Asymptotics and Relative Index on a Cylinder with Conical Cross Section [1]

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Abstract. We study pseudodifferential operators on a cylinder $\mathbb{R} \times B$ with cross section B that has conical singularities. Configurations of that kind are the local model of corner singularities with base spaces B. Operators A in our calculus are assumed to have symbols a which are meromorphic in the complex covariable with values in the space of all cone operators on B. In case a is independent of the axial variable $t \in \mathbb{R}$, we show an explicit formula for solutions of the homogeneous equation. Each non-bijectivity point of the symbol in the complex plane corresponds to a finite-dimensional space of solutions. Moreover, we give a relative index formula.

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Introduction

Ellipticity and asymptotics of solutions for (pseudo-)differential operators on a manifold with corner singularities are in spirit related to Euler's theory of ordinary differential equations, cf. the article [16] for the case of conical singularities with smooth cross section X. A neighbourhood of the singularity is then transformed to an infinite cylinder, and the relation to ordinary differential equations refers to the axial variable.

As is well known, cf. Kondratyev [7], the analysis of differential operators A of Fuchs type on a (stretched) manifold $\mathbb{B}$ with conical singularities (cf. Section 1.1 below) is connected with a two-component symbolic hierarchy $\sigma(A) = (\sigma_\psi(A), \sigma_M(A))$. Here $\sigma_\psi(A)$ is the standard homogeneous principal symbol and $\sigma_M(A)$ the so-called *conormal* symbol of A. Operators in an infinite cylinder with smooth cross section have been studied also by Sternin [18] and later on by other authors under different assumptions at infinity.

Conormal symbols generate an algebra of meromorphic operator functions $m(z), z \in \mathbb{C}$, operating on $X \cong \partial\mathbb{B}$ as (classical parameter-dependent pseudo-) differential operators, cf. [12] or, for the case of boundary value problems, Schrohe and Schulze [11]. General functional analytic background may be found in Gohberg and Sigal [6]. In the elliptic case there is a (generalised) spectrum, consisting of non-bijectivity points ("zeros" $z \in \mathbb{C}$) of corresponding operator families $m(z) : H^s(X) \to H^{s-\mu}(X)$ in Sobolev spaces on X ($\mu = \operatorname{ord} m$). For $m(z) = \sigma_M(A)$ the zeros are responsible for asymptotics of solutions of the equation $Au = f$ near the conical singularities. At the same time, the zeros of conormal symbols (and poles in the pseudo-differential case) determine the relative index of Fredholm operators A in weighted Sobolev spaces under changing weights. A sufficiently developed analysis for conical singularities is crucial for understanding operators on manifolds with edges that are locally described by wedges (Cartesian products of cones with open sets in $\mathbb{R}^q$), cf. [12], or Egorov and Schulze [2].

Geometric singularities (e.g., conical and edge singularities) are of interest in a variety of models in mathematical physics and mechanics, and, at least some further steps in the hierarchy of stratified spaces, e.g., corners or "higher" edges, appear in problems of quite practical relevance, e.g., in describing heat asymptotics in lense-shaped or cubic bodies, to name just some concrete examples. In such cases the time plays the role of an extra (anisotropic) edge-variable, cf. Krainer and Schulze [8] for the case of spatial configurations with conical cross section. In the simplest corner situation (corresponding to a cone with a base which has itself conical singularities) the problem is again to study holomorphic and meromorphic operator functions on a space with conical points; the configuration now has edges, starting from the corner points. Spaces of that kind can be modelled by Riemannian metrics with corresponding singularities, and associated Laplacians (as well as other geometric operators in this context) are degenerate in a typical way, see also [15]. Such operators belong to a rich algebra of parameter-dependent operators analogously to the one in the smooth case, mentioned at the beginning.

The purpose of this paper is to analyse asymptotics of solutions to elliptic equations in an infinite cylinder with conical cross section and to express the relative index with respect to changing corner weights (i.e., weights at the cylindrical ends at $\pm\infty$).

In Section 1 we prepare the necessary tools on parameter-dependent operators on a manifold with conical singularities. The ideas go back to the paper [13]. Here we specify the operator families for the case of constant discrete asymptotics.

Section 2 studies holomorphic families of cone operators. The main points are an explicit construction of such families (Theorem 2.6) and factorisations in the sense of Gohberg and Sigal [6], here in the frame of our algebra of holomorphic cone operator-valued functions (Theorem 2.17).

In Section 3 we investigate equations on an infinite cylinder with conical cross section which is a space with edges tending to $\pm\infty$. Operators are considered in edge Sobolev spaces with exponential weights at the cylindrical ends. As a consequence of the evaluation of kernels and cokernels (Theorem 3.14) we show the explicit form of solutions of the homogeneous equation, associated with the characteristic values of the given cone operator-valued symbol $a(w)$. The new difficulty, compared with the smooth case, treated in [16], is that ellipticity requires additional conditions of trace and potential type on the edges with principal edge symbols acting as operator families in weighted Sobolev spaces on infinite cones. Another essential point in this context is that the ellipticity of subordinate conormal symbols is preserved under kernel cut-offs (cf. relation (29)). Finally, we give a relative index formula in terms of the logarithmic residues of $a(w)$ (Theorem 3.16). This result can be regarded as a complementary information to [3], where the index of an elliptic operator itself is expressed, though with another machinery and without referring to the meromorphic structure of conormal symbols, cf. also the article of Schulze and Tarkhanov [17] for more general corner manifolds, locally modelled on cylinders with singular cross sections.

