $M\times[0,1]$ with the

[3] And hence in topological terms.

operator D defined on it. Consider the family $\mathcal{D}_T$, $0 \le T \le 1$, of spectral boundary value problems for the restrictions of D to the family of cylinders $M \times [0,T]$. By the theorem on the index and the spectral flow, the index variation on the interval $[\varepsilon, T]$ is equal to the spectral flow:

$$\operatorname{ind} \mathcal{D}_T - \operatorname{ind} \mathcal{D}_\varepsilon = \operatorname{sf} \{A_t\}_{t\in[\varepsilon,T]} \,. \tag{3.6}$$

Without loss of generality we can assume that the operator $A_0 = A_1$ is invertible. (This can be achieved by a small deformation that does not change the spectral flow.) The index on the small cylinder of length ε is zero (since the problem is uniquely solvable).

To prove the proposition, by (3.6), we should establish that the index of the spectral problem at $T = 1$ is equal to the index of the operator D on the torus. This reduction of the index of a boundary value problem to the index of an operator on a closed manifold can be carried out as follows. One can verify that the linear homotopy between the spectral boundary condition and the periodic boundary condition

$$u|_{t=0} = u|_{t=1}$$

preserves the ellipticity of the boundary value problem. In particular, it follows that the indices of the operators with these boundary conditions are the same. Now one can readily show that the kernel and cokernel of the problem with the periodic boundary condition are isomorphic to the kernel and cokernel of the operator on the torus, respectively. □

PROPOSITION 3.8. *One has*

$$\operatorname{ind} D = \left\langle \operatorname{ch}\left[\operatorname{Im} \Pi_+ \sigma(\mathcal{A})\right] \operatorname{Td}\left(T^*M \otimes \mathbb{C}\right), \left[S^*M \times \mathbb{S}^1\right] \right\rangle . \tag{3.7}$$

Sketch of proof. The Atiyah–Singer formula

$$\operatorname{ind} D = \left\langle \operatorname{ch}[\sigma(D)] \operatorname{Td}(T^*M \otimes \mathbb{C}), [T^*(M \times \mathbb{S}^1)] \right\rangle \tag{3.8}$$

expresses the index of the operator D via the Chern character of the difference construction $[\sigma(D)] \in K^0_c(T^*(M \times \mathbb{S}^1))$ in K-theory of the cotangent space. This element is determined by the principal symbol of the operator. To rewrite the formula (3.8) in terms of the original family A_t, we note that the element $[\sigma(D)]$ can be expressed via the symbol of the original family by the formula

$$[\sigma(D)] = \partial[\operatorname{Im} \Pi_+ \sigma(\mathcal{A})] \tag{3.9}$$

as the image of the element $[\operatorname{Im} \Pi_+ \sigma(\mathcal{A})] \in K(S^*M \times \mathbb{S}^1)$ under the coboundary mapping

$$\partial : K(S^*M \times \mathbb{S}^1) \longrightarrow K^1_c(T^*M \times \mathbb{S}^1) \simeq K_c(T^*(M \times \mathbb{S}^1)),$$

corresponding to the pair $\mathbb{S}^1 \times S^*M \subset \mathbb{S}^1 \times B^*M$. (Here B^*M is the unit ball bundle.) Transferring the computation of the pairing (3.7) from the cotangent bundle $T^*(M \times \mathbb{S}^1)$ to the product $S^*M \times \mathbb{S}^1$, one can show that the Atiyah–Singer formula acquires the desired form (3.7). □

3.3. The Atiyah–Patodi–Singer Eta Invariant

3.3.1. Definition of the eta invariant

If A is an elliptic self-adjoint operator on a closed manifold M, then the signature of the corresponding quadratic form is undefined, for there are infinitely many eigenvalues.

Now let us view the signature as the sum of the series whose terms are the signs of the eigenvalues of A. In the infinite-dimensional case, we regularize this series by analogy with the ζ-function. Namely, we define the spectral *η-function* of A by the formula

$$\eta\left(A,s\right)=\sum_{\lambda_j\in\operatorname{Spec}A,\lambda_j\neq 0}\frac{\operatorname{sgn}\lambda_i}{\left|\lambda_i\right|^s}\equiv\operatorname{Tr}\left(A\left(A^2\right)^{-s/2-1/2}\right),\tag{3.10}$$

where the λ_j are the nonzero eigenvalues of A. The function (3.10) is analytic in the half-plane $\operatorname{Re}s>\dim M/\operatorname{ord}A$ (where the series converges absolutely).

DEFINITION 3.9. (Atiyah, Patodi and Singer 1975) The *η-invariant* of the operator A is the real number

$$\eta\left(A\right)=\frac{1}{2}\left(\eta\left(A,0\right)+\dim\ker A\right)\in\mathbb{R}.\tag{3.11}$$

Clearly, the η-invariant of an invertible operator in the finite-dimensional case is equal to the signature of the corresponding quadratic form up to the factor $1/2$. However, to make Definition 3.9 meaningful in the general case, we have to continue the η-function analytically to the point $s=0$.

THEOREM 3.10 ((Atiyah, Patodi and Singer 1976), (Gilkey 1981)). *The η-function of an elliptic operator on a smooth closed manifold has a meromorphic continuation into the entire complex plane, possibly, with poles at the points $s_j=\frac{\operatorname{ord}A-j}{\dim M}$, $j\in\mathbb{Z}_+$. There is no pole at the point $s=0$.*

A detailed proof of this difficult theorem can be found in the cited papers (see also (Gilkey 1989*a*)). Hence we only describe the key points of the proof.

1. Meromorphic continuability. For positive definite operators, the η-function coincides with the ζ-function ($\eta(A,s)\equiv\zeta(A,s)$), which is well known to be meromorphic (e.g., see (Seeley 1967)). If the operator is not sign definite, then one can use the expression

$$\eta\left(A,s\right)=\frac{\zeta\left(A_+,s\right)-\zeta\left(A_-,s\right)}{2^s-1}\tag{3.12}$$

of the η-function via the ζ-functions of the positive definite operators $A_\pm=(3\left|A\right|\pm A)/2$.

2. Holomorphy at zero. It follows from formula (3.12) that the η-function may have at most a simple pole at zero, since the ζ-function is holomorphic there. The residue at this point is equal to

$$\operatorname{Res}_{s=0}\eta\left(A,s\right)=\frac{\zeta\left(A_+,0\right)-\zeta\left(A_-,0\right)}{\ln 2}.\tag{3.13}$$

Seeley (Seeley 1967) showed that the ζ-invariants can be represented as integrals over the manifold of some expressions determined by the complete symbol of A. Moreover, the corresponding integrand is in general nonzero. However, it was shown in (Atiyah, Patodi and Singer 1976) for odd-dimensional manifolds and in (Gilkey 1981) for even-dimensional manifolds that the residue is still zero! Thus, the η-function is holomorphic at zero and the η-invariant is well defined.

Remark 3.11. In contrast to the spectral flow, which can be defined in a purely functional-analytic situation, the definition of the η-invariant heavily relies on the fact that the operator is defined on a closed manifold. For example, the η-function of self-adjoint boundary value problems on a manifold with boundary may have a pole at zero (Grubb and Seeley 1996). It is interesting that so far there is no *analytic* proof of the holomorphy of the η-function at zero. (The proofs cited above use global topological methods.) The vanishing of the residue can be verified by a straightforward analytic computation only for Dirac type operators (e.g., see (Bismut and Freed 1986)).

