Toeplitz Operators, and Ellipticity of Boundary Value Problems with Global Projection Conditions

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Abstract. Ellipticity of (pseudo-) differential operators A on a compact manifold X with boundary (or with edges) Y is connected with boundary (or edge) conditions of trace and potential type, formulated in terms of global projections on Y together with an additional symbolic structure. This gives rise to operator block matrices $\mathcal{A}$ with A in the upper left corner. We study an algebra of such operators, where ellipticity is equivalent to the Fredholm property in suitable scales of spaces: Sobolev spaces on X plus closed subspaces of Sobolev spaces on Y which are the range of corresponding pseudo-differential projections. Moreover, we express parametrices of elliptic elements within our algebra and discuss spectral boundary value problems for differential operators.

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Introduction

Ellipticity of differential (and pseudo-differential) operators on a manifold X with boundary $Y = \partial X$ is connected with a specific control of data near the boundary. More generally, ellipticity on a manifold with edges or with higher (say, polyhedral) singularities includes conditions on the lower-dimensional strata (e.g., edges, corners, etc.) of the configuration.

Such conditions may occur as trace or potential operators, linked to the singularities.

It is a common point of view to interpret A together with the trace and potential conditions as operator block matrices $\mathcal{A}$ (with A as upper left corner) and to construct an algebra of such block matrices that contains the parametrices of elliptic elements, see Vishik and Eskin [**50**], Eskin [**11**], Boutet de Monvel [**6**], Rempel and Schulze [**30**], [**31**], or Schulze [**38**]. In many known cases this is a transparent and satisfying concept. The operators $\mathcal{A}$ then have a principal symbolic hierarchy $\sigma(\mathcal{A})$, and in simplest cases, ellipticity is invertibility of all components of $\sigma(\mathcal{A})$, where parametrices belong to $\sigma^{-1}(\mathcal{A})$.

To illustrate phenomena we first consider the case of boundary value problems for elliptic differential operators A on a compact C^∞ manifold X with boundary Y with Shapiro-Lopatinskij elliptic conditions (also called SL-ellipticity in this paper).

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By a boundary value problem in standard form we understand an equation $Au = f$, together with boundary conditions $Tu = g$, where $A : C^\infty(X, E) \to C^\infty(X, F)$ for $E, F \in \mathrm{Vect}(X)$, is a given differential operator of order μ and $T = {}^{\mathrm{t}}(T_1, \dots, T_N)$ a column vector of trace operators $T_j : C^\infty(X, E) \to C^\infty(Y, G_j)$, for $G_j \in \mathrm{Vect}(Y)$, $j = 1, \dots, N$. Here $\mathrm{Vect}(\cdot)$ denotes the set of all smooth complex vector bundles on the space in the brackets. The operators T_j are assumed to be given in the form $T_j = \mathrm{r}'B_j$, where r' is the operator of restriction to the boundary, and $B_j : C^\infty(X, E) \to C^\infty(X, \widetilde{G}_j)$ are differential operators of order μ_j, with $\widetilde{G}_j \in \mathrm{Vect}(X)$, $G_j := \widetilde{G}_j|_Y$, $j = 1, \dots, N$. The column matrix operator

$$\mathcal{A} := \begin{pmatrix} A \\ T \end{pmatrix} : C^\infty(X, E) \to \begin{matrix} C^\infty(X, F) \\ \oplus \\ \bigoplus_{j=1}^N C^\infty(Y, G_j) \end{matrix} \tag{0.1}$$

then extends to continuous operators

$$\mathcal{A} : H^s(X, E) \to \begin{matrix} H^{s-\mu}(X, F) \\ \oplus \\ \bigoplus_{j=1}^N H^{s-\mu_j-\frac{1}{2}}(Y, G_j) \end{matrix} \tag{0.2}$$

between the respective Sobolev spaces of distributional sections for all real $s > \max(\mu_j + \frac{1}{2})$. The principal symbolic hierarchy in this case consists of two components, namely

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})), \tag{0.3}$$

where $\sigma_\psi(\mathcal{A}) := \sigma_\psi(A)$ is the standard homogeneous principal symbol of A of order μ, also called the principal interior symbol, which is a bundle morphism

$$\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F, \tag{0.4}$$

$\pi_X : T^*X \setminus 0 \to X$, and $\sigma_\partial(\mathcal{A})$ is the principal boundary symbol of $\mathcal{A}$. To give a definition of the principal boundary symbol we first look at A. Let us fix a collar neighbourhood $\cong Y \times [0, 1)$ of the boundary in the local splitting of variables $x = (y, t) \in \Omega \times [0, 1)$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, with covariables $\xi = (\eta, \tau)$. (For convenience, transition diffeomorphisms near the boundary are assumed to be independent of t.) We then have a family of continuous operators

$$\sigma_\partial(A)(y, \eta) := \sigma_\psi(A)(y, 0, \eta, D_t) : \mathbb{C}^k \otimes H^s(\mathbb{R}_+) \to \mathbb{C}^l \otimes H^{s-\mu}(\mathbb{R}_+) \tag{0.5}$$

for $(y, \eta) \in T^*\Omega \setminus 0$, acting in Sobolev spaces $H^s(\mathbb{R}_+) := H^s(\mathbb{R})|_{\mathbb{R}_+}$ in normal direction to the boundary; k and l are the fibre dimensions of E and F, respectively. Using the invariance of (0.5) under transition maps we obtain a bundle morphism

$$\sigma_\partial(A) : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* F' \otimes H^s(\mathbb{R}_+), \tag{0.6}$$

$\pi_Y : T^*Y \setminus 0 \to 0$, where prime indicates the restriction of the corresponding bundle to the boundary. In a similar manner we can proceed for the trace operators $T_j = \mathrm{r}'B_j$ and obtain

$$\sigma_\partial(T_j)(y, \eta) := \mathrm{r}'\sigma_\psi(B_j)(y, 0, \eta, D_t) : \mathbb{C}^k \otimes H^s(\mathbb{R}_+) \to \mathbb{C}^{l_j}$$

for $(y, \eta) \in T^*\Omega \setminus 0$, where l_j is the fibre dimension of G_j. Globally, we have again bundle morphisms

$$\sigma_\partial(T_j) : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* G_j,$$

$j = 1, \dots, N$, and we set altogether

$$\sigma_\partial(\mathcal{A}) := \begin{pmatrix} \sigma_\partial(A) \\ \sigma_\partial(T) \end{pmatrix} : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ G \end{pmatrix} \tag{0.7}$$

for $\sigma_\partial(T) := {}^{\mathrm{t}}(\sigma_\partial(T_1), \dots, \sigma_\partial(T_N))$ and $G := \bigoplus_{j=1}^N G_j$.

A boundary value problem $\mathcal{A}$ is called SL-elliptic, if (0.4) is an isomorphism, and if also (0.7) is an isomorphism for any sufficiently large real s. If we talk about the operator A alone, the first condition will also be referred to as σ_ψ-ellipticity of A.

The choice of s is unessential for the condition of SL-ellipticity. Moreover, (0.7) is an isomorphism if and only if

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \to \pi_Y^* \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ G \end{pmatrix} \tag{0.8}$$

is an isomorphism, $\mathcal{S}(\overline{\mathbb{R}}_+) := \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$. Simple examples for SL-elliptic boundary value problems are the Dirichlet or the Neumann problem for Laplace operators.

Let us now observe the specific homogeneity of boundary symbols. For $u(t) \in H^s(\mathbb{R}_+)$ we set

$$(\kappa_\lambda u)(t) := \lambda^{\frac{1}{2}} u(\lambda t), \qquad \lambda \in \mathbb{R}_+. \tag{0.9}$$

In this way we obtain a strongly continuous group $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms on the space $H^s(\mathbb{R}_+)$ (here, if H is a Hilbert space, a group $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : H \to H$, $\lambda \in \mathbb{R}_+$, with $\kappa_\lambda \kappa_\varrho = \kappa_{\lambda\varrho}$ for all $\lambda, \varrho \in \mathbb{R}_+$, is called strongly continuous, if $\kappa_\lambda h \in C(\mathbb{R}_+, H)$ for every $h \in H$).

It can easily be verified that

$$\sigma_\partial(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\partial(A)(y, \eta) \kappa_\lambda^{-1} \tag{0.10}$$

for all $\lambda \in \mathbb{R}_+$. Similarly, we have

$$\sigma_\partial(T_j)(y, \lambda\eta) = \mathrm{r}' \lambda^{\mu_j} \kappa_\lambda \sigma_\partial(B_j)(y, \eta) \kappa_\lambda^{-1} = \lambda^{\mu_j + \frac{1}{2}} \sigma_\partial(T_j)(y, \eta) \kappa_\lambda^{-1} \tag{0.11}$$

for all $\lambda \in \mathbb{R}_+$, where we empoly the relation $\mathrm{r}' \circ \kappa_\lambda = \lambda^{\frac{1}{2}} \mathrm{r}'$ for the corresponding operators on functions in t on the half-axis. This gives us

$$\sigma_\partial(\mathcal{A})(y, \lambda\eta) = \operatorname{diag}(\lambda^\mu \kappa_\lambda, \lambda_1^{\mu_1 + \frac{1}{2}}, \ldots, \lambda_N^{\mu_N + \frac{1}{2}}) \sigma_\partial(\mathcal{A})(y, \eta) \kappa_\lambda^{-1} \tag{0.12}$$

as the homogeneity of boundary symbols.

On the manifold X we fix a Riemannian metric that is equal to the product metric from $Y \times [0, 1)$ in a collar neighbourhood of Y. We then have the absolute values of covectors on X and Y in an invariant way.

It is often convenient to unify the orders of trace operators by composing them from the left by classical pseudo-differential operators R_j on Y of order $\mu - \mu_j - \frac{1}{2}$ with homogeneous principal symbol $|\eta|^{\mu - \mu_j - \frac{1}{2}} \mathrm{id}_{\pi_Y^* G_j}$, such that

$$R_j : H^{s - \mu_j - \frac{1}{2}}(Y, G_j) \to H^{s - \mu}(Y, G_j)$$

are isomorphisms for all s. Such a choice of R_j is always possible; we then talk about a corresponding reduction of orders on the boundary.

Set $\sigma_\partial(R_j T_j)(y, \eta) := |\eta|^{\mu - \mu_j - \frac{1}{2}} \sigma_\partial(T_j)(y, \eta)$. We now replace the former T by ${}^{\mathrm{t}}(R_1 T_1, \ldots, R_N T_N)$ and denote the new boundary value problem again by $\mathcal{A} = \binom{A}{T}$. This gives us continuous operators

$$\mathcal{A} : H^s(X, E) \to \begin{matrix} H^{s-\mu}(X, F) \\ \oplus \\ H^{s-\mu}(Y, G) \end{matrix} \tag{0.13}$$

for all sufficiently large s.

For the boundary symbol $\sigma_\partial(\mathcal{A})$ we have

$$\sigma_\partial(\mathcal{A})(y, \lambda\eta) = \lambda^\mu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \sigma_\partial(\mathcal{A})(y, \eta) \kappa_\lambda^{-1} \tag{0.14}$$

for all $\lambda \in \mathbb{R}_+$, $(y, \eta) \in T^*Y \setminus 0$, where 1 is the identity map in $\pi_Y^* G$.

There is now the following natural question. Given a σ_ψ-elliptic differential operator $A : H^s(X, E) \to H^{s-\mu}(X, F)$ on X, does there always exist an SL-elliptic boundary value problem $\mathcal{A} = \binom{A}{T}$ with a suitable trace operator T? The answer is negative, 'unfortunately' for many interesting geometric operators, e.g., Dirac operators in even dimensions, in particular, for the Cauchy-Riemann operator in the complex plane.

For the existence there is a well known condition of Atiyah and Bott [**3**] that we want to recall here. First note that when A is σ_ψ-elliptic, the boundary symbol

$$\sigma_\partial(A) : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \tag{0.15}$$

is a family of Fredholm operators (surjective in the case of differential operators). Let S^*Y be the unit cosphere bundle induced by T^*Y with the canonical projection $\pi_1 : S^*Y \to Y$. The restriction of (0.15) to S^*Y gives us a family of Fredholm operators parametrised by the compact set S^*Y. As such it represents an index element in the K-group of S^*Y, namely

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A) \in K(S^*Y), \tag{0.16}$$

cf. [**2**] and Section 2.1 below. In the present case, $L_+ := \ker \sigma_\partial(A)$ is a vector bundle on $T^*Y \setminus 0$, and we have $\operatorname{ind}_{S^*Y} \sigma_\partial(A) = [L_+|_{S^*Y}]$, where $[\dots]$ denotes the element in the K-group on S^*Y, represented by the bundle in the brackets. Now if there is an SL-elliptic boundary value problem $\mathcal{A}$ for A, i.e., the corresponding boundary symbol (0.7) is an isomorphism, we have necessarily $\ker_{S^*Y} \sigma_\partial(A) := \bigcup_{(y,\eta)\in S^*Y} \ker \sigma_\partial(A)(y,\eta) \cong \pi_1^* G$, i.e.,

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A) \in \pi_1^* K(Y), \tag{0.17}$$

where $\pi_1^* : K(Y) \to K(S^*Y)$ is the pull-back of K-groups under the projection $\pi_1 : S^*Y \to Y$. In other words, (0.17) is a necessary condition for the existence of an SL-elliptic operator $\mathcal{A}$ for a σ_ψ-elliptic A.

As noted before, the condition (0.17) may be violated. We also talk about a topological obstruction for the existence of SL-elliptic boundary value problems. The criterion of Atiyah and Bott says that (0.17) is also sufficient for the existence of an SL-elliptic $\mathcal{A}$ (in a slightly more general context, connected with a stabilisation that we shall see more explicitly below in connection with pseudo-differential boundary value problems, cf. Section 2.1 below.)

Boundary value problems for the case with non-vanishing obstruction for $\sigma_\psi(A)$ have been studied by many authors before, in particular, by Calderón [**8**], and Seeley [**48**] (more references will be given below), and there have been discovered many relations to other classical areas, in particular, to the Riemann-Hilbert problem (and its various generalisations), Toeplitz operators, index theory, spectral theory, and geometric analysis.

Among the essential observations there is the aspect that Fredholm operators of the form (0.13) (which are adequate for the SL-elliptic case) have to be replaced by Fredholm operators

$$\mathcal{A} : H^s(X, E) \to \begin{array}{c} H^{s-\mu}(X, F) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{L}) \end{array} \tag{0.18}$$

for a new scale of closed subspaces $P^s(Y, \mathbb{L})$ of $H^s(Y, J)$, $s \in \mathbb{R}$, which are defined as the image under a projection $P_+ : H^s(Y, J) \to H^s(Y, J)$ for a suitable element $J \in \operatorname{Vect}(Y)$, where $P_+ \in L^0_{\mathrm{cl}}(Y; J, J)$ is a classical pseudo-differential operator of order zero (acting between distributional sections in J), cf. Schulze, Sternin and Shatalov [**45**]. Here $\mathbb{L}$ denotes the triple (P_+, J, L), where $L \in \operatorname{Vect}(T^*Y \setminus 0)$ is the image of $\pi_Y^* J$ under the projection $p_+ : \pi_Y^* J \to \pi_Y^* J$, the homogeneous principal symbol of P_+; the symbolic calculus of classical pseudo-differential operators gives us $p_+^2 = p_+$ as a consequence of $P_+^2 = P_+$, such that L is a subbundle of $\pi_Y^* J$. A well known theorem says that for every subbundle $L \subset \pi_Y^* J$ and every choice of a projection $p_+ : \pi_Y^* J \to L$ that is homogeneous in η of order zero there is an associated projection $P_+ \in L^0_{\mathrm{cl}}(Y; J, J)$ with p_+ as homogeneous principal symbol (we will give a proof in Section 1.1 below).

In the present exposition we shall extend an approach of the author [**39**], where operators of the form (0.18) have been completed to an algebra of block matrix operators

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{array}{c} H^s(X, E) \\ \oplus \\ P^s(Y, \mathbb{L}_-) \end{array} \to \begin{array}{c} H^{s-\mu}(X, F) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{L}_+) \end{array} \tag{0.19}$$

by conditions of trace and potential type T and K, respectively, connected with suitable triples $\mathbb{L}_\pm = (P_\pm, J_\pm, L_\pm)$ of a similar meaning as before. The operator Q is a Toeplitz operator on Y, see the terminology in Section 1.1 below. Starting point will be an algebra $\mathcal{B}(X)$ of operators of the form

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A & \widetilde{K} \\ \widetilde{T} & \widetilde{Q} \end{pmatrix} : \begin{matrix} H^s(X,E) \\ \oplus \\ H^s(Y,J_-) \end{matrix} \to \begin{matrix} H^{s-\mu}(X,F) \\ \oplus \\ H^{s-\mu}(Y,J_+) \end{matrix} \tag{0.20}$$

that are (classical) pseudo-differential boundary value problems on X with the transmission property at Y, cf. Boutet de Monvel [**6**] or Rempel and Schulze [**30**]. The operators (0.19) will be compositions

$$\mathcal{A} = \begin{pmatrix} A & \widetilde{K}R_- \\ P_+\widetilde{T} & P_+\widetilde{Q}R_- \end{pmatrix}, \tag{0.21}$$

where $R_- : P^{s-\mu}(Y,\mathbb{L}_-) \to H^s(Y,J_-)$ is the canonical embedding, and $P_+ : H^{s-\mu}(Y,J_+) \to P^{s-\mu}(Y,\mathbb{L}_+)$ is one of the chosen projections. Operators of the form (0.21) will be called boundary value problems for A with global projection conditions. In Section 1.2 we give a brief overview on the algebra $\mathcal{B}(X)$ including its symbolic structure. $\mathcal{B}(X)$ will be a subalgebra of the larger algebra of operators (0.21). In fact, for the case $P_\pm = \mathrm{id}$, $L_\pm = J_\pm$ we just recover the case (0.20).

Elliptic operators A on a manifold with boundary from the point of view of global projection conditions are of interest on their own right. On the other hand, manifolds with boundary may be viewed as particular manifolds with edges, where the boundary is the edge and the inner normal the model cone of a wedge. In general, wedges with non-trivial model cones (i.e., when they have a base of dimension > 0) locally describe manifolds with edges. In such a case ellipticity is also connected with edge operators of trace and potential type, and there is again a topological obstruction for the existence of SL-elliptic conditions, cf. [**34**, Section 3.3.5, Proposition 10]. Similar phenomena may be expected on manifolds with higher (say, polyhedral) singularities with a hierarchy of obstructions for SL-elliptic conditions on the lower-dimensional strata. In all those cases it makes sense to construct operators with global projection data when the corresponding obstructions do not vanish. A joint paper of the author with Seiler [**43**] treats the case of manifolds with egdes. A substructure with special features is the case of boundary value problems without the transmission property, contained in the author's joint paper with Seiler [**44**].

A large variety of examples of edge-degenerate differential operators with non-vanishing obstruction is constructed in Nazaikinskij, Schulze, Sternin and Shatalov [**26**]. In another article [**27**] the authors study the K-theoretic nature of the topological obstruction in general.

In the present paper we will not discuss the aspect of edge operators in detail; this is another fascinating branch of the development of operators with global projection conditions and their index theory. Let us conclude this introduction by references for the case of smooth manifolds with boundary, namely Atiyah, Patodi and Singer [**4**], Melrose [**25**], Booss-Bavnbek and Wojciechovski [**5**], Grubb and Seeley [**16**], Schulze, Sternin, and Shatalov [**45**], Schulze and Tarkhanov [**46**].

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1. Elements of the classical calculus of boundary value problems

1.1. Pseudo-differential and Toeplitz operators on a closed manifold. Let M be a closed, compact C^∞ manifold, $m = \dim M$, with a fixed Riemannian metric and an associated measure dx. Complex smooth vector bundles on M are assumed to be equipped with Hermitian metrics. For every $E, F \in \mathrm{Vect}(M)$ we then have the space $L^2(M,E)$ of square integrable sections in E with a corresponding scalar product. Moreover, we have the scale $H^s(M,E)$, $s \in \mathbb{R}$, of distributional sections in E of Sobolev smoothness s, where we identify $H^0(M,E)$ with $L^2(M,E)$.

Let $E \boxtimes F \in \mathrm{Vect}(M \times M)$ be the external tenser product of bundles $E, F \in \mathrm{Vect}(M)$, i.e., $E \boxtimes F := (p_1^* E) \otimes (p_2^* F)$ with the projections $p_j : (x_1, x_2) \to x_j$, $M \times M \to M$, to the j th component, $j = 1, 2$. We identify the space $C^\infty(F \boxtimes E^*)$ (where E^* is the dual bundle of E) with the space $L^{-\infty}(M; E, F)$ of all operators $K : C^\infty(M,E) \to C^\infty(M,F)$ which have kernels $c(x,\widetilde{x})$ in $C^\infty(F \boxtimes E^*)$, i.e.,

$$Ku(x) = \int_M (c(x,\widetilde{x}), u(\widetilde{x}))_E \, d\widetilde{x}, \tag{1.1}$$

where $(\cdot,\cdot)_E$ means the fibrewise pairing between E^* and E.

We now recall some definitions and results from the standard pseudo-differential calculus on a manifold.

Definition 1.1. (i) The space $S^\mu(U \times \mathbb{R}^n)$ for $\mu \in \mathbb{R}$, and $U \subseteq \mathbb{R}^m$ open, is defined to be the subspace of all $a(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le c\langle\xi\rangle^{\mu-|\beta|} \tag{1.2}$$

for all $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$ and $(x,\xi) \in K \times \mathbb{R}^n$ for all $K \Subset U$, with constants $c = c(\alpha,\beta,K) > 0$; here $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$. The elements of $S^\mu(U \times \mathbb{R}^n)$ are called symbols of order μ.

(ii) $S^{(\mu)}(U \times (\mathbb{R}^n \setminus \{0\}))$ denotes the space of all $a_{(\mu)}(x,\xi) \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$ such that $a_{(\mu)}(x,\lambda\xi) = \lambda^\mu a_{(\mu)}(x,\xi)$ for all $\lambda \in \mathbb{R}_+$, $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$.

(iii) An excision function $\chi(\xi)$ in $\mathbb{R}^n$ is any element of $C^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 0$ for $0 \le |\xi| \le c_0$, $\chi(\xi) = 1$ for $c_1 \le |\xi|$ with certain constants $0 < c_0 < c_1$.

(iv) A symbol $a(x,\xi) \in S^\mu(U \times \mathbb{R}^n)$ is said to be classical, if there are elements $a_{(\mu-j)}(x,\xi) \in S^{(\mu-j)}(U \times (\mathbb{R}^n \setminus \{0\}))$, $j \in \mathbb{N}$, such that

$$a(x,\xi) - \sum_{j=0}^{N} \chi(\xi) a_{(\mu-j)}(x,\xi) \in S^{\mu-(N+1)}(U \times \mathbb{R}^n)$$

for every $N \in \mathbb{N}$, where $\chi(\xi)$ is any excision function. Let $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^n)$ denote the space of all classical symbols of order μ.

Let $\Omega \subseteq \mathbb{R}^n$ be open, and set

$$L^\mu_{(\mathrm{cl})}(\Omega) := \{\mathrm{Op}(a) : \; a(x,x',\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)\},$$

where $\mathrm{Op}(a)u(x) := \iint e^{i(x-x')\xi} a(x,x',\xi) u(x') \, dx' đ\xi$, $đ\xi = (2\pi)^{-n} \, d\xi$, subscript '(cl)' means that we talk about classical or non-classical symbols and operators, respectively. As usual, $\mathrm{Op}(a)u(x)$ makes sense in the oscillatory integral sense, first as a continuous map: $\mathrm{Op}(a) : C_0^\infty(\Omega) \to C^\infty(\Omega)$, and then extended to larger function and distribution spaces.

For $A \in L^\mu_{\mathrm{cl}}(\Omega)$ we set

$$\sigma_\psi(A)(x,\xi) = a_{(\mu)}(x,x',\xi)|_{x'=x},$$

(cf. the notation in Definition 1.1 (iv)) called the homogeneous principal symbol of order μ of the operator A.

Let $L^\mu_{(\mathrm{cl})}(\Omega; \mathbb{C}^j, \mathbb{C}^k)$ denote the space of all $k \times j$ matrices of elements in $L^\mu_{(\mathrm{cl})}(\Omega)$, and let E and F be vector bundles on M of fibre dimension j and k, respectively, with trivialisations $\Omega \times \mathbb{C}^j$ and $\Omega \times \mathbb{C}^k$ for open sets $\Omega \subseteq \mathbb{R}^n$, with transition maps

$$\eta_{\widetilde{\Omega}\Omega} : \Omega \times \mathbb{C}^j \to \widetilde{\Omega} \times \mathbb{C}^j, \qquad \vartheta_{\widetilde{\Omega}\Omega} : \Omega \times \mathbb{C}^k \to \widetilde{\Omega} \times \mathbb{C}^k \tag{1.3}$$

for open $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$, with an underlying coordinate diffeomorphism $\kappa : \Omega \to \widetilde{\Omega}$. We then have a push-forward

$$L^\mu_{(\mathrm{cl})}(\Omega; \mathbb{C}^j, \mathbb{C}^k) \to L^\mu_{(\mathrm{cl})}(\widetilde{\Omega}; \mathbb{C}^j, \mathbb{C}^k)$$

of operators $A : C_0^\infty(\Omega, \mathbb{C}^j) \to C^\infty(\Omega, \mathbb{C}^k)$ to operators $\widetilde{A} : C_0^\infty(\widetilde{\Omega}, \mathbb{C}^j) \to C^\infty(\widetilde{\Omega}, \mathbb{C}^k)$ when we set $\widetilde{A} := T_{\widetilde{\Omega}\Omega} A H^{-1}_{\widetilde{\Omega}\Omega}$, where $H_{\widetilde{\Omega}\Omega} : C_0^\infty(\Omega, \mathbb{C}^j) \to C_0^\infty(\widetilde{\Omega}, \mathbb{C}^j)$ and $T_{\widetilde{\Omega}\Omega} : C^\infty(\Omega, \mathbb{C}^k) \to C^\infty(\widetilde{\Omega}, \mathbb{C}^k)$ are the isomorphisms induced by (1.3). To every chart $\chi : D \to \Omega$ on M we thus obtain the spaces $L^\mu_{(\mathrm{cl})}(D; E|_D, F|_D)$

of pseudo-differential operators on D, acting between sections in the bundles E and F. Set

$$(1.4)\quad L^{\mu}_{(\mathrm{cl})}(M;E,F) := \Big\{ \sum_{j=1}^{N} \varphi_j A_j \psi_j + C :\ A_j \in L^{\mu}_{(\mathrm{cl})}(D_j; E|_{D_j}, F|_{D_j}), \quad C \in L^{-\infty}(M;E,F) \Big\},$$

where $\{D_1, \dots, D_N\}$ is an open covering of M by coordinate neighbourhoods (say, diffeomorphic to balls), $\{\varphi_1, \dots, \varphi_N\}$ a subordinate partition of unity, and $\{\psi_1, \dots, \psi_N\}$ a system of functions $\psi_j \in C_0^\infty(D_j)$ that are equal to 1 on $\operatorname{supp} \varphi_j$ for all j.

Remark 1.2. $L^{\mu}_{(\mathrm{cl})}(M;E,F)$ is a Fréchet space in a natural way.

Let $A \in L^{\mu}_{\mathrm{cl}}(M;E,F)$, and let $(A_{pq})_{1\le p\le j, 1\le q\le k}$ be a local representative of A on Ω. Then the system of matrices $(\sigma_\psi(A_{pq}))_{1\le p\le j, 1\le q\le k}$ has an invariant meaning as a bundle morphism

$$(1.5)\qquad \sigma_\psi(A) : \pi_M^* E \to \pi_M^* F,$$

called the homogeneous principal symbol of A; here, $\pi_M : T^*M \setminus 0 \to M$ is the canonical projection of the cotangent bundle of M (minus the zero section) to M.

Let $S^{(\mu)}(T^*M \setminus 0; E, F)$ denote the set of all morphisms $p_{(\mu)} : \pi_M^* E \to \pi_M^* F$ such that $p_{(\mu)}(x, \lambda\xi) = \lambda^\mu p_{(\mu)}(x,\xi)$ for all $(x,\xi) \in T^*M \setminus 0$, $\lambda \in \mathbb{R}_+$.

Proposition 1.3. *The principal symbolic map*

$$(1.6)\qquad \sigma_\psi : L^{\mu}_{\mathrm{cl}}(M;E,F) \to S^{(\mu)}(T^*M \setminus 0; E,F)$$

is surjective, and there is a linear map

$$(1.7)\qquad \mathrm{op} : S^{(\mu)}(T^*M \setminus 0; E,F) \to L^{\mu}_{\mathrm{cl}}(M;E,F)$$

such that $\sigma_\psi \circ \mathrm{op} = \mathrm{id}$.

A choice of (1.7) directly follows from the existence of local operators in $L^{\mu}_{\mathrm{cl}}(\Omega; \mathbb{C}^j, \mathbb{C}^k)$ associated with principal symbols that correspond to local representations of a given element in $S^{(\mu)}(T^*M \setminus 0; E, F)$; a subsequent globalisation, according to the expressions in (1.4), then gives us (1.6).

Theorem 1.4. *Let M be a closed, compact C^∞ manifold.*

(i) *Every $A \in L^\mu(M;E,F)$ for $E, F \in \mathrm{Vect}(M)$ induces continuous operators*

$$(1.8)\qquad A : H^s(M,E) \to H^{s-\mu}(M,F)$$

for all $s \in \mathbb{R}$.

(ii) *$A \in L^{\mu}_{\mathrm{cl}}(M;E,F)$ and $\sigma_\psi(A) = 0$ implies $A \in L^{\mu-1}_{\mathrm{cl}}(M;E,F)$, and hence the operator* (1.8) *is compact.*

(iii) *$A \in L^{\mu}_{(\mathrm{cl})}(M;E_0,F)$, $B \in L^{\nu}_{(\mathrm{cl})}(M;E,E_0)$ for $E, E_0, F \in \mathrm{Vect}(M)$ implies $AB \in L^{\mu+\nu}_{(\mathrm{cl})}(M;E,F)$. In the classical case we have*

$$(1.9)\qquad \sigma_\psi(AB) = \sigma_\psi(A)\sigma_\psi(B).$$

(iv) *Let $A \in L^{\mu}_{(\mathrm{cl})}(M;E,F)$, and let A^* denote the formal adjoint of A (defined by $(Au,v)_{L^2(M,F)} = (u, A^*v)_{L^2(M,E)}$ for all $u \in C^\infty(M,E)$, $v \in C^\infty(M,F)$). Then we have $A^* \in L^{\mu}_{(\mathrm{cl})}(M;F,E)$ and, in the classical case,*

$$(1.10)\qquad \sigma_\psi(A^*) = \sigma_\psi(A)^*,$$

(the adjoint on the right hand side refers to the Hermitian metrics in the bundles.)

In the sequel we mainly concentrate on classical pseudo-differential operators.

Definition 1.5. Let $A \in L^{\mu}_{\mathrm{cl}}(M;E,F)$, $\mu \in \mathbb{R}$, $E, F \in \mathrm{Vect}(M)$.

(i) The operator A is said to be elliptic (of order μ), if $\sigma_\psi(A) : \pi_M^* E \to \pi_M^* F$ is an isomorphism.

(ii) An operator $P \in L_{\mathrm{cl}}^{-\mu}(M; F, E)$ is called a parametrix of A, if P satisfies the following relations:

$$C_l := I - PA \in L^{-\infty}(M; E, E), \qquad C_r := I - AP \in L^{-\infty}(M; F, F), \tag{1.11}$$

where I denotes the corresponding identity operators.

Notice that when P is a parametrix of A, we have $\sigma_\psi(P) = \sigma_\psi(A)^{-1}$.
Moreover, if A is elliptic, so is the formal adjoint A^*.
We call an operator $A \in L_{\mathrm{cl}}^{\mu}(M; E, F)$ underdetermined (overdetermined) elliptic, if $\sigma_\psi(A) : \pi_M^* E \to \pi_M^* F$ is injective (surjective).

Remark 1.6. (i) $A \in L_{\mathrm{cl}}^{\mu}(M; E, F)$ is underdetermined elliptic if and only if $A^* \in L_{\mathrm{cl}}^{\mu}(M; F, E)$ is overdetermined elliptic.

(ii) If A is underdetermined (overdetermined) elliptic, then $A^*A \in L_{\mathrm{cl}}^{2\mu}(M; E, E)$ ($AA^* \in L_{\mathrm{cl}}^{2\mu}(M; F, F)$) is elliptic.

Theorem 1.7. *Let $A \in L_{\mathrm{cl}}^{\mu}(M; E, F)$, $\mu \in \mathbb{R}$, $E, F \in \mathrm{Vect}(M)$.*

(i) *The operator A is elliptic (of order μ), if and only if*

$$A : H^s(M, E) \to H^{s-\mu}(M, F) \tag{1.12}$$

is a Fredholm operator for an $s = s_0 \in \mathbb{R}$.

(ii) *If A is elliptic,* (1.12) *is a Fredholm operator for all $s \in \mathbb{R}$, and* $\dim\ker A$ *and* $\dim\operatorname{coker} A$ *are independent of s. We have $V := \ker A \subset C^\infty(M, E)$, and there is a finite-dimensional subspace $W \subset C^\infty(M, F)$ such that*

$$W + \operatorname{im} A = H^{s-\mu}(M, F)$$

and $W \cap \operatorname{im} A = \{0\}$ for every $s \in \mathbb{R}$.

(iii) *An elliptic operator A has a parametrix $P \in L_{\mathrm{cl}}^{-\mu}(M; F, E)$, cf. Definition* 1.5 (ii)*, and P can be chosen in such a way that the remainders in the relation* (1.11) *are projections*

$$C_l : H^s(M, E) \to V, \qquad C_r : H^{s-\mu}(M, F) \to W$$

to subspaces V and W as in (ii) *for all $s \in \mathbb{R}$.*

As a consequence of Theorem 1.7 (ii) we see that the index of A

$$\operatorname{ind} A := \dim\ker A - \dim\operatorname{coker} A$$

is independent of s.

Theorem 1.7 (iii) yields elliptic regularity of solutions $u \in H^{-\infty}(M, E)$ to an elliptic equation $Au = f \in H^s(M, F)$, $s \in \mathbb{R}$, namely $u \in H^{s+\mu}(M, E)$. In fact, using a parametrix P of A as a left parametrix, we obtain $PAu = Pf \in H^{s+\mu}(M, E)$, but $C_l = I - PA$ implies $PAu = u - C_l u$ where $C_l u \in H^\infty(M, F)$, and it follows that $u = Pf + C_l u \in H^{s+\mu}(M, F)$.

The latter consideration gives us, in particular, $V = \ker A \subset C^\infty(M, E)$. Moreover, the relation $\ker A^* + \operatorname{im} A = H^{s-\mu}(M, F)$ (which is a direct decomposition, i.e., $\ker A^* \cap \operatorname{im} A = \{0\}$) allows us to set $W = \ker A^* \subset C^\infty(M, F)$, cf. Theorem 1.7 (ii).

The assertions of Theorem 1.7 (ii), (iii) have an abstract background which will be useful in other situations below.

We consider operators A between Hilbert spaces that belong to scales $\{H_i^s\}_{s\in\mathbb{R}}$, where first

$$A : H_1^\infty \to H_2^\infty \tag{1.13}$$

is continuous, $H_i^\infty := \bigcap_{s\in\mathbb{R}} H_i^s$. To every A there is an order $\mu \in \mathbb{R}$ and a constant $c > 0$ such that (1.13) extends to continuous operators

$$A_s : H_1^s \to H_2^{s-\mu} \tag{1.14}$$

for all $s > c$ (the aspect with the constant $c = c(A)$ is not relevant for Theorems 1.4 and 1.7, but in boundary value problems it will play a role). If it is clear from the context which s is considered for the operator, we also write A instead of A_s.

We then assume the following properties:

(i) There are continuous embeddings $H_i^{s'} \hookrightarrow H_i^s$ for $s' \geq s$ that are compact for $s' > s$.
(ii) The space H_i^∞ is dense in H_i^s for every $s \in \mathbb{R}$, $i = 1, 2$.
(iii) If $V \subset H_i^\infty$, $i = 1, 2$, is a finite-dimensional subspace and $C_V : H_i^0 \to V$ a projection, then C_V induces continuous operators $C_V : H_i^s \to V$ for all $s > c$ for some $c \in \mathbb{R}$.
(iv) (1.13) extends to a Fredholm operator (1.14) for all $s > c$, and there is a continuous operator

$$P : H_2^\infty \to H_1^\infty$$

that extends to a parametrix

$$P_{s-\mu} : H_2^{s-\mu} \to H_1^s$$

of A_s for every $s > c$, i.e., the remainders

$$C_{s,l} := I - P_{s-\mu} A_s, \qquad C_{s-\mu,r} := I - A_s P_{s-\mu}$$

induce continuous operators

$$C_{s,l} : H_1^s \to H_1^\infty, \qquad C_{s-\mu,r} : H_2^{s-\mu} \to H_2^\infty$$

for all $s > c$.

Remark 1.8. Under the abovementioned conditions the dimensions of $\ker A_s$ and $\operatorname{coker} A_s$ of the Fredholm operator $A : H_1^s \to H_2^{s-\mu}$ are independent of $s > c$, we have $V := \ker A_s \subset H_1^\infty$, and there is a finite-dimensional subspace $W \subset H_2^\infty$ such that $W + \operatorname{im} A_s = H_2^{s-\mu}$, $W \cap \operatorname{im} A_s = \{0\}$ for every $s > c$. Moreover, the parametrix P can be chosen in such a way that the remainders

$$C_l = I - PA, \qquad C_r = I - AP$$

are projections to V and W, respectively.

For references below we now prepare some well known material on parameter-dependent pseudo-differential operators. First we have the Fréchet space $L^{-\infty}(M; E, F)$ of smoothing operators on M, and we set

$$L^{-\infty}(M; E, F; \mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(M; E, F))$$

which is the space of smoothing parameter-dependent operators (between sections of $E, F \in \operatorname{Vect}(M)$). The space

$$L^\mu_{(\mathrm{cl})}(M; E, F; \mathbb{R}^l)$$

of parameter-dependent pseudo-differential operators of order μ is defined in a similar manner as (1.4), now for $C(\lambda) \in L^{-\infty}(M; E, F; \mathbb{R}^l)$ and $A_j(\lambda) \in L^\mu_{(\mathrm{cl})}(D_j; E|_{D_j}, F|_{D_j}; \mathbb{R}^l)$, where the latter spaces are operator pull backs under charts $\chi_j : D_j \to \Omega$ of operators on Ω, with matrices of amplitude functions $a_j(x, x', \xi, \lambda) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{n+l}_{\xi,\lambda})$.

Theorem 1.9. *For every $A(\lambda) \in L^\mu(M; E, F; \mathbb{R}^l)$ and $\nu \geq \mu$ we have*

$$\|A(\lambda)\|_{\mathcal{L}(H^s(X,E), H^{s-\nu}(X,F))} \leq c \begin{cases} (1+|\lambda|)^\mu & \text{for} \quad \nu \geq 0, \\ (1+|\lambda|)^{\mu-\nu} & \text{for} \quad \nu \leq 0 \end{cases},$$

for constants $c = c_{s,\nu} > 0$.

For classical parameter-dependent operators $A(\lambda) \in L^\mu_{\mathrm{cl}}(M; E, F; \mathbb{R}^l)$ we have a parameter-dependent homogeneous principal symbol (1.5), now for the projection

$$\pi_M : (T^*M \times \mathbb{R}^l) \setminus 0 \to M \qquad (0 \text{ means } (\xi, \lambda) = 0).$$

In this case, $\sigma_\psi(A)(x, \xi, \lambda)$ is defined in terms of the homogeneous principal components of local amplitude functions in $(\xi, \lambda) \neq 0$ (at $x = x'$).