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1 Parameter-dependent cone calculus

1.1 Cone Sobolev spaces and Green operators

Let B be a compact manifold with conical singularities, i.e., there is a finite subset $S \subset B$ of conical points and B is modelled near any $v \in S$ by a cone

$$X^{\triangle} := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X),$$

where $X = X(v)$ is a compact C^∞ manifold. In this paper we assume X to be closed. Let $\mathbb{B}$ denote the stretched manifold associated with B, which is a compact C^∞ manifold with boundary $\partial\mathbb{B} \cong \bigcup_{v\in S} X(v)$, invariantly defined by attaching the manifolds $X(v)$ to $B \setminus \{v\}$ for every $v \in S$. Let us fix a Riemannian metric on $\mathbb{B}$ which restricts to the product metric of $[0,1) \times \partial\mathbb{B}$ in a collar neighbourhood of $\partial\mathbb{B}$, for some Riemannian metric on $\partial\mathbb{B}$. Let $(r,x) \in [0,1) \times \partial\mathbb{B}$ denote the corresponding splitting of variables. For convenience, we consider the case that S only consists of one point, i.e., $X := \partial\mathbb{B}$ for a closed compact manifold X of dimension n. The general case is similar; details will be omitted.

Let $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^q)$ denote the space of *classical parameter-dependent* pseudo-differential operators $A(\lambda)$ of order μ on the manifold X, with the parameter $\lambda \in \mathbb{R}^q$ being involved in the local amplitude functions $a(x,\xi,\lambda)$ as a component of the covariables $(\xi,\lambda) \in \mathbb{R}^{n+q}$, $n = \dim X$, while $L^{-\infty}(X;\mathbb{R}^q) := \mathcal{S}(\mathbb{R}^q, L^{-\infty}(X))$. Recall that $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^q)$ is a Fréchet space in a natural way, cf. [14, Section 1.2.2].

An element $A(\lambda) \in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^q)$ is said to be *parameter-dependent elliptic* (of order μ), if the homogeneous principal part $a_{(\mu)}(x,\xi,\lambda)$ is non-zero for all $(x,\xi,\lambda) \in T^*X \times \mathbb{R}^q \setminus 0$. We use the well known fact that for every $\mu \in \mathbb{R}$ there is a parameter-dependent elliptic element $R^\mu(\lambda)$ which induces isomorphisms $H^s(X) \to H^{s-\mu}(X)$ between the standard Sobolev spaces on X for all $\lambda \in \mathbb{R}^q, s \in \mathbb{R}$.

Let $\mathcal{H}^{s,\gamma}(X^\wedge), s,\gamma \in \mathbb{R}$, for $X^\wedge := \mathbb{R}_+ \times X \ni (r,x)$ denote the completion of the space $C_0^\infty(X^\wedge)$ with respect to the norm

$$\left\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} ||R^s(\operatorname{Im} z)Mu(z)||^2_{L^2(X)}dz\right\}^{\frac{1}{2}}.$$

Here M is the Mellin transform on functions $u(r) \in C_0^\infty(\mathbb{R}_+, C^\infty(X))$, $Mu(z) = \int_0^\infty r^{z-1}u(r)dr$ (holomorphic in z), $R^s(\tau) \in L^\mu_{\mathrm{cl}}(X;\mathbb{R}_\tau)$ is an order reducing family of order s in the abovementioned sense, and $\Gamma_\beta := \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$, $n = \dim X$.

In this paper, a *cut-off function* on the half-axis is any real-valued element $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1 in a neighbourhood of $r = 0$. We define a modified scale of spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ on the infinite stretched cone $X^\wedge$ by setting

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge), v \in H^s_{\mathrm{cone}}(X^\wedge)\},$$

where ω is some cut-off function. Here the space $H^s_{\mathrm{cone}}(X^\wedge), s \in \mathbb{R}$, is defined as follows: Choose an open covering $\{U_1,\ldots,U_N\}$ of X by coordinate neighbourhoods and a subordinate partition of unity $\{\varphi_1,\ldots,\varphi_N\}$. Let $\chi_j : U_j \to V_j$ be diffeomorphisms to open subsets $V_j \subset S^n = \{\tilde{x} \in \mathbb{R}^{n+1} : |\tilde{x}| = 1\}, j = 1,\ldots,N$. Moreover, set $\kappa_j(t,x) = t\chi_j(x)$ for $x \in U_j, t \in \mathbb{R}_+$, which defines a diffeomorphism $\kappa_j : U_j^\wedge \to V_j^\wedge = \{\tilde{x} \in \mathbb{R}^{n+1} : \tilde{x}/|\tilde{x}| \in V_j\}, j = 1, \ldots, N$. Then, if $\omega(t)$ is any fixed cut-off function, $H^s_{\mathrm{cone}}(X^\wedge), s \in \mathbb{R}$, denotes the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$||u||_{H^s_{\mathrm{cone}}(X^\wedge)} := \{||\omega u||^2_{H^s(\mathbb{R}_+ \times X)} + \sum_{j=1}^{N} ||(\kappa_j^*)^{-1}(1-\omega)\varphi_j u||^2_{H^s(\mathbb{R}^{n+1})}\}^{\frac{1}{2}},$$

with κ_j^* being the pull back of functions with respect to κ_j, and $H^s(\mathbb{R}_+ \times X) := H^s(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$. This definition is correct, i.e., independent of the system of charts, of the partition of unity and of ω.