EXAMPLE 3.12. Consider the operator family

$$A_t = -i\frac{d}{d\varphi} + t$$

on a circle of length 2π with coordinate φ, where t is a real constant. Let us compute the η-invariant of this operator. The spectrum of A_t is the lattice $t + \mathbb{Z}$. Hence the η-invariant is a 1-periodic function of the parameter t. For $0 < t < 1$, we can arrange the eigenvalues into pairs and transform the η-function to the form

$$\eta\left(A_t, s\right) = \sum_{n\geq 1}\left[(n+t)^{-s} - (n-t)^{-s}\right] + t^{-s}.$$

This series absolutely converges on the half-line $s > 0$, and the limit as $s \to +0$ is equal to $-2t+1$. One can readily establish the latter by substituting the Taylor expansion

$$\left[(n+t)^{-s} - (n-t)^{-s}\right] = -2tsn^{-s} + O\left(\frac{s}{n^{2+s}}\right)$$

into the series. Thus, the η-invariant is equal to

$$\eta\left(A_t\right) = \frac{\eta\left(A_t, 0\right) + \dim\ker A_t}{2} = \frac{1}{2} - \{t\},$$

where $\{\cdot\} \in [0,1)$ is the fractional part.

3.3.2. Variation under deformations of the operator

In the preceding example, the η-invariant of a smooth elliptic family proves to be piecewise smooth. Moreover, the (integer) jumps occur at the parameter values where some eigenvalue changes its sign. It turns out that the η-invariant is a piecewise smooth function of the parameter in the general case as well. More precisely, the following result is valid.

THEOREM 3.13 ((Atiyah, Patodi and Singer 1976)). *Let A_t, $t \in [0,1]$, be a smooth family of elliptic self-adjoint operators. Then the function $\eta\left(A_t\right)$ is piecewise smooth and admits the expansion*

$$\eta\left(A_{t'}\right) - \eta\left(A_0\right) = \operatorname{sf}\left(A_t\right)_{t\in[0,t']} + \int_0^{t'} \omega\left(t_0\right) dt_0 \tag{3.14}$$

into a piecewise constant function (*the spectral flow*) *and a smooth function of the parameter, which can be expressed as*

$$\omega\left(t_0\right) = \left.\frac{d}{dt}\zeta\left(B_{t,t_0}\right)\right|_{t=t_0} \in C^\infty[0,1]$$

via the derivative of the ζ-invariant of the auxiliary family $B_{t,t_0} = |A_{t_0}| + P_{\ker A_{t_0}} + (t - t_0)\dot{A}_{t_0}$. Here $P_{\ker A}$ is the projection on the kernel of A and the ζ-invariant is given by the expression

$$\zeta(A) \stackrel{\text{def}}{=} \zeta(A, 0)/2.$$

Sketch of proof. If the family is invertible, then one can express the derivatives of the η- and ζ-functions by the formulas

$$\frac{d}{dt}\zeta(B_t, s) = -s\mathrm{Tr}\left(\dot{B}_t\, B_t^{-s-1}\right), \qquad \frac{d}{dt}\eta(A_t, s) = -s\mathrm{Tr}\left(\dot{A}_t\, \left(A_t^2\right)^{-\frac{1}{2}(s+1)}\right).$$

This readily implies (3.14) for $s = 0$ and $t = t_0$.

To prove (3.14) in the general case, one should use the broken line occurring in the definition of spectral flow (see Fig. 3.2), which permits one to deal with invertible operators alone. □

This result is of practical importance owing to the following *Seeley formula* for the η-invariant. If A is a positive semidefinite operator whose complete symbol has the asymptotic expansion

$$\sigma(A) \sim a_m + a_{m-1} + a_{m-2} + \ldots, \quad m = \mathrm{ord}\, A,$$

then the ζ-invariant can be computed according to the formula

$$2\zeta(A) = \frac{1}{(2\pi)^{\dim M}\mathrm{ord}A}\int\limits_{S^*M} dx d\xi \int\limits_0^\infty b_{-\dim M - \mathrm{ord}A}(x, \xi, -\lambda)\, d\lambda, \tag{3.15}$$

where the coefficient $b_{-\dim M - \mathrm{ord}A}(x, \xi, -\lambda)$ is determined by the recursion relation

$$b_{-m-j}(x, \xi, \lambda)(a_m(x, \xi) - \lambda) + \sum_{\substack{k+l+|\alpha|=j \\ l>0}} \frac{1}{\alpha!}(-i\partial_\xi)^\alpha b_{-m-k}(x, \xi, \lambda)(-i\partial_x)^\alpha a_{m-l}(x, \xi) = 0, \tag{3.16}$$

and moreover, $b_{-m}(x, \xi, \lambda) = (a_m(x, \xi) - \lambda)^{-1}$.

Thus, by substituting the Seeley formula (3.16) into Theorem 3.13, we obtain a closed-form local expression for the derivative of the η-invariant in terms of the homogeneous components of the complete symbol of the operator.

Remark 3.14. If we transpose the integral in (3.14) to the left-hand side, then we obtain the new definition

$$\mathrm{sf}\{A_t\}_{t\in[0,1]} = \eta(A_1) - \eta(A_0) - \int_0^1 \left(\frac{d}{dt}\eta(A_t)\right) dt$$

of the spectral flow via the η-invariant. This is an analog of the expression of the spectral flow via the signature in the finite-dimensional case. Although at first glance this seems to be an expression of a simpler object via a more complicated one, it often proves useful (e.g., see (Zhang 2000)) if there are independent methods for the computation of the η-invariant.

3.3.3. Homotopy invariance. Examples

The cases in which the η-invariant is homotopy invariant are especially important in applications in geometry and topology. Unfortunately, in contrast to the signature, the η-invariant is not homotopy invariant even in the class of invertible operators. More precisely, for deformations in this class the spectral flow is always zero and only the second, smooth term in formula (3.14) survives. To verify that this component is also zero in some narrower operator classes, it is useful to analyze symmetries of the Seeley formula for the ζ-invariant. Some examples in which such symmetries can be used are considered in the remaining part of this subsection.

Eta invariants and flat bundles. Consider the formula (3.14). It follows from this formula that although the η-invariant itself is not local, its variation modulo the spectral flow already enjoys this property. This observation permits us to conclude that if we take two locally coinciding operators A and B and deform them while maintaining the local coincidence, then the difference of their η-invariants will be constant under this deformation. Of course, we should precisely explain what the "local coincidence of operators" means. Instead of giving any general definition, we consider a special case in which this notion arises.

One can obtain a pair of locally coinciding operators by twisting an operator by a flat bundle.

Recall that a *flat bundle* $\gamma \in \operatorname{Vect}(M)$ is a vector bundle with locally constant transition functions. For a given operator

$$A : C^\infty(M, E) \longrightarrow C^\infty(M, F),$$

we can define an *operator with coefficients in the flat bundle*, denoted by

$$A \otimes 1_\gamma : C^\infty(M, E \otimes \gamma) \longrightarrow C^\infty(M, F \otimes \gamma),$$

by patching together local expressions of the direct sum of $\dim \gamma$ copies of A in the charts with the use of the locally constant transition functions of the bundle. To preserve the self-adjointness, one should also require that the transition functions be unitary matrices. If A is a *pseudo*differential operator, then the corresponding operator with coefficients in the flat bundle is determined modulo smoothing operators.

EXAMPLE 3.15. The operator $-id/d\varphi + t$ is isomorphic to the operator $-id/d\varphi \otimes 1_\gamma$ with coefficients in the bundle γ on the circle with transition function $e^{2\pi i t}$. The isomorphism

$$e^{-it\varphi}\left(-i\frac{d}{d\varphi}\right)e^{it\varphi} = -i\frac{d}{d\varphi} + t$$

is provided by the trivialization $e^{it\varphi}$ of the bundle γ.