Definition 1.10. An $A(\lambda) \in L^\mu_{\mathrm{cl}}(M; E, F; \mathbb{R}^l)$ is called parameter-dependent elliptic (of order μ), if

$$\sigma_\psi(A) : \pi_M^* E \to \pi_M^* F$$

for $\pi_M : (T^*M \times \mathbb{R}^l) \setminus 0 \to M$, is an isomorphism.

THEOREM 1.11. *Let $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(M;E,F;\mathbb{R}^l)$ be parameter-dependent elliptic (of order μ).*

(i) *Then*

$$A(\lambda) : H^s(M,E) \to H^{s-\mu}(M,F) \tag{1.15}$$

is a family of Fredholm operators of index 0. Moreover, there is a constant $C > 0$ such that the operators (1.15) are isomorphisms for all $|\lambda| \geq C$. This holds for all $s \in \mathbb{R}$.

(ii) *$A(\lambda)$ has a parameter-dependent parametrix $P(\lambda) \in L^{-\mu}_{\mathrm{cl}}(M;F,E;\mathbb{R}^l)$, i.e., $I - P(\lambda)A(\lambda) \in L^{-\infty}(M;E,E;\mathbb{R}^l)$, $I - A(\lambda)P(\lambda) \in L^{-\infty}(M;F,F;\mathbb{R}^l)$.*

THEOREM 1.12. *For every $\mu \in \mathbb{R}$ there exists an element $R^{\mu}_E(\lambda) \in L^{\mu}_{\mathrm{cl}}(M;E,E;\mathbb{R}^l)$ that induces isomorphisms*

$$R^{\mu}_E(\lambda) : H^s(X,E) \to H^{s-\mu}(X,E)$$

for all $s \in \mathbb{R}$ and $\lambda \in \mathbb{R}^l$.

We now turn to an extension of the concept of classical pseudo-differential operators $L^{\mu}_{\mathrm{cl}}(M;E,F)$ to so called Toeplitz operators, acting between closed subspaces of Sobolev spaces on M. A crucial technical point is the following result:

THEOREM 1.13. *Let $p : \pi^*_M J \to \pi^*_M J$, $J \in \mathrm{Vect}(M)$, be a projection, i.e., $p^2 = p$, where $p(x,\lambda\xi) = p(x,\xi)$ for all $(x,\xi) \in T^*M \setminus 0$, $\lambda \in \mathbb{R}_+$. Then there exists an element $P \in L^0_{\mathrm{cl}}(M;J,J)$ such that $P^2 = P$, and $\sigma_\psi(P) = p$.*

Moreover, if p satisfies the condition $p = p^*$, there is a choice of the associated pseudo-differential projection $P \in L^0_{\mathrm{cl}}(M;J,J)$ such that $P = P^*$.

COROLLARY 1.14. *To every $J \in \mathrm{Vect}(M)$ and every subbundle $L \subset \pi^*_M J$ there is an element $P \in L^0_{\mathrm{cl}}(M;J,J)$ such that $P^2 = P$, where $\sigma_\psi(P) : \pi^*_M J \to \pi^*_M J$ is a projection to L.*

The proof of Theorem 1.13 is based on the following general construction. Let H be a (complex) Hilbert space, $\mathcal{L}(H)$ the space of linear continuous operators, $\mathcal{K}(H)$ the subspace of compact operators in H. We then have the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$ and the corresponding canonical map $\pi : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{K}(H)$.

LEMMA 1.15. *Let $p \in \mathcal{L}(H)/\mathcal{K}(H)$ be an element with $p^2 = p$, and choose any $Q \in \mathcal{L}(H)$ such that $\pi Q = p$. Then the spectrum $\sigma_{\mathcal{L}(H)}(Q)$ of Q has the property that*

$$\sigma_{\mathcal{L}(H)}(Q) \cap (\mathbb{C} \setminus (\{0\} \cup \{1\}))$$

is discrete.

PROOF. First observe that $p^2 = p$ implies $\sigma_{\mathcal{L}(H)/\mathcal{K}(H}(p) \subseteq \{0\} \cup \{1\}$. In fact, for $\lambda \in \mathbb{C} \setminus (\{0\} \cup \{1\}) =: U$ there exists the inverse $(\lambda e - p)^{-1} = p(\lambda - 1)^{-1} + (e - p)\lambda^{-1}$, where $e \in \mathcal{L}(H)/\mathcal{K}(H)$ is the identity, $e = \pi I$ for the identity $I \in \mathcal{L}(H)$. Now $U \ni \lambda \to \lambda I - Q \in \mathcal{L}(H)$ is a holomorphic Fredholm family in U, and $\lambda I - Q$ is invertible in $\mathcal{L}(H)$ for $|\lambda| > \|Q\|_{\mathcal{L}(H)}$. A well known invertibility result on holomorphic Fredholm families (cf. [**34**, Section 2.2.5] for a proof) implies that $\lambda I - Q$ is invertible for all $\lambda \in U \setminus D$ for a certain discrete subset D (i.e., D is countable and $D \cap K$ finite for every compact subset $K \subset U$). □

PROOF OF THEOREM 1.13. From Lemma 1.15 we see that there exists a $0 < \delta < 1$ such that the circle $C_\delta := \{\lambda \in \mathbb{C} : \ |\lambda - 1| = \delta\}$ does not intersect $\sigma_{\mathcal{L}(H)}(Q)$. Setting

$$P := \frac{1}{2\pi i} \int_{C_\delta} (\lambda I - Q)^{-1} \, d\lambda \tag{1.16}$$

we obtain $P^2 = P$ and $P \in L^0_{\mathrm{cl}}(M;J,J)$ as a consequence of the holomorphic functional calculus for $L^0_{\mathrm{cl}}(M;J,J)$. Moreover, we have

$$\begin{aligned}
\sigma_\psi(P) &= \frac{1}{2\pi i} \int_{C_\delta} (\lambda e - p)^{-1} \, d\lambda \\
&= \left\{ \frac{1}{2\pi i} \int_{C_\delta} \frac{1}{\lambda - 1} \, d\lambda \right\} p + \left\{ \frac{1}{2\pi i} \int_{C_\delta} \frac{1}{\lambda} \, d\lambda \right\} (e - p).
\end{aligned}$$

The second summand on the right hand side vanishes, while the first one is equal to p by the Residue theorem.

To prove the second part of Theorem 1.13 we suppose $p = p^*$. Then, if $P_1 = P_1^2 \in L^0_{\mathrm{cl}}(M; J, J)$ is any choice with $\sigma_\psi(P_1) = p$, also $Q := P_1^* P_1 \in L^0_{\mathrm{cl}}(M; J, J)$ satisfies $\sigma_\psi(Q) = p^*p = p^2 = p$. For Q we have $Q = Q^* \geq 0$. Let η be the spectral measure of Q. Then the projection $P \in L^0_{\mathrm{cl}}(M; J, J)$ defined by the formula (1.16) is equal to the spectral projection

$$\eta(B_\delta(1) \cap \sigma_{\mathcal{L}(L^2(M,J))}(Q)) \quad \text{for } B_\delta = \{\lambda \in \mathbb{C} : \ |\lambda - 1| < \delta\}.$$

In particular, we have $P = P^* = P^2$, and $\sigma_\psi(P) = p$ as before. □

Remark 1.16. The above construction of projections has a more general functional analytic background. If Ψ is a Fréchet operator algebra with a given ideal $\mathcal{I}$, there is a lifting of idempotent elements of $\Psi/\mathcal{I}$ to idempotent elements in Ψ, provided some natural assumptions on the operator algebra are satisfied, cf. Gramsch [**14**]. In particular, for $\Psi = L^0_{\mathrm{cl}}(M; J, J)$ and $\mathcal{I} = L^{-1}_{\mathrm{cl}}(M; J, J)$ the space $\Psi/\mathcal{I}$ is isomorphic to the space of homogeneous symbols of order zero. The general theory gives a characterisation of the space of all idempotent elements $P \in L^0_{\mathrm{cl}}(M; J, J)$ which belong to the connected component of a given idempotent $P_1 \in L^0_{\mathrm{cl}}(M; J, J)$ and have the same homogeneous principal symbol as P_1. The result says that all those P have the form GP_1G^{-1}, where G varies over the connected component of the identity in the group $\{I + K \in \Psi^{-1} : \ K \in L^{-1}_{\mathrm{cl}}(M; J, J)\}$, where Ψ^{-1} denotes the group of invertible elements of $L^0_{\mathrm{cl}}(M; J, J)$.

Proposition 1.17. *Let H be a Hilbert space, and let $P, Q \in \mathcal{L}(H)$ be projections such that $P - Q$ is a compact operator. Then the restrictions of P to $\operatorname{im} Q$ and of Q to $\operatorname{im} P$ are Fredholm operators*

$$P_Q : \operatorname{im} Q \to \operatorname{im} P, \qquad Q_P : \operatorname{im} P \to \operatorname{im} Q$$

between the respective closed subspaces of H, and Q_P is a parametrix (i.e., a Fredholm inverse) of P_Q.

Proof. The operator Q acts as the identity on $\operatorname{im} Q$. Therefore, we have

$$Q_P P_Q - 1 = Q_P P_Q - Q^2 = Q_P(P_Q - Q_P) : \operatorname{im} Q \to \operatorname{im} Q,$$

i.e., $Q_P P_Q - 1$ is a compact operator in $\operatorname{im} Q$. It follows that Q_P is a Fredholm inverse of P_Q, and we see that both P_Q and Q_P are Fredholm operators. □

Let

$$\operatorname{ind}(P, Q) \tag{1.17}$$

denote the index of $P_Q : \operatorname{im} Q \to \operatorname{im} P$. We then have

$$\operatorname{ind}(P, Q) = -\operatorname{ind}(Q, P). \tag{1.18}$$

For every $L \in \operatorname{Vect}(T^*M \setminus 0)$ there is a $J \in \operatorname{Vect}(M)$ and a projection $p : \pi_Y^* J \to L$ with the property $p(x, \lambda, \xi) = p(x, \xi)$ for all $(x, \xi) \in T^*M \setminus 0$, $\lambda \in \mathbb{R}_+$. It suffices to set $J = \mathbb{C}^N$ (the trivial bundle with fibre $\mathbb{C}^N$) for sufficiently large N.

Set

$$\begin{aligned} \mathcal{P}(M) := \{\mathbb{L} := (P, J, L) : \ & P \in L^0_{\mathrm{cl}}(M; J, J), P^2 = P, \\ & J \in \operatorname{Vect}(M), L = \operatorname{im} \sigma_\psi(P)\}. \end{aligned} \tag{1.19}$$

Incidentally, the elements of $\mathcal{P}(M)$ will be called projection data on M.

Example 1.18. (i) For every $J \in \operatorname{Vect}(M)$ we have

$$(\mathrm{id}, J, \pi_M^* J) \in \mathcal{P}(M).$$

(ii) For $\mathbb{L} := (P, J, L)$, $\mathbb{M} := (Q, G, M) \in \mathcal{P}(M)$ we have

$$\mathbb{L} \oplus \mathbb{M} := (P \oplus Q, J \oplus G, L \oplus M) \in \mathcal{P}(M);$$

the direct sum for bundles is defined as usual. We then have $L^0_{\mathrm{cl}}(M, J \oplus G) = L^0_{\mathrm{cl}}(M, J) \oplus L^0_{\mathrm{cl}}(M, G)$, and $P \oplus Q$ in the latter space is again a pseudo-differential projection.

(iii) For every $\mathbb{L} := (P, J, L) \in \mathcal{P}(M)$ we have a complementary element $\mathbb{L}^\perp \in \mathcal{P}(M)$ in the sense that $L \oplus L^\perp = (\mathrm{id}, J, \pi_M^* J)$. In fact, it suffices to write $\pi_M^* J$ in the form $L \oplus L^\perp$ for a subbundle $L^\perp$ of $\pi_M^* J$ such that $L^\perp = (1 - P, J, L^\perp)$.

(iv) For $\mathbb{L} = (P, J, L)$ we can form an adjoint $\mathbb{L}^* := (P^*, J, L^*)$ by defining $P^* \in L^0_{\mathrm{cl}}(M; J, J)$ to be the formal adjoint of P, cf. Theorem 1.2 (iv), which is again a projection, and $L^* := \sigma_\psi(P^*)\pi_Y^* J$.

Every element $\mathbb{L} = (P, J, L) \in \mathcal{P}(M)$ gives rise to continuous operators

$$P : H^s(M, J) \to H^s(M, J),$$

$s \in \mathbb{R}$, and we set

$$P^s(M, \mathbb{L}) := PH^s(M, J) \tag{1.20}$$

which is a closed subspace of $H^s(M, J)$. In fact, $P^s(M, \mathbb{L})$ is equal to the kernel of $I - P : H^s(M, J) \to H^s(M, J)$.

Proposition 1.19. *We have continuous embeddings*

$$P^{s'}(M, \mathbb{L}) \hookrightarrow P^s(M, \mathbb{L}) \tag{1.21}$$

for all $s' \geq s$ that are compact for $s' > s$.

Proof. The inclusion in (1.21) is clear; the compactness follows from the fact that (1.21) may be written as a composition PC with the compact embedding $C : P^{s'}(M, \mathbb{L}) \to H^s(M, J)$ and the continuous projection $P : H^s(M, J) \to P^s(M, \mathbb{L})$. □

Proposition 1.20. (i) *The space $P^\infty(M, \mathbb{L}) = \bigcap_{s \in \mathbb{R}} P^s(M, \mathbb{L})$ is dense in $P^s(M, \mathbb{L})$ for every $s \in \mathbb{R}$.*

(ii) *Let $P^0(M; \mathbb{L})$ be equipped with the scalar product from $H^0(M, J) = L^2(M, J)$, let $V \subset P^\infty(M, \mathbb{L})$ be a finite-dimensional subspace. Then the orthogonal projection $C_V : P^0(M, \mathbb{L}) \to V$ induces continuous operators $C_V : P^s(M, \mathbb{L}) \to V$ for all $s \in \mathbb{R}$, and C_V is compact as an operator $P^s(M, \mathbb{L}) \to P^s(M, \mathbb{L})$ for every $s \in \mathbb{R}$.*

Proof. (i) Every $u \in P^s(M, \mathbb{L})$ can be written as $u = Pv$ for a $v \in H^s(M, J)$. Since $C^\infty(M, J)$ is dense in $H^s(M, J)$, there is a sequence $(v_n)_{n \in \mathbb{N}} \subset C^\infty(M, J)$ such that $v = \lim_{n \to \infty} v_n$ in $H^s(M, J)$. From the continuity of P we get $u = \lim_{n \to \infty} u_n$ for $u_n := Pv_n \in P^\infty(M, \mathbb{L})$, and hence $P^\infty(M, \mathbb{L})$ is dense in $P^s(M, \mathbb{L})$ for every s.

(ii) $V \subset C^\infty(M, J)$ is a finite-dimensional subspace. Hence, every projection $\widetilde{C}_V : H^s(M, J) \to V$ represents a compact operator $\widetilde{C}_V : H^s(M, J) \to H^s(M, J)$ for every $s \in \mathbb{R}$. The projection $C_V : P^s(M, \mathbb{L}) \to P^s(M, \mathbb{L})$ can be extended to a projection $\widetilde{C}_V := C_N P : H^s(M, J) \to H^s(M, J)$ that is a compact operator. Thus, also $\widetilde{\widetilde{C}}_V := P\widetilde{C}_V : H^s(M, J) \to P^s(M, \mathbb{L})$ is compact. The operator C_N itself can be written as a composition $C_V = \widetilde{\widetilde{C}}_V R$, where $R : P^s(M, \mathbb{L}) \to H^s(M, J)$ is the canonical embedding. Hence, also C_V is a compact operator. □

Definition 1.21. Let $\mathbb{L}_+ := (P_+, J_+, L_+)$, $\mathbb{L}_- := (P_-, J_-, L_-) \in \mathcal{P}(M)$, and let $R_- : P^s(M, \mathbb{L}_-) \hookrightarrow H^s(M, J_-)$ denote the canonical embedding (given for every $s \in \mathbb{R}$). Then the composition

$$A := P_+ \widetilde{A} R_- \tag{1.22}$$

for $\widetilde{A} \in L^\mu_{\mathrm{cl}}(M; J_-, J_+)$ is called a Toeplitz operator on M of order μ. Let $T^\mu(M; \mathbb{L}_-, \mathbb{L}_+)$ denote the set of all Toeplitz operators (1.22) associated with $\mathbb{L}_\pm \in \mathcal{P}(M)$. Set

$$T^{-\infty}(M; \mathbb{L}_-, \mathbb{L}_+) := \{P_+ \widetilde{A} R_- : \ \widetilde{A} \in L^{-\infty}(M; J_-, J_+)\}.$$

Remark 1.22. The space $T^\mu(M; \mathbb{L}_-, \mathbb{L}_+)$ can be identified with the quotient space $L^\mu_{\mathrm{cl}}(M; J_-, J_+)/\sim$, where $\widetilde{A}_1 \sim \widetilde{A}_2$ means $P_+ \widetilde{A}_1 P_- = P_+ \widetilde{A}_2 P_-$.

Remark 1.23. Let $\widetilde{A} \in L^{\mu}_{\mathrm{cl}}(M; J_-, J_+)$ and form $\widetilde{\widetilde{A}} := P_+\widetilde{A}P_- \in L^{\mu}_{\mathrm{cl}}(M; J_-, J_+)$. Then we have

$$P_+\widetilde{A}R_- = P_+ \,\widetilde{\widetilde{A}}\, R_-.$$

Remark 1.24. Let $\widetilde{A} \in L^{\mu}_{\mathrm{cl}}(M; J_-, J_+)$, and choose elements $\mathbb{L}_-, \mathbb{L}_-^{\perp}$ and $\mathbb{L}_+, \mathbb{L}_+^{\perp}$ in $\mathcal{P}(M)$ that are complementary in the abovementioned sense. Then, writing $\mathbb{L}_{\pm} = (P_{\pm}, J_{\pm}, L_{\pm})$ and $\mathbb{L}_{\pm}^{\perp} = (P_{\pm}^{\perp}, J_{\pm}, L_{\pm}^{\perp})$, we can decompose $\widetilde{A}$ into a block matrix operator

$$\begin{pmatrix} P_-\widetilde{A}R_+ & P_+\widetilde{A}R_-^{\perp} \\ P_+^{\perp}\widetilde{A}R_- & P_+^{\perp}\widetilde{A}R_-^{\perp} \end{pmatrix} : \begin{matrix} P^s(M, \mathbb{L}_-) \\ \oplus \\ P^s(M, \mathbb{L}_-^{\perp}) \end{matrix} \to \begin{matrix} P^{s-\mu}(M, \mathbb{L}_+) \\ \oplus \\ P^{s-\mu}(M, \mathbb{L}_+^{\perp}) \end{matrix},$$

where $R_- : P^s(Y, \mathbb{L}_-) \to H^s(Y, J_-)$ and $R_-^{\perp} : P^s(Y, \mathbb{L}_-^{\perp}) \to H^s(Y, J_-)$ are the corresponding embedding operators.

Applying that for an operator of the form

$$\widetilde{A} := P_+\widetilde{A}_1P_- + P_+^{\perp}\widetilde{A}_2P_-^{\perp} \in L^{\mu}_{\mathrm{cl}}(M; J_-, J_+)$$

for any $\widetilde{A}_1, \widetilde{A}_2 \in L^{\mu}_{\mathrm{cl}}(M; J_-, J_+)$ we obtain $\widetilde{A}$ in the form

$$\operatorname{diag}(P_+\widetilde{A}_1P_-, P_+^{\perp}\widetilde{A}_2P_-^{\perp}) \in L^{\mu}_{\mathrm{cl}}(M; J_-, J_+). \tag{1.23}$$

Toeplitz operators have been studied in the literature in many variants, see, for instance, Boutet de Monvel [**7**]. Note that

$$P_+\widetilde{A}R_- \in T^{-\infty}(M; \mathbb{L}_-, \mathbb{L}_+) \quad \Leftrightarrow \quad P_+\widetilde{A}P_- \in L^{-\infty}(M; \mathbb{L}_-, \mathbb{L}_+). \tag{1.24}$$

For every $A \in T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+)$ written in the form (1.22) we have a bundle morphism

$$\sigma_{\psi}(A) : L_- \to L_+ \tag{1.25}$$

defined by the composition $\sigma_{\psi}(A)(x,\xi) := p_+(x,\xi)\sigma_{\psi}(\widetilde{A})(x,\xi)r_-(x,\xi)$, where $p_+(x,\xi) = \sigma_{\psi}(P_+)(x,\xi)$, while $r_-(x,\xi) : L_{-,(x,\xi)} \to (\pi_M^* J_-)(x,\xi)$ is the canonical embedding of fibres on (x,ξ). We then have $\sigma_{\psi}(A)(x,\lambda\xi) = \lambda^{\mu}\sigma_{\psi}(A)(x,\xi)$ for all $(x,\xi) \in T^*M \setminus 0$, $\lambda \in \mathbb{R}_+$.

Let $S^{(\mu)}(T^*M \setminus 0; L_-, L_+)$ for $L_{\pm} \in \operatorname{Vect}(T^*M \setminus 0)$ denote the set of all bundle morphisms $p_{(\mu)} : L_- \to L_+$ such that $p_{(\mu)}(x,\lambda\xi) = \lambda^{\mu}p_{(\mu)}(x,\xi)$ for all $(x,\xi) \in T^*M \setminus 0$; $\lambda \in \mathbb{R}_+$ (we hope this notation does not cause confusion in connection with $S^{(\mu)}(T^*M \setminus 0; E, F)$ for $E, F \in \operatorname{Vect}(M)$ which is an abbreviation of $S^{(\mu)}(T^*M \setminus 0; \pi_M^*E, \pi_M^*F)$).

Proposition 1.25. *The principal symbolic map*

$$\sigma_{\psi} : T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+) \to S^{(\mu)}(T^*M \setminus 0; L_-, L_+) \tag{1.26}$$

is surjective, and there is a linear map

$$\operatorname{op} : S^{(\mu)}(T^*M \setminus 0; L_-, L_+) \to T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+) \tag{1.27}$$

such that $\sigma_{\psi} \circ \operatorname{op} = \mathrm{id}$.

Proof. The map (1.26) directly follows from (1.6) when we represent elements of $T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+)$ as $P_+\widetilde{A}P_-$ and write $\sigma_{\psi}(P_+\widetilde{A}R_-)$ as $\sigma_{\psi}(P_+\widetilde{A}P_-)|_{L_-}$, interpreted as a morphism (1.25). To see that (1.26) is surjective we choose a morphism $a_{(\mu)} : \pi_M^*J_- \to \pi_M^*J_+$ which restricts to a given element $t_{(\mu)} \in S^{(\mu)}(T^*M \setminus 0; L_-, L_+)$ (this is always possible), then form an operator $\widetilde{A} := \operatorname{op}(a_{(\mu)}) \in L^{\mu}_{\mathrm{cl}}(M; J_-, J_+)$ such that $\sigma_{\psi}(\widetilde{A}) = a_{(\mu)}$, cf. formula (1.7), and finally set $A := P_+\widetilde{A}R_-$; then $\sigma_{\psi}(A) = t_{(\mu)}$. $\square$

Proposition 1.26. *Every* $A \in T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+)$ *induces continuous operators*

$$A : P^s(M, \mathbb{L}_-) \to P^{s-\mu}(M, \mathbb{L}_+) \tag{1.28}$$

for all $s \in \mathbb{R}$.

This is an immediate consequence of the definition and of Theorem 1.4 (i).

PROPOSITION 1.27. *Let $A \in T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+)$ and suppose $\sigma_\psi(A) = 0$. Then $A \in T^{\mu-1}(M; \mathbb{L}_-, \mathbb{L}_+)$, and the operator* (1.28) *is compact for every $s \in \mathbb{R}$.*

PROOF. We have $P_+\widetilde{A}P_- \in L^{\mu}_{\mathrm{cl}}(M; J, J)$ and $\sigma_\psi(P_+\widetilde{A}P_-) = 0$. Hence $P_+\widetilde{A}P_- : H^s(M, J_-) \to H^{s-\mu}(M, J_+)$ is a compact operator, cf. Theorem 1.4 (ii). Thus $P_+(P_+\widetilde{A}P_-) = P_+\widetilde{A}P_- : H^s(M, J_-) \to P^{s-\mu}(M, \mathbb{L}_+)$ is also compact. Finally, since $R_- : P^s(M, \mathbb{L}_-) \to H^s(M, J_-)$ is continuous, also the operator $P_+\widetilde{A}P_-R_- = P_+\widetilde{A}R_- : P^s(M, \mathbb{L}_-) \to P^{s-\mu}(M, \mathbb{L}_+)$ is compact. These conclusions hold for all $s \in \mathbb{R}$. □

THEOREM 1.28. *$A \in T^{\mu}(M; \mathbb{L}_0, \mathbb{L}_+)$, $B \in T^{\nu}(M; \mathbb{L}_-, \mathbb{L}_0)$ for $\mathbb{L}_-, \mathbb{L}_0, \mathbb{L}_+ \in \mathcal{P}(M)$ implies $AB \in T^{\mu+\nu}(M; \mathbb{L}_-, \mathbb{L}_+)$, and we have*

$$\sigma_\psi(AB) = \sigma_\psi(A)\sigma_\psi(B).$$

PROOF. Writing $\mathbb{L}_\pm = (P_\pm, J_\pm, L_\pm)$, $\mathbb{L}_0 = (P_0, J_0, L_0)$, we have $A = P_+\widetilde{A}R_0$, $B = P_0\widetilde{B}R_-$ for certain $\widetilde{A} \in L^{\mu}_{\mathrm{cl}}(M; J_0, J_+)$, $\widetilde{B} \in L^{\nu}_{\mathrm{cl}}(M; J_-, J_0)$, with obvious meaning of R_0, R_-. Then $AB = P_+\widetilde{A}R_0P_0\widetilde{B}R_- = P_-\widetilde{A}P_0\widetilde{B}R_-$. Since $\widetilde{A}P_0\widetilde{B} \in L^{\mu+\nu}_{\mathrm{cl}}(M; J_-, J_+)$, cf. Theorem 1.4 (iii), we obtain $AB \in T^{\mu+\nu}(M; \mathbb{L}_-, \mathbb{L}_+)$. The symbolic rule is a consequence of relation (1.9). □

Given an operator $A \in T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+)$, $A = P_+\widetilde{A}R_-$, we define the formal adjoint

$$A^* := P_-^*\widetilde{A}^*R_+^*,$$

where $\widetilde{A}^*, P_-^*$ are the formal adjoints in the sense of Theorem 1.4 (iv), and $R_+^* : P^s(M; \mathbb{L}_+^*) \to H^s(M, J_+)$ is the canonical embedding.

THEOREM 1.29. *$A \in T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+)$ entails*

$$A^* \in T^{\mu}(M; \mathbb{L}_+^*, \mathbb{L}_-^*)$$

and

$$\sigma_\psi(A^*) = \sigma_\psi(A)^*,$$

where the adjoint on the right hand side refers to the Hermitian metrics in the bundles $L_+^* := \sigma_\psi(P_+^*)\pi_Y^*J_+$ and $L_-^* := \sigma_\psi(P_-^*)\pi_Y^*J_-$, induced by $\pi_Y^*J_+$ and $\pi_Y^*J_-$, respectively.

PROOF. It suffices to apply Theorem 1.4 (iv) and Definition 1.21. □

DEFINITION 1.30. Let $A \in T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+)$, $\mu \in \mathbb{R}$, $\mathbb{L}_\pm = (P_\pm, J_\pm, L_\pm) \in \mathcal{P}(M)$.

(i) The operator A is said to be elliptic (of order μ), if $\sigma_\psi(A) : L_- \to L_+$ is an isomorphism.
(ii) An operator $B \in T^{-\mu}(M; \mathbb{L}_+, \mathbb{L}_-)$ is called a parametrix of A, if B satisfies the following relations:

$$C_l := I - BA \in T^{-\infty}(M; \mathbb{L}_-, \mathbb{L}_-), \qquad C_r := I - AB \in T^{-\infty}(M; \mathbb{L}_+, \mathbb{L}_+), \tag{1.29}$$

where I denotes the corresponding identity operators.

REMARK 1.31. For every $\mathbb{L} := (P, J, L) \in \mathcal{P}(M)$ and every $\mu \in \mathbb{R}$ there exists an elliptic operator $R^{\mu}_{\mathbb{L}} \in T^{\mu}(M; \mathbb{L}, \mathbb{L})$.

In fact, let $a_{(\mu)} \in S^{(\mu)}(T^*M \setminus 0; J, J)$ be the unique element that restricts to the identity map on π_1^*J, with $\pi_1 : S^*M \to M$ being the canonical projection of the unit cosphere bundle S^*M to M. Set $\widetilde{A} := \mathrm{op}(a_{(\mu)})$, cf. formula (1.7). Then $P\widetilde{A}R$ for the embedding $R : P^s(M, \mathbb{L}) \to H^s(M, J)$ is elliptic because $\sigma_\psi(P\widetilde{A}R) : L \to L$ is an isomorphism.

THEOREM 1.32. *Let $A \in T^{\mu}(M; \mathbb{L}_-, \mathbb{L}_+)$, $\mu \in \mathbb{R}$, $\mathbb{L}_\pm = (P_\pm, J_\pm, L_\pm) \in \mathcal{P}(M)$.*

(i) *The operator A is elliptic (of order μ) if and only if*

$$A : P^s(M, \mathbb{L}_-) \to P^{s-\mu}(M, \mathbb{L}_+) \tag{1.30}$$

is a Fredholm operator for some $s = s_0 \in \mathbb{R}$.

(ii) *If $\mathcal{A}$ is elliptic,* (1.30) *is Fredholm for all $s \in \mathbb{R}$, and* $\dim\ker\mathcal{A}$ *and* $\dim\operatorname{coker}\mathcal{A}$ *are independent of s.*

(iii) *An elliptic operator $A \in T^{\mu}(M;\mathbb{L}_-,\mathbb{L}_+)$ has a parametrix $B \in T^{-\mu}(M;\mathbb{L}_+,\mathbb{L}_-)$, and B can be chosen in such a way that the remainders in the relation* (1.29) *are projections*

$$C_l : P^s(M,\mathbb{L}_-) \to V, \qquad C_l : P^{s-\mu}(M,\mathbb{L}_+) \to W$$

for all $s \in \mathbb{R}$, for $V := \ker A \subset P^{\infty}(M,\mathbb{L}_-)$ and a finite-dimensional subspace $W \subset P^{\infty}(M,\mathbb{L}_+)$ with the property $W + \operatorname{im} A = P^{s-\mu}(M,\mathbb{L}_+)$ and $W \cap \operatorname{im} A = \{0\}$ for every $s \in \mathbb{R}$.

The proof of Theorem 1.32 will be given below. First observe that for arbitrary $\mathbb{L}_\pm = (P_\pm, J_\pm, L_\pm) \in \mathcal{P}(M)$ there exist elements $\mathbb{M}_\pm = (Q_\pm, \mathbb{C}^m, L_\pm) \in \mathcal{P}(M)$ such that

$$T^{\mu}(M;\mathbb{L}_-,\mathbb{L}_+) = T^{\mu}(M;\mathbb{M}_-,\mathbb{M}_+). \tag{1.31}$$

In fact, for m sufficiently large, $J_\pm$ may be represented as subbundles of the trivial bundle $\mathbb{C}^m$, with complementary bundles $J_\pm^{\perp}$ and projections $\pi_M^*\mathbb{C}^m \to \pi_M^* J_\pm$ along $\pi_M^* J_\pm^{\perp}$ and associated pseudo-differential projections

$$\widetilde{P}_\pm \in L^0_{\mathrm{cl}}(M;\mathbb{C}^m;\mathbb{C}^m), \quad \widetilde{P}_\pm : H^s(M,\mathbb{C}^m) \to H^s(M,J_\pm).$$

Moreover, the compositions of projections

$$\sigma_\psi(P_\pm)\sigma_\psi(\widetilde{P}_\pm) := \sigma_\psi(Q_\pm) : \pi_M^*\mathbb{C}^m \to \pi_M^* J_\pm \to L_\pm$$

are again projections, and $Q_\pm := P_\pm \widetilde{P}_\pm \in L^0_{\mathrm{cl}}(M;\mathbb{C}^m,\mathbb{C}^m)$ are associated pseudo-differential projections, $Q_\pm : H^s(M,\mathbb{C}^m) \to \mathcal{P}^s(M,\mathbb{L}_\pm)$. Relation (1.31) is then obvious when we use that every $\widetilde{A} \in L^{\mu}_{\mathrm{cl}}(M;J_-,J_+)$ can be identified with an element $\widetilde{\widetilde{A}} \in L^{\mu}_{\mathrm{cl}}(M;\mathbb{C}^m,\mathbb{C}^m)$ by $\widetilde{\widetilde{A}} = \widetilde{A}\widetilde{P}_-$.

Proposition 1.33. *Let $A \in T^{\mu}(M;\mathbb{L}_-,\mathbb{L}_+)$ be an elliptic operator, and let us represent A as an element $A \in T^{\mu}(M;\mathbb{M}_-,\mathbb{M}_+)$ for $\mathbb{M}_\pm := (Q_\pm,\mathbb{C}^m,L_\pm)$ for a sufficiently large m. Then there exists an elliptic operator $A^{\perp} \in T^{\mu}(M;\mathbb{M}_-^{\perp},\mathbb{M}_+^{\perp})$ for suitable $\mathbb{M}_\pm^{\perp} \in \mathcal{P}(M)$ such that $A \oplus A^{\perp} \in L^{\mu}(M;\mathbb{C}^m,\mathbb{C}^m)$ is elliptic in the sense of Definition* 1.5 (i).

Proof. Let us write the bundles $L_\pm$ as subbundles of the trivial bundle $\mathbb{C}^m$ on $T^*M \setminus 0$ in such a way that $L_{-,(x,\xi)} \cap L_{+,(x,\xi)} = \{0\}$ for every $(x,\xi) \in T^*M \setminus 0$. We then have a $G \in \mathrm{Vect}(T^*M \setminus 0)$ such that $L_- \oplus L_+ \oplus G = \mathbb{C}^m$. Let us define an isomorphism

$$(L_- \oplus L_+ \oplus G)|_{S^*M} \to (L_+ \oplus L_- \oplus G)|_{S^*M} \tag{1.32}$$

by $\operatorname{diag}(\sigma_\psi(A)|_{S^*M}, \sigma_\psi^{-1}(A)|_{S^*M}, \mathrm{id}_{G|_{S^*M}})$, set $L_-^{\perp} := L_+ \oplus G$, $L_+^{\perp} := L_- \oplus G$, and define $\sigma_\psi(A^{\perp}) \in S^{(\mu)}(T^*M \setminus 0; L_-^{\perp}, L_+^{\perp})$ to be the unique element such that $\sigma_\psi(A^{\perp}) = \operatorname{diag}(\sigma_\psi^{-1}(A)|_{S^*M}, \mathrm{id}_{G|_{S^*M}})$. Moreover, set

$$\mathbb{M}_\pm^{\perp} := (Q_\pm^{\perp}, \mathbb{C}^m, L_\pm^{\perp}).$$

By definition, the operator A has the form $A = Q_+\widetilde{A}_1 R_-$ for some element $\widetilde{A}_1 \in L^{\mu}_{\mathrm{cl}}(M;\mathbb{C}^m,\mathbb{C}^m)$, where $R_- : P^s(M,\mathbb{M}_-) \to H^s(M,\mathbb{C}^m)$ is the canonical embedding. Moreover, let $\sigma_\psi(\widetilde{A}_2) \in S^{(\mu)}(T^*M \setminus 0;\mathbb{C}^m,\mathbb{C}^m)$ denote the unique element such that $\sigma_\psi(\widetilde{A}_2)|_{S^*M}$ coincides with (1.32). Then, applying (1.7) to $\sigma_\psi(\widetilde{A}_2)$, we obtain an associated elliptic operator $\widetilde{A}_2 \in L^{\mu}_{\mathrm{cl}}(M;\mathbb{C}^m,\mathbb{C}^m)$ with $R_-^{\perp} : P^s(M,\mathbb{M}_-^{\perp}) \to H^s(M,\mathbb{C}^m)$ being the canonical embedding, where $\sigma_\psi(A^{\perp})$ coincides with the abovementioned symbol. The operator

$$\widetilde{A} := A \oplus A^{\perp} = \operatorname{diag}(Q_+\widetilde{A}_1 R_-, Q_+^{\perp}\widetilde{A}_2 R_-^{\perp})$$

can be identified with

$$Q_+\widetilde{A}_1 P_- + Q_+^{\perp}\widetilde{A}_2 P_-^{\perp} : H^s(M,\mathbb{C}^m) \to H^{s-\mu}(M,\mathbb{C}^m),$$

i.e., as a sum of standard pseudo-differential operators, cf. formula (1.23), and hence belongs to $L^{\mu}_{\mathrm{cl}}(M;\mathbb{C}^m,\mathbb{C}^m)$, and $\widetilde{A}$ is elliptic, since $\sigma_\psi(\widetilde{A}) = \sigma_\psi(\widetilde{A}_2)$. ☐

Proposition 1.34. *Let $A_j \in T^{\mu-j}(M;\mathbb{L}_-,\mathbb{L}_+)$, $j \in \mathbb{N}$, be an arbitrary sequence. Then there exists an $A \in T^{\mu}(M;\mathbb{L}_-,\mathbb{L}_+)$ such that*

$$A - \sum_{j=0}^{N} A_j \in T^{\mu-(N+1)}(M;\mathbb{L}_-,\mathbb{L}_+) \tag{1.33}$$

for every $N \in \mathbb{N}$, and A is unique mod $T^{-\infty}(M;\mathbb{L}_-,\mathbb{L}_+)$.

Proof. By definition, every A_j has the form $A_j = P_+\widetilde{A}_jR_-$ for suitable $\widetilde{A}_j \in L^{\mu-j}_{\mathrm{cl}}(M;J_-,J_+)$. A well known result on standard pseudo-differential operators says that there is an $\widetilde{A} \in L^{\mu}_{\mathrm{cl}}(M;J_-,J_+)$ such that $\tilde{A} - \sum_{j=0}^{N}\tilde{A}_j \in L^{\mu-(N+1)}_{\mathrm{cl}}(M;J_-,J_+)$ for every $N \in \mathbb{N}$, and $\widetilde{A}$ is unique mod $L^{-\infty}(M;J_-,J_+)$. We set $A := P_+\widetilde{A}R_-$; then the relation (1.33) holds. Moreover, if an element $A' \in T^{\mu}(M;\mathbb{L}_-,\mathbb{L}_+)$ satisfies (1.33) with A' in place of A, we have $A - A' = P_+(\widetilde{A} - \widetilde{A}')R_-$, where $P_+(\widetilde{A} - \widetilde{A}')P_- \in \bigcap_{j\in\mathbb{N}} L^{\mu-j}_{\mathrm{cl}}(M;J_-,J_+) = L^{-\infty}(M;J_-,J_+)$, and hence, $A - A' \in T^{-\infty}(M;\mathbb{L}_-,\mathbb{L}_+)$, by virtue of (1.24). □

Proof of Theorem 1.32. First assume that $A \in T^{\mu}(M;\mathbb{L}_-,\mathbb{L}_+)$ is elliptic. According to the relation (1.31), without loss of generality we may assume $\mathbb{L}_\pm = (P_\pm,\mathbb{C}^m,L_\pm)$ for some sufficiently large m. Applying Proposition 1.33 we find an elliptic element $A^\perp \in T^{\mu}(M;\mathbb{L}_+^\perp,\mathbb{L}_-^\perp)$ such that $\widetilde{A} = A \oplus A^\perp$ is elliptic in $L^{\mu}_{\mathrm{cl}}(M;\mathbb{C}^m,\mathbb{C}^m)$. We now choose a parametrix $\widetilde{P} \in L^{-\mu}_{\mathrm{cl}}(M;\mathbb{C}^m,\mathbb{C}^m)$ in the sense of Theorem 1.7 (iii) and set $B_0 := P_-\widetilde{P}R_+ \in T^{-\mu}(M;\mathbb{L}_+,\mathbb{L}_-)$. Writing $A = P_-\widetilde{A}R_-$ we obtain

$$\begin{aligned} C_l^0 := I - B_0A &= I - P_-\widetilde{P}R_+P_+\widetilde{A}R_- \\ &= I - P_-\widetilde{P}P_+\widetilde{A}R_- \end{aligned}$$

which yields $\sigma_\psi(C_l^0) = 0$, i.e., $C_l^0 \in T^{-1}(M;\mathbb{L}_-,\mathbb{L}_-)$. In a similar manner it follows that $C_r^0 := I - AB_0 \in T^{-1}(M;\mathbb{L}_+,\mathbb{L}_+)$.