There are continuous embeddings $\mathcal{K}^{s',\gamma'}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}(X^\wedge)$ for $s' \geq s, \gamma' \geq \gamma$ which are compact for $s' > s, \gamma' > \gamma$.

For future references we recall a terminology on pasting of Fréchet spaces E_0 and E_1 which are embedded in a Hausdorff topological space H. First, let $E_0 + E_1 := \{e_0 + e_1 : e_0 \in E_0, e_1 \in E_1\}$ and endow this space with the Fréchet topology induced by the isomorphism $E_0 + E_1 \cong E_0 \oplus E_1/\{(e, -e) : e \in E_0 \cap E_1\}$ (called the *non-direct* sum of Fréchet spaces). In particular, if E_0, E_1 are Hilbert spaces, we also get a Hilbert space structure in $E_0 + E_1$ by taking the orthogonal complement of $E_0 \cap E_1$ in $E_0 \oplus E_1$. Moreover, if a Fréchet space E is a left module over an algebra A, i.e., the elements $a \in A$ induce (by multiplication $e \to ae$) linear operators $a : E \to E$, we define $[a]E$ to be the completion of $\{ae : e \in E\}$ in the space E. In this sense we can set

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^s_{\mathrm{cone}}(X^\wedge).$$

If B is a manifold with conical singularities, we identify a collar neighbourhood of $\partial\mathbb{B} = X$ with $[0,1) \times X$ and set

$$\mathcal{H}^{s,\gamma}(\mathbb{B}) := \{u \in H^s_{\mathrm{loc}}(\mathrm{int}\,\mathbb{B}) : \omega u \in \mathcal{H}^{s,\gamma}(X^\wedge)\}$$

for some cut-off function ω supported in $[0,1)$.

It will be interesting also to consider subspaces of our weighted Sobolev spaces with discrete asymptotics. For brevity we consider asymptotics in a finite weight interval $\Theta = (\vartheta, 0], \vartheta > -\infty$, relative to a weight $\gamma \in \mathbb{R}$ (our results have a straightforward extension to the case $\vartheta = -\infty$ that follows by a simple projective limit procedure, cf. [14, Section 2.3.1]). First we consider

$$\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) := \varprojlim_{\varepsilon > 0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(X^\wedge)$$

in the Fréchet topology of the projective limit.

A sequence

$$P = \{(p_j, m_j, L_j)\}_{j=0,\ldots,N},$$

$N = N(P) \in \mathbb{N}$, will be called a *discrete asymptotic type* with respect to the weight data (γ, Θ) if $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, and $L_j \subset C^\infty(X)$ are subspaces of finite dimension, and $\pi_{\mathbb{C}} P := \{p_0, \ldots, p_N\}$ satisfies

$$\pi_{\mathbb{C}} P \subset \{z \in \mathbb{C} : \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} z < \frac{n+1}{2} - \gamma\}.$$

The set of all such sequences will be denoted by $\operatorname{As}(X; (\gamma, \Theta))$.

Moreover, $\overline{P}$ will denote the complex conjugate of P, i.e., $\overline{P} = \{(\overline{p}_j, m_j, \overline{L}_j)\}_{j=1,\ldots,N}$ when $P = \{(p_j, m_j, L_j)\}_{j=1,\ldots,N}$.

Remark 1.1 *The (finite-dimensional) space*

$$\mathcal{E}_P^\gamma(X^\wedge) := \Big\{ \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) \omega(r) r^{-p_j} \log^k r : c_{jk} \in L_j, 0 \le k \le m_j, j = 0, \ldots, N \Big\}$$

for $P \in \operatorname{As}(X; (\gamma, \Theta))$ (with a fixed cut-off function $\omega(r)$) is contained in $\mathcal{K}^{\infty,\gamma}(X^\wedge)$, and we have

$$\mathcal{E}_P^\gamma(X^\wedge) \cap \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) = \{0\}$$

for all $s \in \mathbb{R}$.

We set

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) := \{u \in \mathcal{K}^{s,\gamma}(X^\wedge) : u = u_s + u_r \text{ for certain } u_s \in \mathcal{E}_P^\gamma(X^\wedge), u_r \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)\}$$

and

$$\mathcal{H}_P^{s,\gamma}(\mathbb{B}) := \{u \in \mathcal{H}^{s,\gamma}(\mathbb{B}) : \omega u \in \mathcal{K}_P^{s,\gamma}(X^\wedge)\}$$

for a cut-off function ω supported in $[0, 1)$.

The spaces $\mathcal{K}_P^{s,\gamma}(X^\wedge)$ and $\mathcal{H}_P^{s,\gamma}(\mathbb{B})$ are Fréchet in a natural way. For purposes below we also form the spaces

$$\mathcal{S}_P^\gamma(X^\wedge) := \varprojlim_{k \in \mathbb{N}} \langle r \rangle^{-k} \mathcal{K}_P^{k,\gamma}(X^\wedge). \tag{1}$$

Parameter-dependent Green operators on $\mathbb{B}$ will be defined in terms of operator-valued symbols in a neighbourhood of the conical singularities and parameter-dependent smoothing operators elsewhere.

Operator-valued symbols will also play an important role for parameter-dependent cone operators in general.