PROPOSITION 3.16 ((Atiyah, Patodi and Singer 1976)). *The fractional part of the η-invariant is homotopy invariant in the class of direct sums*

$$A \otimes 1_\gamma \oplus (-(\dim \gamma) A)$$

with an arbitrary self-adjoint elliptic operator A, where γ is a given flat bundle.

Proof. The operators $A \otimes 1_\gamma$ and $(\dim \gamma) A$ are locally isomorphic. Hence the derivatives of their η-invariants coincide,

$$\frac{d}{dt}\left\{\eta\left(A_t \otimes 1_\gamma\right)\right\} = \frac{d}{dt}\left\{n\eta\left(A_t\right)\right\},$$

in view of the fact that the derivative of the η-invariant is local. □

Eta invariant and the Gilkey conditions. The preceding example was based on the locality of the ζ-invariant, and now we use another property of the formula (3.15). Namely, the ζ-invariant is obtained by integration over the cosphere bundle. The integral is obviously zero if the integrand is an odd function on the sphere, for the contributions of the antipodal points ξ and $-\xi$ cancel each other. This idea is realized for differential operators. Namely, the components of the complete symbol of a differential operator are homogeneous polynomials and have the symmetry

$$a_k(x,-\xi)=(-1)^k a_k(x,\xi).$$

By substituting these relations into (3.15) and (3.14), we obtain the following result.

THEOREM 3.17 ((Gilkey 1989*b*)). *The fractional part of the spectral η-invariant of an elliptic self-adjoint differential operator A on a manifold M is homotopy invariant if the parity condition*

$$\operatorname{ord} A+\dim M\equiv 1(\operatorname{mod} 2) \tag{3.17}$$

is satisfied.

Sketch of proof. First, consider the case of even-order operators. By induction, we establish the following homogeneity of the coefficients determined by the operators B_t (see the recursion relations (3.16)):

$$b_j\left(x,-\xi,\lambda,t\right)=(-1)^j\, b_j\left(x,\xi,\lambda,t\right). \tag{3.18}$$

Hence we obtain

$$\frac{d}{dt}\left\{\eta\left(A_t\right)\right\}=\frac{d}{dt}\zeta\left(B_t\right)=\mathrm{Const}\frac{d}{dt}\left(\int\limits_{S^*M} dxd\xi\int\limits_0^\infty b_{-\dim M-\operatorname{ord} A}\left(x,\xi,-\lambda,t\right)\right)d\lambda.$$

It follows from the homogeneity (3.18) and the condition that the manifold is odd-dimensional that under the parity condition (3.17) the integrand $b_{-\dim M-\operatorname{ord} A}\left(x,\xi,-\lambda,t\right)$ is an odd function on the sphere S^*_xM. Hence the integral is zero and we obtain the desired relation

$$\frac{d}{dt}\left\{\eta\left(A_t\right)\right\}=0.$$

For odd-order operators, we have the opposite homogeneity

$$b_j\left(x,-\xi,\lambda,-t\right)=(-1)^{j+1}\, b_j\left(x,\xi,\lambda,t\right).$$

By substituting these homogeneous functions into the formula for the ζ-invariant, we obtain

$$\frac{d}{dt}\zeta\left(B_t\right)=\frac{d}{dt}\zeta\left(B_{-t}\right).$$

Thus the derivative is also zero. □

Note that as a by-product we have even proved that the ζ-invariant of even-order differential operators on odd-dimensional manifolds is zero.

3.4. The Eta Invariant of Families with Parameter (Melrose's Theory)

It turns out that one can also arrive at the η-invariant starting from another finite-dimensional invariant, the winding number of a nonzero function, rather than the signature as in Sec. 3.3. In this section, we show how this invariant can be transferred to the infinite-dimensional case.

Recall that if $f(p)$ is a nonvanishing complex-valued function on the real line with coordinate $p \in \mathbb{R}$, then the winding number $w(f) \in \mathbb{Z}$ of f around zero is defined by the formula

$$w(f) = \frac{1}{2\pi i} \int_{\mathbb{R}} f^{-1} df.$$

The winding number is well defined if, say, the function is constant at infinity. Then it is well known that the integral is equal to the number of revolutions of the point $f(p)$ around zero $\mathbb{C} \setminus \{0\}$ as the parameter runs over the real line. A similar formula is valid for invertible matrix functions:

$$w(F) = \frac{1}{2\pi i} \int_{\mathbb{R}} \operatorname{tr}(F^{-1} dF). \tag{3.19}$$

(In this case, one speaks of the winding number of the determinant of the family.)

Let us describe the relationship, which we wish to transfer to the infinite-dimensional case, between the signature and the winding number. If A is an invertible Hermitian matrix, then the family $F = p - iA$ is invertible on the real line. An easy computation shows that the rotation number of this function is equal to the signature of A.

We are interested in an infinite-dimensional generalization of the winding number (3.19) to operator-valued functions. In this case, the expression $F^{-1}dF$ is still well-defined (provided that the family is sufficiently smooth). However, the remaining part of the formula may well be meaningless: the trace may fail to exist, and the integral may diverge. Thus, to define an analog of the winding number, we should generalize these two notions.

It turns out that one can make a beautiful generalization of the trace and integral by considering *families of parameter-dependent operators*, well known in analysis (e.g., see (Shubin 1985)), rather than arbitrary families. This construction will be carried out in the next subsection.

3.4.1. A trace on the algebra of parameter-dependent operators

Parameter-dependent operators. A family of differential *operators of order m with parameter $p \in \mathbb{R}$* is a family $A(p)$ obtained from a translation-invariant mth-order differential operator on the cylinder $X \times \mathbb{R}$ by the Fourier transform with respect to the variable $t \in \mathbb{R}$. (Here t and p are dual variables.) If we denote the operator on the cylinder by

$$A\left(x, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial t}\right),$$

then the corresponding family is $A(x, -i\partial/\partial x, p)$. The *symbol* of the parameter-dependent family is just the symbol $A(x, \xi, p)$ of the original operator.

One can also define parameter-dependent pseudodifferential operators, for which the parameter occurs as an additional covariable in the symbol. The reader can find a detailed exposition, say, in (Shubin 1985). Here we are concise on the subject.

Locally, a parameter-dependent symbol of order m must satisfy the estimates

$$\left|\partial_x^\alpha \partial_{\xi,p}^\beta \sigma(x,\xi,p)\right| \leq C_{\alpha,\beta,\gamma}(1+|\xi|^2+p^2)^{m-|\beta|}$$

for arbitrary multiindices α and β. The corresponding parameter-dependent operator is constructed from the symbol by the standard (Schrödinger) quantization. A parameter-dependent operator is said to be *smoothing* if, viewed as a function of the parameter p, it belongs to the Schwartz space of smooth functions ranging in the space of smoothing operators on X and rapidly decaying at infinity.

By analogy with the case of a single operator, one defines the notion of a *classical* parameter-dependent operator. The symbol $\sigma(x,\xi,p)$ of such an operator admits an asymptotic expansion

$$\sigma(x,\xi,p) \sim \sum_{j\geq 0} \sigma_{m-j}(x,\xi,p)$$

with terms σ_{m-j} homogeneous in the covariables (ξ,p) of degree $m-j$, where $j\in\mathbb{Z}$.

For a closed manifold X, by $\Psi_p(X)$ we denote the set of classical parameter-dependent operators supplemented by smoothing parameter-dependent operators. The set $\Psi_p(X)$ is an algebra with respect to pointwise addition and multiplication of operator families. This algebra is filtered by the subspaces $\Psi_p^m(X)$ of operators of order $\leq m$.