By virtue of Proposition 1.27 the operators $C_l^0 \in \mathcal{L}(P^s(M,\mathbb{L}_-))$ and $C_r^0 \in \mathcal{L}(P^{s-\mu}(M,\mathbb{L}_+))$ are compact. Thus (1.30) is a Fredholm operator for every $s \in \mathbb{R}$. Applying a formal Neumann series argument we find a $K \in T^{-1}(M;\mathbb{L}_-,\mathbb{L}_-)$ such that $K \sim -\sum_{j=1}^{\infty}(C_l^0)^j$, cf. Proposition 1.34.

For $B_l := (I-K)B_0 \in T^{-\mu}(M;\mathbb{L}_+,\mathbb{L}_-)$ we then obtain $C_l := I - B_lA \in T^{-\infty}(M;\mathbb{L}_-,\mathbb{L}_-)$. In an analogous manner we find a $B_r \in T^{-\mu}(M;\mathbb{L}_+,\mathbb{L}_-)$ such that $C_r := I - AB_r \in T^{-\infty}(M;\mathbb{L}_-,\mathbb{L}_-)$. A standard algebraic argument shows that B_l is a two-sided parametrix of A. In fact, $B_lA = I - C_l^1$ implies $AB_lA = A - AC_l$ and $AB_lAB_r = AB_l(I - C_r) = AB_r - AC_l^1B_r$; then Theorem 1.28 and relation $AB_r^1 = I \bmod T^{-\infty}(M;\mathbb{L}_+,\mathbb{L}_+)$ yield $AB_l^1 = I \bmod T^{-\infty}(M;\mathbb{L}_+,\mathbb{L}_+)$. Propositions 1.19 and 1.20 together with the first part of the proof allow us to apply Remark 1.8 to the present situation. This gives us the assertions of Theorem 1.32 (ii), (iii).

It remains to show the second part of Theorem 1.32 (i), namely that the Fredholm property of (1.30) for an $s = s_0 \in \mathbb{R}$ implies the ellipticity of A. Without loss of generality we may assume $s_0 = \mu = 0$. In fact, Remark 1.23 gives us elliptic operators

$$R^{-s_0}_{\mathbb{L}_-} : P^0(M,\mathbb{L}_-) \to P^{s_0}(M,\mathbb{L}_-), R^{s_0-\mu}_{\mathbb{L}_+} : P^{s_0-\mu}(M,\mathbb{L}_+) \to P^0(M,\mathbb{L}_+).$$

These are Fredholm operators by the first part of the proof of Theorem 1.32. Thus, also $A_0 : R^{s_0-\mu}_{\mathbb{L}_+}AR^{-s_0}_{\mathbb{L}_-} : P^0(M,\mathbb{L}_-) \to P^0(M,\mathbb{L}_+)$ is Fredholm. If we show the ellipticity of A_0, we obtain at once the ellipticity of A, because ellipticity remains preserved under compositions. In other words, we may consider the case $A := A_0$,

$$A : P^0(M,\mathbb{L}_-) \to P^0(M,\mathbb{L}_+). \tag{1.34}$$

Moreover, let

$$E : P^0(M,\mathbb{L}_-^\perp) \to P^0(M,\mathbb{L}_-^\perp), \tag{1.35}$$

be the identity operator, cf. the notation in Remark 1.24.

We have

$$L^2(M,J_-) = P^0(M,\mathbb{L}_-) \oplus P^0(M,\mathbb{L}_-^\perp), \tag{1.36}$$

and there are continuous embeddings

$$R_+ : P^0(M,\mathbb{L}_+) \to L^2(M,J_+), \qquad R_-^\perp : P^0(M,\mathbb{L}_-^\perp) \to L^2(M,J_-). \tag{1.37}$$

Then we can pass to the operator

$$B := \begin{pmatrix} R_+ & 0 \\ 0 & R_-^\perp \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix} : L^2(M,J_-) \to L^2(M,J_+ \oplus J_-)$$

which is an element of $L^0_{\mathrm{cl}}(M;J_-,J_+\oplus J_-)$. By assumption, the operator (1.34) is Fredholm. In particular, there is an operator $Q : P^0(M,\mathbb{L}_+) \to P^0(M,\mathbb{L}_-)$ such that $I - QA : P^0(M,\mathbb{L}_-) \to P^0(M,\mathbb{L}_-)$ is compact.

Let $S : L^2(M,J_+ \oplus J_-) \to P^0(M,\mathbb{L}_+) \oplus P^0(M,\mathbb{L}_-^\perp)$ denote a projection. Then $T := \operatorname{diag}(Q,E) \circ S$ has the property that $I - TB =: K$ is compact in $L^2(M,J_-)$. Since $I-K$ is a Fredholm operator, we have $\dim\ker(I-K) < \infty$, and then $\dim\ker B < \infty$, since $Bu = 0$ implies $TBu = 0$ which yields $(I-K)u = 0$, i.e., $\ker B \subseteq \ker(I-K)$.

The operator $\widetilde{B} := B^*B : L^2(M,J_-) \to L^2(M,J_-)$ belongs to $L^0_{\mathrm{cl}}(M,J_-)$ and is self-adjoint and Fredholm.

From Theorem 1.6 (i) we know that $\widetilde{B}$ is elliptic. It follows that $\sigma_\psi(B)$ is injective. Hence, also $\sigma_\psi(A)$ is injective. By passing to adjoint operators, in an analogous manner we can show that $\sigma_\psi(A)$ is also surjective.

This completes the proof of Theorem 1.32. ☐

We now discuss the question to what extent the specific choice of projections $P_\pm$ in an elliptic operator $A \in \mathcal{T}^\mu(M;\mathbb{L}_-,\mathbb{L}_+)$ may affect the index. This is an aspect on more general properties of Fredholm operators and projections in Hilbert spaces, see, for instance, [**44**].

Let H_+ and H_- be Hilbert spaces, and let $P_+, Q_+ \in \mathcal{L}(H_+)$ and $P_-, Q_- \in \mathcal{L}(H_-)$ be continuous projections, such that $P_\pm - Q_\pm$ are compact operators in $H_\pm$.

THEOREM 1.35. *Let $\widetilde{A} \in \mathcal{L}(H_-,H_+)$ such that*

$$A := P_+\widetilde{A} : \operatorname{im} P_- \to \operatorname{im} P_+$$

is a Fredholm operator. Then, also

$$B := Q_+\widetilde{A} : \operatorname{im} Q_- \to \operatorname{im} Q_+$$

is a Fredholm operator, and we have

$$\operatorname{ind} A - \operatorname{ind} B = \operatorname{ind}(P_-,Q_-) - \operatorname{ind}(P_+,Q_+), \tag{1.38}$$

cf. the notation (1.17).

PROOF. From Proposition 1.17 we know that

$$P_- : \operatorname{im} Q_- \to \operatorname{im} P_- \quad \text{and} \quad Q_+ : \operatorname{im} P_+ \to \operatorname{im} Q_+$$

are Fredholm operators, and Q_- is a Fredholm inverse of P_-. Thus $Q_-P_- : \operatorname{im} Q_- \to \operatorname{im} Q_-$ is a Fredholm operator of index zero. An analogous observation holds for the projections Q_+, P_+. Thus, the operator

$$D : \operatorname{im} Q_- \xrightarrow{P_-} \operatorname{im} P_- \xrightarrow{A} \operatorname{im} P_+ \xrightarrow{Q_+} \operatorname{im} Q_+$$

is Fredholm of index

$$\operatorname{ind} D = \operatorname{ind} A + \operatorname{ind}(P_-,Q_-) - \operatorname{ind}(P_+,Q_+), \tag{1.39}$$

cf. the relation (1.18). We have

$$\begin{aligned} D &= (Q_+P_+)B(Q_-P_-) \\ &- Q_+[P_+,Q_+]\widetilde{A}Q_-P_- + Q_+P_+\widetilde{A}(1-Q_-)P_-, \end{aligned}$$

where $[P_+, Q_+]$ is the commutator in H_+ which is compact, since

$$\begin{aligned}[P_+, Q_+] &= P_+Q_+ - Q_+P_+ \\ &= (P_+ - Q_+)(1 - Q_+ - P_+).\end{aligned}$$

Moreover, $(1-Q_-)P_- = (P_- - Q_-)P_- : H_- \to H_-$ is compact. Hence the operator $(Q_+P_+)B(Q_-P_-) - D$ is compact, i.e., $(Q_+P_+)B(Q_-P_-)$ is Fredholm, and we have

$$\operatorname{ind} D = \operatorname{ind}(Q_+P_+)B(Q_-P_-). \tag{1.40}$$

By virtue of Proposition 1.17 the operators $Q_-P_- : \operatorname{im} Q_- \to \operatorname{im} Q_-$ and $Q_+P_+ : \operatorname{im} Q_+ \to \operatorname{im} Q_+$ are Fredholm and of index zero. Therefore, we have $\operatorname{ind}(Q_+P_+)B(Q_-P_-) = \operatorname{ind} B$, and the assertion is a consequence of the relations (1.40) and (1.39). □

Corollary 1.36. *Consider elliptic operators $A \in T^\mu(M; \mathbb{L}_-, \mathbb{L}_+)$ for $\mathbb{L}_\pm := (P_\pm, J_\pm, L_\pm)$ and $B \in T^\mu(M; \mathbb{N}_-, \mathbb{N}_+)$ for $\mathbb{N}_\pm := (Q_\pm, J_\pm, L_\pm)$, and assume that $\sigma_\psi(A) = \sigma_\psi(B)$. Then the Fredholm indices of A and B as operators*

$$A : P^s(M, \mathbb{L}_-) \to P^{s-\mu}(M, \mathbb{L}_+), \qquad B : P^s(M, \mathbb{N}_-) \to P^{s-\mu}(M, \mathbb{N}_+)$$

are related by the formula (1.38).

Remark 1.37. Given projection data $\mathbb{L} := (P, J, L)$ and $\mathbb{M} := (Q, J, L) \in \mathcal{P}(M)$ (with the same J, L but different projections), the operators

$$P : P^s(M, \mathbb{M}) \to P^s(M, \mathbb{L}), \qquad Q : P^s(M, \mathbb{L}) \to P^s(M, \mathbb{N}) \tag{1.41}$$

are Fredholm, and Q is a Fredholm inverse of P (and vice versa), cf. Proposition 1.17. We have

$$P \in T^0(M; \mathbb{M}, \mathbb{L}), \qquad Q \in T^0(M; \mathbb{L}, \mathbb{M}),$$

P and Q are elliptic of order zero, and hence, in particular, the Fredholm indices (as well as the dimensions of kernel and cokernel) of (1.41) are independent of s.

Remark 1.38. Let $A_0, A_1 \in T^\mu(M; \mathbb{L}_-, \mathbb{L}_+)$ be elliptic, and assume that the principal symbols

$$\sigma_\psi(A_i) : L_- \to L_+$$

coincide for $i = 0, 1$. Then we have

$$\operatorname{ind} A_0 = \operatorname{ind} A_1.$$

In fact, Proposition 1.27 gives us $\sigma_\psi(A_0 - A_1) = 0$, i.e., $A_0 - A_1 \in T^{\mu-1}(M; \mathbb{L}_-, \mathbb{L}_+)$, and hence A_0 is equal to A_1 modulo a compact operator.

Let us now assume that

$$\mathbb{L}_\pm(t) := (P_\pm(t), J, L_\pm(t)), \quad 0 \le t \le 1,$$

is a family of elements in $\mathcal{P}(M)$, where

$$P_\pm(\cdot) \in C([0,1], L^0_{\mathrm{cl}}(M; J, J))$$

are families of projections, such that

$$L_\pm(t) = \sigma_\psi(P_\pm(t))\pi_M^* J_\pm$$

are families of subbundles in $\pi_M^* J_\pm$. Let us assume that

$$a_{(\mu)}(t) : L_-(t) \to L_+(t)$$

is a continuous family of isomorphisms, smooth in $(x, \xi) \in T^*M \setminus 0$ and homogeneous of degree μ. We can complete $a_{(\mu)}(t)$ to a continuous family of morphisms

$$\widetilde{a}_{(\mu)} : \pi_M^* J_- \to \pi_M^* J_+$$

such that $a_{(\mu)}(t)$ can be identified with $\sigma_\psi(P_+(t))\widetilde{a}_{(\mu)}(t)\sigma_\psi(P_-(t))$ for every t. Let us set $\widetilde{A}_t := \operatorname{op}(\widetilde{a}_{(\mu)}(t))$, cf. Proposition 1.3, which gives us an element of $C([0,1], L^\mu_{\mathrm{cl}}(M; J_-, J_+))$.

We then obtain a family

$$A_t := P_+(t)\widetilde{A}(t)R_-(t) \in T^\mu(M; \mathbb{L}_-(t), \mathbb{L}_+(t))$$

with $R_-(t) : P_-(t)H^s(M, J_-) \to H^s(M, J_-)$ being the canonical embeddings. The operators A_t are elliptic for all $t \in [0,1]$.

THEOREM 1.39. *Under the abovementioned conditions on $A_t \in T^\mu(M; \mathbb{L}_-(t), \mathbb{L}_+(t))$, $0 \le t \le 1$, we have*

$$\operatorname{ind} A_0 = \operatorname{ind} A_1,$$

where the index of A_t refers to the Fredholm operator

$$A_t : P_-(t)H^s(M, J_-) \to P_+(t)H^{s-\mu}(M, J_+).$$

Theorem 1.39 has a more general functional analytic background. The following considerations up to Remark 1.42 have been contributed to this paper by Thomas Krainer of the University of Potsdam.

THEOREM 1.40. *Let H and $\widetilde{H}$ be Hilbert spaces, and consider families of operators $(A_t)_{0 \le t \le 1} \in C([0,1], \mathcal{L}(H, \widetilde{H}))$,*

$$(P_t)_{0 \le t \le 1} \in C([0,1], \mathcal{L}(\widetilde{H})) \quad \text{and} \quad (Q_t)_{0 \le t \le 1} \in C([0,1], \mathcal{L}(H)).$$

Assume $P_t^2 = P_t$, $Q_t^2 = Q_t$ for all $t \in [0,1]$. Moreover, let

$$P_t A_t Q_t : \operatorname{im} Q_t \to \operatorname{im} P_t$$

be a Fredholm operator for every $t \in [0,1]$. Then we have

$$\operatorname{ind}(P_0 A_0 Q_0 : \operatorname{im} Q_0 \to \operatorname{im} P_0) = \operatorname{ind}(P_1 A_1 Q_1 : \operatorname{im} Q_1 \to \operatorname{im} P_1).$$

To prove this theorem we first show another result. Consider the set

$$\Pi(\widetilde{H}) \times \mathcal{L}(H, \widetilde{H}) \times \Pi(H), \tag{1.42}$$

where $\Pi(\mathcal{L}(H))$ for a Hilbert space H denotes the set of all $P \in \mathcal{L}(H)$ such that $P^2 = P$. Let $\Phi_k(H, \widetilde{H})$ denote the set of all triples (P, A, Q) in (1.42) such that

$$PAQ : \operatorname{im} Q \to \operatorname{im} P$$

is a Fredholm operator of index k.

PROPOSITION 1.41. *For every $k \in \mathbb{Z}$ the set $\Phi_k(H, \widetilde{H})$ is open in* (1.42).

PROOF. As is well known, the set of Fredholm operators of index k between Hilbert spaces L and $\widetilde{L}$ is open in $\mathcal{L}(L, \widetilde{L})$. Applying this to $L := \operatorname{im} Q$ and $\widetilde{L} := \operatorname{im} P$, it follows that for a triple $(P, A, Q) \in \Phi_k(H, \widetilde{H})$ there exists an $\varepsilon_0 > 0$ such that $(P, A + K, Q) \in \Phi_k(H, \widetilde{H})$ for every $K \in \mathcal{L}(H, \widetilde{H})$, $\|K\| < \varepsilon_0$.

We now prove that for every

$$(P, A, Q) \in \Phi_k(H, \widetilde{H}) \tag{1.43}$$

there exist constants $\alpha > 0$, $\varepsilon > 0$, $\beta > 0$, such that

$$(M, B, N) \in \Pi(\widetilde{H}) \times \mathcal{L}(H, \widetilde{H}) \times \Pi(H)$$

and

$$\|M - P\| < \alpha, \ \|B - A\| < \varepsilon, \ \|N - Q\| < \beta$$

imply

$$(M, B, N) \in \Phi_k(H, \widetilde{H}). \tag{1.44}$$

Let $G \in \mathcal{L}(H), \widetilde{G} \in \mathcal{L}(\widetilde{H})$ be invertible elements, such that

$$\|\widetilde{G} - I\| < \delta_1, \qquad \|G - I\| < \delta_2 \tag{1.45}$$

for sufficiently small $\delta_1, \delta_2 < 1$. Set

$$M := \widetilde{G} P \widetilde{G}^{-1}, \qquad N := G Q G^{-1}.$$

We prove that relation (1.44) holds for ε and δ_1, δ_2 so small that

$$\frac{1 + \delta_2}{1 - \delta_1}\varepsilon + \frac{\delta_1 + \delta_2}{1 - \delta_1}\|A\| < \varepsilon_0 \tag{1.46}$$

holds. From Neumann series arguments we obtain

$$\|\widetilde{G}^{-1}\| < \frac{1}{1-\delta_1}, \qquad \|\widetilde{G}^{-1} - I\| < \frac{\delta_1}{1-\delta_1} \tag{1.47}$$

and

$$\|G\| < 1 + \delta_2. \tag{1.48}$$

We now reformulate the operator MBN as follows:

$$MBN = \widetilde{G}P(A+K)QG^{-1} = \widetilde{G}\{PKQ + PAQ\}G^{-1} \tag{1.49}$$

for

$$K := \widetilde{G}^{-1}(B-A)G + (\widetilde{G}^{-1} - I)A + A(G-I) + (\widetilde{G}^{-1} - I)A(G-I).$$

Let us verify that $\|K\| < \varepsilon_0$. In fact, using (1.45), (1.46), (1.47), (1.48), we obtain

$$\begin{aligned}
\|K\| &\leq \|\widetilde{G}^{-1}\| \, \|B-A\| \, \|G\| \\
&+ \|\widetilde{G}^{-1} - I\| \, \|A\| + \|A\| \, \|G-I\| \\
&+ \|\widetilde{G}^{-1} - I\| \, \|A\| \, \|G-I\| \\
&\leq \frac{1}{1-\delta_1}\varepsilon(1+\delta_2) + \frac{\delta_1}{1-\delta_1}\|A\| + \delta_2\|A\| + \frac{\delta_1}{1-\delta_1}\delta_2\|A\| < \varepsilon_0.
\end{aligned}$$

Thus, from the first part of the proof it follows that $(P, A+K, Q) \in \Phi_k(H, \widetilde{H})$. Moreover, (1.49) together with the isomorphisms

$$\widetilde{G} : \operatorname{im} P \to \operatorname{im} M, \qquad G : \operatorname{im} N \to \operatorname{im} Q$$

gives us the relation (1.49).

Let us finally consider the map

$$s : \Pi(\widetilde{H}) \to \mathcal{L}(\widetilde{H}),$$

defined by $s(M) := MP + (I-M)(I-P)$. The map s is continuous, and we have $s(P) = I$. Let us choose $\widetilde{\delta}_1 > 0$ in such a way that $\|s(M) - s(P)\| < \delta_1$ holds when $\|M - P\| < \widetilde{\delta}_1$. Since $\delta_1 < 1$ is very small, we have the inverse $s(M)^{-1} \in \mathcal{L}(\widetilde{H})$ for $\|M - P\| < \widetilde{\delta}_1$. We then obtain $Ms(M) = MP = s(MP)$, i.e., $M = \widetilde{G}P\widetilde{G}^{-1}$ for $\widetilde{G} := s(M)$. In a similar manner it follows that $\|N - Q\| < \widetilde{\delta}_2$ for a suitable small $\widetilde{\delta}_2 > 0$ implies $N = GQG^{-1}$ for an invertible $G \in \mathcal{L}(H)$, $\|G - I\| < \delta_2$. This completes the proof of Proposition 1.41. □

Proof of Theorem 1.40. By virtue of Proposition 1.41 the map $[0,1] \to \mathbb{Z}$ defined by $t \to \operatorname{ind} P_t A_t Q_t$ is continuous and hence constant.

□

Remark 1.42. Theorem 1.40 can be generalised to the case when $[0,1]$ is replaced by any connected topological space X. It follows that the index of Fredholm operators $P_x A_x Q_x : \operatorname{im} Q_x \to \operatorname{im} P_x$ which are continuously parametrised by $x \in X$, is constant.

1.2. Operators with the transmission property at the boundary. In this section we prepare some necessary material on symbols and operators with the transmission property at the boundary and establish a connection with operators on the boundary with operator-valued symbols.

Let $\Omega \subseteq \mathbb{R}^q \ni y$ be open, and let $U := \Omega \times \mathbb{R} \ni x = (y,t), \xi = (\eta, \tau)$.

Set

$$H^s_{\mathrm{loc}(y)}(U) := \big\{u \in H^s_{\mathrm{loc}}(U) \quad : \quad \varphi u \in H^s(\mathbb{R}^{q+1}) \text{ for every } \varphi(y) \in C_0^\infty(\Omega)\big\},$$

and define

$$H^s_{\mathrm{comp}(y)}(U) := \big\{u \in H^s_{\mathrm{loc}}(U) \quad : \quad \psi u \in H^s_{\mathrm{comp}}(\mathbb{R}^{n+1}) \text{ for every } \psi(t) \in C_0^\infty(\mathbb{R})\big\}.$$

Moreover, set

$$\begin{aligned} H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}_\pm) &:= H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R})|_{\Omega\times\mathbb{R}_\pm}, \\ H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_\pm) &:= H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R})|_{\Omega\times\mathbb{R}_\pm}. \end{aligned}$$

DEFINITION 1.43. A symbol $a(x,\xi) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^{q+1})$ for $\mu \in \mathbb{Z}$ is said to have the transmission property at $t = 0$, if the homogeneous components $a_{(\mu-j)}(x,\xi)$ satisfy the following relations:

$$D^k_t D^\alpha_\eta \{a_{(\mu-j)}(y,t,\eta,\tau) - (-1)^{\mu-j} a_{(\mu-j)}(y,t,-\eta,-\tau)\} = 0$$

for $y \in \Omega, t = 0, \eta = 0, \tau \in \mathbb{R}\setminus\{0\}$, for all $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^q$, and for all $j \in \mathbb{N}$.

Let $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^{q+1})_{\mathrm{tr}}$ denote the space of all symbols $a(x,\xi) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^{q+1})$ with the transmission property at $t = 0$. Moreover, set

$$S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_\pm \times \mathbb{R}^{q+1})_{\mathrm{tr}} := \{a|_{\Omega\times\overline{\mathbb{R}}_\pm\times\mathbb{R}^{q+1}} : a \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^{q+1})_{\mathrm{tr}}\}.$$

REMARK 1.44. $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^{q+1})_{\mathrm{tr}}$ is a closed subspace of $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^{q+1})$; analogously, $S^\mu_{\mathrm{cl}}(\Omega\times\overline{\mathbb{R}}_\pm\times\mathbb{R}^{q+1})_{\mathrm{tr}}$ is closed in $S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_\pm \times \mathbb{R}^{q+1})$.

To illustrate the structure of the transmission property we want to have a look at the one-dimensional case, say on $\overline{\mathbb{R}}_+ \times \mathbb{R}$ (the case $\overline{\mathbb{R}}_- \times \mathbb{R}$ is analogous; the transmission property is an invariant condition with respect to the reflection map $t \longrightarrow -t$).

DEFINITION 1.45. Let $a(t,\tau) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R})$, $\mu \in \mathbb{Z}$, and write the component $a_{(\mu-j)}(t,\tau)$ of $a(t,\tau)$ of homogeneity $\mu - j$ in $\tau \neq 0$ in the form

$$a_{(\mu-j)}(t,\tau) = \{\Theta^+(\tau)a^+(t) + \Theta^-(\tau)a^-_j(t)\}\tau^{\mu-j},$$

$j \in \mathbb{N}$; here $\Theta^\pm$ is the characteristic function of $\mathbb{R}_\pm \ni \tau$. The symbol $a(t,\tau)$ is said to have the transmission property at $t = 0$, written $a(t,\tau) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R})_{\mathrm{tr}}$, if

$$D^k_t a^+_j(t)|_{t=0} = D^k_t a^-_j(t)|_{t=0} \quad \text{for all} \quad j,k \in \mathbb{N}. \tag{1.50}$$

REMARK 1.46. Definition 1.45 can also be written as follows. A symbol $a(t,\tau) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R})$, $\mu \in \mathbb{R}$, belongs to $S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R})_{\mathrm{tr}}$ if and only if the coefficients $a^\pm_j(t) \in C^\infty(\overline{\mathbb{R}}_+)$ in the expansion

$$a(t,\tau) \sim \sum_{j=0}^\infty a^\pm_j(t)\tau^{\mu-j} \quad \text{for} \quad \tau \longrightarrow \pm\infty$$

satisfy the condition (1.50).

PROPOSITION 1.47. *Let $a(y,t,\eta,\tau) \in S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{q+1})$, and set*

$$a_{y,\eta}(t,\tau) := a(y,t,\eta,\tau)$$

for every fixed $(y,\eta) \in \Omega \times \mathbb{R}^q$; then we have $a_{y,\eta}(t,\tau) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R})$. The following conditions are equivalent:

(i) $a(y,t,\eta,\tau) \in S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{q+1})_{\mathrm{tr}}$,

(ii) $a_{y,\eta}(t,\tau) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R})_{\mathrm{tr}}$ *for every* $(y,\eta) \in \Omega \times \mathbb{R}^q$.

PROOF. Let us first assume the condition (i). Then the symbol $a_{y,\eta}(t,\tau)$ belongs to $S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R})_{\mathrm{tr}}$ in the sense of Definition 1.45 if and only if $(\chi(\eta,\tau)a_{(\mu-j)}(y,t,\eta,\tau))_{y,\eta}$ has this property for any excision function χ, for all $j \in \mathbb{N}$. Since χ is equal to 1 for $|\eta,\tau| >$ const for some constant > 0, it suffices to show that the symbol $(a_{(\mu-j)}(y,t,\eta,\tau))_{y,\eta}$ for $\eta \neq 0$ has this property. Let $\nu := \mu - j$ for any $j \in \mathbb{N}$. Let us consider, for simplicity, the case of symbols which are independent of (t,y); the general case is easy as well and left to the reader.

By assumption, we have

$$D^\alpha_\eta a_{(\nu)}(\eta,\tau)|_{\eta=0,\tau\neq 0} = D^\alpha_\eta((-1)^\nu a_{(\nu)}(-\eta,-\tau))|_{\eta=0,\tau\neq 0} \tag{1.51}$$

or, equivalently,

$$(D^\alpha_\eta a_{(\nu)})(0,1) = (-1)^{\nu+|\alpha|}(D^\alpha_\eta a_{(\nu)})(0,-1) \tag{1.52}$$

for all $\alpha \in \mathbb{N}^q$. Set $\rho := |\tau|^{-1}$, and consider the expansion of $a_{(\nu)}(\eta,\tau)$ for $\tau \longrightarrow \pm\infty$. We have

$$\begin{aligned} a_{(\nu)}(\eta,\tau) &= a_{(\nu)}(\rho^{-1}\rho\eta,\rho^{-1}) = \rho^{-\nu}a_{(\nu)}(\rho\eta,1) \\ &\sim \rho^{-\nu}\sum_\alpha \frac{\rho^{|\alpha|}\eta^\alpha}{\alpha!}(D^\alpha_\eta a_{(\nu)})(0,1) \\ &= \sum_{k=0}^\infty \Big\{ \sum_{|\alpha|=k} \frac{\eta^\alpha}{\alpha!}(D^\alpha_\eta a_{(\nu)})(0,1)\Big\}\tau^{\nu-k} \qquad \text{for} \qquad \tau > 0, \end{aligned}$$

and

$$\begin{aligned} a_{(\nu)}(\eta,\tau) &= a_{(\nu)}(\rho^{-1}\rho\eta,-\rho^{-1}) = \rho^{-\nu}a_{(\nu)}(\rho\eta,-1) \\ &\sim \rho^{-\nu}\sum_\alpha \frac{\rho^{|\alpha|}\eta^\alpha}{\alpha!}(D^\alpha_\eta a_{(\nu)})(0,-1) \\ &= \sum_{k=0}^\infty \Big\{ \sum_{|\alpha|=k} (-1)^{\nu+|\alpha|}\frac{\eta^\alpha}{\alpha!}(D^\alpha_\eta a_{(\nu)})(0,-1)\Big\}\tau^{\nu-k} \qquad \text{for} \qquad \tau < 0. \end{aligned}$$

By virtue of (1.52) we then obtain (i) $\Longrightarrow$ (ii). On the other hand, (ii) gives us immediately the relation (1.51), i.e., we have also proved (ii) $\Longrightarrow$ (i). □

Let $\mathrm{e}^\pm$ be the extension operators of functions on $\mathbb{R}_\pm$ by zero to the opposite sides (applied to elements in $H^s_{\mathrm{loc}(y)}(\Omega\times\mathbb{R}_\pm)$ for $s > -\frac{1}{2}$). Moreover, let $\mathrm{r}^\pm$ denote the operator of restriction to $\Omega\times\mathbb{R}_\pm$ (applied to distributions in $\Omega\times\mathbb{R}$).

Set

$$\mathrm{Op}^+(a)u(x) := \mathrm{r}^+\,\mathrm{Op}(\widetilde{a})\mathrm{e}^+u(x) \tag{1.53}$$

for $a \in S^\mu_{\mathrm{cl}}(\Omega\times\overline{\mathbb{R}}_+\times\mathbb{R}^{q+1})_{\mathrm{tr}}$, $\widetilde{a} \in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}\times\mathbb{R}^{q+1})_{\mathrm{tr}}$, such that $\widetilde{a}|_{\Omega\times\overline{\mathbb{R}}_+\times\mathbb{R}^{q+1}} = a$. Clearly, (1.53) is then independent of the specific choice of $\widetilde{a}$. We form (1.53) first for $u \in C^\infty_0(\Omega\times\mathbb{R}_+)$ and then for $u \in H^s_{\mathrm{comp}(y)}(\Omega\times\mathbb{R}_+)$, $s > -\frac{1}{2}$, cf. Theorem 1.55 below. Similarly, we consider operator families on the half-axis

$$\mathrm{op}^+(a)(y,\eta) := \mathrm{r}^+\,\mathrm{op}(\widetilde{a})(y,\eta)\mathrm{e}^+, \tag{1.54}$$

where $\mathrm{op}(\widetilde{a})u(t) := \iint e^{i(t-t')\tau}\widetilde{a}(y,t,\eta,\tau)u(t')dt'đ\tau$, $\widetilde{a} \in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}\times\mathbb{R}^{q+1})_{\mathrm{tr}}$.

$\mathrm{op}^+(a)(y,\eta)$ will be regarded as an operator-valued symbol.

Let H be a Hilbert space, and let $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ be a strongly continuous group of isomorphisms $\kappa_\lambda : H \to H$, i.e., $\kappa_\lambda h \in C(\mathbb{R}_+, H)$ for every $h \in H$, and $\kappa_\lambda\kappa_\rho = \kappa_{\lambda\rho}$ for all $\lambda,\rho \in \mathbb{R}_+$. To have a simple notation, we say that H is endowed with a group action. As is known, there are constants $c > 0$ and M such that

$$\|\kappa_\lambda\|_{\mathcal{L}(H)} \le c(\max(\lambda,\lambda^{-1}))^M \qquad \text{for all} \quad \lambda \in \mathbb{R}_+. \tag{1.55}$$

More generally, if F is a Fréchet space, written as a projective limit of Hilbert spaces $F = \varprojlim_{k\in\mathbb{N}} H^k$ with continuous embeddings $H^{k+1} \hookrightarrow H^k$ for all k, and if H^0 is endowed with a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, such that $\{\kappa_\lambda|_{H^k}\}_{\lambda\in\mathbb{R}_+}$ is a group action on H^k for every k, we say that F is endowed with a group action.

Example 1.48. (i) The space $H^s(\mathbb{R}_+)$ $(:= H^s(\mathbb{R})|_{\mathbb{R}_+})$ is endowed with the group action $(\kappa_\lambda u)(t) = \lambda^{\frac{1}{2}}u(\lambda t), \lambda \in \mathbb{R}_+$. The same is true of the spaces $\langle t\rangle^{-k}H^s(\mathbb{R}_+) := \{\langle t\rangle^{-k}u : u \in H^s(\mathbb{R}_+)\}$.

(ii) The Fréchet space $\mathcal{S}(\overline{\mathbb{R}}_+)$ $(:= \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+})$, written as

$$\mathcal{S}(\overline{\mathbb{R}}_+) = \varprojlim_{k\in\mathbb{N}} \langle t\rangle^{-k} H^k(\mathbb{R}_+),$$

is endowed with the group action defined as in (i).

DEFINITION 1.49. Let $\{H, \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}\}$ and $\{\widetilde{H}, \{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}\}$ be Hilbert spaces with group actions, moreover, let $\Omega \subseteq \mathbb{R}^p$ be open, and $\mu \in \mathbb{R}$.

(i) The space $S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ is defined to be the set of all $a(y,\eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ such that

$$\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \le c\langle\eta\rangle^{\mu-|\beta|}$$

for all $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$ and $(y,\eta) \in K \times \mathbb{R}^q$ for all $K \subset\subset \Omega$, with constants $c = c(\alpha,\beta,K) > 0$. The elements of $S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ are called operator-valued symbols of order μ (associated with the given group actions).

(ii) $S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H})$ denotes the space of all $a_{(\mu)}(y,\eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \widetilde{H}))$ such that $a_{(\mu)}(y,\lambda\eta) = \lambda^\mu \widetilde{\kappa}_\lambda a_{(\mu)}(y,\eta)\kappa^{-1}_\lambda$ for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$.

(iii) A symbol $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ is said to be classical, if there are elements $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H})$, $j \in \mathbb{N}$, such that

$$a(y,\eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$$

for every $N \in \mathbb{N}$, where $\chi(\eta)$ is any excision function. Let $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ denote the space of all classical symbols of order μ.

Let us extend this notation to the case of Fréchet spaces with group actions as follows: If $\widetilde{F} := \varprojlim_{j\in\mathbb{N}} \widetilde{H}^j$ is a Fréchet space with group action $\{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, we set

$$S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{F}) := \bigcap_{j\in\mathbb{N}} S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H}^j). \tag{1.56}$$

Moreover, if also $F = \varprojlim_{k\in\mathbb{N}} H^k$ is a Fréchet space with group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, we first fix a function $r : \mathbb{N} \to \mathbb{N}$ and set

$$S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; F, \widetilde{F})_r := \bigcap_{j\in\mathbb{N}} S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H^{r(j)}, \widetilde{H}^j). \tag{1.57}$$

Then we define $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; F, \widetilde{F})$ to be the union of (1.57) over all r. Similarly, we have the spaces $S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); F, \widetilde{F})$.

THEOREM 1.50. *Let $\Omega \subseteq \mathbb{R}^q$ be an open set, and fix a symbol $a(y,t,\eta,\tau) \in S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+)_{\mathrm{tr}}$, $\mu \in \mathbb{Z}$, that is independent of t for $t > c$ for a constant $c > 0$.*

(i) We have

$$\mathrm{op}^+(a)(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+)) \tag{1.58}$$

for every real $s > -\frac{1}{2}$, and

$$\mathrm{op}^+(a)(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)). \tag{1.59}$$

The operator-valued symbol $\mathrm{op}^+(a)(y,\eta)$ is classical, when the symbol a is independent of t.

(ii) Let $a_{(\mu-j)}(y,t,\eta,\tau) \in S^{(\mu-j)}(\Omega \times \overline{\mathbb{R}}_+ \times (\mathbb{R}^{q+1} \setminus \{0\}))$ be the homogeneous component of $a(y,t,\eta,\tau)$ of order $\mu - j$. Then we have

$$\mathrm{op}^+(a_{(\mu-j)}|_{t=0})(y,\eta) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H^s(\mathbb{R}_+), H^{s-\mu+j}(\mathbb{R}_+))$$

for every $s > -\frac{1}{2}$, and

$$\mathrm{op}^+(a_{(\mu-j)}|_{t=0})(y,\eta) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)).$$

The technicalities to prove Theorem 1.50 can be found in [**38**].

We now turn to pseudo-differential operators associated with symbols $a(y,y',\eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; H, \widetilde{H})$ (for simplicity, generalities will be formulated for the case of Hilbert spaces $H, \widetilde{H}$ with group actions; the case with Fréchet spaces is analogous and will tacitly be used below).

Let

$$L^\mu_{(\mathrm{cl})}(\Omega; H, \widetilde{H}) := \{\mathrm{Op}(a) : a(y,y',\eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; H, \widetilde{H})\}, \tag{1.60}$$

where $\mathrm{Op}(a)u(y) := \iint e^{i(y,-y')\eta} a(y,y',\eta)u(y')dy'đ\eta$, $đ\eta = (2\pi)^{-q}d\eta$.

Definition 1.51. Let $\{H, \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}\}$ be a Hilbert space with group action. Then $\mathcal{W}^s(\mathbb{R}^q, H)$ for $s \in \mathbb{R}$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, H)$ (or, equivalently, of $C_0^\infty(\mathbb{R}^q, H)$) with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} = \left\{\int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H d\eta\right\}^{\frac{1}{2}},$$

($\hat{u}(\eta)$ denotes the Fourier transform of u with respect to $y \in \mathbb{R}^q$).

The spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ have been introduced in [**33**] in connection with operators on manifolds with edges, see also [**32**], and their properties are studied, for instance, in [**34**], [**38**], [**20**], [**49**]. Let us summarise some results in the following theorems.

Theorem 1.52. *Let H be a Hilbert space with group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, and let $s \in \mathbb{R}$.*

(i) *We have*

$$\mathcal{W}^s(\mathbb{R}^q, H) = \left\{u \in \mathcal{S}'(\mathbb{R}^q, H) : \langle\eta\rangle^s \kappa^{-1}_{\langle\eta\rangle}\widehat{u}(\eta) \in L^2(\mathbb{R}^q, H)\right\}$$

where $\mathcal{S}'(\mathbb{R}^q, H) := \mathcal{L}(\mathcal{S}(\mathbb{R}^q), H)$. The space $\mathcal{W}^s(\mathbb{R}^q, H)$ is a Hilbert space with the scalar product

$$(u,v)_{\mathcal{W}^s(\mathbb{R}^q,E)} = \left(\langle\eta\rangle^s \kappa^{-1}_{\langle\eta\rangle}\widehat{u}(\eta), \langle\eta\rangle^s \kappa^{-1}_{\langle\eta\rangle}\widehat{u}(\eta)\right)_{L^2(\mathbb{R}^q,H)}.$$

(ii) *The operator of multiplication $\mathcal{M}_\varphi$ by a function $\varphi \in \mathcal{S}(\mathbb{R}^q)$ induces a continuous operator*

$$\mathcal{M}_\varphi : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^s(\mathbb{R}^q, H),$$

and $\varphi \to \mathcal{M}_\varphi$ represents a continuous operator $\mathcal{S}(\mathbb{R}^q) \to \mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H))$.