If E is a Hilbert space and $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ a strongly continuous group of isomorphisms $\kappa_\lambda : E \to E$ (i.e., $\{\kappa_\lambda e\}_{\lambda \in \mathbb{R}_+} \in C(\mathbb{R}_+, E)$ for every $e \in E$, and $\kappa_\lambda \kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \mathbb{R}_+$), we say that E is endowed with a group action. More generally, if $E = \varprojlim E^k$ is a projective limit of Hilbert spaces E^k with continuous embeddings $E^{k+1} \hookrightarrow E^k$ for all k, and if $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a group action on E^0 which restricts to a group action on E^k for every k, we say that E is endowed with a group action.

Example 1.2 (i) *The space* $E := \langle r\rangle^{-\rho}\mathcal{K}^{s,\gamma}(X^\wedge)$ *is endowed with the group action*

$$(\kappa_\lambda u)(r,x) := \lambda^{\frac{n+1}{2}} u(\lambda r, x)$$

for every $\rho, s, \gamma \in \mathbb{R}$, *where* $n = \dim X$.

(ii) *The Fréchet space* $\mathcal{S}^\gamma_P(X^\wedge)$ *in the representation* (1) *is endowed with a group action.*

Definition 1.3 *Let* $(E, \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+})$ *and* $(\tilde{E}, \{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+})$ *be Hilbert spaces with group actions. Then the space* $S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ *for* $U \subset \mathbb{R}^p$ *open,* $\mu \in \mathbb{R}$, *is defined to be the set of all* $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E,\tilde{E}))$ *such that*

$$||\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}||_{\mathcal{L}(E,\tilde{E})} \le c\langle\eta\rangle^{\mu-|\beta|}$$

for all multi-indices $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$ *and all* $(y,\eta) \in K \times \mathbb{R}^q$ *for every* $K \subset\subset U$, *with constants* $c = c(\alpha,\beta,K) > 0$; *here* $\langle\eta\rangle := (1+|\eta|^2)^{\frac{1}{2}}$.

The elements of $S^\mu(U\times\mathbb{R}^q; E, \tilde{E})$ are called *operator-valued* symbols of order μ.

Let $S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ denote the set of all $a_{(\mu)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E,\tilde{E}))$ such that

$$a_{(\mu)}(y,\lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a_{(\mu)}(y,\eta)\kappa^{-1}_\lambda \tag{2}$$

for all $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\}), \lambda \in \mathbb{R}_+$.

A symbol $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ is called *classical*, if there are elements $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ such that

$$a(y,\eta) - \chi(\eta)\sum_{j=0}^N a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; E, \tilde{E})$$

for all $N \in \mathbb{N}$; here $\chi(\eta)$ is any *excision function* in $\mathbb{R}^q$ (i.e., $\chi \in C^\infty(\mathbb{R}^q), \chi(\eta) = 0$ for $|\eta| < c_0, \chi(\eta) = 1$ for $|\eta| > c_1$ for certain $0 < c_0 < c_1$).

$a_{(\mu-j)}(y,\eta)$ is called the *homogeneous* component of $a(y,\eta)$ of order $\mu - j$.

If a notation or relation is valid both for classical and general symbols we write "(cl)" as subscript. Let $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$ denote the corresponding subspaces of y-independent elements.

Similarly to the spaces of scalar symbols which are included as special cases for $E = \tilde{E} = \mathbb{C}$ with trivial group action (i.e., identities for all $\lambda \in \mathbb{R}_+$), the spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})$ are Fréchet spaces in a natural way.

We also employ the generalisation of symbol spaces for the case of Fréchet spaces E or $\tilde{E}$. The definition for the case of a Fréchet space $\tilde{E} = \varprojlim_{k\in\mathbb{N}} \tilde{E}^k$

(for Hilbert spaces $\tilde{E}^k$) and a Hilbert space E, both endowed with group actions, is

$$S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E}) := \varprojlim S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E}^k).$$

Finally, if both E and $\tilde{E}$ are Fréchet, with group actions $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+}$, respectively, we fix a function $\mathrm{r} : \mathbb{N} \to \mathbb{N}$, form the space $S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})_{\mathrm{r}} := \varprojlim S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q; E^{\mathrm{r}(j)}, \tilde{E}^j)$, and set

$$S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E}) := \bigcup_{\mathrm{r}} S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})_{\mathrm{r}}.$$

Green symbols in the parameter-dependent cone calculus are motivated by the symbolic structure of Green's function in boundary value problems (which corresponds to the case when the cone is the inner normal to the boundary, interpreted as the edge of a corresponding wedge). In our case the (stretched) wedge is $X^\wedge \times \mathbb{R}^q$ with the open stretched cone $X^\wedge = \mathbb{R}_+ \times X$ and edge $\mathbb{R}^q$. We will employ this mainly in the case $q = 1$.