The formula (3.19) for the winding number contains the functional

$$\int_{\mathbb{R}} \operatorname{tr} A(p)dp.$$

In the remaining part of this section, we show how one can extend this functional, originally defined only for families $A(p)$ lying in the subalgebra $\Psi_p^{-n-2}(X)$, where $n=\dim X$, to families of arbitrary order. This continuation permits us to define the η-invariant of families in the next subsection.

Obviously, we obtain the desired continuation of the functional once we are able to define the notions of *trace* and *integral* for families of arbitrary order. Let us perform these tasks successively.

Regularization of the trace. The regularization procedure for the trace is based on the following two facts: a) the trace of a family is well defined if the family has a sufficiently large negative order; b) the order of the family can be diminished by differentiation with respect to the parameter. Thus the *regularized trace* can be defined as

$$(\operatorname{TR} A)(p) \overset{def}{=} \int_0^p \int_0^{p_1} \ldots \int_0^{p_{k-1}} \operatorname{tr}\left[\left(\frac{\partial}{\partial q}\right)^k A(q)\right] dq dp_{k-1}\ldots dp_1. \tag{3.20}$$

The resulting function of a real variable is well defined for $k>m+n$ and has asymptotics of a special form at infinity:

LEMMA 3.18. *One has the asymptotic expansion*

$$(\operatorname{TR} A)(p) \sim \sum_{i\leq k} p^i c_i^{\pm} + \sum_{j=0}^{k} p^j d_j^{\pm} \ln|p| \tag{3.21}$$

as $p\to\pm\infty$.

The proof of the expansion (3.21) is based on the fact that the trace takes a classical parameter-dependent operator to a classical symbol depending on the covariable p and raises the order by the dimension n of the manifold X. One can verify this using the local expression for the trace. Now the relation (3.21) is obtained by k-fold integration of the classical symbol.

One can readily verify that the traces $\mathrm{TR}\,A$ corresponding to various numbers k differ by polynomials. Thus if by $S_{log}(\mathbb{R})$ we denote the space of smooth functions possessing asymptotic expansions of the form (3.21) for some k, then Lemma 3.18 shows that there is a well-defined mapping

$$\mathrm{TR} : \Psi_p(X) \longrightarrow S_{log}(\mathbb{R})/\mathcal{P},$$

where $\mathcal{P}$ is the subalgebra of polynomials. Note that the space $S_{log}(\mathbb{R})$ is invariant with respect to differentiation and integration.

Regularization of the integral. The regularized trace may prove to be a function growing at infinity. To integrate such functions, we use the Eisenstein and Hadamard rules.

The integral of a function belonging to the space $S_{log}(\mathbb{R})$ over the interval $(-T, T)$ viewed as a function of the parameter T has an asymptotic expansion of the form (3.21). We define the *regularized integral* as the constant term in this asymptotic expansion as $T \to +\infty$:

$$⨍_{\mathbb{R}} f(p)dp \stackrel{\mathrm{def}}{=} c_0, \text{ where } \int_{-T}^{T} f(p)dp \sim \sum_{i\le k} T^i c_i + \sum_{j=0}^{k} T^j d_j \ln T. \tag{3.22}$$

It follows from the definition that the regularized integral coincides with the ordinary integral on functions that decay at infinity at least as p^{-2}. Note the following simple properties of the regularized integral: it vanishes on all odd functions as well as all polynomials.

By combining (3.21) with (3.22), we arrive at the following assertion, which is the main result of this subsection.

PROPOSITION 3.19. *The expression*

$$\overline{\mathrm{Tr}}A \stackrel{\mathrm{def}}{=} ⨍_{\mathbb{R}} (\mathrm{TR}\,A)\,dp$$

is a trace (*i.e., satisfies* $\overline{\mathrm{Tr}}AB = \overline{\mathrm{Tr}}BA$) *on the algebra of parameter-dependent operators and an extension of the trace*

$$\int_{\mathbb{R}} \mathrm{tr}\,A(p)dp$$

from the subalgebra $\Psi_p^{-n-2}(X)$.

3.4.2. Definition of the Melrose eta invariant

By analogy with ordinary operators, we introduce the notion of ellipticity for parameter-dependent operators.

DEFINITION 3.20. A parameter-dependent operator $A(p)$ is said to be *elliptic* if its principal symbol $A(x, \xi, p)$ is invertible for $|\xi|^2 + p^2 \neq 0$.

The ellipticity of a parameter-dependent operator implies not only that the operators in the family are Fredholm but also that they are invertible for sufficiently large parameter values (e.g., see (Shubin 1985)).

Let $A(p)$ be an invertible elliptic parameter-dependent operator. Then the *Melrose η-invariant* of A is defined as

$$\eta(A) = \frac{1}{2\pi i}\overline{\mathrm{Tr}}(A^{-1}\dot{A}).$$

PROPOSITION 3.21. 1) *The η-invariant has the logarithmic property*

$$\eta(AB) = \eta(A) + \eta(B).$$

2) *If $1 + S(p)$ is an invertible family where $S(p)$ is a family of finite-dimensional operators vanishing at infinity, then the η-invariant of the family is equal to the winding number*:

$$\eta(1+S) = \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{tr}((1+S)^{-1}dS).$$

Proof. The logarithmic property is a special case of the following algebraic fact: if some algebra $\mathcal{A}$ is equipped with a trace $T : \mathcal{A} \to \mathbb{C}$ and a derivation d, then the functional $T(A^{-1}dA)$ defined on invertible elements possesses the logarithmic property. The second property immediately follows from the definition. □

3.4.3. Relationship with the Atiyah–Patodi–Singer eta invariant

The term "η-invariant" in the preceding subsection is not occasional: it turns out that the Atiyah–Patodi–Singer η-invariant is a special case of the Melrose η-invariant. Namely, if A is an elliptic self-adjoint first-order differential operator, then $D(p) = p - iA$ is an elliptic parameter-dependent operator. This family is invertible if we additionally assume that so is A.

THEOREM 3.22.

$$\eta(A) = \eta(p - iA).$$

The proof is rather cumbersome (see (Lesch and Pflaum 2000)). Hence we restrict ourselves to heuristic computations showing why this is the case.

Consider the integral representation

$$\eta(A,s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty t^{(s-1)/2}\mathrm{tr}(Ae^{-tA^2})dt$$

of the η-function. In a number of cases (say, for the Dirac operator), the integral converges absolutely even for $s = 0$. By formally substituting the factor

$$1 = (\sqrt{\pi})^{-1}\int_{\mathbb{R}} t^{1/2}e^{-t\tau^2}d\tau$$

into the integral and by reversing the order of integration, we obtain the "expression"

$$\eta(A) = \frac{\eta(A,0)}{2} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}\left(A(A^2+\tau^2)^{-1}\right)d\tau \tag{3.23}$$

for the η-invariant. On the other hand, for the Melrose η-invariant we have

$$\eta(p-iA)=\frac{1}{2\pi i}\fint_{\mathbb{R}}\mathrm{TR}((p-iA)^{-1})dp=\frac{1}{2\pi}\fint_{\mathbb{R}}\mathrm{TR}(A(p^2+A^2)^{-1})dp. \tag{3.24}$$

In the last equation, we have used the fact that the regularized integral vanishes on odd functions.

The expressions (3.23) and (3.24) obviously coincide.

3.4.4. Locality of the derivative of the eta invariant. Examples

Just as with the classical η-invariant, the derivative of the Melrose η-invariant also has the locality property. Let us obtain a formula for the derivative of the η-invariant in terms of symbols of the operators involved.