(iii) *$(\chi_\lambda u)(y) := \kappa_\lambda \lambda^{q/2} u(\lambda y)$ for $\lambda \in \mathbb{R}_+$, $u \in \mathcal{S}(\mathbb{R}^q, H)$, extends to a group action on $\mathcal{W}^s(\mathbb{R}^q, H)$, and we have*

$$\mathcal{W}^s(\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)) = \mathcal{W}^s(\mathbb{R}^{p+q}, H), \tag{1.61}$$

where the space on the left of relation (1.61) *refers to $\{\chi_\lambda\}_{\lambda\in\mathbb{R}_+}$ on $\mathcal{W}^s(\mathbb{R}^q, H)$ and that on the right of* (1.61) *to $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ on H.*

(iv) *For $H := H^s(\mathbb{R}^q)$ with the group action $(\kappa_\lambda u)(y) := \lambda^{q/2}u(\lambda y)$ we have*

$$\mathcal{W}^s(\mathbb{R}^p, H^s(\mathbb{R}^q)) = H^s(\mathbb{R}^{p+q}).$$

Let us set for any open $\Omega \subseteq \mathbb{R}^q$, $s \in \mathbb{R}$,

$$\mathcal{W}^s_{\mathrm{loc}}(\Omega, H) := \big\{u \in \mathcal{D}'(\Omega, H) \ : \ \varphi u \in \mathcal{W}^s(\mathbb{R}^q, H) \text{ for every } \varphi \in C_0^\infty(\Omega)\big\} \tag{1.62}$$

and

(1.63) $$\mathcal{W}^s_{\mathrm{comp}}(\Omega, H) := \{u \in \mathcal{W}^s_{\mathrm{loc}}(\Omega, H) : \operatorname{supp} u \quad \text{compact}\}.$$

The space (1.62) is Fréchet, and (1.63) is an inductive limit of Fréchet spaces.

In particular, we have

(1.64) $$H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}) = \mathcal{W}^s_{\mathrm{loc}}(\Omega, H^s(\mathbb{R})), H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}) = \mathcal{W}^s_{\mathrm{comp}}(\Omega, H^s(\mathbb{R}))$$

for any open set $\Omega \subseteq \mathbb{R}^q$, as well as

(1.65) $$H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}) = \mathcal{W}^s_{\mathrm{loc}}(\Omega, H^s(\mathbb{R}_\pm)), H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_\pm) = \mathcal{W}^s_{\mathrm{comp}}(\Omega, H^s(\mathbb{R}_\pm)).$$

THEOREM 1.53. *Let $a(y, y', \eta) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^q; H, \widetilde{H})$, $\Omega \subseteq \mathbb{R}^q$ open, $\mu \in \mathbb{R}$, with H and $\widetilde{H}$ being Hilbert spaces with group actions. Then*

$$\mathrm{Op}(a)u(y) := \iint e^{i(y-y')\eta} a(y, y', \eta) u(y') dy' đ\eta$$

induces a continuous operator

$$\mathrm{Op}(a) : C_0^\infty(\Omega, H) \to C^\infty(\Omega, \widetilde{H})$$

which extends to continuous operators

$$\mathrm{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, H) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{H})$$

for all $s \in \mathbb{R}$. In particular, for $a(\eta) \in S^\mu(\mathbb{R}^q; H, \widetilde{H})$ we obtain continuous operators

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H})$$

for all $s \in \mathbb{R}$.

REMARK 1.54. The notation (1.62) and (1.63) as well as the results of Theorems 1.52, 1.53 extend in a natural way to the case of Fréchet spaces H or $\widetilde{H}$ with group actions.

THEOREM 1.55. *Let $a(y, t, \eta, \tau) \in S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+)_{\mathrm{tr}}$ be a symbol which is independent of t for $t > c$ for some $c > 0$. Then $\mathrm{Op}^+(a) := \mathrm{r}^+ \mathrm{Op}(\tilde{a})\mathrm{e}^+$ (for any $\tilde{a} \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R})_{\mathrm{tr}}$ with $a = \tilde{a}$ for $t \geq 0$) induces continuous operators*

$$\mathrm{Op}^+(a) : H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+) \to H^{s-\mu}_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}_+)$$

for all $s \in \mathbb{R}$, $s > -\frac{1}{2}$.

This result is a consequence of Theorem 1.50 (i), the relations (1.65), and Theorem 1.53.

Let X be a compact C^∞ manifold with boundary Y, and let $2X$ denote the double of X, obtained by gluing together two copies X_+ and X_- of X along their common boundary Y by the identity map; we then identify X with X_+. Moreover, let e^+ denote the operator of extension of functions on $\operatorname{int} X_+$ by zero to the opposite side X_-, and let r^+ denote the operator of restriction of distributions on $2X$ to $\operatorname{int} X_+$; analogously, we have operators e^- and r^- with respect to the minus-side of $2X$. Let $2X$ be equipped with a Riemannian metric that equals the product metric of $Y \times (-1, +1)$ in a neighbourhood of Y for some Riemannian metric on Y.

Given an $E \in \mathrm{Vect}(X)$ we fix any $\widetilde{E} \in \mathrm{Vect}(2X)$ such that $E = \widetilde{E}|_X$. For E and F in $\mathrm{Vect}(X)$ with fibre dimensions l and k, respectively, we now consider the space $L^\mu_{\mathrm{cl}}(2X; \widetilde{E}, \widetilde{F})$. For every chart $\chi : V \to U$ on $2X$, $U \subseteq \mathbb{R}^n$ open, and trivialisations $\widetilde{E}|_V \cong U \times \mathbb{C}^l$, $\widetilde{F}|_V \cong U \times \mathbb{C}^k$, the push-forward $\chi_* A$ of an operator $A \in L^\mu_{\mathrm{cl}}(2X; \widetilde{E}, \widetilde{F})$ belongs to $L^\mu_{\mathrm{cl}}(U; \mathbb{C}^l, \mathbb{C}^k)$. By notation, the push-forward χ_* also takes into account the chosen trivialisations of $\widetilde{E}|_V$ and $\widetilde{F}|_V$; for simplicity they are not explicity indicated (this should not cause confusion).

Let $V \cap Y \neq \emptyset$, $V := V' \times (-1, +1)$, where V' is a coordinate neighbourhood on the boundary Y, and assume that χ restricts to charts $\chi_\pm : V_\pm \to \Omega \times \overline{\mathbb{R}}_\pm$ on $X_\pm$, $V_\pm := X_\pm \cap V$, and to a chart $\chi' := \chi|_{V'} : V' \to \Omega$ on Y, $\Omega \subseteq \mathbb{R}^{n-1}$.

Definition 1.56. The space $L^\mu_{\mathrm{cl}}(2X;\widetilde{E};\widetilde{F})_{\mathrm{tr}}$ for $\mu\in\mathbb{Z}$ is defined to be the set of all elements $\widetilde{A}\in L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})$ such that for every $\varphi,\psi\in C_0^\infty(\Omega\times\mathbb{R})$ and $e_\xi:=e^{ix\xi}$

$$\widetilde{a}(x,\xi):=e_{-\xi}\{\varphi(\chi_*A)\psi\}e_\xi$$

is an $k\times l$ matrix of elements in $S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R})_{\mathrm{tr}}$; here, χ is an arbitrary chart of the described kind.

Moreover, set

$$L^\mu_{\mathrm{cl}}(X;E,F)_{\mathrm{tr}}:=\{\mathrm{r}^+\widetilde{A}\mathrm{e}^+ : \widetilde{A}\in L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})_{\mathrm{tr}}\}. \tag{1.66}$$

Remark 1.57. The space $L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})_{\mathrm{tr}}$ is a closed subspace of $L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})$, cf. Remark 1.2. Moreover, the space

$$\{\widetilde{A}\in L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})_{\mathrm{tr}} : \mathrm{r}^+\widetilde{A}\mathrm{e}^+=0\} \tag{1.67}$$

is closed in $L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})_{\mathrm{tr}}$, and we have

$$L^\mu_{\mathrm{cl}}(X;E,F)_{\mathrm{tr}}=L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})/\sim, \tag{1.68}$$

where $/\sim$ indicates the quotient space with respect to (1.67). In this way, in the space $L^\mu_{\mathrm{cl}}(X;E,F)_{\mathrm{tr}}$ we obtain a natural Fréchet topology.

Theorem 1.58. *Every $A\in L^\mu_{\mathrm{cl}}(X;E,F)_{\mathrm{tr}}$ induces continuous operators*

$$A:H^s(X,E)\to H^{s-\mu}(X,F)$$

for all $s\in\mathbb{R}$, $s>-\frac{1}{2}$.

Proof. Let $\omega,\widetilde{\omega}$ be a functions supported in a collar neighbourhood of Y such that $\widetilde{\omega}\equiv 1$ on $\operatorname{supp}\omega$. Then A can be written as $A=\omega A\widetilde{\omega}+(1-\omega)A(1-\widetilde{\omega})+C$, where C has a smooth kernel up to the boundary. The continuity $C:H^s(X,E)\to C^\infty(X,F)$ for $s>-\frac{1}{2}$ is then evident, while the continuity of $(1-\omega)A(1-\widetilde{\omega}):H^s(X,E)\to H^{s-\mu}(X,F)$ for all s is a consequence of Theorem 1.4 (i). The continuity of $\omega A\widetilde{\omega}:H^s(X,E)\to H^{s-\mu}(X,F)$ for $s>-\frac{1}{2}$ follows from Theorem 1.55 by a simple partition of unity argument and from Theorems 1.52, 1.53. $\square$

Let us look at a special kind of order reducing symbols. Choose a function $\varphi\in\mathcal{S}(\mathbb{R})$ such that $\varphi(0)=1$ and $\operatorname{supp}F^{-1}\varphi\subset\mathbb{R}_-$ (here F is the Fouier transform on the real line). Set

$$r^\mu_-(\eta,\tau):=\Big(\varphi\Big(\frac{\tau}{C\langle\tau\rangle}\Big)-i\tau\Big)^\mu \tag{1.69}$$

for any $\mu\in\mathbb{R}$ and a constant $C>0$. We then have $r^\mu_-(\eta,\tau)\in S^\mu_{\mathrm{cl}}(\mathbb{R}^n)$, and the symbol $r^\mu_-(\eta,\tau)$ is elliptic for a sufficiently large $C>0$, cf. Grubb [**15**]. It is well known that

$$\mathrm{r}^+\operatorname{Op}(\mathrm{r}^\mu_-)\mathrm{e}^+_s : H^s(\mathbb{R}^n_+)\to H^{s-\mu}(\mathbb{R}^n_+) \tag{1.70}$$

is continuous for every $s,\mu\in\mathbb{R}$, with $\mathrm{e}^+_s:H^s(\mathbb{R}^n_+)\to H^s(\mathbb{R}^n)$ being a continuous extension operator, i.e., $\mathrm{r}^+\mathrm{e}^+_s=\mathrm{id}_{H^s(\mathbb{R}^n_+)}$. Moreover, for $s>-\frac{1}{2}$ we may replace e^+_s by e^+, and (1.70) is independent of the choice of the specific extension operator. For $\mu\in\mathbb{Z}$ we have $r^\mu_-(\eta,\tau)\in S^\mu_{\mathrm{cl}}(\mathbb{R}^n)_{\mathrm{tr}}$. Order reducing operators also may be formulated globally on a compact C^∞ manifold X with boundary.

Theorem 1.59. *The space $L^\mu_{\mathrm{cl}}(X;E,E)_{\mathrm{tr}}$ for $E\in\operatorname{Vect}(X)$ and arbitrary $\mu\in\mathbb{Z}$ contains an element $R^\mu_E:=\mathrm{r}^+\widetilde{R}^\mu_{\widetilde{E}}\mathrm{e}^+$ for some $\widetilde{R}^\mu_{\widetilde{E}}\in L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{E})_{\mathrm{tr}}$, $E=\widetilde{E}|_X$, such that*

$$R^\mu_E:H^s(X,E)\to H^{s-\mu}(X,F)$$

is an isomorphism for every $s>-\frac{1}{2}$.

This result can be proved in different ways, cf. Boutet de Monvel [**6**], or Grubb [**15**]; the latter paper employs order reducing symbols of the kind (1.69). Such symbols for arbitrary $\mu\in\mathbb{R}$ have been used for an analogous order reducing result for the case without the transmission property in the author's joint paper with Harutjunjan [**19**]. Other constructions in this direction (more general in different ways) may be found in Duduchava and Speck [**9**]. A completely different method to reduce orders in general boundary value problems is given in [**42**], based on ideas of the edge pseudo-differential calculus.

REMARK 1.60. The operators in [**19**] are obtained in a slightly more general form, namely as $\mathrm{r}^+\widetilde{R}^\mu_{\widetilde{E}}\mathrm{e}^+$ for operators $\widetilde{R}^\mu_{\widetilde{E}} \in L^\mu_{\mathrm{cl}}(M;\widetilde{E},\widetilde{E})$ on any closed compact C^∞ manifold M which contains $X =: X_+$ and $X_- := M \setminus \operatorname{int} X_+$ as submanifolds with common boundary Y, with an evident generalisation of the meaning of operators r^+, e^+ (and $\mathrm{r}^-, \mathrm{e}^-$ with respect to the minus side). We can also construct an operator $\widetilde{S}^\mu_{\widetilde{E}}$ of similar meaning for X_-, such that

$$\mathrm{r}^-\widetilde{S}^\mu_{\widetilde{E}}\mathrm{e}^- : H^s(X_-, E|_{X_-}) \to H^{s-\mu}(X_-, E|_{X_-})$$

are isomorphisms for $s > -\frac{1}{2}$. The local symbols of $\widetilde{S}^\mu_{\widetilde{E}}$ in a collar neighbourhood of Y are just $\mathrm{r}^\mu_+(y,\eta) := \overline{\mathrm{r}^\mu_+(y,\eta)}$, the complex conjugate of (1.69).

Given an operator $A \in L^\mu_{\mathrm{cl}}(X;E,F)_{\mathrm{tr}}$, $A := \mathrm{r}^+\widetilde{A}\mathrm{e}^+$ for an $\widetilde{A} \in L^\mu_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})_{\mathrm{tr}}$, cf. formula (1.66), we have the homogeneous principal symbol $\sigma_\psi(\widetilde{A}) : \pi^*_{2X}\widetilde{E} \to \pi^*_{2X}\widetilde{F}$, cf. (1.5), and we set $\sigma_\psi(A) := \sigma_\psi(\widetilde{A})|_{T^*X\setminus 0}$,

$$\sigma_\psi(A) : \pi^*_X E \to \pi^*_X F, \tag{1.71}$$

$\pi_X : T^*X \setminus 0 \to X$. With (1.71) we associate a family of operators

$$\begin{aligned} \sigma_\partial(A)(y,\eta) \quad &:= \quad \mathrm{r}^+\sigma_\psi(A)(y,0,\eta,D_t)\mathrm{e}^+ \\ &:= \quad \mathrm{r}^+ \operatorname{op}(\sigma_\psi(A)|_{t=0})(y,\eta)\mathrm{e}^+ \end{aligned} \tag{1.72}$$

for $(y,\eta) \in T^*Y \setminus 0$. This gives us a bundle morphism

$$\sigma_\partial(A) : \pi^*_Y E' \otimes H^s(\mathbb{R}_+) \to \pi^*_Y F' \otimes H^{s-\mu}(\mathbb{R}_+)$$

for every fixed $s \in \mathbb{R}$, $s > -\frac{1}{2}$, $\pi_Y : T^*Y \setminus 0 \to Y$, where $E' := E|_Y$, $F' := F|_Y$. Alternatively, we consider $\sigma_\partial(A)$ as a family of maps

$$\sigma_\partial(A) : \pi^*_Y E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \to \pi^*_Y F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+), \tag{1.73}$$

$\mathcal{S}(\overline{\mathbb{R}}_+) := \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$. We call $\sigma_\psi(A)$ the principal interior symbol, $\sigma_\partial(A)$ the principal boundary symbol of the operator A.

REMARK 1.61. For $(\kappa_\lambda u)(t) := \lambda^{\frac{1}{2}}u(\lambda t)$, $\lambda \in \mathbb{R}_+$, we have

$$\sigma_\partial(A)(y,\lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\partial(A)(y,\eta)\kappa^{-1}_\lambda \tag{1.74}$$

for all $\lambda \in \mathbb{R}_+$.

We now pass to the algebra of boundary value problems on X with trace and potential conditions, cf. [**6**]. First, for $E, F \in \operatorname{Vect}(X)$, $J_-, J_+ \in \operatorname{Vect}(Y)$, $\boldsymbol{v} := (E,F;J_-,J_+)$, by $\mathcal{B}^{-\infty,0}(X;\boldsymbol{v})$ we denote the space of all

$$\mathcal{G} := \begin{pmatrix} G & K \\ T & Q \end{pmatrix} : \begin{matrix} C^\infty(X,E) \\ \oplus \\ C^\infty(Y,J_-) \end{matrix} \to \begin{matrix} C^\infty(X,F) \\ \oplus \\ C^\infty(Y,J_+) \end{matrix}$$

such that $\mathcal{G}$ and $\mathcal{G}^*$ extend to continuous operators

$$\mathcal{G} : \begin{matrix} H^s(X,E) \\ \oplus \\ H^s(Y,J_-) \end{matrix} \to \begin{matrix} C^\infty(X,F) \\ \oplus \\ C^\infty(Y,J_+) \end{matrix}, \qquad \mathcal{G}^* : \begin{matrix} H^s(X,F) \\ \oplus \\ H^s(Y,J_+) \end{matrix} \to \begin{matrix} C^\infty(X,E) \\ \oplus \\ C^\infty(Y,J_-) \end{matrix}$$

for all $s \in \mathbb{R}$, $s > -\frac{1}{2}$. Here $\mathcal{G}^*$ is the formal adjoint of $\mathcal{G}$ in the sense

$$(u, \mathcal{G}^*v)_{L^2(X,E)\oplus L^2(Y,J_-)} = (\mathcal{G}u, v)_{L^2(X,F)\oplus L^2(Y,J_+)} \tag{1.75}$$

for all $u \in C^\infty(X,E) \oplus C^\infty(Y,J_-)$, $v \in C^\infty(X,F) \oplus C^\infty(Y,J_+)$; the L^2 - scalar products refer to the chosen Riemannian metrics on X and Y and to the Hermitian metrics in the respective vector bundles.

For every $E \in \mathrm{Vect}(X)$ we fix a first order differential operator $T : C^\infty(X,E) \to C^\infty(X,E)$ that is equal to $\partial_t \otimes \mathrm{id}_E$ in a collar neighbourhood of the boundary, with t being the normal variable. Then $\mathcal{B}^{-\infty,d}(X;\boldsymbol{v})$ for $d \in \mathbb{N}\setminus\{0\}$ is defined to be the space of all operators

$$\mathcal{G} = \mathcal{G}_0 + \sum_{j=1}^{d} \mathcal{G}_j \operatorname{diag}(T^j,0) \tag{1.76}$$

for arbitrary $\mathcal{G}_j \in \mathcal{B}^{-\infty,0}(X;\boldsymbol{v})$. The elements of $\mathcal{B}^{-\infty,d}(X;\boldsymbol{v})$ are called smoothing operators of type d.

We now turn to Green symbols that are operator-valued symbols in the sense of Definition 1.49 (iii) for Hilbert spaces

$$H = L^2(\mathbb{R}_+,\mathbb{C}^l) \oplus \mathbb{C}^{j_-}, \qquad \widetilde{H} = L^2(\mathbb{R}_+,\mathbb{C}^k) \oplus \mathbb{C}^{j_+}$$

or Schwartz subspaces, where κ_λ as well as $\widetilde{\kappa}_\lambda$ are defined by $u \oplus c \to \lambda^{\frac{1}{2}} u(\lambda t) \oplus c$, $\lambda \in \mathbb{R}_+$, for a vector-valued function $u(t)$ and a vector c of complex numbers.

Definition 1.62. Let $l,k,j_-,j_+ \in \mathbb{N}$, $\mu \in \mathbb{R}$, and $\Omega \subseteq \mathbb{R}^q$ open. An element

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; L^2(\mathbb{R}_+,\mathbb{C}^l) \oplus \mathbb{C}^{j_-}, L^2(\mathbb{R}_+,\mathbb{C}^k) \oplus \mathbb{C}^{j_+})$$

is called a Green symbol of order μ and type 0, if it has the properties

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; L^2(\mathbb{R}_+,\mathbb{C}^l) \oplus \mathbb{C}^{j_-}, \mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^k) \oplus \mathbb{C}^{j_+})$$

and

$$g^*(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; L^2(\mathbb{R}_+,\mathbb{C}^k) \oplus \mathbb{C}^{j_+}, \mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^l) \oplus \mathbb{C}^{j_-}),$$

where $g^*(y,\eta)$ denotes the pointwise adjoint in the sense

$$(u, g^*(y,\eta)v)_{L^2(\mathbb{R}_+,\mathbb{C}^l)\oplus\mathbb{C}^{j_-}} = (g(y,\eta)u, v)_{L^2(\mathbb{R}_+,\mathbb{C}^k)\oplus\mathbb{C}^{j_+}} \tag{1.77}$$

for all $u \in L^2(\mathbb{R}_+,\mathbb{C}^l) \oplus \mathbb{C}^{j_-}$, $v \in L^2(\mathbb{R}_+,\mathbb{C}^k) \oplus \mathbb{C}^{j_+}$. Let $\mathcal{R}^{\mu,0}_G(\Omega \times \mathbb{R}^q;\boldsymbol{w})$, $\boldsymbol{w} = (l,k;j_-,j_+)$, denote the space of all such symbols.

Remark 1.63. It can be proved that every $g(y,\eta) \in \mathcal{R}^{\mu,0}_G(\Omega \times \mathbb{R}^q;\boldsymbol{w})$ induces elements

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+,\mathbb{C}^l) \oplus \mathbb{C}^{j_-}, \mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^k) \oplus \mathbb{C}^{j_+})$$

for all $s \in \mathbb{R}$, $s > -\frac{1}{2}$.

This is a consequence of a kernel characterisation of [**36**, Theorem 3.1], cf. also [**38**, Section 4.2.3].

Note that the differential operator ∂_t^j on $\mathbb{R}_+$ represents an operator-valued symbol

$$\partial_t^j \in S^j_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+,\mathbb{C}^l), H^{s-j}(\mathbb{R}_+,\mathbb{C}^l))$$

for every $s \in \mathbb{R}$; there is in this case no dependence on $(y,\eta) \in \Omega \times \mathbb{R}^q$.

Definition 1.64. By $\mathcal{R}^{\mu,d}_G(\Omega \times \mathbb{R}^q;\boldsymbol{w})$ for $\mu \in \mathbb{R}$, $d \in \mathbb{N}$, we denote the space of all operator functions

$$g(y,\eta) := g_0(y,\eta) + \sum_{j=1}^{d} g_j(y,\eta) \operatorname{diag}(\partial_t^j,0),$$

for arbitrary $g_j(y,\eta) \in \mathcal{R}^{\mu-j,d}(\Omega \times \mathbb{R}^q,\boldsymbol{w})$. The elements of $\mathcal{R}^{\mu,d}_G(\Omega \times \mathbb{R}^q;\boldsymbol{w})$ are called Green symbols of order μ and type d.

Notice that $g(y,\eta) \in \mathcal{R}^{\mu,d}_G(\Omega \times \mathbb{R}^q;\boldsymbol{w})$ implies

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+,\mathbb{C}^l) \oplus \mathbb{C}^{j_-}, \mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^k) \oplus \mathbb{C}^{j_+}) \tag{1.78}$$

for every $s \in \mathbb{R}$, $s > d - \frac{1}{2}$.

The nature of Green symbols as operator-valued symbols in the sense of Definition 1.62 has been first observed in [**36**, Theorem 3.1]. A similar characterisation holds for Green operators belonging to boundary value problems without the transmission property, see [**37**]. In this case, the space $\mathcal{S}(\overline{\mathbb{R}}_+)$

is to be replaced by a space of functions with more general conormal asymptotics rather than Taylor asymptotics at $t = 0$ (and Schwartz function behaviour for $t \to \infty$).

THEOREM 1.65. *For every $g(y, \eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$, $\boldsymbol{w} = (l, k; j_-, j_+)$, the associated (so called Green operator) $\mathcal{G} := \mathrm{Op}(g)$ induces continuous operators*

$$\begin{aligned} \mathcal{G} : H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+, \mathbb{C}^l) \oplus H^s_{\mathrm{comp}}(\Omega, \mathbb{C}^{j_-}) \\ \to H^{s-\mu}_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}_+, \mathbb{C}^k) \oplus H^{s-\mu}_{\mathrm{loc}}(\Omega, \mathbb{C}^{j_+}) \end{aligned} \tag{1.79}$$

for all $s \in \mathbb{R}$, $s > d - \frac{1}{2}$.

PROOF. We have

$$g(y, \eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_-}, H^{s-\mu}(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_+})$$

(which is more crude than (1.78) but sufficient for the moment), and then

$$\mathcal{G} : \mathcal{W}^s_{\mathrm{comp}}(\Omega, H^s(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_-}) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, H^{s-\mu}(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_+})$$

is continuous for $s > d - \frac{1}{2}$, cf. Theorem 1.53 (the $\mathcal{W}^s$-spaces refer to the same group actions as in Definition 1.62.) Now we have

$$\mathcal{W}^s_{\mathrm{comp}}(\Omega, H^s(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_-}) = H^s_{\mathrm{comp}(y)}(\Omega, \times \mathbb{R}_+, \mathbb{C}^l) \oplus H^s_{\mathrm{comp}}(\Omega, \mathbb{C}^{j_-})$$

and, similarly, which subscripts 'loc', cf. Theorem 1.52 (iv). ☐

By definition, the symbol $g(y, \eta)$ has a homogeneous principal component $g_{(\mu)}(y, \eta)$ of order μ, cf. Definition 1.49 (iii). It will be interpreted as the boundary symbol

$$\sigma_\partial(\mathcal{G})(y, \eta) : H^s(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_-} \to \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_+} \tag{1.80}$$

of the associated pseudo-differential operator $\mathcal{G} = \mathrm{Op}(g)$. Alternatively, we also write

$$\sigma_\partial(\mathcal{G})(y, \eta) : \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_-} \to \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_+}. \tag{1.81}$$

Now let $E, F \in \mathrm{Vect}(X)$ and $J_-, J_+ \in \mathrm{Vect}(Y)$, and consider trivialisations

$$E|_V \cong \Omega \times \overline{\mathbb{R}}_+ \times \mathbb{C}^l, \qquad F|_V \cong \Omega \times \overline{\mathbb{R}}_+ \times \mathbb{C}^k, \tag{1.82}$$

$$J_\pm|_V \cong \Omega \times \mathbb{C}^{j_\pm} \tag{1.83}$$

on a coordinate neighbourhood V on X near the boundary Y, such that $V' := V \cap Y \neq \emptyset$, and let $\chi : V \to \Omega \times \overline{\mathbb{R}}_+$ and $\chi' : V' \to \Omega$ be corresponding charts on X and Y, respectively.

Green operators (1.79), interpreted for a moment as operators

$$\mathcal{G} : C_0^\infty(\Omega \times \overline{\mathbb{R}}_+, \mathbb{C}^l) \oplus C_0^\infty(\Omega, \mathbb{C}^{j_-}) \to C^\infty(\Omega \times \overline{\mathbb{R}}_+, \mathbb{C}^k) \oplus C^\infty(\Omega, \mathbb{C}^{j_+})$$

can be pulled back to X with respect to the mapping (1.82), (1.83) as operators

$$\mathcal{G}_V : C_0^\infty(V, E|_V) \oplus C_0^\infty(V', J_-|_{V'}) \to C^\infty(V, F|_V) \oplus C^\infty(V', J_+|_{V'}).$$

Let us write, for simplicity, $\mathcal{G}_V = (\chi^{-1})_* \mathrm{Op}(g)$; the pull back also refers to the cocycles of transition maps of the involved bundles. Let us now fix a finite system $\{V_j\}_{j=1,\dots,L}$ of such coordinate neighbourhoods on X near Y such that $\{V_1'\}_{j=1,\dots,L}$ form an open covering of Y, and choose functions $\varphi_j, \psi_j \in C_0^\infty(V_j)$, $j = 1, \dots, L$ such that $\sum_{j=1}^L \varphi_j \equiv 1$ in a collar neighbourhood of Y and $\psi_j \equiv 1$ on $\mathrm{supp}\, \varphi_j$ for all j, and set $\varphi_j' := \varphi_j|_{V'}$, $\psi_j' := \psi_j|_{V'}$.

DEFINITION 1.66. (i) The space $\mathcal{B}_G^{\mu,d}(X; \boldsymbol{v})$ for $\boldsymbol{v} := (E, F; J_-, J_+)$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, is defined to be the set of all operators

$$\mathcal{G} := \sum_{j=1}^L \mathrm{diag}(\varphi_j, \varphi_j^{'})(\chi_j^{-1})_* \mathrm{Op}(g_j) \,\mathrm{diag}(\psi_j, \psi_j^{'}) + \mathcal{C}$$

for arbitrary $g_j(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \overline{\mathbb{R}}_+;\boldsymbol{w})$, $1 \le j \le L$, and $\mathcal{C} \in \mathcal{B}^{-\infty,d}(X;\boldsymbol{v})$. The elements of $\mathcal{B}_G^{\mu,d}(X;\boldsymbol{v})$ are called Green operators on X of order μ and type d. The families of maps (1.80) or (1.81), applied to localisations of Green operators $\mathcal{G}$ in a coordinate neighbourhood V, have an invariant meaning as bundle morphisms

$$\sigma_\partial(\mathcal{G}) : \pi_Y^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix}, \tag{1.84}$$

$\pi_Y : T^*Y \setminus 0 \to Y$, $s > d - \frac{1}{2}$, (alternatively, we may write $\mathcal{S}(\overline{\mathbb{R}}_+)$ instead of $H^{s-\mu}(\mathbb{R}_+)$ on the right hand side, or $\mathcal{S}(\overline{\mathbb{R}}_+)$ on both sides).

(ii) The space $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, is defined as the set of all operators

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} \tag{1.85}$$

for arbitrary $A \in L_{\mathrm{cl}}^\mu(X;E,F)_{\mathrm{tr}}$ and $\mathcal{G} \in \mathcal{B}_G^{\mu,d}(X;\boldsymbol{v})$. The elements of $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ are called (pseudo-differential) boundary value problems for the operator A, of order μ and type d.

(iii) We set $\sigma_\psi(\mathcal{A}) := \sigma_\psi(A)$, called the (homogeneous principal) interior symbol of $\mathcal{A}$ of order μ, and

$$\sigma_\partial(\mathcal{A}) := \begin{pmatrix} \sigma_\partial(A) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\partial(\mathcal{G}),$$

called the (homogeneous principal) boundary symbol of $\mathcal{A}$. Set

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})), \tag{1.86}$$

called the principal symbol of the operator $\mathcal{A}$.

The homogeneity of $\sigma_\psi(\mathcal{A})$ is as usual, i.e., $\sigma_\psi(\mathcal{A})(x,\lambda\xi) = \lambda^\mu \sigma_\psi(\mathcal{A})(x,\xi)$ for all $\lambda \in \mathbb{R}_+$, $(x,\xi) \in T^*X \setminus 0$. For $\sigma_\partial(\mathcal{A})$ we have

$$\sigma_\partial(\mathcal{A})(y,\lambda\eta) = \lambda^\mu \operatorname{diag}(\kappa_\lambda, \mathrm{id}) \sigma_\partial(\mathcal{A})(y,\eta) \operatorname{diag}(\kappa_\lambda^{-1}, \mathrm{id}) \tag{1.87}$$

for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in T^*Y \setminus 0$.

By

$$B_G^{\mu,d}(X;E,F) \quad \text{and} \quad B^{\mu,d}(X;E,F) \tag{1.88}$$

we denote the space of upper left cornes of 2×2 block matrices in $B_G^{\mu,d}(X;\boldsymbol{v})$ and $B^{\mu,d}(X;\boldsymbol{v})$, respectively. Given an element $\mathcal{A} \in B^{\mu,d}(X;\boldsymbol{v})$ in the form $\mathcal{A} = (A_{ij})_{i,j=1,2}$, we also set $A_{11} := \mathrm{ulc}\ \mathcal{A}$. The operator A_{21} is often called a trace and A_{12} a potential operator in $B^{\mu,d}(X;\boldsymbol{v})$. Note that A_{22} belong to $L_{\mathrm{cl}}^\mu(Y;J_-,J_+)$.

Remark 1.67. The following conditions are equivalent:

(i) $G \in B^{\mu,d}(X;E,F)$ and $\varphi G \psi \in L^{-\infty}(\mathrm{int}\, X;E,F)$ for every $\varphi,\psi \in C_0^\infty(\mathrm{int}\, X)$,

(ii) $G \in B_G^{\mu,d}(X;E,F)$.

Remark 1.68. Every $G \in B_G^{\mu,d}(X;E,F)$ has a unique representation

$$G = G_0 + \sum_{j=0}^{d-1} K_j \circ T_j \tag{1.89}$$

for a $G_0 \in B_G^{\mu,0}(X;E,F)$ and potential operators $K_j \in \mathcal{B}^{\mu-j-\frac{1}{2},0}(X;0,F;E',0)$ and $T_j u = \mathrm{r}' T^j$, cf. the notation of (1.76).

Theorem 1.69. *Let X be a compact C^∞ manifold with boundary Y.*

(i) *Every $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ for $\boldsymbol{v} := (E,F;J_-,J_+)$ induces continuous operators*

$$\mathcal{A} : \begin{matrix} H^s(X,E) \\ \oplus \\ H^s(Y,J_-) \end{matrix} \to \begin{matrix} H^{s-\mu}(X,F) \\ \oplus \\ H^{s-\mu}(Y,J_+) \end{matrix} \tag{1.90}$$

for all $s \in \mathbb{R}$, $s > d - \frac{1}{2}$.

(ii) $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ *and* $\sigma(\mathcal{A}) = 0$ *imply* $\mathcal{A} \in \mathcal{B}^{\mu-1,d}(X;\boldsymbol{v})$, *and hence the operator* (1.90) *is compact.*

(iii) $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ *for* $\boldsymbol{v} := (E_0, F; J_0, J_+)$ *and* $\mathcal{B} \in \mathcal{B}^{\nu,e}(X;\boldsymbol{w})$ *for* $\boldsymbol{w} := (E, E_0; J_-, J_0)$ *implies* $\mathcal{AB} \in \mathcal{B}^{\mu+\nu,h}(X;\boldsymbol{v}\circ\boldsymbol{w})$ *for* $\boldsymbol{v}\circ\boldsymbol{w} := (E, F; J_-, J_+)$ *and* $h = \max(\nu + d, e)$, *and we have*

$$\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

(*with componentwise multiplication*).

(iv) *The formal adjoint* $\mathcal{A}^*$ (*cf. formula* (1.75)) *of an operator* $\mathcal{A} \in \mathcal{B}^{0,0}(X;\boldsymbol{v})$ *for* $\boldsymbol{v} := (E, F; J_-, J_+)$ *belongs to* $\mathcal{B}^{0,0}(X;\boldsymbol{v}^*)$ *for* $\boldsymbol{v}^* := (F, E; J_+, J_-)$, *and we have*

$$\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})^*$$

(*with componentwise adjoint, cf. formulas* (1.10) *and* (1.77)).

THEOREM 1.70. *Let* $\boldsymbol{v} := (E, F; J_-, J_+)$ *for* $E, F \in \mathrm{Vect}(X)$, $J_-, J_+ \in \mathrm{Vect}(Y)$, *and let* $\mathcal{A}_j \in \mathcal{B}^{\mu-j,d}(X;\boldsymbol{v})$, $j \in \mathbb{N}$, *be an arbitrary sequence. Then there exists an element* $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ *such that*

$$\mathcal{A} - \sum_{j=0}^{N} \mathcal{A}_j \in \mathcal{B}^{\mu-(N+1),d}(X;\boldsymbol{v})$$

for every $N \in \mathbb{N}$, *and* $\mathcal{A}$ *is unique* $\mod \mathcal{B}^{-\infty,d}(X;\boldsymbol{v})$.

1.3. SL-elliptic boundary value problems.

DEFINITION 1.71. Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, $\boldsymbol{v} := (E, F; J_-, J_+)$ for $E, F \in \mathrm{Vect}(X)$, $J_-, J_+ \in \mathrm{Vect}(Y)$.

(i) The operator $\mathcal{A}$ is said to be SL-elliptic (of order μ), if both

$$\sigma_\psi(\mathcal{A}) : \pi_X^* E \to \pi_X^* F, \tag{1.91}$$

$\pi_X : T^*X \setminus 0 \to X$, and

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J_+ \end{pmatrix} \tag{1.92}$$

$\pi_Y : T^*Y \setminus 0 \to Y$, are isomorphisms.

(ii) An operator $\mathcal{P} \in \mathcal{B}^{-\mu,e}(X;\boldsymbol{v}^{-1})$ for some $e \in \mathbb{N}$, $\boldsymbol{v}^{-1} := (F, E; J_+, J_-)$, is called a parametrix of $\mathcal{A}$, if $\mathcal{P}$ satisfies the following relations:

$$\mathcal{C}_l := \mathcal{I} - \mathcal{PA} \in \mathcal{B}^{-\infty,d_l}(X;\boldsymbol{v}_l), \quad \mathcal{C}_r := \mathcal{I} - \mathcal{AP} \in \mathcal{B}^{-\infty,d_r}(X;\boldsymbol{v}_r) \tag{1.93}$$

for certain $d_l, d_r \in \mathbb{N}$, $\boldsymbol{v}_l := (E, E; J_-, J_-)$, $\boldsymbol{v}_r := (F, F; J_+, J_+)$, where $\mathcal{I}$ denotes corresponding identity operators.

REMARK 1.72. If (1.91) is an isomorphism, a bundle morphism (1.92) for $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ is an isomorphism, if and only if

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix} \tag{1.94}$$

is an isomorphism for any fixed $s = s_0 \in \mathbb{R}$, $s_0 > \max(\mu, d) - \frac{1}{2}$ (or, equivalently, for all $s > \max(\mu, d) - \frac{1}{2}$).

REMARK 1.73. If $A \in L^\mu_{\mathrm{cl}}(X; E, F)_{\mathrm{tr}}$ is elliptic, i.e., (1.91) an isomorphism, we have

$$E' \cong F'. \tag{1.95}$$

In fact, let us consider the composition $A_0 := AR_E^{-\mu}$, cf. Theorem 1.59. Then we have $A_0 \in B^{\mu,0}(X; E, F)$, and $\sigma_\psi(A_0) : \pi_X^* E \to \pi_X^* F$ also is an isomorphism. Let $\Xi := S^*X|_Y \cup N$ with the conormal interval $\{(y, (\eta, \tau)) : \eta = 0, \quad -1 \leq \tau \leq +1\}$.

The transmission property of symbols of order zero has the consequence that

$$\sigma_\psi(A_0)|_{S^*X|_Y} : \pi_2^* E' \to \pi_2^* F'$$

for $\pi_2 : S^*X|_Y \to Y$ extends to an isomorphism $\pi_3^* E' \to \pi_3^* F$, where $\pi_3 : \Xi \to Y$ is the canonical projection of $\Xi := S^*X|_Y \cup N$ to Y.