To introduce Green symbols it suffices to specify the spaces E and $\tilde{E}$ in the general definition above. We employ non-degenerate sesquilinear pairings

$$(\cdot, \cdot) : (\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^N) \times (\mathcal{K}^{-s,-\gamma}(X^\wedge) \oplus \mathbb{C}^N) \to \mathbb{C}$$

induced by

$$(u \oplus u', v \oplus v') := (u, v)_{\mathcal{K}^{0,0}(X^\wedge)} + (u', v')_{\mathbb{C}^N}$$

for all $u, v \in C_0^\infty(X^\wedge), u', v' \in \mathbb{C}^N$, with the scalar product

$$(u, v)_{\mathcal{K}^{0,0}(X^\wedge)} = \int \int u(r, x) \overline{v(r, x)} r^n dr dx$$

of the space $\mathcal{K}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(X^\wedge)$ (with the measure $drdx$). We now fix dimensions $j_-, j_+ \in \mathbb{N}$, choose weight data (γ, δ, Θ) for reals $\gamma, \delta \in \mathbb{R}$ and a weight interval $\Theta = (\vartheta, 0], -\infty < \theta < 0$, and consider the spaces

$$E := \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}, \quad \tilde{E} := \mathcal{S}^{\delta}_{P}(X^\wedge) \oplus \mathbb{C}^{j_+}$$

for $P \in \mathrm{As}(X, (\delta, \Theta))$, and

$$F := \mathcal{K}^{s,-\delta}(X^\wedge) \oplus \mathbb{C}^{j_+}, \quad \tilde{F} := \mathcal{S}^{-\gamma}_{Q}(X^\wedge) \oplus \mathbb{C}^{j_-}$$

for a $Q \in \mathrm{As}(X, (-\gamma, \Theta))$, where the group action in spaces of the type $\langle r \rangle^{-\rho} \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^j$ are given as $\mathrm{diag}(\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}, \mathrm{id}_{\mathbb{C}^j})$.

$\mathcal{R}^{\mu}_{G}(\mathbb{R}^q; (\gamma, \delta, \Theta), j_-, j_+)_{P,Q}$ is defined to be the space of all

$$g(\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q; E, \tilde{E}) \tag{3}$$

such that the η- wise formal adjoint $g^*(\eta)$ with respect to the above sesquilinear pairings defines an element

$$g^*(\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q; F, \tilde{F}), \tag{4}$$

where $P \in \mathrm{As}(X, (\delta, \Theta))$ and $Q \in \mathrm{As}(X, (-\gamma, \Theta))$ are fixed asymptotic types which depend on g, and relations (3) and (4) are required for all $s \in \mathbb{R}$.

For $j_- = j_+ = 0$ we simply write $\mathcal{R}^{\mu}_G(\mathbb{R}^q; (\gamma, \delta, \Theta))_{P,Q}$.

We set by $\mathcal{R}^{\mu}_G(\mathbb{R}^q; (\gamma, \delta, \Theta), j_-, j_+)$ the union of all spaces $\mathcal{R}^{\mu}_G(\mathbb{R}^q; (\gamma, \delta, \Theta), j_-, j_+)_{P,Q}$ over all P, Q.

In this paper $[\eta] \in C^{\infty}(\mathbb{R}^q_{\eta})$ will be denoted any strictly positive function such that $[\eta] = |\eta|$ for $|\eta| > c$ for a constant $c > 0$.

Example 1.4 *Let* $k \in (\mathcal{S}^{\delta}_P(X^{\wedge}_{r,x}) \oplus \mathbb{C}^{j_+}) \hat{\otimes}_{\pi} (\mathcal{S}^{-\gamma}_{\overline{Q}}(X^{\wedge}_{r',x'}) \oplus \mathbb{C}^{j_-})$, *for* $P \in \mathrm{As}(X, (\delta, \Theta)), Q \in \mathrm{As}(X, (-\gamma, \Theta))$, *be an arbitrary element regarded as a* 2×2- *block matrix of functions* $(k_{ij})_{i,j=1,2}$, *and let*

$$g(\eta) := (g_{ij}(\eta))_{i=1,2},$$

where

$$\begin{aligned}
g_{11}(\eta)u_1(r,x) &:= [\eta]^{\mu+1} \int\int k_{11}(r[\eta], x, r'[\eta], x')u_1(r', x')dr'dx', \\
g_{12}(\eta)u_2(r,x) &:= [\eta]^{\mu+\frac{n+1}{2}} k_{12}(r[\eta], x)u_2, \\
g_{21}(\eta)u_1 &:= [\eta]^{\mu+\frac{1-n}{2}} \int k_{21}(r'[\eta], x')u_1(r', x')dr'dx', \\
g_{22}(\eta)u_2 &:= [\eta]^{\mu} k_{22}u_2
\end{aligned}$$

for $u = (u_1, u_2) \in \mathcal{K}^{s,\gamma}(X^{\wedge}) \oplus \mathbb{C}^{j_-}$. *Then* $g(\eta)$ *defines an element in* $\mathcal{R}^{\mu}_G(\mathbb{R}^q; (\gamma, \delta, \Theta), j_-, j_+)_{P,Q}$.

We now consider $2\mathbb{B}$, the double of the stretched manifold $\mathbb{B}$, obtained by gluing together two copies $\mathbb{B}_-$ and $\mathbb{B}_+$ of $\mathbb{B}$ along their common boundary $\partial\mathbb{B}$, where we identify $\mathbb{B}$ with $\mathbb{B}_+$. Then $2\mathbb{B}$ is a closed compact C^{∞} manifold of dimension $n+1$. On $2\mathbb{B}$ we have the scale $H^s(2\mathbb{B}), s \in \mathbb{R}$, of standard Sobolev spaces, and we consider the space

$$L^{-\infty}(2\mathbb{B}; j_-, j_+) := \bigcap_{s \in \mathbb{R}} \mathcal{L}(H^s(2\mathbb{B}) \oplus \mathbb{C}^{j_-}, C^{\infty}(2\mathbb{B}) \oplus \mathbb{C}^{j_+})$$

which is Fréchet in a natural way. Then we form the space

$$\mathcal{S}(\mathbb{R}^q, L^{-\infty}(2\mathbb{B}; j_-, j_+))$$

of Schwartz functions with values in that space. Functions in $C^{\infty}_0(\mathrm{int}\mathbb{B})$ will also be interpreted as a functions on $2\mathbb{B}$ by extension by zero on the opposite side. In particular, if $\omega \in C^{\infty}(\mathbb{B})$ is a cut-off function on $\mathbb{B}$, i.e., ω

is supported in a collar neighbourhood $\cong [0,1) \times \partial\mathbb{B}$ of $\partial\mathbb{B}$ and is equal to 1 close to $\partial\mathbb{B}$, we also identify $1-\omega$ with a function on $2\mathbb{B}$.