Let $A_t(p)$ be a smooth homotopy of invertible elliptic parameter-dependent operators. Then the derivative of the η-invariant with respect to t is given by the expression

$$\frac{d}{dt}\eta(A_t)=\frac{1}{2\pi i}\overline{\mathrm{Tr}}\Big(\frac{\partial}{\partial p}\left(A_t^{-1}\dot{A}_t\right)\Big).$$

EXERCISE 3.23. Derive this formula.

A straightforward computation shows that the composition of the trace $\overline{\mathrm{Tr}}$ with differentiation with respect to p is also a trace, which will be denoted by

$$\widetilde{\mathrm{Tr}}\overset{\mathrm{def}}{=}\overline{\mathrm{Tr}}\circ\frac{\partial}{\partial p}.$$

This trace vanishes on operators of large negative order (where the regularization of the integral and the trace is not needed). A convenient formula was obtained by Melrose.

THEOREM 3.24. *Let B be a parameter-dependent operator defined in some coordinate neighborhood on a manifold X. Then*

$$\widetilde{\mathrm{Tr}}B=\lim_{L\to\infty}\int_{|\eta|\le L}\left[\mathrm{tr}\,b_{-\dim X}(y,\eta,1)-\mathrm{tr}\,b_{-\dim X}(y,\eta,-1)\right]\frac{\omega^n}{n!},\quad (y,\eta)\in T^*X,$$

where ω is the symplectic form on the cotangent bundle and b_k is the degree k component of the complete symbol of B.

Sketch of proof. 1. Since the trace is local, it suffices to prove the formula for homogeneous symbols. It also suffices to consider components of homogeneity degree $\ge -\dim X$. Such components $b(y,\eta,p)$ are locally integrable on $T^*X\times\mathbb{R}$ for all p, and so in the construction of the operator from the symbol they require no smoothing at $\eta=p=0$.

2. Consider a homogeneous component $b(x,\eta,p)$ of the symbol. Then the value of the trace $\widetilde{\mathrm{Tr}}$ coincides with the constant term in the asymptotic expansion as $p\to\infty$ of the integral

$$\int_{-p}^{p}\int_{-1}^{p_{k-1}}\dots\int_{-1}^{p_1}\int_{\mathbb{R}^{2n}}\frac{\partial^k}{\partial q^k}b(y,\eta,q)\,dyd\eta dqdp_1\dots dp_{k-1}.$$

The inner integral over (y,η) is a homogeneous distribution of degree $\operatorname{ord} b + \dim X - k$. Hence the remaining integration gives a homogeneous distribution of degree $\operatorname{ord} b + \dim X$ plus a polynomial. The polynomial does not contribute to the regularized integral, while the homogeneous function contributes if and only if $\operatorname{ord} b = -\dim X$. Thus it remains to consider the latter case.

3. In the computation of the contribution of the homogeneous component of degree $-\dim X$, it suffices to take $k=1$. Then, by Newton–Leibniz,

$$\fint_{\mathbb{R}} \int_{\mathbb{R}^{2n}} \frac{\partial b}{\partial q} dy d\eta dq = \operatorname{reg-}\lim_{p\to\infty} \int_{\mathbb{R}^{2n}} [b(y,\eta,p) - b(y,\eta,-p)] dy d\eta,$$

where $\operatorname{reg-}\lim$ is the regularized limit (the constant term in the asymptotic expansion). After the change of variables $\eta = p\eta'$, we see that this limit is equal to the integral

$$\int_{\mathbb{R}^{2n}} [b(y,\eta,1) - b(y,\eta,-1)] dy d\eta.$$

The proof of the formula is complete. □

Using the formula for the derivative of the η-invariant, one can construct classes of operators in which the η-invariant is homotopy invariant. These results are parallel to those in Sec. 3.3.3. Let us state them in the form of examples.

EXERCISE 3.25. The fractional part of the Melrose η-invariant is homotopy invariant in the class of direct sums

$$D(p) \otimes 1_\gamma \oplus (-(\dim \gamma) D(p))$$

for elliptic family $D(p)$ with a given flat bundle γ. Here the family $D \otimes 1_\gamma$ is defined by twisting with γ in a standard manner.

EXERCISE 3.26. The fractional part of the Melrose η-invariant is homotopy invariant in the class of operators whose complete symbols $b \sim b_m + b_{m-1} + \dots$ possess the symmetry

$$b_k(-\xi,-p) = (-1)^k b_k(\xi,p)$$

provided that the manifold on which the family is defined is even-dimensional.

EXERCISE 3.27. The fractional part of the Melrose η-invariant is homotopy invariant in the class of operators whose complete symbols $b \sim b_m + b_{m-1} + \dots$ possess the symmetry

$$b_k(-\xi,p) = (-1)^k b_k(\xi,p),$$

provided that the manifold on which the family is defined is odd-dimensional.

◀ In the last two exercises, one should note first that these symmetries single out subalgebras in the algebra of classical parameter-dependent operators. Then, using the Melrose formula, one verifies that the symbolic trace $\widetilde{\operatorname{Tr}}$ vanishes on these subalgebras. ▶

3.5. The Spectral Flow of Families of Parameter-Dependent Operators

Recall that for self-adjoint operators we know the definition of the spectral flow via the η-invariant (see Remark 3.14). Needless to say, the "working" definition is the one in which the spectral flow is described as the number of eigenvalues passing through zero. For parameter-dependent operators, we have also defined the η-invariant. Using this η-invariant, we can construct an analog of the spectral flow. What does this new spectral flow count? Can it be defined without using the η-invariant? In the present section, we shall answer these questions. For the time being, let us make several simple but important observations.

Recall that the ordinary spectral flow counts the number of eigenvalues passing through zero. At the same time, we have noted above that the self-adjoint theory can be embedded in the theory of parameter-dependent operators by assigning the family $p - iA$ to a self-adjoint operator A. This family is analytic, and the spectral flow is just the number of zeros of the family (counted according to their multiplicities) crossing the real axis.

Thus, to define the spectral flow for general families, we should introduce the notion of "multiplicity of zero" for analytic operator functions. From the viewpoint of complex analysis, it is more natural to deal not only with *holomorphic* but also with *meromorphic* operator functions. In this case, one should define the multiplicity of a singular point of a meromorphic operator function. This is done in the next subsection.

3.5.1. Meromorphic operator functions. Multiplicities of singular points

DEFINITION 3.28.
1. An operator function $D(p) : H \longrightarrow G$ (where H and G are Hilbert spaces) in a domain $U \subset \mathbb{C}$ is said to be *finitely meromorphic* if it is meromorphic and the coefficients of the principal parts of its Laurent series expansions at all poles are finite rank operators.
2. An operator function $D(p)$ is said to be *strongly finitely meromorphic* if both $D(p)$ and $D^{-1}(p)$ is finitely meromorphic in U.

In a sense, strongly finitely meromorphic operator functions are invertible neglecting a finite-dimensional subspace.

PROPOSITION 3.29. *Let $D(p)$ be a strongly finitely meromorphic operator function in a neighborhood of a point p_0. Then there exists direct sum decompositions*

$$H = H_0 \oplus H_0^{\perp}, \quad G = G_0 \oplus G_0^{\perp}$$

with finite-dimensional subspaces H_0 and G_0 such that in the corresponding block matrix representation

$$D(p) = \begin{pmatrix} D_{11}(p) & D_{12}(p) \\ D_{21}(p) & D_{22}(p) \end{pmatrix} \tag{3.25}$$

of $D(p)$ the only infinite-dimensional component $D_{22}(p)$ is holomorphic and invertible[4] *in a neighborhood of p_0.*

[4] Recall that if an operator function is holomorphic and invertible, then the inverse is also holomorphic.