Since $Y \subset \Xi$, the latter isomorphism then restricts to an isomorphism (1.95).

THEOREM 1.74. *Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, $\boldsymbol{v} := (E,F;J_-,J_+)$.*

(i) *The operator $\mathcal{A}$ is SL-elliptic (of order μ), if and only if (1.90) is a Fredholm operator for an $s = s_0 \in \mathbb{R}$, $s_0 > \max(\mu,d) - \frac{1}{2}$.*

(ii) *If $\mathcal{A}$ is elliptic, (1.90) is a Fredholm operator for all $s > \max(\mu,d) - \frac{1}{2}$, and $\dim\ker\mathcal{A}$ and $\dim\operatorname{coker}\mathcal{A}$ are independent of s.*

(iii) *An elliptic operator $\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,(d-\mu)^+}(X;\boldsymbol{v}^{-1})$ ($\rho^+ := \max(\rho,0)$ for any $\rho \in \mathbb{R}$) which can be chosen in such a way that the remainders in the relation (1.93) are projections*

$$\begin{aligned} \mathcal{C}_l &: H^s(X,E) \oplus H^s(Y,J_-) \to V, \\ \mathcal{C}_r &: H^{s-\mu}(X,F) \oplus H^{s-\mu}(Y,J_+) \to W \end{aligned}$$

for all $s > \max(\mu,d) - \frac{1}{2}$ and are of type $d_l = \max(\mu,d)$ and $d_r = (d-\mu)^+$, respectively, for $V := \ker\mathcal{A} \subset C^\infty(X,E) \oplus C^\infty(Y;J_-)$, and some finite-dimensional subspace $W \subset C^\infty(X,F) \oplus C^\infty(Y;J_+)$ such that

$$W + \operatorname{im}\mathcal{A} = H^{s-\mu}(X,F) \oplus H^{s-\mu}(Y;J_+)$$

and $W \cap \operatorname{im}\mathcal{A} = \{0\}$ for every $s > \max(\mu,d) - \frac{1}{2}$.

An operator $A \in B^{\mu,d}(X;E,F)$ will be called σ_ψ-elliptic (of order μ), if (1.91) is an isomorphism.

REMARK 1.75. Let $A \in B^{\mu,d}(X;E,F)$ be σ_ψ-elliptic. Then

$$\sigma_\partial(A)(y,\eta) : E'_y \otimes H^s(\mathbb{R}_+) \to F'_y \otimes H^{s-\mu}(\mathbb{R}_+) \tag{1.96}$$

is a family of Fredholm operators for all $s > \max(\mu,d) - \frac{1}{2}$, and $\ker\sigma_\partial(A)(y,\eta)$, $\operatorname{coker}\sigma_\partial(A)(y,\eta)$ are independent of s. For every $G \in B_G^{\mu,d}(X;E,F)$ we have $\operatorname{ind}\sigma_\partial(A)(y,\eta) = \operatorname{ind}\sigma_\partial(A+G)(y,\eta)$ for all $(y,\eta) \in T^*Y \setminus 0$. Moreover, by virtue of the homogeneity (1.87) it follows that $\operatorname{ind}\sigma_\partial(A)(y,\eta) = \operatorname{ind}\sigma_\partial(A)(y,\eta/|\eta|)$.

Thus it makes sense to interpret (1.96) as a family of Fredholm operators, parametrised by $(y,\eta) \in S^*Y$, the unit cosphere bundle. As such there is an index element

$$\operatorname{ind}_{S^*Y}\sigma_\partial(A) \in K(S^*Y), \tag{1.97}$$

cf. Atiyah and Bott [**3**], Boutet de Monvel [**6**].

REMARK 1.76. Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ be SL-elliptic. Then, in the notation of Definition 1.71 for $A := \operatorname{ulc}\mathcal{A}$ we have

$$\operatorname{ind}_{S^*Y}\sigma_\partial(A) = [\pi_1^* J_+] - [\pi_1^* J_-], \tag{1.98}$$

where $\pi_1 : S^*Y \to Y$ is the canonical projection. In other words, SL-ellipticity of $\mathcal{A}$ entails the relation

$$\operatorname{ind}_{S^*Y}\sigma_\partial(A) \in \pi_1^* K(Y). \tag{1.99}$$

2. Ellipticity with global projection conditions

2.1. The index obstruction. We first discuss the problem, whether a σ_ψ-elliptic operator $A \in B^{\mu,d}(X;E,F)$ admits SL-elliptic boundary conditions.

THEOREM 2.1. *Let $A \in B^{\mu,d}(X;E,F)$ be σ_ψ-elliptic. Then the following conditions are equivalent:*

(i) *There is an SL-elliptic element $\mathcal{A} \in B^{\mu,d}(X;\boldsymbol{v})$, $\boldsymbol{v} := (E,F;J_-,J_+)$ for certain $J_\pm \in \mathrm{Vect}(Y)$ such that $A = \mathrm{ulc}\,\mathcal{A}$.*

(ii) *A satisfies the relation* (1.99).

Remark 1.76 shows that (1.99) is necessary for the existence of an SL-elliptic operator $\mathcal{A}$ with A as upper left corner. For the converse direction we first establish a result on general families of Fredholm operators.

In the following consideration we assume M to be a compact topological space. In this connection by $\mathrm{Vect}(M)$ we understand the set of complex vector bundles on M in the continuous category, i.e., with continuous transition maps between local trivialisations. Moreover, let H_1 and H_2 be separable infinite-dimensional Hilbert spaces. For every function

$$a \in C(M,\mathcal{L}(H_1,H_2))$$

with values in the set $\mathcal{F}(H_1,H_2)$ of Fredholm operators between H_1,H_2 there is an index element

$$\mathrm{ind}_M\, a \in K(M). \tag{2.1}$$

The construction is based on the following observation. There exists a finite-dimensional vector space and an injective linear map $k : W \to H_2$ such that

$$\begin{pmatrix} a(m) & k \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ W \end{matrix} \to H_2 \tag{2.2}$$

is surjective for all $m \in M$. Then (2.2) is again a continuous family of Fredholm operators, now surjective. Hence the family of kernels $\{\ker(a(m) \;\; k) : m \in M\}$ represents a finite-dimensional subbundle $\widetilde{V}$ of $M \times (H_1 \oplus W)$. In other words, there is a $V \in \mathrm{Vect}(M)$ and a continuous family of isomorphisms

$$v_m : \ker(a(m) \;\; k) \to V_m \tag{2.3}$$

(with V_m being the fibre of V over m). Let $p_m : H_1 \oplus W \to \ker(a(m) \;\; k)$ denote the orthogonal projection and set $(t(m) \;\; g(m)) := v_m \circ p_m$.

Then

$$\begin{pmatrix} a(m) & k \\ t(m) & g(m) \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ W_m \end{matrix} \to \begin{matrix} H_2 \\ \oplus \\ V_m \end{matrix}$$

is a continuous family of isomorphisms, and we define

$$\mathrm{ind}_M\, a := [V] - [W] \in K(M), \tag{2.4}$$

where W is identified with the trivial bundle $M \times \mathbb{C}^{\dim W}$.

REMARK 2.2. Let $a \in C(M,\mathcal{F}(H_1,H_2))$ be as before, and let $V,W \in \mathrm{Vect}(M)$ such that there is a continuous family of isomorphisms

$$\begin{pmatrix} a(m) & k(m) \\ t(m) & g(m) \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ W_m \end{matrix} \to \begin{matrix} H_2 \\ \oplus \\ V_m \end{matrix}, \qquad m \in M$$

for suitable operator functions k,t and g. Then we have

$$\mathrm{ind}_M\, a = [V] - [W].$$

In other words, the element $\mathrm{ind}_M\, a$ is independent of the choice of the bundles V,W and of the operator families k,t,q.

Remark 2.3. Let $c \in C(M, \mathcal{L}(H_1, H_2))$ be a family such that $c(m)$ is a compact operator for every $m \in M$. Then we have

$$\operatorname{ind}_M(a + c) = \operatorname{ind}_M a.$$

For purposes below we need the following more specific construction.

Proposition 2.4. *Let $a \in C(M, \mathcal{F}(H_1, H_2))$ be a Fredholm function, and let $L_\pm \in \operatorname{Vect}(M)$ be a fixed choice of vector bundles such that $\operatorname{ind}_M a = [L_+] - [L_-]$. Then there exists an element $c \in C(M, \mathcal{L}(H_1, H_2))$ with values in operators of finite rank such that $\widetilde{a} := a + c$ has the following properties:*

(i) $\ker \widetilde{a} \cong L_+$, $\operatorname{coker} \widetilde{a} \cong L_-$, *i.e., there are subbundles $\widetilde{L}_+ \subset M \times H_1$, $\widetilde{L}_+ \cong L_+$ and $\widetilde{L}_- \subset M \times H_2$, $\widetilde{L}_- \cong L_-$, such that $\widetilde{L}_{+,m} = \ker \widetilde{a}(m)$, $\widetilde{L}_{-,m} + \operatorname{im} \widetilde{a}(m) = H_2$, and $\widetilde{L}_{-,m} + \operatorname{im} \widetilde{a}(m) = \{0\}$ for all $m \in M$*

(ii) *There are (continuous) bundle morphisms*

$$\widetilde{k} : L_- \to M \times H_2, \qquad \widetilde{t} : M \times H_1 \to L_+ \tag{2.5}$$

such that

$$\begin{pmatrix} \widetilde{a} & \widetilde{k} \\ \widetilde{t} & 0 \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ L_- \end{matrix} \to \begin{matrix} H_2 \\ \oplus \\ L_+ \end{matrix} \tag{2.6}$$

is an isomorphism.

Proof. As before we first pass to the surjective family (2.2). By assumption, we have

$$\operatorname{ind}_M a = [V] - [W] = [L_+] - [L_-]$$

in $K(M)$. We can choose $\dim W$ as large as we want, and we now replace W by $W \oplus W_1$ for another finite-dimensional W_1, and together with k we choose a $k_1 : W_1 \to H_2$, and such that $k^0 := \operatorname{diag}(k, k_1) : W \oplus W_1 \to H_2$ is injective. Let $p : H_2 \to \operatorname{im} k^0$ denote the orthogonal projection, and write $a^0 := (1-p)a$. Then, for $W^0 := W \oplus W_1$ the operator family

$$\begin{pmatrix} a^0 & k^0 \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ W^0 \end{matrix} \to H_2$$

is surjective, and $\widetilde{V}^0 := \ker_M a^0$ is a subbundle of H_1 isomorphic to a bundle $V^0 \in \operatorname{Vect}(M)$ where $V^0 \cong V \oplus W_1$ (here vector spaces F are identified with the respective trivial bundles $M \times F$). By the formula (2.4) applied to a^0 we obtain

$$\operatorname{ind}_M a^0 = [V^0] - [W^0] = [L_+] - [L_-] \tag{2.7}$$

for $V^0 := V \oplus W_1$. We may assume that the given bundles $L_\pm$ are both subbundles of the trivial bundle W_1 for a sufficienty large choice of $\dim W_1$.

There are then complementary bundles $L_\pm^\perp$ of $L_\pm$ in V^0 and W^0, respectively, i.e., we have

$$L_+ \oplus L_+^\perp = V^0, \quad L_- \oplus L_-^\perp = W^0.$$

Then the relation (2.7) implies $[L_+^\perp] = [L_-^\perp]$ in $K(M)$, i.e., there is an R such that $L_+^\perp \oplus \mathbb{C}^R \cong L_-^\perp \oplus \mathbb{C}^R$. Replacing W_1 by $W_1 \oplus \mathbb{C}^R$ in the construction before and returning to the former notation, we thus obtain $L_+^\perp \cong L_-^\perp$. By construction there are subbundles $\widetilde{L}_-, \widetilde{L}_-^\perp \subset H_2$ such that $\widetilde{L}_- \cong L_-$, $\widetilde{L}_-^\perp \cong L_-^\perp$ wich $\widetilde{L}_- \oplus \widetilde{L}_-^\perp = \operatorname{im} k^0$, and subbundles $\widetilde{L}_+, \widetilde{L}_+^\perp \subset H_1$ such that $\widetilde{L}_+ \cong L_+$, $\widetilde{L}_+^\perp \cong L_+^\perp$ and $\widetilde{L}_+ \oplus \widetilde{L}_+^\perp = \ker a^0$. Choose any isomorphism $\lambda : \widetilde{L}_+^\perp \to \widetilde{L}_-^\perp$, and let $\pi^\perp : H_1 \to \widetilde{L}_+^\perp$ denote the orthogonal projection, $\iota^\perp : \widetilde{L}_-^\perp \to H_2$ the canonical embedding. Then $q := \iota^\perp \circ \lambda \circ \pi^\perp : H_1 \to H_2$ is a continuous family of operators of finite rank, and $a^0 + q$ satisfies the relations

$$\ker_M(a^0 + q) \cong L_+, \quad \operatorname{coker}_M(a^0 + q) \cong L_-.$$

Because of $a^0 = (1-p)a$ we may set $\widetilde{a} := a + c$ for $c := -pa + q$. Then $\widetilde{a}$ satisfies the relations of Proposition 2.4 (i). To construct the isomorphism (2.6) it suffices to choose isomorphisms $h : L_- \to \widetilde{L}_-$

and $l : \widetilde{L}_+ \to L_+$, and to set $\widetilde{k} := \iota h, \widetilde{t} := l\pi$, where $\iota : \widetilde{L}_- \to H_2$ is the canonical embedding, $\pi : H_1 \to \widetilde{L}_+$ the orthogonal projection. □

REMARK 2.5. Let $A \in B^{\mu,d}(X; E, F)$ be σ_ψ-elliptic, and consider the Fredholm operators (1.96) for any fixed $s > \max(\mu, d) - \frac{1}{2}$. Then there is a subbundle $\widetilde{W} \subset \pi_1^* F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+)$ of finite fibre dimension such that

$$\widetilde{W}_{y,\eta} + \operatorname{im} \sigma_\partial(A)(y, \eta) = F'_y \otimes H^{s-\mu}(\mathbb{R}_+)$$

for all $(y, \eta) \in S^*Y$.

Choose a vector bundle $W \in \mathrm{Vect}(S^*Y)$, and let

$$k : W \to \widetilde{W}$$

be an isomorphism. Then

$$(\sigma_\partial(A)(y,\eta) \quad k(y,\eta)) : \begin{pmatrix} E'_y \otimes H^s(\mathbb{R}_+) \\ \oplus \\ W_{y,\eta} \end{pmatrix} \to F'_y \otimes H^{s-\mu}(\mathbb{R}_+) \tag{2.8}$$

is a surjective family of Fredholm operators. W can be chosen as the pull-back of a bundle on Y with respect to $\pi_1 : S^*Y \to Y$ (in fact, we may assume that it is trivial).

Let

$$p : \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \to \widetilde{W}$$

be a projection (orthogonal with respect to the $F'_y \otimes L^2(\mathbb{R}_+)$ - scalar product in the fibres). Then

$$(1-p)\sigma_\partial(A) : \pi_1^* E' \otimes H^s(\mathbb{R}_+) \to \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+)$$

is again a Fredholm family such that

$$\widetilde{V} := \ker_{S^*Y}(1-p)\sigma_\partial(A) \subset \pi_1^* E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+)$$

is a subbundle, and

$$\operatorname{ind}_{S^*Y}(1-p)\sigma_\partial(A) = [V] - [W]$$

for any $V \in \mathrm{Vect}(S^*Y)$ which is isomorphic to $\widetilde{V}$.

PROPOSITION 2.6. *Let $A \in B^{\mu,d}(X; E, F)$ be σ_ψ-elliptic, and let $L_\pm \in \mathrm{Vect}(S^*Y)$ such that*

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A) = [L_+] - [L_-]. \tag{2.9}$$

Then there exists an element $G \in B_G^{\mu,0}(X; E, F)$ such that

$$\ker_{S^*Y}(\sigma_\partial(A+G)) \cong L_+, \qquad \operatorname{coker}_{S^*Y}(\sigma_\partial(A+G)) \cong L_-. \tag{2.10}$$

PROOF. The construction of G may be given along the lines of Proposition 2.4 (i). Let us fix some $s \in \mathbb{R}$ such that $s > \max(\mu, d) - \frac{1}{2}$ (the specific choice of s will be unessential), and let

$$a : \pi_1^* E' \otimes H^s(\mathbb{R}_+) \to \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+)$$

for $\pi_1 : S^*Y \to Y$ denote the restriction of $\sigma_\partial(A)$ to S^*Y. According to Remark 2.5 there is a surjective bundle morphism

$$(a \quad k) : \begin{pmatrix} \pi_1^* E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ W \end{pmatrix} \to \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+)$$

for a $W \in \mathrm{Vect}(S^*Y)$, where $k : W \to \widetilde{W}$ is an isomorphism to some subbundle $\widetilde{W}$ of $\pi_1^* F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+)$. Without loss of generality, we assume W to be trivial. Let $p : \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \to \widetilde{W}$ be a projection that is orthogonal in the fibres with respect to the scalar products of $F'_y \otimes L^2(\mathbb{R}_+)$. By adding, if necessary, another finite-dimensional subbundle to W (and denoting the new bundle again by W) we obtain the following properties: There are subbundles $\widetilde{L}_- \subset \widetilde{W}$ and

$$\widetilde{L}_+ \subset \widetilde{V} := \ker_{S^*Y}((1-p)a) \subset \pi_1^* E' \otimes H^s(\mathbb{R}_+),$$

such that $\widetilde{L}_- \cong L_-$, $\widetilde{L}_+ \cong L_+$. In addition, choosing complements $\widetilde{L}_-^\perp$ in $\widetilde{W}$ and $\widetilde{L}_+^\perp$ in $\widetilde{V}$, we have $\widetilde{L}_-^\perp \cong \widetilde{L}_+^\perp$, provided the fibre dimension of W is sufficiently large. If $\lambda : \widetilde{L}_+^\perp \to \widetilde{L}_-^\perp$ is an isomorphism, and if

$$\iota^\perp : \widetilde{L}_-^\perp \to \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+)$$

is the canonical embedding, and

$$\pi^\perp : \pi_1^* E' \otimes H^s(\mathbb{R}_+) \to \widetilde{L}_+^\perp$$

the orthogonal projection, the operator family $a_0 := (1-p)a + q$ for $q := \iota^\perp \circ \lambda \circ \pi^\perp$ has the property

$$L_+ \cong \ker_{S^*Y} a_0, \quad L_- \cong \operatorname{coker}_{S^*Y} a_0. \tag{2.11}$$

The operator function $g := -pa+q : \pi_1^* E' \otimes H^s(\mathbb{R}_+) \to \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+)$ can be extended by homogeneity μ to a morphism

$$g_{(\mu)} : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* F' \otimes H^{s-\mu}(\mathbb{R}_+),$$

i.e., $g_{(\mu)}(y, \lambda\eta) = \lambda^\mu \kappa_\lambda g_{(\mu)}(y,\eta)\kappa_\lambda^{-1}$ for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in T^*Y \setminus 0$, and $g_{(\mu)}|_{S^*Y} = g$. Now we may set

$$G := \sum_{j=1}^{L} \varphi_j (\chi_j^{-1})_* \operatorname{Op}(g_j)\psi_j, \tag{2.12}$$

cf. Definition 1.66, where $g_j(y,\eta)$ are local Green symbols of order μ which have $g_{(\mu)}(y,\eta)$ as homogeneous principal components (in local coordinates it suffices to set $g_j(y,\eta) := \chi(\eta) g_{(\mu)}(y,\eta)$ for any excision function $\chi(\eta)$). Because of $a_0 = \sigma_\partial(A+G)|_{S^*Y}$, the assertion follows from the relations (2.11). $\square$

Theorem 2.7. *Let $A \in B^{\mu,d}(X; E, F)$ be a σ_ψ-elliptic operator. Then there exist vector bundles*

$$J_\pm \in \operatorname{Vect}(Y) \quad \text{and} \quad L_\pm \in \operatorname{Vect}(T^*Y \setminus 0)$$

and an operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ for $\boldsymbol{v} := (E, F; J_-, J_+)$ such that (1.94) *restricts to an isomorphism*

$$\begin{matrix} \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ L_- \end{matrix} \to \begin{matrix} \pi_Y^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ L_+ \end{matrix}. \tag{2.13}$$

Proof. If $A \in B^{\mu,d}(X; E, F)$ is σ_ψ-elliptic, the boundary symbol $\sigma_\partial(A)$ represents a family of Fredholm operators (1.96), and there is an index element (1.97). Choose any $L_\pm \in \operatorname{Vect}(S^*Y)$ such that

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A) = [L_+] - [L_-].$$

For abbreviation, $L_\pm$ will also denote the pull-backs of these bundles to $T^*Y \setminus 0$ under the canonical projection $T^*Y \setminus 0 \to S^*Y$. Applying Proposition 2.6 we find a Green operator $G \in B_G^{\mu,0}(X; E, F)$ such that the relations (2.10) hold.

Choose arbitrary bundle morphisms

$$k_1 : L_- \to \pi_1^* F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+), \quad t_1 : \pi_1^* E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \to L_+,$$

such that k_1 represents an isomorphism $L_- \to \widetilde{L}_-$, and t_1 restricts to an isomorphism $\widetilde{L}_+ \to L_+$, cf. the notation in the proof of Proposition 2.6. Then the block matrix

$$\begin{pmatrix} \sigma_\partial(A+G) & k_1 \\ t_1 & 0 \end{pmatrix} : \begin{matrix} \pi_1^* E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ L_- \end{matrix} \to \begin{matrix} \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ L_+ \end{matrix} \tag{2.14}$$

is an isomorphism for every $s > \max(\mu, d) - \frac{1}{2}$. Now let $J_\pm \in \operatorname{Vect}(Y)$ be arbitrary bundles such that $L_\pm$ are subbundles of $\pi_1^* J_\pm$ (for $J_\pm$ we may always take trivial bundles of sufficiently large fibre dimension). We then obtain a bundle morphism

$$\begin{pmatrix} \sigma_\partial(A+G) & k_0 \\ t_0 & 0 \end{pmatrix} : \begin{matrix} \pi_1^* E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ \pi_1^* J_- \end{matrix} \to \begin{matrix} \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \pi_1^* J_+ \end{matrix} \tag{2.15}$$

when we set $k_0 := k_1 \circ \pi_-$ for a bundle projection $\pi_- : \pi_1^* J_- \to L_-$ and $t_0 := \iota_+ \circ t_1$ for the canonical embedding $\iota_+ : L_+ \to \pi_1^* J_+$. By construction, (2.15) restricts to the isomorphism (2.14). Next we extend (2.15) by κ_λ-homogeneity μ to a boundary symbol of the form (1.94), which has the form

$$\begin{pmatrix} \sigma_\partial(A+G) & k_{(\mu)} \\ t_{(\mu)} & 0 \end{pmatrix}$$

for unique $k_{(\mu)}(y,\eta)$ and $t_{(\mu)}(y,\eta)$ satisfying $k_{(\mu)}(y,\eta/|\eta|) = k_1(y,\eta/|\eta|)$ and $t_{(\mu)}(y,\eta/|\eta|) = t_1(y,\eta/|\eta|)$, respectively. Similarly to the construction of (2.12) we find potential and trace operators K and T, respectively, such that $\sigma_\partial(K) = k_{(\mu)}$, $\sigma_\partial(T) = t_{(\mu)}$.

Setting

$$\mathcal{A} = \begin{pmatrix} A+G & K \\ T & 0 \end{pmatrix} \tag{2.16}$$

we obtain an element in $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ as desired. □

REMARK 2.8. For $\operatorname{ind}_{S^*Y} \sigma_\partial(A) \in \pi_1^* K(Y)$ we can carry out the construction in the latter proof with bundles $\pi_1^* J_\pm$ for suitable $J_\pm \in \operatorname{Vect}(Y)$ in place of $L_\pm$. Then (2.13) shows that the operator (2.16) is SL-elliptic.

PROOF OF THEOREM 2.1, (ii) ⇒ (i). If the relation (1.99) is satisfied, the family of Fredholm operators (1.96) for any fixed $s > \max(\mu,d) - \frac{1}{2}$ can be completed to a block matrix (1.94) that is an isomorphism. For the construction we first restrict $\sigma_\partial(A)$ to S^*Y (denote it again by $\sigma_\partial(A)$) and set $J_- := \mathbb{C}^{j_-}$. For a sufficiently large j_- there is an injective bundle morphism

$$k_0 : \pi_1^* J_- \to \pi_1^* F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+)$$

such that

$$\begin{pmatrix} \sigma_\partial(A) & k_0 \end{pmatrix} : \begin{matrix} \pi_1^* E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ \pi_1^* J_- \end{matrix} \to \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+)$$

is surjective. Then $\ker_{S^*Y}\begin{pmatrix} \sigma_\partial(A) & k_0 \end{pmatrix}$ is a finite-dimensional subbundle of

$$\begin{matrix} \pi_1^* E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \pi_1^* J_- \end{matrix} . \tag{2.17}$$

As we saw by the above constructions, for a sufficiently large choice of j_- the bundle $\ker_{S^*Y}\begin{pmatrix} \sigma_\partial(A) & k_0 \end{pmatrix}$ is isomorphic to $\pi_1^* J_+$ for some $J_+ \in \operatorname{Vect}(S^*Y)$.

Now let

$$\kappa_0 : \ker_{S^*Y}\begin{pmatrix} \sigma_\partial(A) & k_0 \end{pmatrix} \to \pi_1^* J_+$$

by any isomorphism, and let

$$\pi_0 : \begin{matrix} \pi_1^* E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ \pi_1^* J_- \end{matrix} \to \ker_{S^*Y}\begin{pmatrix} \sigma_\partial(A) & k_0 \end{pmatrix} \tag{2.18}$$

be the orthogonal projection with respect to the $(E'_y \otimes L^2(\mathbb{R}_+)) \oplus J_{-,y}$-scalar products in the fibres, first for $s \geq \max(\mu,d)$ and then extended by continuity to all $s > \max(\mu,d) - \frac{1}{2}$.

Setting $\begin{pmatrix} t_0 & q_0 \end{pmatrix} := \kappa_0 \circ \pi_0$ we obtain an isomorphism

$$\begin{pmatrix} \sigma_\partial(A) & k_0 \\ t_0 & q_0 \end{pmatrix} : \quad \pi_1^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi_1^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix} . \tag{2.19}$$

There is then a unique extension to an isomorphism

$$\sigma_\partial(\mathcal{A}) := \begin{pmatrix} \sigma_\partial(A) & k_{(\mu)} \\ t_{(\mu)} & q_{(\mu)} \end{pmatrix} : \quad \pi_Y^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix},$$

homogeneous in the sense

$$\sigma_\partial(\mathcal{A})(y, \lambda\eta) = \lambda^\mu \operatorname{diag}(1, \kappa_\lambda)\sigma_\partial(\mathcal{A})(y,\eta) \operatorname{diag}(1, \kappa_\lambda^{-1})$$

for all $(y,\eta) \in T^*Y \setminus 0$ and all $\lambda \in \mathbb{R}_+$.

We finally pass to an element

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$$

for $\boldsymbol{v} = (E, F; J_-, J_+)$ that has $\sigma_\partial(\mathcal{A})$ as homogeneous principal symbol. The construction of the entries T, K and Q in terms of $t_{(\mu)}, k_{(\mu)}$ and $q_{(\mu)}$, respectively, is analogous to that for (2.16). ☐

2.2. A Toeplitz algebra of boundary value problems. For every $L \in \operatorname{Vect}(T^*Y \setminus 0)$ there exists an element $J \in \operatorname{Vect}(Y)$ such that L is a subbundle of $\pi_Y^* J$. In fact, we may choose J as a trivial bundle $Y \times \mathbb{C}^N$ (also written as $\mathbb{C}^N$) for a sufficiently large N. Let

$$p_{(0)} : \pi_Y^* J \to L$$

be a bundle morphism that is a projection to L, such that $p_{(0)}(y, \lambda\eta) = p_{(0)}(y,\eta)$ for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in T^*Y \setminus 0$. Then, by Theorem 1.13 there exists a $P \in L^0_{\mathrm{cl}}(Y; J, J)$ such that $P^2 = P$ and $p_{(0)} = \sigma_\psi(P)$. Recall that triples of the form

$$\mathbb{L} := (P, J, L) \tag{2.20}$$

are called global projection data and recall that $\mathcal{P}(Y)$ denotes the set of all such triples, cf. (1.19).

Definition 2.9. Let

$$\mathbb{L}_+ := (P_+, J_+, L_+), \quad \mathbb{L}_- := (P_-, J_-, L_-) \in \mathcal{P}(Y) \tag{2.21}$$

be projection data, and let $R_- : P^s(Y, \mathbb{L}_-) \to H^s(Y, J_-)$ denote the canonical embedding, $s \in \mathbb{R}$. Moreover, let $\boldsymbol{v} := (E, F; J_-, J_+)$, and set $\boldsymbol{l} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$. Then $\mathcal{S}^{\mu,d}(X; \boldsymbol{l})$ for $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, is defined as the set of all operators

$$\mathcal{A} := \begin{pmatrix} 1 & 0 \\ 0 & P_+ \end{pmatrix} \widetilde{\mathcal{A}} \begin{pmatrix} 1 & 0 \\ 0 & R_- \end{pmatrix} \tag{2.22}$$

for arbitrary $\widetilde{\mathcal{A}} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$.

Theorem 2.10. *Every $\mathcal{A} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{l})$ (with notation of Definition 2.9) induces continuous operators*

$$\mathcal{A} : \begin{matrix} H^s(X, E) \\ \oplus \\ P^s(Y, \mathbb{L}_-) \end{matrix} \to \begin{matrix} H^{s-\mu}(X, F) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{L}_+) \end{matrix} \tag{2.23}$$

for all $s \in \mathbb{R}$, $s > d - \frac{1}{2}$.

Proof. It suffices to apply Theorem 1.69 to the operator $\widetilde{\mathcal{A}}$ in (2.22) and to employ the definitions of R_- and P_+, cf. (1.20). ☐

We now introduce the principal symbolic structure of $\mathcal{S}^{\mu,d}(X; \boldsymbol{l})$. By definition, elements in that space are 2×2 block matrices $\mathcal{A} = (A_{ij})_{i,j=1,2}$ with $A_{11} = \operatorname{ulc} \mathcal{A} \in B^{\mu,d}(X; E, F)$. We then call

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(\operatorname{ulc} \mathcal{A}) : \pi_X^* E \to \pi_X^* F \tag{2.24}$$

the (homogeneous principal) interior symbol of $\mathcal{A}$. Furthermore, the operator family

$$\sigma_\partial(\mathcal{A}) := \begin{pmatrix} 1 & 0 \\ 0 & p_+ \end{pmatrix} \sigma_\partial(\widetilde{\mathcal{A}}) \begin{pmatrix} 1 & 0 \\ 0 & r_- \end{pmatrix} : \begin{matrix} \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ L_- \end{matrix} \to \begin{matrix} \pi_Y^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ L_+ \end{matrix} \tag{2.25}$$

is called the (homogeneous principal) boundary symbol of $\mathcal{A}$. Here $p_+(y,\eta)$ is the homogeneous principal symbol of order zero of the projection $P_+ \in L^0_{\mathrm{cl}}(Y;J_+,J_+)$, and $r_- : L_- \to \pi_Y^* J_-$ is the canonical embedding.

Let us set

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})). \tag{2.26}$$

REMARK 2.11. Let $\widetilde{\mathcal{A}} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ for $\boldsymbol{v} = (E,F;J_-,J_+)$, and form the operator

$$\tilde{\widetilde{\mathcal{A}}} := \begin{pmatrix} 1 & 0 \\ 0 & P_+ \end{pmatrix} \widetilde{\mathcal{A}} \begin{pmatrix} 1 & 0 \\ 0 & P_- \end{pmatrix}$$

(with notation of Definition 2.9). Then we also have $\tilde{\widetilde{\mathcal{A}}} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$, if we interpret P_+ as a map $H^{s-\mu}(Y,J_+) \to H^{s-\mu}(Y,J_+)$ (not as $H^{s-\mu}(Y,J_+) \to P^{s-\mu}(Y,\mathbb{L}_+)$ as in (2.22)), and the operator (2.22) can also be written as

$$\mathcal{A} := \begin{pmatrix} 1 & 0 \\ 0 & P_+ \end{pmatrix} \tilde{\widetilde{\mathcal{A}}} \begin{pmatrix} 1 & 0 \\ 0 & R_- \end{pmatrix} \tag{2.27}$$

(with $P_+ : H^{s-\mu}(Y,J_+) \to P^{s-\mu}(Y,\mathbb{L}_+)$). Then $\sigma(\tilde{\widetilde{\mathcal{A}}}) = 0$ in the sense of $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$, cf. (1.86), is equivalent to $\sigma(\mathcal{A}) = 0$ in the sense of $\mathcal{S}^{\mu,d}(X;\boldsymbol{l})$.

THEOREM 2.12. (i) *$\mathcal{A} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{l})$ (cf. Definition 2.9) and $\sigma(\mathcal{A}) = 0$ imply $\mathcal{A} \in \mathcal{S}^{\mu-1,d}(X;\boldsymbol{l})$, and (2.23) is compact for every $s > d - \frac{1}{2}$.*

(ii) *$\mathcal{A} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{l}_0)$ for $\boldsymbol{l}_0 := (E_0,F;\mathbb{L}_0,\mathbb{L}_+)$, $\mathcal{B} \in \mathcal{S}^{\nu,e}(X;\boldsymbol{l}_1)$ for $\boldsymbol{l}_1 := (E,E_0;\mathbb{L}_-,\mathbb{L}_0)$, $\mathbb{L}_0,\mathbb{L}_\pm \in \mathcal{P}(Y)$, implies $\mathcal{A}\mathcal{B} \in \mathcal{S}^{\mu+\nu,h}(X;\boldsymbol{l}_0 \circ \boldsymbol{l}_1)$ for $\boldsymbol{l}_0 \circ \boldsymbol{l}_1 := (E,F;\mathbb{L}_-,\mathbb{L}_+)$, and $h = \max(\nu+d,e)$, and we have*

$$\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

(with componentwise multiplication).

(iii) *$\mathcal{A} \in \mathcal{S}^{0,0}(X;\boldsymbol{l})$ for $\boldsymbol{l} := (E,F;\mathbb{L}_-,\mathbb{L}_+)$, implies $\mathcal{A}^* \in \mathcal{S}^{0,0}(X;\boldsymbol{l}^*)$ for $\boldsymbol{l}^* := (F,E;\mathbb{L}_+^*,\mathbb{L}_-^*)$ in the sense of*

$$(u,\mathcal{A}^* v)_{L^2(X,E)\oplus P^0(Y,\mathbb{L}_-)} = (\mathcal{A}u,v)_{L^2(X,F)\oplus P^0(Y,\mathbb{L}_+)}$$

for all $u \in L^2(X,E) \oplus P^0(Y,\mathbb{L}_-)$, $v \in L^2(X,F) \oplus P^0(Y,\mathbb{L}_+)$, and we have $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})^*$ (with componentwise adjoint, cf. Theorem 1.69 and Theorem 1.29).

PROOF. (i) Let us write $\mathcal{A}$ in the form (2.27) and apply Remark 2.11. Then we have $\sigma(\tilde{\widetilde{\mathcal{A}}}) = 0$. From Theorem 1.69 (ii) we then obtain $\tilde{\widetilde{\mathcal{A}}} \in \mathcal{B}^{\mu-1,d}(X;\boldsymbol{v})$ which implies $\mathcal{A} \in \mathcal{S}^{\mu-1,d}(X;\boldsymbol{l})$. Moreover, $\tilde{\widetilde{\mathcal{A}}}$ is compact and so is $\mathcal{A}$.

(ii) The operators $\mathcal{A}$ and $\mathcal{B}$ can be written in the form

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & P_+ \end{pmatrix} \widetilde{\mathcal{A}} \begin{pmatrix} 1 & 0 \\ 0 & R_0 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & P_0 \end{pmatrix} \widetilde{\mathcal{B}} \begin{pmatrix} 1 & 0 \\ 0 & R_- \end{pmatrix}$$

with P_+ from $\mathbb{L}_+ = (P_+,J_+,L_+)$, P_0 from $\mathbb{L}_0 = (P_0,J_0,L_0)$, and $R_- : P^s(Y,\mathbb{L}_-) \to H^s(Y,J_-)$ and the corresponding canonical embeddings $R_0 : P^{s-\nu}(Y,\mathbb{L}_0) \to H^{s-\nu}(Y,J_0)$, moreover, $\widetilde{\mathcal{A}} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v}_0)$, $\boldsymbol{v}_0 := (E_0,F;J_0,J_+)$, and $\widetilde{\mathcal{B}} \in \mathcal{B}^{\nu,d}(X;\boldsymbol{v}_1)$, $\boldsymbol{v}_1 := (E,E_0;J_-,J_0)$. Then it follows that

$$\mathcal{A}\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & P_+ \end{pmatrix} \widetilde{\mathcal{A}} \begin{pmatrix} 1 & 0 \\ 0 & P_0 \end{pmatrix} \widetilde{\mathcal{B}} \begin{pmatrix} 1 & 0 \\ 0 & R_- \end{pmatrix}$$

From $\widetilde{\mathcal{E}} := \widetilde{\mathcal{A}} \operatorname{diag}(1,P_0)\widetilde{\mathcal{B}} \in \mathcal{B}^{\mu+\nu,h}(X;\boldsymbol{v}_0 \circ \boldsymbol{v}_1)$, cf. Theorem 1.69, we obtain $\mathcal{A}\mathcal{B} \in \mathcal{S}^{\mu+\nu,h}(X;\boldsymbol{l}_0 \circ \boldsymbol{l}_1)$. In addition, we have

$$\sigma(\widetilde{\mathcal{E}}) = \sigma(\widetilde{\mathcal{A}}) \operatorname{diag}(1,p_0)\sigma(\widetilde{\mathcal{B}})$$

where p_0 is the homogeneous principal symbol of P_0, and hence $\sigma_\psi(\mathcal{A}\mathcal{B}) = \sigma_\psi(\mathcal{A})\sigma_\psi(\mathcal{B})$. Moreover, from

$$\begin{aligned} \sigma_\partial(\widetilde{\mathcal{E}}) &= \sigma_\partial(\widetilde{\mathcal{A}}) \operatorname{diag}(1,p_0)\sigma_\partial(\widetilde{\mathcal{B}}) \\ &= \sigma_\partial(\widetilde{\mathcal{A}}) \operatorname{diag}(1,r_0) \operatorname{diag}(1,p_0)\sigma_\partial(\widetilde{\mathcal{B}}) \end{aligned}$$

with $r_0 : L_0 \to \pi_Y^* J_0$ being the canonical embedding, it follows that

$$\begin{aligned}\sigma_\partial(\mathcal{A}\mathcal{B}) &= \operatorname{diag}(1,p_+)\sigma_\partial(\widetilde{\mathcal{A}})\operatorname{diag}(1,r_0)\operatorname{diag}(1,p_0)\sigma_\partial(\widetilde{\mathcal{B}})\operatorname{diag}(1,r_-)\\ &= \sigma_\partial(\mathcal{A})\sigma_\partial(\mathcal{B}).\end{aligned}$$

(iii) Writing $\mathcal{A}$ in the form (2.22) for $\widetilde{\mathcal{A}} \in \mathcal{B}^{0,0}(X;\boldsymbol{v})$, $\boldsymbol{v} := (E,F;J_-,J_+)$, we have

$$\mathcal{A}^* = \begin{pmatrix}1 & 0\\ 0 & P_-^*\end{pmatrix}\widetilde{\mathcal{A}}^*\begin{pmatrix}1 & 0\\ 0 & R_+^*\end{pmatrix}$$

with $\widetilde{\mathcal{A}}^* \in \mathcal{B}^{0,0}(X;\boldsymbol{v}^*)$ as in Theorem 1.69 (iv) and $P_\pm^*$ as in Theorem 1.29, where $R_+^* : P^s(Y,L_+^*) \to H^s(Y,J_+)$ is the canonical embedding. This yields $\mathcal{A}^* \in \mathcal{S}^{0,0,}(X;\boldsymbol{l}^*)$, cf. also the notation of Theorem 1.29. Moreover, we have

$$\sigma_\partial(\mathcal{A}^*) = \operatorname{diag}(1,p_-^*)\sigma_\partial(\widetilde{\mathcal{A}})^*\operatorname{diag}(1,r_+^*),$$

where p_-^* is the homogeneous principal symbol of the projection P_-^* and $r_+^* : L_+^* \hookrightarrow \pi_Y^* J_+$ the canonical embedding. From $\sigma_\partial(\widetilde{\mathcal{A}})^* = \sigma_\partial(\widetilde{\mathcal{A}}^*)$ we then obtain the assertion. □

THEOREM 2.13. *Let $\boldsymbol{l} := (E,F;\mathbb{L}_-,\mathbb{L}_+)$ for $E,F \in \operatorname{Vect}(X)$, $\mathbb{L}_\pm \in \mathcal{P}(Y)$, and let $\mathcal{A}_j \in \mathcal{S}^{\mu-j,d}(X;\boldsymbol{l})$, $j \in \mathbb{N}$, be an arbitrary sequence. Then there exists an element $\mathcal{A} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{l})$ such that*

$$\mathcal{A} - \sum_{j=0}^{N}\mathcal{A}_j \in \mathcal{S}^{\mu-(N+1),d}(X;\boldsymbol{l})$$

for every $N \in \mathbb{N}$, and $\mathcal{A}$ is unique $\bmod\ \mathcal{S}^{-\infty,d}(X;\boldsymbol{l})$.