Let us fix $P \in \mathrm{As}(X;(\delta,\Theta)), Q \in \mathrm{As}(X;(-\gamma,\Theta))$, and let $\mathcal{C}_G(\mathbb{B};(\gamma,\delta,\Theta),j_-,j_+)_{P,Q}$ denote the space of all operators c such that

$$c: \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{B}) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,\delta}_P(\mathbb{B}) \\ \oplus \\ \mathbb{C}^{j_+} \end{array}$$

and the formal adjoints

$$c^*: \begin{array}{c} \mathcal{H}^{s,-\delta}(\mathbb{B}) \\ \oplus \\ \mathbb{C}^{j_+} \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,-\gamma}_Q(\mathbb{B}) \\ \oplus \\ \mathbb{C}^{j_-} \end{array}$$

are continuous for all $s \in \mathbb{R}$. The space $\mathcal{C}_G(\mathbb{B};(\gamma,\delta,\Theta),j_-,j_+)_{P,Q}$ is Fréchet in a canonical way, and we set

$$\mathcal{C}^{-\infty}(\mathbb{B};(\gamma,\delta,\Theta),j_-,j_+;\mathbb{R}^q)_{P,Q} = \mathcal{S}(\mathbb{R}^q,\mathcal{C}_G(\mathbb{B};(\gamma,\delta,\Theta),j_-,j_+)_{P,Q}). \quad (5)$$

Moreover, let $\mathcal{C}^{-\infty}(\mathbb{B};(\gamma,\delta,\Theta),j_-,j_+;\mathbb{R}^q)$ denote the union of all spaces (5) over P, Q.

Definition 1.5 *We define*

$$\mathcal{C}^{\mu}_G(\mathbb{B};(\gamma,\delta,\Theta),j_-,j_+;\mathbb{R}^q) \quad (6)$$

to be the space of all operator families of the form

$$g(\eta) := \mathrm{diag}(\omega,1)g_0(\eta)\mathrm{diag}(\omega_1,1)+\mathrm{diag}((1-\omega),1)g_1(\eta)\mathrm{diag}((1-\omega_2),1)+c(\eta)$$

for arbitrary $g_0 \in \mathcal{R}^{\mu}_G(\mathbb{R}^q;(\gamma,\delta,\Theta),j_-,j_+)$, $g_1 \in \mathcal{S}(\mathbb{R}^q,L^{-\infty}(2\mathbb{B};j_-,j_+))$, *and* $c \in \mathcal{C}^{-\infty}(\mathbb{B};(\gamma,\delta,\Theta),j_-,j_+;\mathbb{R}^q)$, *where* ω,ω_1,ω_2 *are cut-off functions on* $\mathbb{B}$ *such that* $\omega\omega_1=\omega$, $\omega\omega_2=\omega_2$. *The elements of* (6) *are called parameter-dependent Green operators on the* (*stretched*) *manifold* $\mathbb{B}$ *with conical singularities.*

Remark 1.6 *Definition* 1.5 *is correct in the sense that it is independent of the specific choice of the cut-off functions* ω,ω_1,ω_2.

1.2 Mellin operator families with asymptotics

We now turn to another typical part of the parameter-dependent cone calculus, so-called *Mellin operators with discrete asymptotics.* Similarly to the case of Green operators the essential contribution comes from a neighbourhood of $\partial\mathbb{B}$. Thus we first consider the open stretched cone $X^\wedge$.

A sequence

$$R = \{(p_j,m_j,N_j)\}_{j\in\mathbb{Z}}$$

will be called a discrete asymptotic type of Mellin symbols, if $p_j \in \mathbb{C}, m_j \in \mathbb{N}$, and $N_j \subset L^{-\infty}(X)$ are finite-dimensional spaces of operators of finite rank, and $\pi_{\mathbb{C}} R := \{p_j\}_{j\in\mathbb{Z}}$ has the property that $\pi_{\mathbb{C}} R \cap \{z : c \leq \operatorname{Re} z \leq c'\}$ is a finite set for every $c \leq c'$. Let $\underline{\mathrm{As}}(X)$ denote the set of all such sequences.

A function $\chi \in C^\infty(\mathbb{C})$ is called an $\pi_{\mathbb{C}} R$ - excision function, if $\chi(z) \equiv 0$ for $\operatorname{dist}(z, \pi_{\mathbb{C}} R) < \varepsilon_0$, $\chi(z) \equiv 1$ for $\operatorname{dist}(z, \pi_{\mathbb{C}} R) > \varepsilon_1$ for certain $0 < \varepsilon_0 < \varepsilon_1$.