Remark 3.30. This is an operator function counterpart of the following well known fact: *a Fredholm operator becomes invertible when restricted to appropriate subspaces of finite codimension.*

Proof. 1. First, let us ensure the holomorphy. For brevity, we set $p_0 = 0$.

Consider the decompositions $H = H_1 \oplus H_1^\perp$ and $G = G_1 \oplus G_1^\perp$, where G_1 is the (finite-dimensional) linear span of ranges of the coefficients of the principal part of the Laurent series

$$D(p) = \frac{D_{-N}}{p^N} + \ldots + \frac{D_{-1}}{p} + D_0 + D_1 p + \ldots$$

and H_1 is the corresponding subspace for the inverse function $D^{-1}(p)$.

In this decomposition, our operator functions have the representations

$$D(p) = \begin{pmatrix} D_{11}(p) & D_{12}(p) \\ D_{21}(p) & D_{22}(p) \end{pmatrix}, \quad D^{-1}(p) = \begin{pmatrix} C_{11}(p) & C_{12}(p) \\ C_{21}(p) & C_{22}(p) \end{pmatrix},$$

where only the blocks in the first row may be nonholomorphic at p_0. Let us show that $D_{22}(p)$ is Fredholm up to $p = p_0$. This easily follows from the obvious identities

$$D_{22}(p)C_{22}(p) = 1 - D_{21}(p)C_{12}(p), \quad C_{22}(p)D_{22}(p) = 1 - C_{21}(p)D_{12}(p).$$

Indeed, here the left-hand sides are holomorphic. Hence so are the products of finite rank operator functions on the right-hand sides. Letting $p \to p_0$, we find that $C_{22}(p_0)$ is an almost inverse of $D_{22}(p_0)$.

2. To ensure invertibility, from the subspaces $H_1^\perp$ and $G_1^\perp$ we subtract the kernel and cokernel, respectively, of the Fredholm operator $D_{22}(p_0)$. Thus we finally set $H_0 = H_1 \oplus \ker D_{22}(p_0)$ and $G_0 = G_1 \oplus \operatorname{coker} D_{22}(p_0)$. □

EXERCISE 3.31. Using this proposition, prove that a strongly finitely meromorphic operator function $D(p)$ can be represented in a neighborhood of an arbitrary point (even a pole!) as

$$D(p) = (1 + K(p))Q(p), \tag{3.26}$$

where $Q(p)$ is holomorphic and invertible and $K(p)$ has a uniformly finite rank: $\operatorname{rank} K(p) \le N < \infty$.

◀ *Hint.* Use the invertibility of D_{22} and factorize the operator function (3.25) as the product of upper- and lower-triangular matrices. ▶

The poles of $D(p)$ and $D^{-1}(p)$ will be called *singular points* of $D(p)$. Now let us define the notion of multiplicity for singular points.

DEFINITION 3.32. The *multiplicity of a singular point* $p = p_0$ of a strongly finitely meromorphic operator function $D(p)$ is the integer

$$m_D(p_0) = \operatorname{Tr} \operatorname{Res}_{p=p_0} \left(D^{-1} \frac{dD}{dp} \right).$$

It is also sometimes useful to represent the multiplicity by the Cauchy type integral

$$m_D(p_0) = \frac{1}{2\pi i} \operatorname{Tr} \left(\int_{|p-p_0|=\varepsilon} D^{-1} \frac{dD}{dp} dp \right) \tag{3.27}$$

over a circle of small radius ε centered at p_0.

These expressions are obviously well defined (since the coefficients of the Laurent series are of finite rank). It is also clear that for scalar functions they indeed give the multiplicity of the singular point. However, the fact that the multiplicity is always *integer* requires a subtler proof. The following theorem comprises the main properties of multiplicities, including the fact that they are integer.

THEOREM 3.33. *The multiplicities of singular points have the following properties.*

1) *The logarithmic property*:

$$m_{D_1 D_2}(p_0) = m_{D_1}(p_0) + m_{D_2}(p_0).$$

2) *For a strongly finitely meromorphic operator function with a triangular block matrix*[5]

$$D(p) = \begin{pmatrix} D_1(p) & * \\ 0 & D_2(p) \end{pmatrix}, \tag{3.28}$$

one has $m_D(p_j) = m_{D_1}(p_j) + m_{D_2}(p_j)$.

3) *Integrality*: $m_D(p_0) \in \mathbb{Z}_+$.

Proof. First, note that the functional $\operatorname{Tr}\operatorname{Res}_{p=p_0}$ on the set of finitely meromorphic operator functions is a trace, i.e., satisfies

$$\operatorname{Tr}\operatorname{Res}_{p=p_0}(AB) = \operatorname{Tr}\operatorname{Res}_{p=p_0}(BA).$$

This property permits one to exchange the operator functions D and D^{-1} in the expression for the multiplicity (the sign of the expression then changes to the opposite) and change the order of factors (without change in the sign).

1. The logarithmic property can be verified by a straightforward algebraic computation with the use of the fact that $\operatorname{Tr}\operatorname{Res}_{p=p_0}$ is a trace.

2. The formula (3.28) for the multiplicity of a singular point of a triangular operator function can also be obtained by a straightforward computation. On the one hand, the trace of an operator with a triangular matrix can be computed via the diagonal entries. On the other hand, the diagonal entries in the product $D^{-1}dD/dp$ are just $D_{1,2}^{-1}dD_{1,2}/dp$.

3. Now let us prove that the multiplicities are integer. To this end, we use the factorization (3.26). By the logarithmic property, it suffices to prove the desired assertion for the multiplicity $m_{1+K}(p_0)$ of the strongly finitely meromorphic operator function $1 + K(p)$ with meromorphic function $K(p)$ whose rank is bounded by some number N.

In this finite-dimensional case, the multiplicity is given by the formula

$$m_{1+K}(p_0) = w_{p_0}(\det(1 + K(p))) \in \mathbb{Z}, \tag{3.29}$$

where $w_{p_0}(f)$ is the degree of the singular point $p = p_0$ of a function f. To prove this, it suffices to note that the integrand in the expression of the multiplicity by the Cauchy integral (3.27) is just the logarithmic derivative of the determinant:

$$\operatorname{Tr}((1 + K(p))^{-1}\dot{K}(p)) = \frac{d}{dp}\ln\det(1 + K(p)).$$

By integrating this relation, we obtain (3.29).

Thus we have proved that the multiplicity is an integer. □

[5]With respect to some direct sum decompositions $H = H_1 \oplus H_2$ and $G = G_1 \oplus G_2$.

Homotopy invariance of the multiplicities. The multiplicities possess an important property: they are homotopy invariant (and hence constant) under continuous deformations.

Consider a family $D_t(p)$, $t \in [0,1]$, of strongly finitely meromorphic operator functions in a domain $U \subset \mathbb{C}$.

PROPOSITION 3.34. *The sum of multiplicities of singular points of $D_t(p)$ in a subdomain $V \Subset U$ with smooth boundary $\partial\overline{V}$ is independent of the parameter t provided that the following conditions are satisfied*:

1) *the family $D_t(p)$ is continuous in $(t,p) \in [0,1] \times \partial\overline{V}$*;
2) *no singular point of $D_t(p)$ lies on the boundary $\partial\overline{V}$ for any t*;
3) *the dimension of the linear span of the ranges of the coefficients (and their adjoints) in the principal parts of the Laurent series of $D_t(p)$ and $D_t^{-1}(p)$ at the singular points lying in V does not exceed some finite number N independent of t.*

Proof. The sum of multiplicities over V is an integer, and so to establish the desired invariance property it suffices to show that the sum continuously depends on t.