PROOF. The assertion is an immediate consequence of Theorem 1.70 and Definition 2.9. □

Given $\mathcal{A} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{l})$, $\mathcal{B} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{m})$ for

$$\boldsymbol{l} := (E,F;\mathbb{L}_-,\mathbb{L}_+),\quad \mathbb{L}_\pm := (P_\pm,J_\pm,L_\pm),$$

and

$$\boldsymbol{m} := (V,W;\mathbb{M}_-,\mathbb{M}_+),\quad \mathbb{M}_\pm := (Q_\pm,G_\pm,M_\pm),$$

$\mathbb{L}_\pm,\mathbb{M}_\pm \in P(Y)$, we can form the direct sum

$$\mathcal{A} \oplus \mathcal{B} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{l}\oplus\boldsymbol{m}) \tag{2.28}$$

for

$$\boldsymbol{l}\oplus\boldsymbol{m} := (E\oplus V, F\oplus W;\mathbb{L}_-\oplus\mathbb{M}_-,\mathbb{L}_+\oplus\mathbb{M}_+),$$

$$\mathbb{L}_\pm \oplus \mathbb{M}_\pm := (P_\pm\oplus Q_\pm, J_\pm\oplus G_\pm, L_\pm\oplus M_\pm).$$

We then have

$$\sigma_\psi(\mathcal{A}\oplus\mathcal{B}) = \sigma_\psi(\mathcal{A})\oplus\sigma_\psi(\mathcal{B}),\quad \sigma_\partial(\mathcal{A}\oplus\mathcal{B}) = \sigma_\partial(\mathcal{A})\oplus\sigma_\partial(\mathcal{B})$$

with an evident meaning of '$\oplus$' for the symbolic components.

2.3. Ellipticity, parametrices, and the Fredholm property. Our next objective is to study ellipticity with global projection conditions.

DEFINITION 2.14. An operator $\mathcal{A} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{l})$ for $\boldsymbol{l} := (E,F;\mathbb{L}_-,\mathbb{L}_+)$, $\mathbb{L}_\pm := (P_\pm,J_\pm,L_\pm)$, is called elliptic, if

(i) the interior symbol

$$\sigma_\psi(\mathcal{A}) : \pi_X^* E \to \pi_X^* F \tag{2.29}$$

is an isomorphism, and

(ii) the boundary symbol

$$\sigma_\partial(\mathcal{A}) : \begin{array}{c} \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ L_- \end{array} \to \begin{array}{c} \pi_Y^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ L_+ \end{array} \tag{2.30}$$

is an isomorphism for every $s \in \mathbb{R}$, $s > \max(\mu, d) - \frac{1}{2}$.

REMARK 2.15. Condition (ii) in Definition 2.14 holds if and only if it is satisfied for any fixed $s_0 > \max(\mu, d) - \frac{1}{2}$. Moreover, this is equivalent to the bijectivity of

$$\sigma_\partial(\mathcal{A}) : \begin{array}{c} \pi_Y^* E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ L_- \end{array} \to \begin{array}{c} \pi_Y^* F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ L_+ \end{array}. \tag{2.31}$$

THEOREM 2.16. *For every operator $A \in L^\mu_{\mathrm{cl}}(X; E, F)_{\mathrm{tr}}$ (cf. notation (1.66)) such that $\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F$ is an isomorphism there exist projection data $\mathbb{L}_\pm \in \mathcal{P}(Y)$ and an element $\mathcal{A} \in \mathcal{S}^{\mu,0}(X; \boldsymbol{l})$ for $\boldsymbol{l} = (E, F; \mathbb{L}_-, \mathbb{L}_+)$ which is elliptic in the sense of Definition 2.14.*

PROOF. Let us choose elements $L_\pm \in \mathrm{Vect}(S^*Y)$ such that the relation (2.9) holds, and let $G \in B_G^{\mu,0}(X; E, F)$ be an operator as in Proposition 2.6. We can apply the construction of the proof of Theorem 2.7 and denote the operator (2.16) by $\widetilde{\mathcal{A}}$ instead of $\mathcal{A}$. This shows that the operator (2.16) has the asserted properties. ☐

PROPOSITION 2.17. *For every $\mu \in \mathbb{Z}$, $E \in \mathrm{Vect}(X)$ and $\mathbb{L} \in \mathcal{P}(Y)$ there exists an elliptic element $\mathcal{R}^\mu_{E,\mathbb{L}} \in \mathcal{S}^{\mu,0}(X; \boldsymbol{l})$ for $\boldsymbol{l} := (E, E; \mathbb{L}, \mathbb{L})$ which induces Fredholm operators*

$$\mathcal{R}^\mu_{E,\mathbb{L}} : \begin{array}{c} H^s(X, E) \\ \oplus \\ P^s(Y, \mathbb{L}) \end{array} \to \begin{array}{c} H^{s-\mu}(X, E) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{L}) \end{array}$$

for all $s > \max(\mu, 0) - \frac{1}{2}$, such that $(\mathcal{R}^\mu_{E,\mathbb{L}})^{-1} \in \mathcal{S}^{-\mu,(-\mu)^+}(X; \boldsymbol{l})$.

PROOF. It suffices to set

$$\mathcal{R}^\mu_{E,\mathbb{L}} := \begin{pmatrix} R^\mu_E & 0 \\ 0 & R^\mu_{\mathbb{L}} \end{pmatrix}$$

with R^μ_E from Theorem 1.59 and $R^\mu_{\mathbb{L}}$ from Remark 1.31. ☐

REMARK 2.18. Let R^μ_E be as in Theorem 1.59. Then

$$\sigma_\partial(R^\mu_E) : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* E' \otimes H^{s-\mu}(\mathbb{R}_+)$$

is an isomorphism, $s > \max(\mu, 0) - \frac{1}{2}$. This implies

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(R^\mu_E) = 0.$$

Similarly to R^μ_E we can form an operator $S^\mu_E \in L^\mu_{\mathrm{cl}}(X; E, E)_{\mathrm{tr}}$ in terms of the local symbols $\mathrm{r}^\mu_+(\eta, \tau) := \overline{\mathrm{r}^\mu_-(\eta, \tau)}$ (the complex conjugate), cf. Remark 1.60. The operator $S^\mu_E \in B^{\mu,0}(X; E, E)$ can be chosen in such a way that

$$\sigma_\partial(S^\mu_E) : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* E' \otimes H^{s-\mu}(\mathbb{R}_+)$$

is surjective and $\ker_{S^*Y} \sigma_\partial(S^\mu_E) = \mu[\pi_1^* E']$; this yields

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(S^\mu_E) = \mu[\pi_1^* E']. \tag{2.32}$$

THEOREM 2.19. *For every elliptic operator $\mathcal{A} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{l})$, $\boldsymbol{l} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$, there exists an elliptic operator $\mathcal{B} \in \mathcal{S}^{\mu,0}(X; \boldsymbol{m})$ for $\boldsymbol{m} := (F, E; \mathbb{M}_-, \mathbb{M}_+)$, for certain projection data $\mathbb{M}_\pm \in \mathcal{P}(Y)$ of the form $\mathbb{M}_\pm := (Q_\pm, \mathbb{C}^N, M_\pm)$ for some $N \in \mathbb{N}$, such that $\mathcal{A} \oplus \mathcal{B} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, $\boldsymbol{v} := (E \oplus F, F \oplus E; \mathbb{C}^N, \mathbb{C}^N)$ is SL-elliptic*

PROOF. Choose any $s \in \mathbb{N}$, $s - \mu \geq 0$, and form the operator $A_0 := R_F^{s-\mu} A R_E^{-s} : L^2(X, E) \to L^2(X, F)$, cf. Theorem 1.59. Then we have $A_0 \in B^{0,0}(X; E, F)$ and

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A_0) = \operatorname{ind}_{S^*Y} \sigma_\partial(A) = [L_+] - [L_-].$$

For the L^2-adjoint $A_0^* \in B^{0,0}(X; F, E)$, cf. Theorem 1.69 (iv), we have

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A_0^*) = [L_-] - [L_+]$$

as well as $\operatorname{ind}_{S^*Y} \sigma_\partial(B_1) = [L_-] - [L_+]$ for $B_1 := R_E^{-s+\mu} A_0^* R_F^s \in B^{\mu,e}(X; F, E)$ with some type $e \in \mathbb{N}$. The operator B_1 can be written as $B + G_1$ for a certain $B \in B^{\mu,0}(X; F, E)$ and a Green operator $G_1 \in B_G^{\mu,e}(X; F, E)$, and then

$$\operatorname{ind}_{S^*Y} \sigma_\partial(B) = [L_-] - [L_+],$$

because $\sigma_\partial(G_1)$ is a family of compact operators in the respective Sobolev spaces on $\mathbb{R}_+$.

There are bundles $M_+, M_- \in \operatorname{Vect}(S^*Y)$ such that $M_- \oplus L_- \cong M_+ \oplus L_+ \cong \mathbb{C}^N$, and we obtain

$$\operatorname{ind}_{S^*Y} \sigma_\partial(B) = [M_+] - [M_-].$$

Applying Theorem 2.7 and the proof of Theorem 2.16 we find an elliptic operator $\mathcal{B} \in S^{\mu,0}(X; \boldsymbol{m})$ for $\boldsymbol{m} = (F, E; \mathbb{M}_-, \mathbb{M}_+)$, $\mathbb{M}_\pm = (Q_\pm, \mathbb{C}^N, M_\pm)$, such that $\ker_{S^*Y} \sigma_\partial(\mathcal{B}) \cong M_+$, $\operatorname{coker}_{S^*Y} \sigma_\partial(\mathcal{B}) \cong M_-$. The operator $\mathcal{B}$ is then as desired. $\square$

DEFINITION 2.20. Let $\mathcal{A} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{l})$ for $\boldsymbol{l} = (E, F; \mathbb{L}_-, \mathbb{L}_+)$, $\mathbb{L}_\pm \in \mathcal{P}(Y)$. An operator $\mathcal{P} \in \mathcal{T}^{-\mu,e}(X; \boldsymbol{l}^{-1})$ for $\boldsymbol{l}^{-1} := (F, E; \mathbb{L}_+, \mathbb{L}_-)$ and some $e \in \mathbb{N}$ is called a parametrix of $\mathcal{A}$, if the operators

$$\mathcal{C}_l := \mathcal{I} - \mathcal{P}\mathcal{A} \quad \text{and} \quad \mathcal{C}_r := \mathcal{I} - \mathcal{A}\mathcal{P} \tag{2.33}$$

belong to $\mathcal{S}^{-\infty,d_l}(X; \boldsymbol{m}_l)$ and $\mathcal{S}^{-\infty,d_r}(X; \boldsymbol{m}_r)$, respectively, for $\boldsymbol{m}_l := (E, E; \mathbb{L}_-, \mathbb{L}_-)$ and $\boldsymbol{m}_r := (F, F; \mathbb{L}_+, \mathbb{L}_+)$, and certain $d_l, d_r \in \mathbb{N}$.

THEOREM 2.21. *Let $\mathcal{A} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{l})$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, $\boldsymbol{v} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$ for $E, F \in \operatorname{Vect}(X)$, $\mathbb{L}_\pm \in \mathcal{P}(Y)$.*

(i) *The operator $\mathcal{A}$ is elliptic if and only if*

$$\mathcal{A} : \begin{array}{c} H^s(X, E) \\ \oplus \\ P^s(Y, \mathbb{L}_-) \end{array} \to \begin{array}{c} H^{s-\mu}(X, F) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{L}_+) \end{array} \tag{2.34}$$

is a Fredholm operator for an $s = s_0 \in \mathbb{R}$, $s_0 > \max(\mu, d) - \frac{1}{2}$.

(ii) *If $\mathcal{A}$ is elliptic,* (2.34) *is a Fredholm operator for all $s > \max(\mu, d) - \frac{1}{2}$, and* $\dim\ker\mathcal{A}$ *and* $\dim\operatorname{coker}\mathcal{A}$ *are independent of s.*

(iii) *An elliptic operator $\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{S}^{-\mu,(d-\mu)^+}(X; \boldsymbol{l}^{-1})$ (in the sense of Definition* 2.20*) for $d_l = \max(\mu, d)$, $d_r = (d-\mu)^+$, and $\mathcal{P}$ can be chosen in such a way that the remainders in the relation* (2.33) *are projections*

$$\mathcal{C}_l : H^s(X, E) \oplus P^s(Y, \mathbb{L}_-) \to V, \quad \mathcal{C}_r : H^{s-\mu}(X, F) \oplus P^{s-\mu}(Y, \mathbb{L}_+) \to W$$

for all $s > \max(\mu, d) - \frac{1}{2}$, for $V := \ker\mathcal{A} \subset C^\infty(X, E) \oplus P^\infty(Y, \mathbb{L}_-)$ and a finite-dimensional subspace $W \subset C^\infty(X, F) \oplus P^\infty(Y, \mathbb{L}_+)$ with the property $W + \operatorname{im}\mathcal{A} = H^{s-\mu}(X, F) \oplus P^{s-\mu}(Y, \mathbb{L}_+)$, $W \cap \operatorname{im}\mathcal{A} = \{0\}$ for every $s > \max(\mu, d) - \frac{1}{2}$.

PROOF. Let $\mathcal{A} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{l})$, $\boldsymbol{l} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$, be elliptic. Choose an elliptic operator $\mathcal{B} \in \mathcal{S}^{\mu,0}(X; \boldsymbol{m})$, $\boldsymbol{m} := (F, E; \mathbb{M}_-, \mathbb{M}_+)$ as in Theorem 2.19 such that $\mathcal{A} \oplus \mathcal{B} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ is SL-elliptic. Applying Theorem 1.74 (iii) we find a parametrix $(\mathcal{A} \oplus \mathcal{B})^{(-1)} \in \mathcal{B}^{-\mu,(d-\mu)^+}(X; \boldsymbol{v}^{-1})$, such that the remainders $\mathcal{I} - (\mathcal{A} \oplus \mathcal{B})^{(-1)}(\mathcal{A} \oplus \mathcal{B})$ and $\mathcal{I} - (\mathcal{A} \oplus \mathcal{B})(\mathcal{A} \oplus \mathcal{B})^{(-1)}$ are of type $d_l = \max(\mu, d)$ and $d_r = (d-\mu)^+$, respectively. For the principal symbolic components of $(\mathcal{A} \oplus \mathcal{B})^{(-1)}$ we have

$$\sigma_\psi((\mathcal{A} \oplus \mathcal{B})^{(-1)}) = \sigma_\psi(\mathcal{A})^{-1} \oplus \sigma_\psi(\mathcal{B})^{-1}$$

and

$$\sigma_\partial((\mathcal{A} \oplus \mathcal{B})^{(-1)}) = \sigma_\partial(\mathcal{A})^{-1} \oplus \sigma_\partial(\mathcal{B})^{-1}.$$

In particular,

$$\sigma_\partial\Big((\mathcal{A}\oplus\mathcal{B})^{(-1)}\Big) : \begin{matrix}\pi_Y^* F'\otimes H^{s-\mu}(\mathbb{R}_+)\\ \oplus\\ L_+\\ \oplus\\ \pi_Y^* E'\otimes H^{s-\mu}(\mathbb{R}_+)\\ \oplus\\ M_+\end{matrix} \to \begin{matrix}\pi_Y^* E'\otimes H^{s}(\mathbb{R}_+)\\ \oplus\\ L_-\\ \oplus\\ \pi_Y^* F'\otimes H^{s}(\mathbb{R}_+)\\ \oplus\\ M_-\end{matrix} \tag{2.35}$$

is an isomorphism which induces isomorphisms $\sigma_\partial(\mathcal{A})^{-1}$ and $\sigma_\partial(\mathcal{B})^{-1}$ between the respective bundles separately. In particular, by omitting the third row and column of (2.35) we obtain a morphism

$$\sigma_\partial(\widetilde{\mathcal{P}}_0) : \begin{matrix}\pi_Y^* F'\otimes H^{s-\mu}(\mathbb{R}_+)\\ \oplus\\ \mathbb{C}^N\end{matrix} \to \begin{matrix}\pi_Y^* E'\otimes H^{s}(\mathbb{R}_+)\\ \oplus\\ \mathbb{C}^N\end{matrix} \tag{2.36}$$

which is the boundary symbol of an operator $\widetilde{\mathcal{P}}_0 \in \mathcal{B}^{-\mu,(d-\mu)^+}(X;\boldsymbol{w})$ for $\boldsymbol{w} := (F, E; \mathbb{C}^N, \mathbb{C}^N)$ such that $\sigma_\psi(\widetilde{\mathcal{P}}_0) = \sigma_\psi(\mathcal{A})^{-1}$, and $(\sigma_\partial(\mathcal{P}_0))$ restricts to

$$\sigma_\partial(\mathcal{A})^{-1} : \begin{matrix}\pi_Y^* F'\otimes H^{s-\mu}(\mathbb{R}_+)\\ \oplus\\ L_+\end{matrix} \to \begin{matrix}\pi_Y^* E'\otimes H^{s}(\mathbb{R}_+)\\ \oplus\\ L_-\end{matrix}.$$

Thus, if we set

$$\mathcal{P}_0 := \begin{pmatrix}1 & 0\\ 0 & P_{L_-}\end{pmatrix}\widetilde{\mathcal{P}}_0\begin{pmatrix}1 & 0\\ 0 & R_{L_+}\end{pmatrix}$$

with the projection $P_{L_-} : H^s(Y,\mathbb{C}^N) \to P^s(Y,\mathbb{L}_-)$ and the canonical embedding $R_{L_+} : P^{s-\mu}(Y,\mathbb{L}_+) \to H^s(Y,\mathbb{C}^N)$, we obtain an operator $\mathcal{P}_0 \in \mathcal{S}^{-\mu,(d-\mu)^+}(X;\boldsymbol{l}^{-1})$ such that

$$\sigma_\psi(\mathcal{P}_0) = \sigma_\psi(\mathcal{A})^{-1}, \qquad \sigma_\partial(\mathcal{P}_0) = \sigma_\partial(\mathcal{A})^{-1}.$$

Thus we have $\mathcal{C}_l^0 := \mathcal{I} - \mathcal{P}_0\mathcal{A} \in \mathcal{S}^{-1,d_l}(X;\boldsymbol{m}_l)$, and $\mathcal{C}_r^0 := \mathcal{I} - \mathcal{A}\mathcal{P}_0 \in \mathcal{S}^{-1,d_r}(X;\boldsymbol{m}_r)$, cf. Theorem 2.12 (i), (ii).

Let us form $\mathcal{K} \sim -\sum_{j=1}^\infty(\mathcal{C}_l^0)^j$, cf. Theorem 2.13, and set $\mathcal{P}_l := (\mathcal{I} - \mathcal{K})\mathcal{P}_0$ which belongs to $\mathcal{S}^{-\mu,d_l}(X;\boldsymbol{l}^{-1})$. Then it follows that $\mathcal{I} - \mathcal{P}_l\mathcal{A} \in \mathcal{S}^{-\infty,d_l}(X;\boldsymbol{m}_l)$. In an analogous manner we find an operator $\mathcal{P}_r \in \mathcal{S}^{-\infty,d_r}(X;\boldsymbol{l}^{-1})$ such that $\mathcal{I} - \mathcal{A}\mathcal{P}_r \in \mathcal{S}^{-\infty,d_r}(X;\boldsymbol{m}_r)$. In other words, there is a left parametrix $\mathcal{P}_l$ and a right parametrix $\mathcal{P}_r$ of $\mathcal{A}$, i.e., we may set $\mathcal{P} := \mathcal{P}_l$. From Theorem 2.12 we thus obtain that (2.34) is a Fredholm operator for every $s > \max(\mu, d) - \frac{1}{2}$. The assertions of Theorem 2.21 (ii) and (iii) now follow in a similar manner as the analogous ones of Theorem 1.32 (ii), (iii), again by applying Remark 1.8 to the present situation. Thus, to complete the proof of Theorem 1.32 it remains to show that the Fredholm property of (2.34) for an $s_0 > \max(\mu, d) - \frac{1}{2}$ entails the ellipticity.

If (2.34) is a Fredholm operator for $s = s_0 > \max(\mu, d) - \frac{1}{2}$, also

$$\mathcal{A}_0 := R^{s_0-\mu}_{F,\mathbb{L}_+}\mathcal{A}\Big(R^{s_0}_{E,\mathbb{L}_-}\Big)^{-1} \in \mathcal{S}^{0,0}(X;\boldsymbol{l})$$

is a Fredholm operator

$$\mathcal{A}_0 : \begin{matrix}L^2(X,E)\\ \oplus\\ P^0(Y,\mathbb{L}_-)\end{matrix} \to \begin{matrix}L^2(X,F)\\ \oplus\\ P^0(Y,\mathbb{L}_+)\end{matrix}, \tag{2.37}$$

cf. the notation in Proposition 2.17. If we show the ellipticity of $\mathcal{A}_0$, we also obtain the ellipticity of $\mathcal{A}$ itself, because the order reducing operators are elliptic, and compositions of elliptic operators are again elliptic. To simplify notation we write $\mathcal{A} := \mathcal{A}_0$. We now proceed in a similar manner as in the proof of

Theorem 1.32 and use the same notation as in (1.35), (1.36), (1.37) (with Y in place of M). Then we can form the operator

$$\mathcal{B} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & R_+ & 0 \\ 0 & 0 & R_-^{\perp} \end{pmatrix} \begin{pmatrix} \mathcal{A} & 0 \\ 0 & E \end{pmatrix} : \begin{matrix} L^2(X,E) \\ \oplus \\ L^2(Y,J_-) \end{matrix} \to \begin{matrix} L^2(X,F) \\ \oplus \\ L^2(Y,J_+ \oplus J_-) \end{matrix}$$

The operator $\mathcal{A}$ is Fredholm as a map (2.37). Hence there is a

$$\mathcal{Q} : \begin{matrix} L^2(X,F) \\ \oplus \\ P^0(Y,\mathbb{L}_+) \end{matrix} \to \begin{matrix} L^2(X,E) \\ \oplus \\ P^0(Y,\mathbb{L}_-) \end{matrix}$$

such that

$$\mathcal{I} - \mathcal{Q}\mathcal{A} : \begin{matrix} L^2(X,E) \\ \oplus \\ P^0(Y,\mathbb{L}_-) \end{matrix} \to \begin{matrix} L^2(X,E) \\ \oplus \\ P^0(Y,\mathbb{L}_-) \end{matrix}$$

in compact. Let

$$\mathcal{S} : \begin{matrix} L^2(X,F) \\ \oplus \\ L^2(Y,J_+ \oplus J_-) \end{matrix} \to \begin{matrix} L^2(X,F) \\ \oplus \\ P^0(Y,\mathbb{L}_+) \oplus P^0(Y,\mathbb{L}_-^{\perp}) \end{matrix}$$

denote a projection. Then

$$\mathcal{T} := \begin{pmatrix} \mathcal{Q} & 0 \\ 0 & E \end{pmatrix} \circ \mathcal{S} : \begin{matrix} L^2(X,F) \\ \oplus \\ L^2(Y,J_+ \oplus J_-) \end{matrix} \to \begin{matrix} L^2(X,E) \\ \oplus \\ L^2(Y,J_-) \end{matrix}$$

has the property that $\mathcal{I} - \mathcal{T}\mathcal{B} =: \mathcal{K}$ is compact in the space $L^2(X,E) \oplus L^2(Y,J_-)$. Since $\mathcal{I} - \mathcal{K}$ is a Fredholm operator in the latter space, it follows that $\dim\ker(\mathcal{I} - \mathcal{K}) < \infty$, and hence $\dim\ker\mathcal{B} < \infty$ since

$$\ker\mathcal{B} \subset \ker(\mathcal{I} - \mathcal{K}).$$

The operator $\mathcal{B}^*\mathcal{B} : L^2(X,E) \oplus L^2(Y,J_-) \to L^2(X,E) \oplus L^2(Y,J_-)$ belongs to $\mathcal{B}^{0,0}(X;\boldsymbol{w})$ for $\boldsymbol{w} := (E,E;J_-,J_-)$ and is Fredholm. From Theorem 1.74 (i) we know that $\mathcal{B}^*\mathcal{B}$ is elliptic. It follows that both $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ are injective. By passing to adjoint operators in an analogous manner we can show that $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ are also surjective. This completes the proof of Theorem 2.21. ☐

2.4. Reduction to the boundary. Let

$$\mathcal{A}_i = \begin{pmatrix} D \\ T_i \end{pmatrix} \in \mathcal{S}^{\mu,d}(X;\boldsymbol{l}) \tag{2.38}$$

for $\boldsymbol{l} := (E,F;0,\mathbb{L}_i)$, $\mathbb{L}_i \in \mathcal{P}(Y)$, $i = 1,2$, be two elliptic boundary value problems for the same σ_ψ-elliptic operator $D \in B^{\mu,d}(X;E,F)$ (without loss of generality we assume the type d in $\mathcal{A}_i$ to be independent of i).

We want to reduce the operator $\mathcal{A}_2$ to the boundary by means of $\mathcal{A}_1$. By virtue of Theorem 2.21 (iii) there are parametrices $\mathcal{P}_i \in \mathcal{S}^{-\mu,(d-\mu)^+}(X;\boldsymbol{l}^{-1})$. They have the form of row matrices

$$\mathcal{P}_i =: (G_i \quad K_i), \tag{2.39}$$

where G_i is an analogue of Green's function of the boundary value problem $\mathcal{A}_i$, and K_i is a potential operator. Using

$$\mathcal{A}_1\mathcal{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod \mathcal{S}^{-\infty,(d-\mu)^+}(X;\boldsymbol{m}_r)$$

for $\boldsymbol{m}_r := (F,F;\mathbb{L}_1,\mathbb{L}_1)$, we obtain

$$\mathcal{A}_2\mathcal{P}_1 = \begin{pmatrix} 1 & 0 \\ T_2G_1 & T_2K_1 \end{pmatrix} \mod \mathcal{S}^{-\infty,(d-\mu)^+}(X;\boldsymbol{m}), \tag{2.40}$$

for $\boldsymbol{m} := (F, F; \mathbb{L}_1, \mathbb{L}_2)$, and we have

$$Q := T_2 K_1 \in \mathcal{T}^0(Y; \mathbb{L}_1, \mathbb{L}_2),$$

cf. Definition 1.21. Because both $\mathcal{A}_2$ and $\mathcal{P}_1$ are Fredholm operators, also

$$Q : P^s(Y, \mathbb{L}_1) \to P^s(Y, \mathbb{L}_2)$$

is Fredholm, and $\operatorname{ind} R$ is independent of s, cf. Remark 1.37.

The operator R is called the reduction of $\mathcal{A}_2$ to the boundary by means of $\mathcal{A}_1$. The following result is an analogue of the Agranovich-Dynin index formula.

THEOREM 2.22. *Let $\mathcal{A}_i$, $i = 1, 2$, be two elliptic operators* (2.38)*. Then we have*

$$\operatorname{ind} \mathcal{A}_2 - \operatorname{ind} \mathcal{A}_1 = \operatorname{ind} Q.$$

PROOF. The assertion follows from the relation (2.40), using the fact that smoothing remainders are compact operators and $\operatorname{ind} \mathcal{P}_1 = -\operatorname{ind} \mathcal{A}_1$. $\square$

REMARK 2.23. Let $\mathcal{A}_i$, be two elliptic operators (2.38) for $\mathbb{L}_i = (P_i, J, L)$ with the same bundles J, L but different projections P_i, $i = 1, 2$, and assume $T_i = P_i \widetilde{T}$ for the same trace operator $\widetilde{T}$. Then, if $R_1 : P^s(Y, \mathbb{L}_1) \to H^s(Y, J)$ is the canonical embedding, we have (in the notation of (2.40))

$$T_2 K_1 = P_2 R_1$$

modulo a compact operator and hence, using Proposition 1.17 and notation (1.17)

$$\operatorname{ind} \mathcal{A}_2 - \operatorname{ind} \mathcal{A}_1 = \operatorname{ind}(P_1, P_2).$$

This is a consequence of the fact that when $\widetilde{K}$ denotes the potential operator that appears in the parametrix construction for $\mathcal{A}_1$ as in the proof of Theorem 2.21 we have $\sigma_\partial(\widetilde{T})\sigma_\partial(\widetilde{K}) = \mathrm{id}$ as a map $L \to L$.

REMARK 2.24. The procedure to reduce elliptic elements of $\mathcal{S}^{\mu,d}(X, \boldsymbol{l}_i)$ with the same upper left corners to the boundary can be generalised to arbitrary $\boldsymbol{l} = (E, F; \mathbb{L}_{-,i}, \mathbb{L}_{+,i})$, $i = 1, 2$. The algebraic technique is the same as in [**30**, Section 3.2.1.3]. There is then an immediate analogue of Theorem 2.22 for the general case.

3. Transmission operators and Cauchy data spaces

3.1. Transmission operators. Let M be a closed compact C^∞ manifold which is subdivided into compact C^∞ manifolds X_+ and X_- with common C^∞ boundary Y, i.e.,

$$M = X_+ \cup X_-, \quad Y = X_+ \cap X_-.$$

An example is $M = 2X$, the double of X, where two copies of X are glued together along $Y = \partial X$.

Given an elliptic operator $A \in L^\mu_{\mathrm{cl}}(X; V, W)$, $V, W \in \operatorname{Vect}(M)$, we can consider the restrictions

$$A_\pm := A|_{\operatorname{int} X_\pm} \in L^\mu_{\mathrm{cl}}(\operatorname{int} X_\pm; V_\pm, \mathbb{W}_\pm),$$

$V_\pm := V|_{\operatorname{int} X_\pm}$, $W_\pm := W|_{\operatorname{int} X_\pm}$. We want to study the question to what extent the index of the Fredholm operator

$$A : H^s(X, V) \to H^{s-\mu}(X, W)$$

can be compared with Fredholm indices of elliptic boundary value problems for $A_\pm$ on the $\pm$-sides $X_\pm$ with respect to Y.

At first glance, such a problem appears very natural, for instance, when A is an elliptic differential operator. However, in the pseudo-differential case there is a basic analytic problem: The operators $A_\pm$ have not necessarily the transmission property at the boundary Y (which is, in fact, the exception). Moreover, in general there do not exist Shapiro-Lopatinskij-elliptic boundary conditions for $A_\pm$ (also for differential operators), although there are always elliptic projection conditions, cf. Theorem 2.16, when the transmission property is satisfied.

For convenience, we first consider the case $M = 2X$ and assume that A has the transmission property at Y. Let $\mathrm{r}^\pm$ denote the operator of restriction from M to $\operatorname{int} X_\pm$ and $\mathrm{e}^\pm$ the operator of extension by zero from $\operatorname{int} X_\pm$ to M. Moreover, let $\varepsilon : M$ be the reflection map that maps a point $x_+ \in X_+$ to its counterpart $x_- \in X_-$ and conversely; then Y remains fixed. We use ε as a diffeomorphism $\varepsilon : X_+ \to X_-$ as well as $\varepsilon : X_- \to X_+$ (this should not cause confusion).

We then have

$$\mathrm{r}^+ A\mathrm{e}^+ \in B^{\mu,0}(X_+; V_+, W_+), \mathrm{r}^- A\mathrm{e}^- \in B^{\mu,0}(X_-; V_-, W_-). \tag{3.1}$$

and

$$\mathrm{r}^+ A\mathrm{e}^- \varepsilon^*,\ \varepsilon^* \mathrm{r}^- A\mathrm{e}^+ \in B^{\mu,0}_G(X_+; V_+, W_+), \tag{3.2}$$

$$\mathrm{r}^- A\mathrm{e}^+ \varepsilon^*,\ \varepsilon^* \mathrm{r}^+ A\mathrm{e}^- \in B^{\mu,0}_G(X_-; V_-, W_-). \tag{3.3}$$

Let us first assume that $A_0 \in L^0_{\mathrm{cl}}(M; V, W)$ is an elliptic operator. The ellipticity of A_0 is equivalent to the Fredholm property of the operator

$$A_0 : L^2(M, V) \to L^2(M, W),$$

cf. Theorem 1.7 (i), or, equivalently, of

$$\begin{pmatrix} \mathrm{r}^+ A_0 \mathrm{e}^+ & \mathrm{r}^+ A_0 \mathrm{e}_- \\ \mathrm{r}^- A_0 \mathrm{e}^+ & \mathrm{r}^- A_0 \mathrm{e}_- \end{pmatrix} : \begin{matrix} L^2(X_+, V_+) \\ \oplus \\ L^2(X_-, V_-) \end{matrix} \to \begin{matrix} L^2(X_+, V_+) \\ \oplus \\ L^2(X_-, V_-) \end{matrix}.$$

Writing $X := X_+$, $E_1 := V_+$, $E_2 := \varepsilon^* V_-$, $F_1 := W_+$, $F_2 := \varepsilon^* W_-$ this is equivalent to the Fredholm property of

$$\mathbb{A} := \begin{pmatrix} \mathrm{r}^+ A_0 \mathrm{e}^+ & \mathrm{r}^+ A_0 \mathrm{e}^- \varepsilon^* \\ \varepsilon^* \mathrm{r}^- A_0 \mathrm{e}^+ & \varepsilon^* \mathrm{r}^- A_0 \mathrm{e}^- \varepsilon^* \end{pmatrix} : \begin{matrix} L^2(X, E_1) \\ \oplus \\ L^2(X, E_2) \end{matrix} \to \begin{matrix} L^2(X, F_1) \\ \oplus \\ L^2(X, F_2) \end{matrix}. \tag{3.4}$$

By assumption, A_0 has the transmission property at Y; recall that this condition is symmetric with respect to both sides $X_\pm$. Thus we have $\mathbb{A} \in B^{0,0,}(X; E_1 \oplus E_2, F_1 \oplus F_2)$. Because of the Fredholm property of (3.4) the operator $\mathbb{A}$ is elliptic in $B^{0,0}(X; E_1 \oplus E_2, F_1 \oplus F_2)$, cf. Theorem 1.74 (i). In other words, the symbols $\sigma_\psi(\mathbb{A}) : \pi^*_X(E_1 \oplus E_2) \to \pi^*_X(F_1 \oplus F_2)$ and

$$\sigma_\partial(\mathbb{A}) : \pi^*_Y(E' \oplus E') \otimes L^2(\mathbb{R}_+) \to \pi^*_Y(F' \oplus F') \otimes L^2(\mathbb{R}_+) \tag{3.5}$$

are isomorphisms. Here we use that $E' := E'_1 \cong E'_2$ and $F' := F'_1 \cong F'_2$. Let us write $\mathbb{A} := (\mathbb{A}_{ij})_{i,j=1,2}$. The boundary symbols $\sigma_\partial(\mathbb{A}_{12}) : \pi^*_Y E' \otimes L^2(\mathbb{R}_+) \to \pi^*_Y F' \otimes L^2(\mathbb{R}_+)$ and

$$\sigma_\partial(\mathbb{A}_{21}) : \pi^*_Y E' \otimes L^2(\mathbb{R}_+) \to \pi^*_Y F' \otimes L^2(\mathbb{R}_+)$$

take values in compact operators, because

$$\mathbb{A}_{12} \in B^{0,0}_G(X; E_1, F_2), \quad \mathbb{A}_{21} \in B^{0,0}_G(X; E_2, F_1),$$

cf. the relations (3.2) and (3.3). Since (3.5) is a family of isomorphisms, we therefore have

$$\operatorname{ind}_{S^*Y} \sigma_\partial(\mathbb{A}) = \operatorname{ind}_{S^*Y} \sigma_\partial(\mathbb{A}_{11}) + \operatorname{ind}_{S^*Y} \sigma_\partial(\mathbb{A}_{22}) = 0. \tag{3.6}$$

Let us also consider the boundary symbols $\sigma_{\partial,(\pm)}(\cdot)$ of $\mathrm{r}^\pm A_0 \mathrm{e}^\pm$ with respect to the plus - and the minus - side, i.e., in the sense of $B^{0,0}(X_+; E_1, F_1)$ and $B^{0,0}(X_-; E_2, F_2)$, respectively.

Let ε_* denote the operator push-forward under the reflection diffeomorphism $\varepsilon : M \to M$. We have

$$\mathbb{A}_{11} = \mathrm{r}^+ A_0 \mathrm{e}^+ \quad \text{and} \quad \mathbb{A}_{22} = \mathrm{r}^+(\varepsilon_* A_0)\mathrm{e}^+$$

and

$$\operatorname{ind}_{S^*Y} \sigma_{\partial,(+)}(\mathrm{r}^+(\varepsilon_* A_0)\mathrm{e}^+) = \operatorname{ind}_{S^*Y} \sigma_{\partial,(-)}(\mathrm{r}^- A_0 \mathrm{e}^-).$$

Together with the relation (3.6) we thus obtain the following result.

Proposition 3.1. *Let $A_0 \in L^0_{\mathrm{cl}}(M; V, W)$ be an operator with the transmission property at Y. Then we have*

$$\operatorname{ind}_{S^*Y} \sigma_{\partial,(+)}(\mathrm{r}^+ A_0 \mathrm{e}^+) + \operatorname{ind}_{S^*Y} \sigma_{\partial,(-)}(\mathrm{r}^- A_0 \mathrm{e}^-) = 0 \tag{3.7}$$

((3.7) is interpreted as a relation in $K(S^*Y)$).