The space $M_R^{-\infty}(X)$ for $R \in \underline{\mathrm{As}}(X)$ is defined to be the set of all functions $f \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} R, L^{-\infty}(X))$ such that

(i) for any $\pi_{\mathbb{C}} R$ - excision function χ we have

$$\chi(z) f(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L^{-\infty}(X))$$

for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for every $c \leq c'$,

(ii) close to every $p_j \in \pi_{\mathbb{C}} R$ the function f has a representation

$$f(z) = \sum_{k=0}^{m_j} c_{jk}(z - p_j)^{-(k+1)} + h(z)$$

with coefficients $c_{jk} \in N_j, 0 \leq k \leq m_j$, where $h(z)$ is holomorphic near p_j with values in $L^{-\infty}(X)$.

Given an element

$$f(r, r', z) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu_{\mathrm{cl}}(X; \Gamma_{\frac{1}{2}-\delta}))$$

we form the associated Mellin pseudo-differential operator

$$\mathrm{op}_M^\delta(f) u(r) := \int_{\mathbb{R}} \int_0^\infty \left(\frac{r}{r'}\right)^{-(\frac{1}{2}-\delta+i\tau)} f(r, r', \frac{1}{2} - \delta + i\tau) u(r') \frac{dr'}{r'} đ\tau$$

first on $u(r) \in C_0^\infty(\mathbb{R}_+, C^\infty(X))$ and then we extend it to weighted Sobolev spaces.

In this section we assume f to be independent of r, r', and $f \in M_R^{-\infty}(X)$ for some $R \in \underline{\mathrm{As}}(X)$. Then $\mathrm{op}_M^\delta(f)$ can be defined for every $\delta \in \mathbb{R}$ such that $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{1}{2}-\delta} = \emptyset$.

Remark 1.7 *Let $f \in M_R^{-\infty}(X)$, $R \in \underline{\mathrm{As}}(X)$, and choose a $\gamma \in \mathbb{R}$ such that $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$. Moreover, fix cut-off functions $\omega, \tilde{\omega}$, and let $\nu \in \mathbb{R}$ and $\alpha \in \mathbb{N}^q$. Then the η - dependent family of operators*

$$m(\eta) := r^{-\nu} \omega(r[\eta]) \mathrm{op}_M^{\gamma - \frac{n}{2}}(f) \eta^\alpha \tilde{\omega}(r[\eta])$$

represents operator-valued symbols

$$m(\eta) \in S_{\mathrm{cl}}^{\nu-|\alpha|}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\nu}(X^\wedge))$$

as well as

$$m(\eta) \in S_{\mathrm{cl}}^{\nu-|\alpha|}(\mathbb{R}^q; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{\infty,\gamma-\nu}(X^\wedge))$$

for every $P \in \mathrm{As}(X, (\gamma, \Theta))$ *with some resulting* $Q \in \mathrm{As}(X, (\gamma - \nu, \Theta))$, *for every* $s \in \mathbb{R}$.

Remark 1.8 *Let* $m(\eta)$ *be given as in Remark* 1.7 *for* $\nu = \mu - j, j \in \mathbb{N}, |\alpha| \le j$, *and form*

$$m_1(\eta) := r^{-\mu+j}\omega_1(r[\eta])\mathrm{op}_M^{\gamma_1-\frac{n}{2}}(f)\eta^\alpha\tilde{\omega}_1(r[\eta])$$

with the same $f \in M_R^{-\infty}(X)$ *but another choice of cut-off functions* $\omega_1, \tilde{\omega}_1$ *and of the weight* γ_1 *such that* $\pi_{\mathbb{C}}R \cap \Gamma_{\frac{n+1}{2}-\gamma_1} = \emptyset$. *Then we have*

$$m(\eta) - m_1(\eta) \in \mathcal{R}_G^{\mu-j}(\mathbb{R}^q; (\tilde{\gamma}, \tilde{\gamma} - \mu + j, \Theta)).$$

for $\tilde{\gamma} = \min(\gamma, \gamma_1)$.

In the sequel, for abbreviation, we write $\boldsymbol{g} := (\gamma, \gamma - \mu, \Theta)$.

The space $\mathcal{R}_{M+G}^\mu(\mathbb{R}^q; \boldsymbol{g})$ for $\gamma, \mu \in \mathbb{R}$ and $\Theta := (-(k+1), 0], k \in \mathbb{N}$, is defined to be the set of all families of operators

$$m(\eta) + g(\eta) \tag{7}$$

for arbitrary $g(\eta) \in \mathcal{R}_G^\mu(\mathbb{R}^q; \boldsymbol{g})$ and

$$m(\eta) = r^{-\mu}\omega(r[\eta]) \sum_{j=0}^{k} r^j \sum_{|\alpha| \le j} \mathrm{op}_M^{\gamma_{j\alpha}-\frac{n}{2}}(m_{j\alpha})\eta^\alpha\tilde{\omega}(r[\eta]), \tag{8}$$

where $m_{j\alpha} \in M_{R_{j\alpha}}^{-\infty}(X)$, $R_{j\alpha} \in \underline{\mathrm{As}}(X)$ and $\gamma_{j\alpha} \in \mathbb{R}$, $\pi_{\mathbb{C}}R_{j\alpha} \cap \Gamma_{\frac{n+1}{2}-\gamma_{j\alpha}} = \emptyset$, $\gamma - j \le \gamma_{j\alpha} \le \gamma$ for all j, α.

More generally, $\mathcal{R}_{M+G}^\mu(\mathbb{R}^q; \boldsymbol{g}, j_-, j_+)$ denotes the set of all

$$f_0(\eta) := \begin{pmatrix} m(\eta) & 0 \\ 0 & 0 \end{pmatrix} + g(\eta)$$

for arbitrary $m(\eta)$ of the form (8), and $g(\eta) \in \mathcal{R}_G^\mu(\mathbb{R}^q; \boldsymbol{g}, j_-, j_+)$.