Indeed, the sum of multiplicities is given by the Cauchy integral (3.27) over ∂V. By assumptions 1) and 2) of the theorem, the integrand continuously depends on the parameter t in the *operator norm*. However, to prove the continuous dependence of the multiplicity on t, we should prove that the integral itself is continuous in the *trace norm*. To this end, we shall prove that the rank of the integral is bounded uniformly in t. Since the operator and trace norms are equivalent on the set of operators with rank uniformly bounded by some number M, this will imply the desired assertion.

Let us establish the uniform boundedness of the rank of the integral

$$\frac{1}{2\pi i}\int_{\partial\overline{V}} D^{-1}dD.$$

Consider some parameter value $t = t_0$. Then the family $D_{t_0}(p)$ has finitely many singular points $p_1, \dots, p_M$ in V. The integral in question is the sum of residues at these points. Consider the Laurent series of our operator functions at a singular point (for brevity, we assume that this singular point is zero):

$$\begin{aligned}\frac{d}{dp}D(p) &= \frac{D_{-N}}{p^N} + \dots + \frac{D_{-1}}{p} + D_0 + D_1 p + \dots,\\ D^{-1}(p) &= \frac{C_{-N}}{p^N} + \dots + \frac{C_{-1}}{p} + C_0 + C_1 p + \dots\end{aligned}$$

Then the contribution to the integral of this point is the sum

$$\sum_{i+j=-1} C_i D_j.$$

We split the sum into two parts, one with negative i and the other with negative j. The rank of the sum of the first parts over all singular points is bounded by N by virtue of the third assumption of the theorem. (The dimension of the linear span of the ranges of all C_i, $i < 0$, is bounded by N.) The rank of the sum of second parts can be estimated in a similar way. (Here one uses the fact that the linear span of the ranges of all D_i^*, $i < 0$, has a dimension than does not exceed N.)

Thus, the rank of the integral is uniformly bounded. The proof is complete. □

Remark 3.35. Note that the last assumption of the theorem is automatically satisfied if the operator function in question is finite-dimensional or holomorphic in the entire V.

◀ One can prove that this condition holds for a holomorphic operator function by analyzing the proof of Gokhberg's theorem (e.g., see (Egorov and Schulze 1997)) saying that a Fredholm holomorphic family invertible at least at one point is strongly finitely meromorphic. More precisely, one observes that if the operator function continuously depends on a parameter, then the number of poles of the inverse function, their orders, and their ranks are bounded locally uniformly with respect to the parameter. ▶

3.5.2. Definition of the spectral flow

Now that we have studied the multiplicities of singular points of general strongly finitely meromorphic operator functions, we can define the spectral flow of families of such operator functions.

Let M be a smooth closed manifold. We consider holomorphic parameter-dependent pseudodifferential operators $D(p)$ defined in a neighborhood of a sector $|\arg p| \leq c < \pi/2$ with nonzero angle and elliptic with parameter p in the sector.

It is well known (e.g., see (Egorov and Schulze 1997)) that such an operator function is finitely meromorphic and each strip $|\operatorname{Re} p| < \lambda$ of finite width (lying in the domain where the operator function is defined) contains only finitely many singular points of the operator function. Thus the multiplicities of singular points considered in the preceding subsection are well defined for parameter-dependent elliptic operators.

The *spectral flow* of a continuous homotopy $D_t(p), t \in [0,1]$, of parameter-dependent elliptic operators is the integer equal to the net number of singular points of the family crossing the line $\operatorname{Im} p = 0$ upwards in the course of the homotopy. One can give a rigorous definition of this notion using the methods of Section 3.2. Namely, we take an admissible broken line with vertices (t_i, γ_i),

$$0 = t_0 < t_1 < \cdots < t_N = 1,$$

i.e. such that the operator function $D_t(p)$ is holomorphic and invertible on the line $\operatorname{Im} p = \gamma_i$ for all $t \in [t_{i-1}, t_i]$.

DEFINITION 3.36. The *spectral flow* of the homotopy $\{D_t\}_{t\in[0,1]}$ is the number

$$\operatorname{sf}\{D_t\}_{t\in[0,1]} = \sum_{i=1}^{N-1} N_i, \tag{3.30}$$

where

$$N_i = \begin{cases} -\sum\limits_{\operatorname{Im} p_j \in (\gamma_i, \gamma_{i+1})} m_{D_{t_i}}(p_j) & \text{if } \gamma_i < \gamma_{i+1}, \\ \sum\limits_{\operatorname{Im} p_j \in (\gamma_{i+1}, \gamma_i)} m_{D_{t_i}}(p_j) & \text{if } \gamma_i \geq \gamma_{i+1}. \end{cases}$$

Here the p_j are the singular points of the conormal symbol D_{t_i} in the strip between the weight lines $\operatorname{Im} p = \gamma_i$ and $\operatorname{Im} p = \gamma_{i+1}$.

THEOREM 3.37. *The spectral flow has the following properties.*

1) *It is independent of the choice of an admissible broken line.*

2) *The spectral flow depends only on the homotopy class of a path D_t with fixed beginning and end. It also remains unchanged under deformations of D_t such that the beginning and the end are not fixed but the families D_0 and D_1 have no singular points on the real axis for any values of the homotopy parameter.*

The proof is similar to that of Theorem 3.3 on the ordinary spectral flow. □

EXERCISE 3.38. Check that the spectral flow of a homotopy $\{A_t\}_{t\in[0,1]}$ of elliptic self-adjoint operators coincides with the spectral flow of the corresponding homotopy of operator functions $p - iA_t$:

$$\mathrm{sf}\{A_t\}_{t\in[0,1]} = \mathrm{sf}\{p - iA_t\}_{t\in[0,1]}.$$

Remark 3.39. The reader possibly noticed that the definition of the spectral flow in this section, as well as Theorem 3.37, remains valid for abstract strongly finitely meromorphic operator functions. One has only to require that these operator functions have at most finitely many singular points in a neighborhood of the real axis $\mathbb{R} \subset \mathbb{C}$.

3.6. Higher Spectral Flows

The spectral flow and the η-invariant are nearly as universal as the index itself, which, we recall, occurs in various disguises depending on the geometric context: for a single operator, it is an integer; for group-invariant operators, it is a virtual representation, for operator families, it is an element of a K-group, and so on. Similar generalizations also exist for the invariants considered in this chapter. By way of example, in this section we consider the notion of *higher spectral flow*, i.e. the spectral flow with values in a K-group. This will be necessary in subsequent parts of the book. The reader can find other generalizations in special literature.

There is a more powerful approach to the definition of spectral flow than that in the preceding sections. It is based on the idea of transversality. The notion of a spectral section is a milestone of this approach.

3.6.1. Spectral sections

Let $A = \{A_x\}_{x\in X}$ be a continuous family of elliptic self-adjoint operators on a closed manifold with compact parameter space X. For some parameter values, the operators in this family may not be invertible. Accordingly, the corresponding family of spectral projections is not continuous in the general case.

DEFINITION 3.40. A *spectral section* of a family $\{A_x\}_{x\in X}$ is a continuous family $\{P_x\}_{x\in X}$ of projections that differ from the spectral projections by compact operators:

$$P_x - \Pi_+(A_x) \in \mathcal{K}.$$

EXERCISE 3.41. Prove that there exists a spectral section if and only if there exists a perturbation of the family $\{A_x\}_{x\in X}$ by a family of compact operators such that the perturbed family is invertible.