Let us now consider an arbitrary elliptic operator $A \in L^{\mu}_{\mathrm{cl}}(M;V,W)_{\mathrm{tr}}$. We use the fact that there is an operator $\widetilde{R}^{\mu}_{V} \in L^{\mu}_{\mathrm{cl}}(M;V,V)_{\mathrm{tr}}$ such that

$$R^{\mu}_{V_+} = \mathrm{r}^+\widetilde{R}^{\mu}_{V}\mathrm{e}^+ : H^{\mu}(X_+,V_+) \to L^2(X_+,V_+)$$

is an isomorphism with the inverse $\mathrm{r}^+\widetilde{R}^{-\mu}_{V}\mathrm{e}^+$, cf. Remark 1.60. Let us set $A_0 := AR^{-\mu}_{V}$. We then have

$$\mathrm{r}^+A_0\mathrm{e}^+ = (\mathrm{r}^+A\mathrm{e}^+)(\mathrm{r}^+R^{-\mu}_{V}\mathrm{e}^+) + G$$

for some $G \in B^{0,0}_{G}(X_+;V_+,W_+)$. Because of $\mathrm{ind}_{S^*Y}\,\sigma_\partial(\mathrm{r}^+R^{-\mu}_{V}\mathrm{e}^+) = 0$, cf. Remark 2.18, and since $\sigma_\partial(G)(y,\eta)$ is compact for every $(y,\eta) \in S^*Y$, it follows that

$$\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(\mathrm{r}^+A_0\mathrm{e}^+) = \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(\mathrm{r}^+A\mathrm{e}^+). \tag{3.8}$$

On the other hand, we have

$$\mathrm{r}^-A_0\mathrm{e}^- = (\mathrm{r}^-A\mathrm{e}^-)(\mathrm{r}^-R^{-\mu}_{V}\mathrm{e}^-)$$

which implies

$$\begin{aligned}\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-A_0\mathrm{e}^-) &= \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-A\mathrm{e}^-)\\ &+ \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-R^{-\mu}_{V}\mathrm{e}^-)\\ &= \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-A\mathrm{e}^-) - \mu[\pi_1^*E'],\end{aligned} \tag{3.9}$$

cf. (2.32). Moreover, the relations (3.8) and (3.9) yield

$$\begin{aligned}&\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(\mathrm{r}^+A_0\mathrm{e}^+) + \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-A_0\mathrm{e}^-)\\ &\quad = \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(\mathrm{r}^+A\mathrm{e}^+) + \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-A\mathrm{e}^-) - \mu[\pi_1^*E'].\end{aligned}$$

Together with Proposition 3.1 we thus proved the following theorem.

THEOREM 3.2. *For every elliptic operator $A \in L^{\mu}_{\mathrm{cl}}(M;V,W)_{\mathrm{tr}}$ we have*

$$\mathrm{ind}_{S^*Y}\,\sigma_{\partial,(+)}(\mathrm{r}^+A\mathrm{e}^+) + \mathrm{ind}_{S^*Y}\,\sigma_{\partial,(-)}(\mathrm{r}^-A\mathrm{e}^-) = \mu[\pi_1^*E']$$

for $E' := V|_Y$.

We want to specify the latter result for the case that $A := D$ is an elliptic differential operator

$$D : H^s(M,V) \to H^{s-\mu}(M,W) \tag{3.10}$$

of order $\mu \in \mathbb{N}$ on M. In this case we know that

$$\sigma_{\partial,(+)}(D) : \pi_Y^*E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^*F' \otimes H^{s-\mu}(\mathbb{R}_+) \tag{3.11}$$

and

$$\sigma_{\partial,(-)}(D) : \pi_Y^*E' \otimes H^s(\mathbb{R}_-) \to \pi_Y^*F' \otimes H^{s-\mu}(\mathbb{R}_+) \tag{3.12}$$

are both surjective, $s-\mu > -\frac{1}{2}$. (Recall, cf. Remark 1.73, that we always have $E' \cong F'$.) The kernels

$$\ker_{S^*Y}\sigma_{\partial,(+)}(D), \quad \ker_{S^*Y}\sigma_{\partial,(-)}(D)$$

are then subbundles of π_1^*J for $J := E' \oplus \ldots \oplus E'$ (μ summands).

PROPOSITION 3.3. *The Cauchy data spaces*

$$\begin{aligned}L_+(y,\eta) &:= \Big\{\Big(D_t^j u|_{t=0}\Big)_{j=0,\ldots,\mu-1} :\\ &\qquad u \in E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+), \quad \sigma_{\partial,(+)}(D)(y,\eta)u = 0\Big\}\end{aligned}$$

for, $D_t := \frac{1}{i}\frac{\partial}{\partial t}$ and

$$\begin{aligned}L_-(y,\eta) &:= \Big\{\Big(D_t^j u|_{t=0}\Big)_{j=0,\ldots,\mu-1} :\\ &\qquad u \in E' \otimes \mathcal{S}(\overline{\mathbb{R}}_-), \quad \sigma_{\partial,(-)}(D)(y,\eta)u = 0\Big\},\end{aligned}$$

$(y,\eta) \in S^*Y$ form complementary subbundles $L_\pm$ of $\oplus_{j=0}^{\mu-1}E'$, i.e. we have

$$\ker_{S^*Y} \sigma_{\partial,(+)}(D) \oplus \ker_{S^*Y} \sigma_{\partial,(-)}(D) = \oplus_{j=1}^{\mu} \pi_1^* E'. \tag{3.13}$$

The proof of this result will be a consequence of Lemma 3.4 below. In the splitting of variables $x = (y,t) \in Y \times (-1,1)$ in a tubular neighbourhood of Y we write the operator D in the form

$$D = \sum_{j=0}^{\mu} a_j(t) D_t^j \tag{3.14}$$

with coefficients $a_j(t) \in C^\infty((-1,1), \mathrm{Diff}^{\mu-j}(Y; E', F'))$, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$. We then have

$$\sigma_{\partial,(\pm)}(D)(y,\eta) = \sum_{j=0}^{\mu} \sigma_\psi(a_j(0))(y,\eta) D_t^j \tag{3.15}$$

on $\mathbb{R}_\pm \ni t$, where $\sigma_\psi(a_j(0))$ is the homogeneous principal symbol of order $\mu - j$ of the operator $a_j(0) \in \mathrm{Diff}^{\mu-j}(Y; E', F')$, $(y,\eta) \in T^*Y \setminus 0$, $j = 0, \ldots, \mu$.

Lemma 3.4. *Let*

$$A := \sum_{k=0}^{\mu} b_k D_t^k,$$

be an $m \times m$ system of operators on $\mathbb{R}$ with constant coefficients. Assume that

(i) $b_\mu \in GL(m, \mathbb{C})$,
(ii) $\sum_{j=0}^{\mu} b_j \tau^j$ *is invertible for all* $\tau \in \mathbb{R}$.

Then for

$$L_\pm := \Big\{ D_t^k u(0))_{k=0,\ldots,\mu-1} \in \mathbb{C}^{m\mu} : u \in \mathcal{S}(\overline{\mathbb{R}}_\pm, \mathbb{C}^m), Au = 0 \Big\}$$

we have $L_+ \oplus L_- = \mathbb{C}^{m\mu}$.

Proof. Without loss of generality we assume $b_\mu = \mathrm{id}_{\mathbb{C}^m}$, otherwise we pass to a new system with coefficients $b_\mu^{-1} b_k$. The equation $Au = 0$ is equivalent to the system

$$D_t u_{\mu-1} + \sum_{k=0}^{\mu-1} b_k u_k = 0,$$

$$D_t u_j - u_{j+1} = 0 \quad \text{for} \quad j = 0, \ldots, \mu - 2,$$

or

$$\Big(D_t - \mathcal{A}\Big) U = 0 \quad \text{for} \quad U := {}^{\mathrm{t}}(u_0, \ldots, u_{\mu-1}),$$

for

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -b_0 & -b_1 & -b_2 & \ldots & -b_{\mu-1} \end{pmatrix}.$$

We then have

$$L_\pm = \Big\{ U|_{t=0} \in \mathbb{C}^{m\mu} : U \in \mathcal{S}(\overline{\mathbb{R}}_\pm, \mathbb{C}^{m\mu}), \Big(D_t - \mathcal{A}\Big) U = 0 \Big\}.$$

The matrix $\tau - \mathcal{A}$ is invertible for all $\tau \in \mathbb{R}$ if and only if condition (ii) is satisfied, i.e., if $\mathrm{spec}(\mathcal{A}) \cap \mathbb{R} = \emptyset$. Let $\lambda \in \mathrm{spec}(\mathcal{A})$ and set $S_\rho(\lambda) := \{z \in \mathbb{C} : |z - \lambda| = \rho\}$. We have $\Big(D_t - \mathcal{A}\Big)U = 0$ and $U|_{t=0} =: U_0$ if and only if $U(t) = e^{it\mathcal{A}} U_0$, and

$$e^{it\mathcal{A}} U_0 = \sum_{\lambda \in \mathrm{spec}(\mathcal{A})} \Big(\frac{1}{2\pi i} \int_{S_\rho(\lambda)} e^{it\zeta} (\zeta - \mathcal{A})^{-1} d\zeta \Big) U_0$$

for every sufficiently small $0<\rho<1$. Let us set $\operatorname{spec}_\pm(\mathcal{A})=\{\lambda\in\operatorname{spec}(\mathcal{A}):\operatorname{Im}\lambda\gtrless 0\}$; then $\operatorname{spec}(\mathcal{A})=\operatorname{spec}_+(\mathcal{A})\cup\operatorname{spec}_-(\mathcal{A})$. Since, by assumption, $\operatorname{spec}(\mathcal{A})\cap\mathbb{R}=\emptyset$, we have

$$\begin{aligned} U(t) &= \sum_{\lambda\in\operatorname{spec}_+(\mathcal{A})}\Big(\frac{1}{2\pi i}\int_{S_\rho(\lambda)} e^{it\zeta}(\zeta-\mathcal{A})^{-1}d\zeta\Big)U_0 \\ &+ \sum_{\lambda\in\operatorname{spec}_-(\mathcal{A})}\Big(\frac{1}{2\pi i}\int_{S_\rho(\lambda)} e^{it\zeta}(\zeta-\mathcal{A})^{-1}d\zeta\Big)U_0 \end{aligned}$$

and

$$U(0)=U_0=\sum_{\lambda\in\operatorname{spec}_+(\mathcal{A})}\Pi_\lambda U_0+\sum_{\lambda\in\operatorname{spec}_-(\mathcal{A})}\Pi_\lambda U_0,$$

where $\Pi_\lambda:=(2\pi i)^{-1}\int_{S_\rho(\lambda)}(\zeta-\mathcal{A})^{-1}d\zeta$ is the projection to the eigenspace of $\mathcal{A}$ to the eigenvalue λ.

We have

$$\Big(D_t-\mathcal{A}\Big)\Big\{\frac{1}{2\pi i}\int_{S_\rho(\lambda)} e^{it\zeta}(\zeta-\mathcal{A})^{-1}d\zeta\Big\}=0$$

for every $\lambda\in\operatorname{spec}(\mathcal{A})$ and

$$\frac{1}{2\pi i}\int_{S_\rho(\lambda)} e^{it\zeta}(\zeta-\mathcal{A})^{-1}d\zeta=e^{i\lambda t}p(t)$$

for a suitable polynominal $p(t)$ (with $m\mu\times m\mu$ matrix-valued coefficients) of order $m(\lambda)-1$ where $m(\lambda)$ is the multiplicity of λ. This gives us

$$L_+=\Big(\sum_{\lambda\in\operatorname{spec}_+(\mathcal{A})}\Pi_\lambda\Big)\mathbb{C}^{m\mu},\quad L_-=\Big(\sum_{\lambda\in\operatorname{spec}_-(\mathcal{A})}\Pi_\lambda\Big)\mathbb{C}^{m\mu}$$

and hence $L_+\oplus L_-=\mathbb{C}^{m\mu}$. □

Remark 3.5. We also have

$$(D_t-\mathcal{A})\left\{\frac{1}{2\pi i}\int_{S_\rho(\lambda)} e^{it\zeta}(\zeta-\mathcal{A})^{-1}\frac{d\zeta}{\zeta}\right\}=0 \tag{3.16}$$

for every $\lambda\in\operatorname{spec}(\mathcal{A})$ and $0<\rho<\operatorname{dist}(\lambda,0)$, i.e.,

$$\Pi_\lambda=(2\pi i)^{-1}\int_{S_\rho(\lambda)}(\zeta-\mathcal{A})^{-1}d\zeta=(2\pi i)^{-1}\mathcal{A}\int_{S_\rho(\lambda)}(\zeta-\mathcal{A})^{-1}\frac{d\zeta}{\zeta}. \tag{3.17}$$

Thus, if we set $\Gamma_\pm:=\{\zeta\in\mathbb{C}:\operatorname{Im}\zeta=\pm\delta\}$ for some sufficiently small $\delta>0$, $\Gamma_+(\Gamma_-)$ oriented with increasing (decreasing) $\operatorname{Re}\zeta$, it follows that

$$\mathcal{P}_\pm:=\sum_{\lambda\in\operatorname{spec}_\pm(\mathcal{A})}\Pi_\lambda=\frac{1}{2\pi i}\mathcal{A}\int_{\Gamma_\pm}(\zeta-\mathcal{A})^{-1}\frac{d\zeta}{\zeta}. \tag{3.18}$$

Moreover, every solution of

$$(D_t-\mathcal{A})U=0,\quad U|_{t=0}=U_0 \tag{3.19}$$

can be written in the form

$$U(t)=\frac{1}{2\pi i}\mathcal{A}\int_{\Gamma_+} e^{it\zeta}(\zeta-\mathcal{A})^{-1}\frac{d\zeta}{\zeta}U_0+\frac{1}{2\pi i}\mathcal{A}\int_{\Gamma_-} e^{it\zeta}(\zeta-\mathcal{A})^{-1}\frac{d\zeta}{\zeta}U_0. \tag{3.20}$$

Now let D be as before an elliptic differential operator of order μ, regarded as a map

$$D: H^\mu(M,V)\to L^2(M,W), \tag{3.21}$$

cf. formula (3.10). Choose arbitrary elliptic elements

$$\mathcal{D}_\pm:=\begin{pmatrix}D_\pm\\ T_\pm\end{pmatrix}\in\mathcal{S}^{\mu,\mu}(X_\pm;\boldsymbol{l}_\pm),$$

for $D_\pm := \mathrm{r}^\pm D\mathrm{e}^\pm$, $\boldsymbol{l}_+ := (V_+, W_+; 0, \mathbb{L}_+)$, $\boldsymbol{l}_- := (V_-, W_-; 0, \mathbb{L}_-)$ for suitable projection data $\mathbb{L}_\pm = (P_\pm, J, L_\pm)$; according to Theorem 2.16 there always exist such operators $T_\pm$. Recall that trace operators $T_\pm$ have the form $T_\pm = P_\pm \widetilde{T}_\pm$ for suitable trace operators $\widetilde{T}_\pm \in \mathcal{B}^{\mu,\mu}(X_\pm; V_\pm, W_\pm; 0, J)$. In view of the ellipticity we have the Fredholm operators (3.21) and

$$\mathcal{D}_\pm : H^\mu(X_\pm, V_\pm) \to \begin{array}{c} L^2(X_\pm, W_\pm) \\ \oplus \\ P^0(Y, \mathbb{L}_\pm) \end{array}. \tag{3.22}$$

We now derive a relation between their indices. To this end we consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^\mu(M,V) & \xrightarrow{i} & \mathcal{M}_+ \oplus \mathcal{M}_- & \xrightarrow{j} & L^2(Y,J) & \longrightarrow & 0 \\ & & \Big\downarrow D & & \mathcal{D}\Big\downarrow\Big\uparrow\mathcal{B} & & \Big\uparrow R & & \\ 0 & \longleftarrow & L^2(M,V) & \xleftarrow{b} & \mathcal{N}_+ \oplus \mathcal{N}_- & \xleftarrow{a} & L^2(Y,J) & \longleftarrow & 0 \end{array} \tag{3.23}$$

for $\mathcal{M}_\pm := H^\mu(X_\pm, V_\pm)$, and $\mathcal{N}_\pm := L^2(X_\pm, W_\pm) \oplus P^0(Y, \mathbb{L}_\pm)$. The maps i and j are defined as follows:

$$i(u) := u|_{X_+} \oplus u|_{X_-}, \quad j(u_+ \oplus u_-) := \oplus_{k=0}^{\mu-1} \Delta_{E'}^{-\mu+k+1/2}(\gamma_+^k u_+ - \gamma_-^k u_-)$$

for $\gamma_\pm^k f := \partial_t^k f|_{Y_\pm}$, with ∂_t^k being the derivative in normal direction to Y and $|_{Y_\pm}$ the restriction to Y from the $\pm$ side. The symbol Δ_E^ν, denotes any element of $L_{\mathrm{cl}}^\nu(Y; E', E')$ that induces isomorphisms $H^s(Y, E') \to H^{s-\nu}(Y, E')$ for all $s \in \mathbb{R}$. The map a is the canonical embedding, where we use

$$H^s(Y, J) = P^s(Y, \mathbb{L}_+) \oplus P^s(Y, \mathbb{L}_-)$$

for $s = 0$. The map b is the canonical projection, using $L^2(M, W) = L^2(X_+, W_+) \oplus L^2(X_-, W_-)$. Moreover, we set

$$\mathcal{D} := \mathcal{D}_+ \oplus \mathcal{D}_- \quad \text{and} \quad \mathcal{B} := \mathcal{B}_+ \oplus \mathcal{B}_-, \tag{3.24}$$

where $\mathcal{B}_\pm \in \mathcal{S}^{-\mu,0}(X_\pm; \boldsymbol{l}_\pm^{-1})$ are parametrices of $\mathcal{D}_\pm$. Finally, we define

$$R := j \circ \mathcal{B} \circ a : L^2(Y, J) \to L^2(Y, J) \tag{3.25}$$

which is an elliptic pseudo-differential operator on Y and, as such, Fredholm. The rows of the diagram (3.23) are exact, and we have

$$D = b \circ \mathcal{D} \circ i.$$

The assumptions of an abstract lemma (see, e.g. Rempel and Schulze [**30**, Section 3.1.1.3]) are satisfied, and we thus obtain the following index formula.

THEOREM 3.6. *With the notation of* (3.25) *we have*

$$\operatorname{ind} D = \operatorname{ind} \mathcal{D}_+ + \operatorname{ind} \mathcal{D}_- + \operatorname{ind} R.$$

3.2. Examples. We now turn to a number of specific observations and examples. Consider the unit circle

$$S^1 := \{w \in \mathbb{C} : |w| = 1\}$$

in the complex plane, and write the space $L^2(S^1)$ (with the scalar product $(u,v)_{L^2(\mathbb{R}_+)} = \int_0^{2\pi} u(\varphi)\overline{v(\varphi)}d\varphi$) as a direct orthogonal sum

$$L^2(S^1) = W^+ \oplus W^-$$

for

$$W^+ := [[w^j : j \in \mathbb{N}\,]], \quad \mathbb{W}^- := [[w^{-j-1} : j \in \mathbb{N}\,]]$$

(where [[...]] denotes closed subspaces spanned by the elements in the brackets.)

THEOREM 3.7. *The orthogonal projections*

$$P_\pm : L^2(S^1) \to W^\pm$$

are elements of $L_{\mathrm{cl}}^0(S^1)$.

PROOF. Let us set

$$V^{+} := \{(F_{t\to\tau}\mathrm{e}^{+}f_{+})(\tau) : f_{+}(t) \in L^{2}(\mathbb{R}_{+})\},$$
$$V^{-} := \{(F_{t\to\tau}\mathrm{e}^{-}f_{-})(\tau) : f_{-}(t) \in L^{2}(\mathbb{R}_{-})\},$$

where $F = F_{t\to\tau}$ is the standard Fourier transform on $\mathbb{R}$. The canonical projections

$$\Pi^{\pm} : L^{2}(\mathbb{R}) \to V^{\pm},$$

can be written in the form

$$\Pi^{\pm} = F\Theta^{\pm}F^{-1}$$

with the characteristic function $\Theta^{\pm}$ of $\mathbb{R}_{\pm}$. If $\chi(t) \in C^{\infty}(\mathbb{R})$ is any excision function in t (i.e., $\chi(t) = 0$ for $|t| < c_0$, $\chi(t) = 1$ for $|t| > c_1$ for certain $0 < c_0 < c_1$) we have

$$\Pi^{+} = F\chi\Theta^{\pm}F^{-1} + F(1-\chi)\Theta^{\pm}F^{-1},$$

where the operators $F(1-\chi)\Theta^{\pm}F^{-1}$ are smoothing. Since $\chi\Theta^{\pm} \in S^{0}_{\mathrm{cl}}(\mathbb{R})$, it follows that

$$\Pi^{\pm} \in L^{0}_{\mathrm{cl}}(\mathbb{R}). \tag{3.26}$$

We now consider the isomorphism

$$T : L^{2}(S^{1}) \to L^{2}(\mathbb{R})$$

by setting $(Tu)(\tau) := 2(1+i\tau)^{-1}u(w(\tau))$ for

$$w(\tau) := \frac{1-i\tau}{1+i\tau}.$$

We then have

$$T(w^{j}) = 2\frac{(1-i\tau)^{j}}{(1+i\tau)^{j+1}}, \qquad T(w^{-j-1}) = 2\frac{(1+i\tau)^{j}}{(1-i\tau)^{j+1}},$$

for $j \in \mathbb{N}$, $TW^{\pm} = V^{\pm}$, and

$$P_{\pm} = T^{-1}\Pi^{\pm}T. \tag{3.27}$$

Using (3.26) we see that the operators $P_{\pm}$ belong to $L^{0}_{\mathrm{cl}}(S^{1})$ because of the invariance of pseudo-differential operators under diffeomorphisms. □

REMARK 3.8. Let $a(\tau) \in S^{0}_{\mathrm{cl}}(\mathbb{R})$ be a symbol with constant coefficients. We then have continuous operators

$$\mathrm{op}^{+}(a) := \mathrm{r}^{+}\,\mathrm{op}(a)\mathrm{e}^{+} : L^{2}(\mathbb{R}_{+}) \to L^{2}(\mathbb{R}_{+}) \tag{3.28}$$

and

$$\mathrm{op}^{-}(a) := \mathrm{r}^{-}\,\mathrm{op}(a)\mathrm{e}^{-} : L^{2}(\mathbb{R}_{-}) \to L^{2}(\mathbb{R}_{-}).$$

The operator (3.28) can equivalently be formulated as

$$\Pi^{\pm}\mathcal{M}_{a} : V^{\pm} \to V^{\pm},$$

where $\mathcal{M}_{a}$ is the operator of multiplication by the function a. In particular, together with the following Remark 3.9 we see that boundary symbols (1.54) can be reformulated as ((y,η)-dependent families of) Toeplitz operators on the circle.

REMARK 3.9. An operator of the form

$$A := P_{+}\mathcal{M}_{\widetilde{a}}P_{+} : W^{+} \to W^{+} \tag{3.29}$$

for an $\widetilde{a}(w) \in C^{\infty}(S^{1})$ is a Toeplitz operator (in classical notation). $\mathcal{M}_{\widetilde{a}}$ is the operator of multiplication by $\widetilde{a}$ in $L^{2}(S^{1})$. We have (in the terminology of Definition 1.21) a canonical identification of (3.29) with

$$P\widetilde{A}R \in T^{0}(S^{1};\mathbb{L},\mathbb{L}). \tag{3.30}$$

Here, $\widetilde{A} := \mathcal{M}_{\widetilde{a}}$, $P := P_{+}$, $R : W^{+} \to L^{2}(S^{1})$ is the canonical embedding, and $\mathbb{L} = (P,\mathbb{C},L) \in \mathcal{P}(S^{1})$ for a bundle $L \in \mathrm{Vect}(T^{*}S^{1}\setminus 0)$ that is isomorphic to $(S^{1}\times\mathbb{R}_{+})\times\mathbb{C}$, where $S^{1}\times\mathbb{R}_{+}$ is the plus-component of $T^{*}S^{1}\setminus 0 \cong (S^{1}\times\mathbb{R}_{+})\cup(S^{1}\times\mathbb{R}_{-})$, and $P^{0}(S^{1},\mathbb{L}) = W^{+}$. Ellipticity of (3.29) in the sense of Definition 1.30 (i) is equivalent to $\widetilde{a}(w) \neq 0$ for all $w \in S^{1}$.

Clearly, if we admit arbitrary operators $\widetilde{A} \in L^{\mu}_{\mathrm{cl}}(S^1)$, we obtain much more general operators than (3.30) (also for $\mu = 0$).

3.3. Spectral boundary value problems. We now consider general (non-homogeneous) elliptic boundary value problems for differential operators as a special case of our pseudo-differential calculus, where we have more explicit information (the material of this section is based on the author's joint papers [**45**], [**29**] with Nazaikinskij, Sternin, and Shatalov.) They are a natural generalisation of (homogeneous) boundary value problems, studied (in an L^2 set-up) by Atiyah, Patoti, and Singer [**4**].

Let X be a compact C^∞ manifold with boundary Y, $n = \dim X$, and let A be an elliptic differential operator on X of order μ with smooth coefficients up to the boundary,

$$A : C^\infty(X, E) \to C^\infty(X, F) \tag{3.31}$$

for $E, F \in \mathrm{Vect}(X)$. In a collar neighbourhood of Y in the splitting of variables $x = (y, t) \in Y \times [0, 1)$ the operator A can be written in the form

$$A = \sum_{j=0}^{\mu} A_j(t) D_t^j, \tag{3.32}$$

$D_t := \frac{1}{i}\frac{\partial}{\partial t}$, with coefficients $A_j \in C^\infty([0,1), \mathrm{Diff}^{\mu-j}(Y))$. The ellipticity of A implies that

$$A_\mu(0) : E' \to F' \tag{3.33}$$

(for $E' := E|_Y,\ F' := F|_Y$) is an isomorphism. The boundary symbol

$$\sigma_\partial(A)(y,\eta) = \sum_{j=0}^{\mu} \sigma_\psi(A_j(0))(y,\eta) D_t^j : H^s(\mathbb{R}_+, E') \to H^{s-\mu}(\mathbb{R}_+, F') \tag{3.34}$$

(with $\sigma_\psi(A_j(0)) : \pi_Y^* E' \to \pi_Y^* F'$ being the homogeneous principal symbol of order $\mu - j$ of the operator $A_j(0) \in \mathrm{Diff}^{\mu-j}(Y; E', F')$) is a surjective family of Fredholm operators, parametrised by $(y, \eta) \in T^*Y \setminus 0$. Thus, as in the general calculus of boundary value problems, there is a kernel bundle

$$L_+ := \ker \sigma_\partial(A) \in \mathrm{Vect}(T^*Y \setminus 0). \tag{3.35}$$

Let us consider the family of differential operators on Y

$$\sigma_c(A)(w) := \sum_{j=0}^{\mu} A_j(0) w^j, \tag{3.36}$$

parametrised by the complex variable $w \in \mathbb{C}$.

Remark 3.10. We have

$$\sigma_c(A)(\beta + i\gamma) \in L^{\mu}_{\mathrm{cl}}(Y; E', F'; \mathbb{R}_\beta)$$

for every $\gamma \in \mathbb{R}$, and $\sigma_c(A)(\beta + i\gamma)$ is parameter-dependent elliptic with parameter $\beta \in \mathbb{R}$, cf. Agranovich and Vishik [**1**]. Moreover, there exists a countable set $D \subset \mathbb{C}$ such that

$$D \cap \{w :\ c < \mathrm{Im}\, w < c'\}$$

is finite for every $c \le c'$ and

$$\sigma_c(A)(w) : H^s(Y, E') \to H^{s-\mu}(Y, F') \tag{3.37}$$

is an isomorphism for every $w \in \mathbb{C} \setminus D$ and $s \in \mathbb{R}$.

The bijectivity of (3.37) for large $|\mathrm{Re}\, w|$ is a well-known phenomenon of parameter-dependent elliptic operators, holomorphically dependent on w. First, (3.37) is a holomorphic family of Fredholm operators (kernels and cokernels are independent of s), and for large $|\mathrm{Re}\, w|$ the operators are isomorphisms, cf. Theorem 1.11 (i). Then there is a countable set $D \subset \mathbb{C}$ of non-bijectivity points with the asserted properties, cf. also [**34**, Section 2.2.5].

Let us set

$$B := \sum_{j=0}^{\mu} A_j(0) D_t^j, \tag{3.38}$$

regarded as a differential operator on the infinite cylinder $Y \times \mathbb{R}$. The coefficient $A_j(0) : E' \to F'$ is an isomorphism. For convenience, we set $J := E' = F'$ and assume $A_\mu(0) = 1$ (otherwise we compose (3.38) from the left by $A_\mu^{-1}(0)$). Let us write B in the form

$$B = D_t^\mu + \sum_{j=0}^{\mu-1} b_j D_t^j \tag{3.39}$$

for $b_j := A_j(0) \in \mathrm{Diff}^{\mu-j}(Y)$. The pull-back of the bundle J to $Y \times \mathbb{R}$ will be denoted again by J.

Let $H^s(Y \times \mathbb{R}, J)$ denote the (cylindrical) Sobolev space on $Y \times \mathbb{R}$, of smoothness $s \in \mathbb{R}$, defined as the completion of $C_0^\infty(Y \times \mathbb{R}, J)$ with respect to the norm

$$\Big\{ \int \|R^s(\tau)\widehat{u}(\tau)\|^2_{L^2(Y,J)} d\tau \Big\}^{1/2},$$

where $R^s(\tau) \in L^s_{\mathrm{cl}}(Y; J, J; \mathbb{R}_\tau)$ is any classical parameter-dependent elliptic pseudo-differential operator of order s on Y, with parameter $\tau \in \mathbb{R}$. Moreover, set

$$H^{s,\gamma}(Y \times \mathbb{R}, J) := \{ e^{\gamma t} u(y,t) : u \in H^s(Y \times \mathbb{R}, J) \}$$

for every $\gamma \in \mathbb{R}$. The operator B then defines continuous maps

$$B : H^{s,\gamma}(Y \times \mathbb{R}, J) \to H^{s-\mu,\gamma}(Y \times \mathbb{R}, J) \tag{3.40}$$

for all $s, \gamma \in \mathbb{R}$.

Let us reformulate the equation $Bu = f$ as

$$D_t U - \mathcal{A} U = F \tag{3.41}$$

for $U := {}^{\mathrm{t}}(u_0, \ldots, u_{\mu-1})$, $F := {}^{\mathrm{t}}(0, \ldots, 0, f)$ and $u_j := D_t^j u$, $j = 0, \ldots, \mu-1$, and

$$\mathcal{A} := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -b_0 & -b_1 & -b_2 & \ldots & -b_{\mu-1} \end{pmatrix}. \tag{3.42}$$

Note that $\det(w - \mathcal{A}) = w^\mu + \sum_{j=0}^{\mu-1} b_j w^j$.

Lemma 3.11. *The operator*

$$w - \mathcal{A} : \bigoplus_{j=0}^{\mu-1} H^{s-j}(Y, J) \to \bigoplus_{k=1}^{\mu} H^{s-k}(Y, J) \tag{3.43}$$

is invertible for a $w \in \mathbb{C}$ if and only if so is

$$w^\mu + \sum_{j=0}^{\mu-1} b_j w^j : H^s(Y, J) \to H^{s-\mu}(Y, J) \tag{3.44}$$

for any $s \in \mathbb{R}$. In other words, we have $\mathrm{spec}\,\mathcal{A} = \{ w \in \mathbb{C} : \sigma_c(A)(w)$ *is bijective*$\}$. *Moreover, we have*

$$(w - \mathcal{A})^{-1} = \sigma_c(A)^{-1}(w) \mathcal{Q}(w) \tag{3.45}$$

for a matrix $\mathcal{Q}(w) = (Q_{jk}(w))_{\substack{k=1,\ldots,\mu \\ j=0,\ldots,\mu-1}}$, of differential operators polynomially dependent on w, where $Q_{jk}(w)$ is parameter-dependent with parameter β (for $w = \beta + i\gamma$) of order $\mu - k + j$ for $k = 1, \ldots, \mu$, $j = 0, \ldots, \mu - 1$.

Proof. Formula (3.45) is elementary, and the proof gives the asserted characterisation of the entries of the matrix Q.

As a consequence of (3.45) we see that the operator (3.43) is invertible if so is (3.44). Conversely, the inverbility of (3.43) entails that of (3.44) because $\sigma_c(A)^{-1}(w)f$ is equal to the first component of $(w-\mathcal{B})^{-1\,\mathrm{t}}(0,\dots,0,f)$. □

Remark 3.12. The set $\mathrm{spec}(\mathcal{A})$ is countable, and every strip $\{w\in\mathbb{C}: c<\operatorname{Im} w<c'\}$ only contains finitely many elements of $\mathrm{spec}(\mathcal{A})$ for arbitrary reals $c<c'$. The operator function $(w-\mathcal{A})^{-1}$ is meromorphic with poles of finite multiplicity at the points of $\mathrm{spec}(\mathcal{A})$, and the Laurent coefficients of $(w-\mathcal{A})^{-1}$ at $(w-p)^{-(k+1)},\ k\in\mathbb{N}$, are operators in $L^{-\infty}(Y;J,J)$ of finite rank.

In fact, $\sigma_c(A)(w)$ is parameter-dependent elliptic with parameter $\operatorname{Re} w$ on every line $\operatorname{Im} w=\text{const}$, and invertible for large $|\operatorname{Re} w|$. Then the asserted properties follow from the relation (3.45) together with classical results of Agranovich and Vishik [**1**], cf. also [**38**, Section 1.2.4].

Similarly to the considerations in Section 3.1 we want to formulate a relation between the spectral points of the operator $\mathcal{A}$ (which is the same as the set of non-bijectivity points of (3.37)) and Cauchy data spaces of Sobolev distributions on X at Y. To this end we interpret (3.39) as a continuous operator

$$B: H^s(Y\times\mathbb{R}_+,J)\to H^{s-\mu}(Y\times\mathbb{R}_+,J),$$

where $H^s(Y\times\mathbb{R}_+,J):=H^s(Y\times\mathbb{R},J)|_{\mathbb{R}_+\times Y}$, and $H^s(Y\times\mathbb{R},J)$ is the cylindrical Sobolev space on $Y\times\mathbb{R}$. The latter space is defined as the completion of $C_0^\infty(Y\times\mathbb{R},J)$ with respect to the norm

$$\left\{\int \|R^s(\tau)\hat{u}(\tau)\|^2_{L^2(Y,J)}d\tau\right\}^{\frac{1}{2}}$$

for any choice of an order reducing family $R^s(\tau)\in L^s_{\mathrm{cl}}(Y;J,J;\mathbb{R})$ in the sense of Theorem 1.12. An equivalent definition for $s\in\mathbb{N}$ is

$$\begin{aligned}H^s(Y\times\mathbb{R},J)=\{&u(y,t)\in L^2(Y\times\mathbb{R},J):\\ &D_y^\alpha D_t^k u(y,t)\in L^2(Y\times\mathbb{R},J)\quad\text{for all}\quad |\alpha|+k\le s\}.\end{aligned}$$

Here D_y^α runs over the set of all differential operators on Y of order $|\alpha|$, acting between sections of J.

For convenience, we assume that the set D of Remark 3.10 does not intersect the real line $\operatorname{Im} w=0$. Otherwise, we can pass to a translated operator $(D_t-i\gamma)^\mu+\sum_{j=0}^{\mu-1}b_j(D_t-i\gamma)^j$ for a suitable real γ with a corresponding shifted set D_γ in the complex plane which does not intersect the real line.

For any integer $s\ge\mu$ we form the Cauchy data space

$$\mathcal{C}^{s,\mu}(Y,J):=\{(D_t^k u(y,0))_{k=0,\dots,\mu-1}: u\in H^s(Y\times\mathbb{R}_+)\}.$$

We then have

$$\mathcal{C}^{s,\mu}(Y,J)=\bigoplus_{k=0}^{\mu-1}H^{s-k-\frac{1}{2}}(Y,J), \tag{3.46}$$

and the operator $\mathcal{T}^\mu:={}^{\mathrm{t}}(\mathrm{r}',\mathrm{r}'D_t,\dots,\mathrm{r}'D_t^{\mu-1})$ with $\mathrm{r}'u:=u|_{t=0}$ defines a continuous map

$$\mathcal{T}^\mu: H^s(Y\times\mathbb{R}_+,J)\to\mathcal{C}^{s,\mu}(Y,J). \tag{3.47}$$

We want to formulate results on the solvability of the boundary value problem

$$Bu_+=f_+\in H^{s-\mu}(Y\times\mathbb{R}_+,J), \tag{3.48}$$

$$\mathcal{T}^\mu u_+=g_+\in\mathcal{C}^{s,\mu}(Y,J). \tag{3.49}$$

It turns out that, in general, not the whole space $\mathcal{C}^{s,\mu}(Y,J)$ of boundary data g_+ on the right hand side of (3.49) is induced by solutions $u_+\in H^s(Y\times\mathbb{R}_+,J)$, but a subspace which is the image under a suitable pseudo-differential projection $\mathcal{P}_+$.

Similarly to (3.48), (3.49) we can also consider a boundary value problem on the negative half-cylinder

$$Bu_-=f_-\in H^{s-\mu}(Y\times\mathbb{R}_-,J), \tag{3.50}$$

$$\mathcal{T}^{\mu} u_{-} = g_{-} \in \mathcal{C}^{s,\mu}(Y,J). \tag{3.51}$$

The admissible boundary data g_{-} are then determined by the corresponding complementary projection $\mathcal{P}_{-}$.

The projections $\mathcal{P}_{\pm}$ are obtained as follows. First recall that the spectrum $\operatorname{spec}(\mathcal{A}) = D$ (in the notation of Remark 3.10) does not intersect the real line $\operatorname{Im} w = 0$. Therefore, we have

$$\operatorname{spec}(\mathcal{A}) \cap \{w : -c \leq \operatorname{Im} w \leq c\} = \emptyset \tag{3.52}$$

for some $c > 0$. Let us fix some $0 < \varepsilon < c$, set

$$\Gamma_{\pm} := \{w = \tau \pm i(c-\varepsilon) : \tau \in \mathbb{R}\},$$

oriented in direction of increasing τ on Γ_{+}, and descreasing τ on Γ_{-}, and form

$$\mathcal{P}_{\pm} = \frac{1}{2\pi i}\, \mathcal{A} \int_{\Gamma_{\pm}} (w-\mathcal{A})^{-1} \frac{dw}{w}. \tag{3.53}$$

Note that the operator $\mathcal{A}$ can be written in the form

$$\mathcal{A} = \mathcal{R}\mathcal{A}_1\mathcal{R}^{-1} \tag{3.54}$$

for

$$\mathcal{R} = \operatorname{diag}(\mathcal{R}_J^0, \mathcal{R}_J^1, \ldots, \mathcal{R}_J^{(\mu-1)}) \tag{3.55}$$

where $\mathcal{R}_J^k : H^s(Y,J) \to H^{s-k}(Y,J)$ is an order reducing operator of order $-k$ in the sense of Theorem 1.12, and $\mathcal{A}_1$ is a system of operators of order 1. Then $\mathcal{P}_{\pm}$ takes the form

$$\mathcal{P}_{\pm} = \mathcal{R}\left\{\frac{1}{2\pi i}\mathcal{A}_1 \int_{\Gamma_{\pm}} (w-\mathcal{A}_1)^{-1}\frac{dw}{w}\right\}\mathcal{R}^{-1}. \tag{3.56}$$

Lemma 3.13. (i) *The integral* (3.53) *stronly converges in* $\mathcal{C}^{s,\mu}(Y,J)$ *on the dense subset* $\mathcal{C}^{s+1,\mu}(Y,J)$ *for every* $s \in \mathbb{R}$.

(ii) *The operators* $\mathcal{P}_{\pm}$ *form a matrix* $(\mathcal{P}_{\pm,jk})_{0\leq j\leq\mu-1,0\leq k\leq\mu-1}$ *of elements of* $L_{\mathrm{cl}}^{j-k}(Y;J,J)$. *Thus* $\mathcal{P}_{\pm}$ *extend to continuous operators*

$$\mathcal{P}_{\pm} : \mathcal{C}^{s,\mu}(Y,J) \to \mathcal{C}^{s,\mu}(Y,J)$$

for all $s \in \mathbb{R}$.

Proof. (i) From Lemma 3.11 we know that the entries of the matrix $(w-\mathcal{A})^{-1}$ are classical parameter-dependent pseudo-differential operators of order ≤ -1, with parameter $\tau \in \mathbb{R}$ (when we identify $\mathbb{R}$ with Γ_{+} or Γ_{-} via $\tau \to w = \tau \pm i(c-\varepsilon)$). In fact, $\sigma_c(A)^{-1}(w)$ is parameter-dependent of order $-\mu$, and the orders of the entries of $Q(w)$ are $\leq \mu - 1$, cf. formula (3.45). By virtue of Theorem 1.9 the $\mathcal{L}(H^{s+1}(Y,J), H^{s+1}(Y,J))$ - norm of every entry of $(w-\mathcal{A})^{-1}$ can be estimated by $c_s(1+|w|)^{-1}$, $w \in \Gamma_{\pm}$, for a constant $c_s > 0$.

This gives us immediately assertion (i).

(ii) Let us write $\mathcal{P}_{\pm}$ in the form (3.56). Then it suffices to observe that

$$\frac{1}{2\pi i}\int_{\Gamma_{\pm}} (w-\mathcal{A}_1)^{-1}\frac{dw}{w}$$

is a matrix of classical pseudo-differential operators of order -1. The technique of the proof is similar to [**47**]. □

Let us write the operator (3.42) in the form $\mathcal{A} = (\mathcal{A}_{kj})_{1\leq k\leq\mu,0\leq j\leq\mu-1}$,

$$\mathcal{A} : \bigoplus_{j=0}^{\mu-1} H^{s-j}(Y,J) \to \bigoplus_{k=1}^{\mu} H^{s-k}(Y,J),$$

and interpret the orders $\operatorname{ord}\mathcal{A}_{kj} = k - j$ in the Douglis-Nirenberg sense, with homogeneous principal symbols $\sigma_\psi(\mathcal{A}_{kj})(y,\eta)$ of order $k-j$. Setting

$$\sigma_\psi(\mathcal{A})(y,\eta) = (\sigma_\psi(\mathcal{A}_{kj})(y,\eta))_{1\leq k\leq\mu,0\leq j\leq\mu-1}$$

we then have

$$\sigma_\psi(\mathcal{A})(y,\eta) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\sigma_\psi(b_0)(y,\eta) & -\sigma_\psi(b_1)(y,\eta) & -\sigma_\psi(b_2)(y,\eta) & \dots & -\sigma_\psi(b_{\mu-1})(y,\eta) \end{pmatrix}.$$

Moreover, set

$$L_\pm(y,\eta) := \Big\{u(0) : u(t) \in \mathcal{S}(\overline{\mathbb{R}}_\pm, E_y), \Big(D_t - \sigma_\psi(\mathcal{A})(y,\eta)\Big)u = 0\Big\},$$

$E = \oplus_{k=0}^{\mu-1} J$.

THEOREM 3.14. (i) *The operators $\mathcal{P}_\pm$ are complementary projections, i.e., $\mathcal{P}_\pm^2 = \mathcal{P}_\pm$, $\mathcal{P}_+ + \mathcal{P}_- = 1$, and have the property*

$$\mathcal{P}_\pm \mathcal{A} = \mathcal{A}\mathcal{P}_\pm. \tag{3.57}$$

(ii) *The homogeneous principal symbols $\sigma_\psi(\mathcal{P}_{\pm,jk})$ of order $j-k$ of $\mathcal{P}_{\pm,jk} \in L_{\mathrm{cl}}^{j-k}(Y;E,E)$ (cf. Lemma* 3.13 (ii)) *form projections*

$$\Big(\sigma_\psi(\mathcal{P}_{\pm,jk})(y,\eta)\Big)_{0\le j\le \mu-1, 0\le k\le \mu-1} =: \sigma_\psi(\mathcal{P}_\pm)(y,\eta) : E_y \to L_\pm(y,\eta)$$

along $L_\mp(y,\eta)$.

(iii) *The operator functions $(w-\mathcal{A})^{-1}\mathcal{P}_\pm$ are holomorphic in* $\mathrm{spec}(\mathcal{A}) \cap \{w : \mathrm{Im}\, w \gtrless 0\}$.

PROOF. (i) First note that the relation (3.57) is evident. Moreover, to verify that $\mathcal{P}_+ + \mathcal{P}_- = 1$, it suffices to observe that

$$\mathcal{P}_+ + \mathcal{P}_- = \mathcal{A}\frac{1}{2\pi i}\int_{\Gamma_0} (w-\mathcal{A})^{-1}\frac{dw}{w}$$

for a small circle Γ_0 clockwise surrounding the origin; by Cauchy's residue theorem the integral is equal to $2\pi i \mathcal{A}^{-1}$.

Let us now calculate $\mathcal{P}_+^2$ (the consideration for $\mathcal{P}_-$ is analogous and left to the reader). Set $\Gamma_+' := \Gamma_+ + i\varepsilon$ for some sufficiently small $\varepsilon > 0$. Then, using the relation (3.57) and the resolvent identity we have

$$\begin{aligned} \mathcal{P}_+^2 &= -\left(\frac{\mathcal{A}}{2\pi}\right)^2 \int_{\Gamma_+'} \left\{\int_{\Gamma_+} \frac{1}{w'w}(w'-\mathcal{A})^{-1}(w-\mathcal{A})^{-1} dw\right\} dw' \\ &= -\left(\frac{\mathcal{A}}{2\pi}\right)^2 \int_{\Gamma_+'} \left\{\int_{\Gamma_+} \frac{1}{w'w(w-w')}\left[(w'-\mathcal{A})^{-1} - (w-\mathcal{A})^{-1}\right] dw\right\} dw' \\ &= I_0 + I_1 \end{aligned}$$

for

$$\begin{aligned} I_0 &= -\left(\frac{\mathcal{A}}{2\pi}\right)^2 \int_{\Gamma_+'} \left\{\int_{\Gamma_+} (w-\mathcal{A})^{-1}\frac{dw}{w(w-w')}\right\}\frac{dw'}{w'} \\ &= -\left(\frac{\mathcal{A}}{2\pi}\right)^2 \int_{\Gamma_+} (w-\mathcal{A})^{-1}\left\{\int_{\Gamma_+'} \frac{dw'}{w'(w-w')}\right\}\frac{dw}{w} \end{aligned}$$

which vanishes, since the inner integral is zero, and

$$
\begin{aligned}
I_1 &= -\left(\frac{\mathcal{A}}{2\pi}\right)^2 \int_{\Gamma'_+} (w'-\mathcal{A})^{-1}\left\{\int_{\Gamma_+} \frac{dw}{w(w-w')}\right\}\frac{dw'}{w'} \\
&= -\left(\frac{\mathcal{A}}{2\pi}\right)^2 2\pi i \int_{\Gamma'_+} (w'-\mathcal{A})^{-1}\frac{dw'}{(w')^2} \\
&= \frac{\mathcal{A}}{2\pi i}\left\{\int_{\Gamma_+} (w'-\mathcal{A})^{-1}\frac{dw'}{w'} + \int_{\Gamma_+} \frac{dw'}{(w')^2}\right\} = \mathcal{P}_+ .
\end{aligned}
$$

In the first line of the latter relation we inserted $\int_{\Gamma_+}\frac{dw}{w(w-w')} = 2\pi i (w')^{-1}$, and in the second line $\mathcal{A} = \mathcal{A} - w' + w'$.

(ii) is simple after the considerations in the proof of Lemma 3.4.

(iii) Let $\operatorname{Im} w > 0$. Then we have

$$
\begin{aligned}
(w-\mathcal{A})^{-1}\mathcal{P}_+ &= \frac{\mathcal{A}}{2\pi i}\int_{\Gamma_+} (w'-\mathcal{A})^{-1}(w-\mathcal{A})^{-1}\frac{dw'}{w'} \\
&= \frac{\mathcal{A}}{2\pi i}\int_{\Gamma_+} (w'-\mathcal{A})^{-1}\frac{dw'}{w'(w-w')} - \frac{\mathcal{A}(w-\mathcal{A})^{-1}}{2\pi i}\int_{\Gamma_+}\frac{dw'}{w'(w-w')}.
\end{aligned}
$$

The second integral on the right hand side vanishes, and the first one is holomorphic for $\operatorname{Im} w > 0$. □

In the following we employ the Fourier-Laplace transform

$$(Fu)(\zeta) = \widetilde{u}(\zeta) = \int_0^\infty e^{-it\zeta}u(t)dt,$$

$\operatorname{Im}\zeta < 0$, with the inverse

$$(F^{-1}\widetilde{u})(t) = \frac{1}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta} e^{ity}\widetilde{u}(y)dy, \quad t>0$$

for $\delta < 0, y = \operatorname{Re}\zeta$.

Let us define the maps

$$E : \mathbb{C} \to \mathbb{C}^\mu, \quad Ez := (0,\dots,0,z)$$

and

$$Q : \mathbb{C}^\mu \to \mathbb{C}, \quad Q(z_0,\dots,z_{\mu-1}) := z_0 .$$

THEOREM 3.15. *The boundary value problem*

$$Bu = f \in H^{s-\mu}(Y\times\mathbb{R}_+, J) \tag{3.58}$$

$$\mathcal{P}_+\mathcal{T}^\mu u = g \in \mathcal{P}_+\mathcal{C}^{s,\mu}(Y,J) \tag{3.59}$$

has a unique solution $u \in H^s(Y\times\mathbb{R}_+, J)$, *given by*

$$u = QF^{-1}\left\{(w-\mathcal{A})^{-1}\left[E(\widetilde{f}(w)) - ig - \frac{1}{2\pi}\int_{\Gamma_-}(z-\mathcal{A})^{-1}E(\widetilde{f}(z))dz\right]\right\}. \tag{3.60}$$

The corresponding map $\mathcal{R} : (f,g) \to u$ *is continuous as an operator*

$$\mathcal{R} : H^{s-\mu}(Y\times\mathbb{R}_+, J) \oplus \mathcal{P}_+\mathcal{C}^{s,\mu}(Y,J) \to H^s(Y\times\mathbb{R}_+, J). \tag{3.61}$$

PROOF. We show that $\mathcal{R}$ is a right inverse of the operator ${}^{\mathrm{t}}(B, P_+\mathcal{T}^\mu)$; the consideration for the left inverse is left to the reader. Concerning the continuity (3.61) we refer to [**29**].

It is convenient to pass from B to the operator $\mathcal{D} := D_t - \mathcal{A}$ with $\mathcal{A}$ being given by (3.42) and the vector function U as in (3.41). Then we can omit the mappings Q and E which only single out the first components of U and replace $\mathcal{T}^\mu$ by r', the restriction to $t=0$.

Let us set

$$\mathcal{R} = (\mathcal{S} \quad \mathcal{K})$$

for

$$\begin{aligned}\mathcal{S}f &:= F^{-1}(w-\mathcal{A})^{-1}\left\{\widetilde{f}(w)-\frac{1}{2\pi}\int_{\Gamma_-}(z-\mathcal{A})^{-1}\widetilde{f}(z)dz\right\}\\ \mathcal{K}g &:= -i\lim_{t\to+0}F^{-1}(w-\mathcal{A})g.\end{aligned}$$

We then show the relations

$$\mathcal{D}\mathcal{S}f=f,\quad \mathcal{D}\mathcal{K}g=0, \tag{3.62}$$

$$\mathcal{P}_+\lim_{t\to+0}\mathcal{S}f=0,\quad \mathcal{P}_+\mathcal{K}g=g \tag{3.63}$$

for f and g belonging to the respective spaces.

We have

$$\begin{aligned}\mathcal{D}\mathcal{S}f &= \{F^{-1}(w-\mathcal{A})F\}\{F^{-1}(w-\mathcal{A})\widetilde{f}(w)\}\\ &- \frac{1}{2\pi}\mathcal{D}F^{-1}(w-\mathcal{A})^{-1}\int_{\Gamma_-}(z-\mathcal{A})^{-1}\widetilde{f}(z)dz.\end{aligned} \tag{3.64}$$

The first summand on the right hand side of (3.64) is equal to f. So it remains to observe that

$$\mathcal{D}F^{-1}(w-\mathcal{A})^{-1}h=0,$$

$h=-\frac{1}{2\pi}\int_{\Gamma_-}(z-\mathcal{A})^{-1}\widetilde{f}(z)dz$ (the latter vector is independent of w). For similar reasons it follows that $\mathcal{D}\mathcal{K}g=0$.

For $P_+\mathcal{K}$ we have

$$\begin{aligned}\mathcal{P}_+\mathcal{K}g &= -i\lim_{t\to+0}\mathcal{P}_+F^{-1}(w-\mathcal{A})^{-1}g\\ &= \lim_{t\to+0}\frac{1}{2\pi i}\int_{\Gamma_-}e^{iwt}\mathcal{P}_+(w-\mathcal{A})^{-1}g\,dw.\end{aligned} \tag{3.65}$$

For $t>0$ the exponent in e^{iwt} has a negative real part in the upper complex w half-plane. Therefore, we can deform the contour of integration to a curve 'surrounding' $\mathrm{spec}_+(\mathcal{A}):=\{\lambda\in\mathrm{spec}(\mathcal{A}):\mathrm{Im}\,\lambda>0\}$ (which is a countable set which intersects every strip $\{w\in\mathbb{C}:c<\mathrm{Im}\,w<c'\}$ in a finite set for arbitrary $c<c'$, cf. Remark 3.12. An elementary consideration shows that the limes on the right hand side of (3.65) is just $\mathcal{P}_+g$. By assumption we have $\mathcal{P}_+g=g$, so it follows the second relation of (3.63). In remains to check the first equation of (3.63):

$$\begin{aligned}&\mathcal{P}_+\lim_{t\to+0}F^{-1}(w-\mathcal{A})^{-1}\left\{\widetilde{f}(w)-\frac{1}{2\pi}\int_{\Gamma_-}(z-\mathcal{A})^{-1}\widetilde{f}(z)dz\right\}\\ &=F^{-1}\mathcal{P}_+(w-\mathcal{A})^{-1}\widetilde{f}(w)|_{t=0}-\frac{1}{2\pi}\int_{\Gamma_-}\mathcal{P}_+(z-\mathcal{A})^{-1}\widetilde{f}(z)dz=0.\end{aligned}$$

□

Let us now consider the operator (3.31) on a compact C^∞ manifold with boundary Y, where we fix a collar neighbourhood $\cong Y\times[0,1)\ni(x,t)$ of Y. As in the cylindrical situation we then have for the Cauchy data spaces $\mathcal{C}^{s,\mu}(Y,E')$ for any integer $s\geq\mu$, $E'=E|_Y$, a continuous operator

$$\mathcal{T}^\mu:H^s(X,E)\to\mathcal{C}^{s,\mu}(Y,E'),$$

and a pseudo-differential projection

$$\mathcal{P}_+:\mathcal{C}^{s,\mu}(Y,E')\to\mathcal{C}^{s,\mu}(Y,E').$$

Concerning the non-bijectivity points of the induced operator family (3.37) we make the same assumptions as before, i.e., that there are no such points on the line $\mathrm{Re}\,w=0$.

THEOREM 3.16. *The spectral boundary value problem*

$$Au = f \in H^{s-\mu}(X,F), \tag{3.66}$$

$$\mathcal{P}_+\mathcal{T}^\mu u = g \in \mathcal{P}_+\mathcal{C}^{s,\mu}(Y,E') \tag{3.67}$$

defines a Fredholm operator

$$\begin{pmatrix} A \\ \mathcal{P}_+\mathcal{T}^\mu \end{pmatrix} : H^s(X,E) \to \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ \mathcal{P}_+\mathcal{C}^{s,\mu}(Y,E') \end{array} \tag{3.68}$$

for every $s \geq \mu$.

More general boundary value problems with a modification of the trace operators $\mathcal{P}_+\mathcal{T}^\mu$ are studied in [**45**]. Boundary value problems of type (3.68) as well as those of [**45**] are special cases of elliptic problems of Section 2.3, up to a slight modification of the orders of the trace operators which may consist of components of different order. In fact, similarly as (3.56) we can form

$$\mathcal{P}_\pm = \mathcal{R}_0 \frac{1}{2\pi i}\int_{\Gamma_\pm} \mathcal{A}_0(w-\mathcal{A}_0)^{-1}\frac{dw}{w}\mathcal{R}_0^{-1}$$

with a diagonal matrix $\mathcal{R}_0 = \operatorname{diag}\left(\mathcal{R}_{E'}^{k+\frac{1}{2}}\right)_{k=0,\dots,\mu-1}$ and $\mathcal{A}_0 = \mathcal{R}_0^{-1}\mathcal{A}\mathcal{R}_0$.

Set

$$P_\pm = \frac{1}{2\pi i}\int_{\Gamma_\pm} \mathcal{A}_0(w-\mathcal{A}_0)^{-1}\frac{dw}{w}$$

which are pseudo-differential projections

$$P_\pm : \bigoplus_{0\leq k\leq \mu-1} H^s(Y,E') \to \bigoplus_{0\leq k\leq \mu-1} H^s(Y,E')$$

of the class $L^0_{\mathrm{cl}}\left(Y; \bigoplus_{0\leq k\leq\mu-1} E', \bigoplus_{0\leq k\leq\mu-1} E'\right)$. The operator $\mathcal{R}_0$

$$\mathcal{R}_0 : \bigoplus_{0\leq k\leq \mu-1} H^s(Y,E') \to \mathcal{C}^{s,\mu}(Y,E'),$$

and (3.68) can be transformed to an equivalent boundary value problem

$$\begin{pmatrix} A \\ T \end{pmatrix} : H^s(X,E) \to \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ P_+H^s\left(Y, \oplus_{0\leq k\leq\mu-1}E'\right) \end{array} \tag{3.69}$$

for $T = \mathcal{R}_0^{-1}\mathcal{P}_+\mathcal{T}^\mu$.

Setting

$$\mathbb{L}_+ = \left(P_+, \bigoplus_{0\leq k\leq\mu-1} E', \quad \operatorname{im}\sigma_\psi \bigoplus_{0\leq k\leq\mu-1} E'\right)$$

we have

$$P_+H^s\left(Y, \bigoplus_{0\leq k\leq\mu-1} E'\right) = P^s(Y,\mathbb{L}_+),$$

and the operator (3.69) is elliptic in the sense of Definition 2.14. Thus, Theorem 3.16 is an immediate consequence of Theorem 2.21 (i).

3.4. Calderón-Seeley projections. In the present section we discuss another way of constructing pseudo-differential projections to Cauchy data spaces of solutions to elliptic differential equations, cf. also the author's joint paper [**46**] with Tarkhanov.

Let

$$A : C^\infty(X, E) \to C^\infty(X, F)$$

be a differential operator on a compact C^∞ manifold X with boundary Y; $E, F \in \mathrm{Vect}(X)$. Moreover, let M be a C^∞ manifold (countable at infinity with a Riemannian metric) containing X as a submanifold with boundary. Let $\widetilde{E}, \widetilde{F} \in \mathrm{Vect}(M)$ such that $E = \widetilde{E}|_X, F = \widetilde{F}|_X$, and let

$$\widetilde{A} : C^\infty(M, \widetilde{E}) \to C^\infty(M, \widetilde{F})$$

be a differential operator with $A = \widetilde{A}|_X$.

Let $\Omega := \operatorname{int} X$, and interpret the characteristic function χ_Ω as an operator of multiplication

$$\chi_\Omega : H^s_{\mathrm{loc}}(M, \widetilde{E}) \to \mathcal{D}'(M, \widetilde{E})$$

for $s \geq \mu = \operatorname{ord} \widetilde{A}$. Then the distributional kernel of the commutator

$$[\widetilde{A}, \chi_\Omega] = \widetilde{A}\chi_\Omega - \chi_\Omega \widetilde{A}$$

(as a map $C_0^\infty(M, \widetilde{E}) \to \mathcal{D}'(M, \widetilde{F})$) is supported on $Y \times Y$. Moreover, $f \in H^s_{\mathrm{loc}}(M, \widetilde{E})$, $s \geq \mu$, and $\mathcal{T}^\mu f = 0$ imply $[\widetilde{A}, \chi_\Omega] f = 0$.

We now assume that there is an operator $\widetilde{P} \in L^{-\mu}_{\mathrm{cl}}(M; \widetilde{F}, \widetilde{E})$ such that

$$\widetilde{A}\widetilde{P} f = f, \qquad \widetilde{P}\widetilde{A} u = u \tag{3.70}$$

for all (distributional sections in the respective bundles) f and u, supported in an ε-neighbourhood of X in M for all $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$.

In the following we employ the operators e^+ and r^+ in the same meaning as in Section 3.1. Set

$$A := \mathrm{r}^+ \widetilde{A} \mathrm{e}^+, \qquad P := \mathrm{r}^+ \widetilde{P} \mathrm{e}^+.$$

PROPOSITION 3.17. *Let $\widetilde{A}$ be an elliptic differential operator with the abovementioned properties. Then*

$$G := 1 - PA$$

(*as an operator on $H^s(X, E)$, $s \geq \mu = \operatorname{ord} \widetilde{A}$*) *belongs to the space $B_G^{0;\mu}(X; E, F)$* (*cf. the notation* (1.88)) *and satisfies*

$$G^2 = G \tag{3.71}$$

PROOF. By virtue of Theorem 1.69 we have $G \in \mathcal{B}^{0,\mu}(X; E, F)$. We then obtain $G \in \mathcal{B}_G^{0;\mu}(X; E, F)$ from Remark 1.67.

Let us now verify (3.71). First we have the relation

$$AP = 1 \tag{3.72}$$

as a consequence of (3.70) and of the fact that $\widetilde{A}$ is a differential operator. Thus $P : H^{s-\mu}(X, F) \to H^s(X, E)$ is a right inverse of A. This yields a projection PA, and G is just the complementary projection. $\square$

REMARK 3.18. Let $s \geq \mu$, and set

$$\ker_s A = \{u \in H^s(X, E) : Au = 0\}.$$

Then

$$G : H^s(X, E) \to \ker_s A \tag{3.73}$$

is a projection to $\ker_s A$.

In fact, $u \in \ker_s A$ gives us $Gu = u$, and for arbitrary $f \in H^s(X, E)$ we have $AGf = A(1 - PA)f = Af - (AP)Af = 0$.

PROPOSITION 3.19. *The operator* (3.73) *can be written in the form*

$$G = \sum_{j=0}^{\mu-1} K_j \circ T_j \tag{3.74}$$

with (unique) potential operators $K_j \in \mathcal{B}^{0,-j-\frac{1}{2}}(X;0,F;E',0)$ (*cf. Remark* 1.68). *Thus* $f \in H^s(X,E)$ *and* $\mathcal{T}^\mu f = 0$ *entails* $Gf = 0$.

PROOF. We have $G = 1 - PA \in \mathcal{B}_G^{0,\mu}(X;E,F)$, and Remark 1.68 gives us a representation (1.89). Since $PA\varphi = \varphi$ for every $\varphi \in C_0^\infty(\Omega, E)$, the operator G_0 vanishes. This yields the relation (3.74) which implies the second assertion. □

THEOREM 3.20. *Let* $\widetilde{A}$ *satisfy the abovementioned assumptions. Moreover, let* $\mathcal{T}^{-\mu}$ *be any right inverse of the map*

$$\mathcal{T}^\mu : H^s(X,E) \to \mathcal{C}^{s,\mu}(Y,E'), \tag{3.75}$$

cf. (3.46), (3.47) *for* $E' = J$.
Then

$$\Pi := \mathcal{T}^\mu G \mathcal{T}^{-\mu} : \mathcal{C}^{s,\mu}(Y,E') \to \mathcal{C}^{s,\mu}(Y,E')$$

is a projection to the Cauchy data space of solutions of $Au = 0$, $u \in H^s(X,E)$ (*called the Calderón-Seeley projection*).

PROOF. Let $g = \mathcal{T}^\mu u$ for some $u \in \ker_s A$. Then we have $\mathcal{T}^\mu(u - \mathcal{T}^{-\mu}g) = 0$ and hence, by Proposition 3.19, $G(u - \mathcal{T}^{-\mu}g) = 0$, which entails $G\mathcal{T}^{-\mu}g = Gu = u$ (by Remark 3.18) and $\Pi g = \mathcal{T}^\mu u = g$. For arbitrary $h \in \mathcal{C}^{s,\mu}(Y,E')$ we have $\mathcal{T}^{-\mu}h \in H^s(X,E)$ and $G\mathcal{T}^{-\mu}h \in \ker_s A$ by Remark 3.18. This yields $\Pi h \in \mathcal{T}^\mu \ker_s A$. □

REMARK 3.21.
(i) The operator Π is independent of the choice of $\mathcal{T}^{-\mu}$. In fact, if $\mathcal{T}_1^{-\mu}$ is another right inverse of $\mathcal{T}^\mu$ we have

$$\mathcal{T}^\mu G \mathcal{T}^{-\mu} - \mathcal{T}^\mu G \mathcal{T}_1^{-\mu} = \mathcal{T}^\mu G(\mathcal{T}^{-\mu} - \mathcal{T}_1^{-\mu}) = 0$$

because $\mathcal{T}^\mu(\mathcal{T}^{-\mu} - \mathcal{T}_1^{-\mu}) = 0$, cf. the second assertion of Proposition 3.19.

(ii) The operator Π is also independent of the specific choice of $\widetilde{P}$, because for another $\widetilde{P}_1$ the associated projection $G_1 = 1 - P_1 A$ has the property that $G_1\mathcal{T}^{-\mu}g \in \ker_s A$ has the Cauchy data g; the same is true of $G\mathcal{T}^{-\mu}g$, i.e., $\mathcal{T}^\mu(G - G_1)\mathcal{T}^{-\mu} = 0$.

REMARK 3.22. Assume that the operator A has the unique continuation property of solutions. Then

$$\mathcal{T}^\mu : \ker_s A \to \mathcal{C}^{s,\mu}(Y,E')$$

is injective, i.e., there is a unique solution $u \in \ker_s A$ for every $g \in \Pi\mathcal{C}^{s,\mu}(Y,E')$ such that $\mathcal{T}^\mu u = g$.

4. Remarks on the edge calculus with global projection data

4.1. Boundary value problems without the transmission property. In this section we want to make some remarks on the role of the transmission property of boundary value problems in connection with Chapter 2 . Pseudo-differential operators that appear as parametrices of elliptic differential operators have always the transmission property, cf. Definition 1.43. On the other hand, there are interesting cases of pseudo-differential operators on a manifold with boundary, where the transmission property is not satisfied, for instance, for operators obtained by a reduction to the boundary of some mixed elliptic problem.

A classical example of a mixed problem is the Zaremba problem for the Laplace operator with jumping boundary conditions, with Dirichlet conditions on one part, Neumann conditions on the other part of the boundary.

In mixed problems the boundary is subdivided into (smooth) submanifolds with an interface of codimension 1 as the common boundary. The solvability can be discussed in terms of boundary value problems (or transmission problems) on the boundary with respect to the interface. Typical operators in this context have $|\eta|$ (the absolute value of the covariable η on the boundary) as the principal symbol; those fail to have the transmission property at the interface. In general, operators with principal symbols $|\eta|^\mu$, $\mu \in \mathbb{R}$, have the transmission property with respect to any interface of codimension 1 if and only if μ is an even integer.

This shows, in particular, in a situation as at the beginning of Section 3.1, that the transmission property of the operators $A_\pm$ only holds in exceptional cases (although always in the case of differential operators).

The general program to construct an operator algebra of boundary value problems for the case without the transmission property and with global projection conditions at the boundary is carried out in Schulze and Seiler [**44**]. This paper also contains an analogue of Theorem 2.21 for elliptic operators without the transmission property.

The pseudo-differential analysis of boundary value problems and parametrices for the case of Shapiro-Lopatinskij ellipticity are studied in Rempel and Schulze [**31**], and further in [**37**] and [**42**]. Another (earlier) approach of Vishik and Eskin [**50**], [**51**], [**11**], is not organised in terms of operator algebras with complete and smooth symbolic structures.

The algebra property for boundary value problems with or without the transmission property fits into the concept of edge problems with a specific kind of operator-valued symbols, see also Section 4.2 below. This point of view has been developed in [**37**], cf. also [**32**]. Let us also note that mixed and crack problems have been systematically investigated in a new monograph jointly with Kapanadze [**21**] and in the author's joint papers with Harutjunjan [**17**], [**18**]. More details in the context of parabolic mixed and transmission problems may be found in Krainer, Schulze and Zhou [**23**].

4.2. Edge problems. Boundary value problems for differential (and pseudo-differential) operators on a smooth manifold with boundary have much in common with problems for operators on a manifold with edges. This is based on the fact that the 'half-space' $\Omega \times \overline{\mathbb{R}}_+$ for an open set $\Omega \subseteq \mathbb{R}^q$ can be regarded as a wedge with model cone $\overline{\mathbb{R}}_+$ and edge Ω. To illustrate this, let $A = \sum_{|\alpha|\le\mu} a_\alpha(\widetilde{x}) D_{\widetilde{x}}^\alpha$ be a differential operator in $\Omega \times \mathbb{R}_+$ with coefficients $a_\alpha(\widetilde{x}) \in C^\infty(\Omega \times \overline{\mathbb{R}}_+)$. Inserting $\widetilde{x} = (y,t)$ for $y \in \Omega$, $t \in \overline{\mathbb{R}}_+$, the operator A takes the form

$$A = t^{-\mu} \sum_{|\beta|+j\le\mu} a_{\beta j}(y,t)(tD_y)^\alpha \left(-t\frac{\partial}{\partial t}\right)^j \tag{4.1}$$

with coefficients $a_{\beta j}(y,t) \in C^\infty(\Omega \times \overline{\mathbb{R}}_+)$ for all j, β.

A differential operator A in $\Omega \times \mathbb{R}_+$ of the form (4.1) will be called edge-degenerate. Clearly, the class of such operators is much larger than that induced by operators with smooth coefficients.

One may ask, whether the calculus of boundary value problems with the transmission property at $t = 0$ in the sense of Section 1.3 has an analogue for edge-degenerate operators, with an adequate substitute of the standard Sobolev spaces, a corresponding generalisation of trace and potential boundary conditions at $t = 0$, Shapiro-Lopatinskj ellipticity, etc. Such a calculus is possible, indeed, and for the wedge $\Omega \times \overline{\mathbb{R}}_+$, or, more generally, a C^∞ manifold X with boundary, it is just a special case of a corresponding calculus on a manifold W with edge Y. Geometrically, such a W is a space such that both $W \setminus Y$ and Y are C^∞ manifolds ($\dim Y = q$, $\dim W \setminus Y = n+1+q$), and every point $y \in Y$ has a neighbourhood V which can be represented (via a 'singular chart') in the form $\Omega \times N^\Delta$ for an open set $\Omega \subseteq \mathbb{R}^q$ and a model cone

$$N^\Delta := (N \times \overline{\mathbb{R}}_+)/(N \times \{0\})$$

with base N which is a closed compact C^∞ manifold. The nature of transition maps and other details may be found, for instance, in [**41**].

Note that the class of edge-degenerate operators of order 2 contains the Laplace-Betrami operators (as well as other geometric operators) when $W \setminus Y$ in the local splitting of variables into (y,t,x) is equipped with a 'wedge metric', e.g.,

$$dt^2 + t^2 g_N(t,y) + dy^2,$$

where $g_N(t,y)$ is a family of Riemannian metrics on N smootly depending on (t,y) up to $t = 0$.

An example is $W = X$ for a C^∞ manifold X with boundary ∂X. In this case we have $Y = \partial X$, and N is a single point, i.e., $N^\Delta = \overline{\mathbb{R}}_+$.

The pseudo-differential algebra of edge problems (say, for the case that W is compact) consists of 2×2 block matrix operators

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(W,E) \\ \oplus \\ H^{s-\frac{n+1}{2}}(Y,J_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(W,F) \\ \oplus \\ H^{s-\mu-\frac{n+1}{2}}(Y,J_+) \end{array} \tag{4.2}$$

of order $\mu \in \mathbb{R}$. Compared with (1.90) the shift of smoothness by $(n+1)/2$ on the edge is not essential for the calculus; a reduction of order on Y allows us unify the orders of the spaces on Y to $H^s(Y,J_-)$ and $H^{s-\mu}(Y,J_+)$ respectively. The spaces $\mathcal{W}^{s,\gamma}(W)$ (for the trivial bundle E on W with fibre $\mathbb{C}$) are contained in $H^s_{\text{loc}}(W \setminus Y)$. Near Y in the variables $(y,x,t) \in \mathbb{R}^q \times N \times \mathbb{R}_+$ the spaces $\mathcal{W}^{s,\gamma}(W)$ are of the form

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(N^\wedge)), \tag{4.3}$$

cf. Definition 1.51, for a certain scale of weighted Sobolev spaces $\mathcal{K}^{s,\gamma}(N^\wedge)$ on the infinite stretched cone $N^\wedge = N \times \mathbb{R}_+$, cf. [**34**], [**38**], or [**10**]. The group action on the space $\mathcal{K}^{s,\gamma}(N^\wedge)$ is given by $\kappa_\lambda : u(x,t) \to \lambda^{\frac{n+1}{2}} u(x,\lambda t)$, $\lambda \in \mathbb{R}_+$. An easy modification of this construction for the case of arbitrary $E \in \text{Vect}(W)$ then gives us the spaces $\mathcal{W}^{s,\gamma}(W,E)$. The operators (4.2) form a so called edge-algebra that contains all edge-degenerate pseudo-differential operators in the upper left corners, cf. [**33**], [**38**], [**10**].

If $W = X$ is a C^∞ manifold X with boundary, the edge-algebra contains all pseudo-differential boundary value problems without the transmission property as a subalgebra, cf. [**37**] or [**42**]. Similarly to the calculus in Sections 1.2 and 1.3, the 2×2 block matrix structure reflects additional trace and potential conditions with respect to the boundary which satisfy an analogue of the Shapiro-Lopatinskij condition in the elliptic case.

In the general case, on a manifold W with edge Y, we have corresponding edge conditions of trace and potential type, again with an analogue of the Shapiro-Lopatinskij condition as an ellipticity condition for the so called principal edge symbol $\sigma_\wedge(\mathcal{A})$ which is the second component of the principal symbolic hierarchy

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\wedge(\mathcal{A}))$$

in the edge algebra (the first component $\sigma_\psi(\mathcal{A})$ is the edge-degenerate principal interior symbol of $\mathcal{A}$). Ellipticity of $\mathcal{A}$ in the edge algebra requires the bijecticity of both components.

The edge symbol $\sigma_\wedge(\mathcal{A})$ is an analogue of the boundary symbol $\sigma_\partial(\mathcal{A})$ from the situation of boundary value problems. In the edge algebra it consists of a family of operators

$$\sigma_\wedge(\mathcal{A})(y,\eta) : \begin{array}{c} E'_y \otimes \mathcal{K}^{s,\gamma}(N^\wedge) \\ \oplus \\ J_- \end{array} \to \begin{array}{c} F'_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \\ \oplus \\ J_+ \end{array} \tag{4.4}$$

parametrised by $(y,\eta) \in T^*Y \setminus 0$; here $E' = E|_Y$ $F' = F|_Y$. Similarly to the relation (1.87) we have homogeneity of order μ in the sense

$$\sigma_\wedge(\mathcal{A})(y,\lambda\eta) = \lambda^\mu \,\text{diag}(\kappa_\lambda, \lambda^{\frac{n+1}{2}}\,\text{id}) \sigma_\wedge(\mathcal{A})(y,\eta)\, \text{diag}(\kappa_\lambda^{-1}, \lambda^{-\frac{n+1}{2}}\,\text{id}) \tag{4.5}$$

for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in T^*Y \setminus 0$.

The Shapiro-Lopatinskij condition is just the bijectivity of (4.4) for any $s \in \mathbb{R}$ (it entails the bijectivity for all $s \in \mathbb{R}$; the weight $\gamma \in \mathbb{R}$ is kept fixed). We also talk about $\sigma_\wedge$-ellipticity of the corresponding operator $\mathcal{A}$.

Now, for a σ_ψ-elliptic operator

$$A : \mathcal{W}^{s,\gamma}(W,E) \to \mathcal{W}^{s-\mu,\gamma-\mu}(W,F) \tag{4.6}$$

we can ask the existence of a 2×2 block matrix operator $\mathcal{A}$ in the edge algebra, containing A in the upper left corner, with suitable bundles $J_\pm \in \text{Vect}(Y)$, such that $\mathcal{A}$ is $\sigma_\wedge$-elliptic.

The answer is similar as in the case of boundary value problems, cf. the Introduction. First, because of the relation (4.5), it suffices to look at $(y,\eta) \in S^*Y$. Then

$$\sigma_\wedge(\mathcal{A})(y,\eta) : E'_y \otimes \mathcal{K}^{s,\gamma}(N^\wedge) \to F'_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \tag{4.7}$$

is necessarily a family of Fredholm operators, parametrised by the compact space S^*Y. An analogue of the Atiyah-Bott condition (0.17) in the present case is then

$$\mathrm{ind}_{S^*Y}\,\sigma_\wedge(A) \in \pi_1^* K(Y), \tag{4.8}$$

$\pi_1 : S^*Y \to Y$. We then have the following theorem.

THEOREM 4.1 ([**34**]). *Let* (4.6) *be a σ_ψ-elliptic operator for which* (4.7) *is a family of Fredholm operators. There is then a $\sigma_\wedge$-elliptic operator $\mathcal{A}$ in the edge algebra containing A in the upper left corner if and only if condition* (4.8) *is fulfilled.*

THEOREM 4.2 ([**38**]). (i) *Every $(\sigma_\psi, \sigma_\wedge)$-elliptic operator $\mathcal{A}$ in the edge algebra has a parametrix within the calculus.*

(ii) *The ellipticity of $\mathcal{A}$ entails the Fredholm property of the operator* (4.2) *for every $s \in \mathbb{R}$.*

Index formulas for elliptic operators in the edge algebra have been constructed in Fedosov, Schulze, and Tarkhanov [**12**], and Nazaikinskij, Savin, Schulze, and Sternin [**28**].

Nazaikiskij, Savin Schulze, and Sternin [**26**], [**27**] constructed large classes of σ_ψ-elliptic operators on manifolds with edge for which the condition (4.8) is not satisfied.

It is now again an interesting problem to extend the edge algebra to an edge-Toeplitz algebra such that every σ_ψ-elliptic operator A admits an elliptic problem with global projection conditions on the edge Y.

Such an algebra was constructed in Schulze and Seiler [**43**]. All essential elements of the calculus of Section 1.3 have a natural analogue in the edge-Toeplitz algebra.

4.3. Analysis on manifolds with singularities. Ellipticity of (pseudo-) differential operators and parametrix constructions within a calculus with symbolic structures are an interesting program also on manifolds with higher (say 'polyhedral') singularities. Locally, such spaces can be generated by iteratively forming cones and wedges, starting from smooth compact base manifolds. The corresponding analysis refers to specific geometric properties, for instance, whether edges and corners are regular or cuspidal. Let us make here a few remarks on the case of regular polyhedra with their system of lower dimensional edges. An example is a cube in $\mathbb{R}^3$ with its two-dimensional faces, one-dimensional edges, and corner points. Outside the corner points close to the one-dimensional edges the configuration is a manifold with edges (with boundary), outside corners and one-dimensional edges it is a C^∞ manifolds with boundary. Thus, a calculus in the cube should contain both edge and boundary conditions.

Operator algebras on manifolds with 'higher' singularities have been constructed and investigated in many variants, cf. [**35**], [**13**], [**40**], [**41**], [**22**], [**24**]. The operators are connected with trace and potential operators on the lower-dimensional strata, and ellipticity of corresponding higher edge/corner problems contains a hierarchy of Shapiro-Lopatinskij conditions.

There is then also a hierarchy of topological obstructions for such edge conditions, and it would be necessary to study Toeplitz extensions of the corresponding operator algebras, in order not to rule out ('most of the') interesting operators.

In this connection, the basic questions are still open, and the analysis of elliptic operators and their index theory on stratified spaces is an awarding task for future activities.

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