It turns out that there exists a *topological obstruction* to the existence of spectral sections. Indeed, according to (Atiyah and Singer 1969) a family of elliptic self-adjoint operators has a well-defined "index," which is an element of the K^1-group of the parameter space. In the simplest way, this index is defined by the formula

$$\operatorname{ind} A \stackrel{\text{def}}{=} \operatorname{ind}(p - iA) \in K_c(X \times \mathbb{R}) \simeq K^1(X)$$

as the ordinary index of the family $p - iA$. This family of Fredholm operators is parametrized by the product $X \times \mathbb{R}$ and invertible outside a compact subset of this space. Consequently, its index indeed belongs to the K-group with compact supports.

This index is zero for an invertible family and is preserved under perturbations of the family by compact operators. Thus we obtain an obstruction to the existence of spectral sections. It turns out that this is the only obstruction.

THEOREM 3.42. *There exists a spectral section of a family A if and only if the index* $\operatorname{ind} A \in K^1(X)$ *is zero.*

The proof can be found in (Melrose 1995). □

EXAMPLE 3.43. If the parameter space is the interval $[0, 1]$, then there always exists a spectral section.

EXAMPLE 3.44. If the parameter space is the circle $\mathbb{S}^1$, i.e., we deal with a periodic family, then there exists a single integer-valued obstruction to the existence of a spectral section. By way of example, the reader can verify that the obstruction coincides in this case with the spectral flow of the periodic family.

Let us use the notion of spectral section to give a new definition of the spectral flow.

Definition of the spectral flow via spectral sections. Let $A = \{A_t\}_{t\in[0,1]}$ be a homotopy of elliptic self-adjoint operators. By P_t we denote some spectral section of this homotopy.

DEFINITION 3.45. The *spectral flow* of the homotopy $\{A_t\}_{t\in[0,1]}$ is the number

$$\operatorname{sf}\{A_t\}_{t\in[0,1]} = -\operatorname{ind}(\Pi_+(A_0), P_0) + \operatorname{ind}(\Pi_+(A_1), P_1).$$

EXERCISE 3.46. Check that this definition coincides with the definition of the spectral flow in Sec. 3.2.

◀ *Hint.* Use the homotopy invariance of the relative index of projections: if P_t and Q_t are continuous families of projections with compact difference $P_t - Q_t$, then the relative index remains constant: $\operatorname{ind}(P_t, Q_t) = \text{const}$. ▶

3.6.2. Spectral flow of homotopies of families of self-adjoint operators

Using the notion of a spectral section, one can readily give the definition of the spectral flow for the case of families.

Let $\{A_{t,x}\}, (t, x) \in [0, 1] \times X$ be a homotopy of families of elliptic self-adjoint operators. Suppose that for $t = 0$ and $t = 1$ the families are invertible. Then by Theorem 3.42 there exists a spectral section $P_{t,x}$ on the product $[0, 1] \times X$.

DEFINITION 3.47. The *spectral flow* of the family $A = \{A_{t,x}\}$, $(t,x) \in [0,1] \times X$, is the element

$$\mathrm{sf}\{A_t\}_{t\in[0,1]} = -\operatorname{ind}(\Pi_+(A_0), P_0) + \operatorname{ind}(\Pi_+(A_1), P_1) \in K(X)$$

of the even K-group.

The spectral flow thus defined possesses all properties of the ordinary integer spectral flow (see Theorem 3.3). In other words, the spectral flow is independent of the choice of a spectral section, additive, and homotopy invariant.

Note that in this case one can also give a definition in terms of admissible broken lines (Nazaikinskii, Savin, Schulze and Sternin 2002). Using broken lines whose links depend on the parameters, one can quite explicitly construct a virtual bundle representing the spectral flow. Unfortunately, the construction is complicated and will not be presented here. We restrict ourselves to the following simplest case.

EXERCISE 3.48. Show that if a family admits a broken line with vertices (τ_i, λ_i) *independent* of the parameters $x \in X$, then the spectral flow can be computed by the well-known formula (3.2):

$$\mathrm{sf}\,\{A_\tau\}_{\tau\in[0,1]} = \sum_{i=1}^{N-1} \operatorname{ind}(\Pi_{\lambda_i}(A_{\tau_i}), \Pi_{\lambda_{i-1}}(A_{\tau_i})) \in K(X).$$

Here ind is the relative index of families of projections.

3.6.3. Spectral flow of homotopies of families of parameter-dependent operators

One can introduce and successfully use a notion similar to that of a spectral section for families of parameter-dependent operators.

Spectral sections. Consider a family $D_x(p)$ of operators elliptic with parameter p in a sector of nonzero angle and holomorphic in p in a neighborhood of the sector. We assume that the variable x ranges over a compact set X.

We define a *spectral section* of the family as an invertible family $\widetilde{D}_x(p)$ that differs from $D_x(p)$ by a family of smoothing operators. By analogy with the self-adjoint case, there exists a topological obstruction to the existence of spectral sections. It is given by the index

$$\operatorname{ind} D \in K_c(X \times \mathbb{R}) \simeq K^1(X).$$

THEOREM 3.49. *A family $D_x(p)$ admits a spectral section if and only if*

$$\operatorname{ind} D = 0 \in K^1(X).$$

Proof. The necessity is obvious. The proof of sufficiency can be found in (Nistor 2003). □

Definition of the spectral flow. Let $D_{t,x}(p)$ be a homotopy of families of parameter-dependent elliptic operators. We require that the family be invertible for $t = 0$ and $t = 1$. Then the index of the family is zero, and by Theorem 3.49 there exists a spectral section, which we denote by $\widetilde{D}_{t,x}(p)$.

DEFINITION 3.50. The *spectral flow* of the homotopy $D_{t,x}(p)$ of families of parameter-dependent operators is the element

$$\operatorname{sf} D = -[D_0\widetilde{D}_0^{-1}] + [D_1\widetilde{D}_1^{-1}] \in K_c^1(X \times \mathbb{R}) \simeq K(X)$$

of the even K-group.

Let us comment on the last formula. The families $D_t\widetilde{D}_t^{-1}$ have the form "1 plus a compact-valued family." Recall that the space of invertible operators of the form $1 + K$ has the homotopy type of the infinite-dimensional unitary group[6] $U(\infty)$, i.e., is the classifying space of the K^1-group (see (Atiyah 1989)). Since the quotient $D_t\widetilde{D}_t^{-1}$ tends to unity at infinity, it follows that it defines an element of the K-group with compact supports. Such elements are denoted by $[\cdot]$.

The reader may state and prove the properties of the spectral flow in this case. In particular, in the absence of additional parameters ($X = pt$) the spectral flow coincides with the similar notion defined in Sec. 3.5.2.

Note that the paper (Nazaikinskii, Savin, Schulze and Sternin 2002) contains an explicit construction of the spectral flow not based on spectral sections.

3.7. Bibliographical Remarks

The spectral flow and η-invariant of elliptic self-adjoint operators were introduced in the remarkable series of papers (Atiyah, Patodi and Singer 1975). The study of the η-invariant from the viewpoint of analysis in this chapter is based on the paper (Seeley 1967). This paper contained some inaccuracies, which were later rectified in (Agranovich 1994, Grubb and Seeley 1996). A detailed exposition of these invariants can be found in the monographs (Gilkey 1995, Gilkey 1989*a*, Booß-Bavnbek and Wojciechowski 1993).

The η-invariant of parameter-dependent elliptic operators was constructed in (Melrose 1995) and subsequently studied in (Lesch and Pflaum 2000). The spectral flow in this context was considered in (Nazaikinskii and Sternin Juli 1999).

The notion of a spectral section was introduced in (Melrose and Piazza 1997). Higher spectral flows were considered in (Dai and Zhang 1998).

[6]One can readily prove this by approximating compact operators by finite-dimensional ones.

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