Note, as a consequence of Remark 1.7, that $\mathcal{R}_{M+G}^\mu(\mathbb{R}^q; \boldsymbol{g})$ is a subspace of $S_{\mathrm{cl}}^\mu(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$, and that every element $a(\eta) \in \mathcal{R}_{M+G}^\mu(\mathbb{R}^q; \boldsymbol{g})$ belongs to a space $S_{\mathrm{cl}}^\mu(\mathbb{R}^q; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{\infty,\gamma-\mu}(X^\wedge))$ for every $P \in \mathrm{As}(X, (\gamma, \Theta))$ with some a - dependent $Q \in \mathrm{As}(X, (\gamma - \mu, \Theta))$; all this is true for every $s \in \mathbb{R}$ (clearly, P and Q are independent of s).

Definition 1.9 *We define* $\mathcal{C}^{\mu}_{M+G}(\mathbb{B};\boldsymbol{g},j_-,j_+;\mathbb{R}^q)$ *to be the set of all* 2×2 *- block matrix families of operators*

$$f(\eta) := \mathrm{diag}(\omega,1)f_0(\eta)\mathrm{diag}(\omega_1,1)+\mathrm{diag}((1-\omega),1)f_1(\eta)\mathrm{diag}((1-\omega_2),1)+k(\eta) \tag{9}$$

for cut-off functions ω,ω_1,ω_2 *on* $\mathbb{B}$ *such that* $\omega\omega_1=\omega$, $\omega\omega_2=\omega_2$ *and elements* $f_0(\eta)\in\mathcal{R}^{\mu}_{M+G}(\mathbb{R}^q;\boldsymbol{g},j_-,j_+)$, $f_1(\eta)\in\mathcal{S}(\mathbb{R}^q,L^{-\infty}(2\mathbb{B};j_-,j_+))$ *as in Definition* 1.5 *and* $k(\eta)\in\mathcal{C}^{-\infty}(\mathbb{B};\boldsymbol{g},j_-,j_+;\mathbb{R}^q)$.

Let us introduce the principal symbolic structure of elements in $\mathcal{C}^{\mu}_{M+G}(\mathbb{B};\boldsymbol{g},j_-,j_+;\mathbb{R}^q)$. Given $f(\eta)$ as in Definition 1.9 we set

$$\sigma^{\mu}_{\wedge}(f)(\eta) := \sigma^{\mu}_{\wedge}(f_0)(\eta) := \begin{pmatrix} \sigma^{\mu}_{\wedge}(m)(\eta) & 0 \\ 0 & 0 \end{pmatrix} + \sigma^{\mu}_{\wedge}(g)(\eta),$$

where $\sigma^{\mu}_{\wedge}(g)(\eta)$ is the homogeneous component of $g(\eta)$ of order μ of the corresponding classical operator-valued symbol, cf. (3), while

$$\sigma^{\mu}_{\wedge}(m)(\eta) := r^{-\mu}\omega(r|\eta|)\sum_{j=0}^{k} r^j \sum_{|\alpha|=j} \mathrm{op}_M^{\gamma_{j\alpha}-\frac{n}{2}}(m_{j\alpha})\eta^{\alpha}\tilde{\omega}(r|\eta|). \tag{10}$$

$\sigma^{\mu}_{\wedge}(f)(\eta)$ is interpreted as a family of operators

$$\sigma^{\mu}_{\wedge}(f)(\eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^{\wedge}) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \rightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}) \\ \oplus \\ \mathbb{C}^{j_+} \end{array} \tag{11}$$

for any fixed $s\in\mathbb{R}$, parametrised by $\eta\in\mathbb{R}^q\setminus\{0\}$. Clearly, $\sigma^{\mu}_{\wedge}(f)(\eta)$ defines mappings to $\mathcal{K}^{\infty,\gamma-\mu}(X^{\wedge})\oplus\mathbb{C}^{j_+}$, but for the ellipticity below it is reasonable to choose larger spaces than in (11). Recall that homogeneity means

$$\sigma^{\mu}_{\wedge}(f)(\lambda\eta) = \lambda^{\mu}\mathrm{diag}(\kappa_{\lambda},1)\sigma^{\mu}_{\wedge}(f)(\eta)\mathrm{diag}(\kappa_{\lambda}^{-1},1) \tag{12}$$

for all $\lambda\in\mathbb{R}_+,\eta\in\mathbb{R}^q\setminus\{0\}$.

Proposition 1.10 *Composition of operators induces a map*

$$\begin{aligned}\mathcal{C}^{\mu}_{M+G}(\mathbb{B};(\gamma-\nu,\gamma-\nu-\mu,\Theta),l,j_+;\mathbb{R}^q) \quad &\times \quad \mathcal{C}^{\nu}_{M+G}(\mathbb{B};(\gamma,\gamma-\nu,\Theta),j_-,l;\mathbb{R}^q) \\ &\rightarrow \quad \mathcal{C}^{\mu+\nu}_{M+G}(\mathbb{B};(\gamma,\gamma-\nu-\mu,\Theta),j_-,j_+;\mathbb{R}^q),\end{aligned}$$

and we have

$$\sigma^{\mu+\nu}_{\wedge}(fg) = \sigma^{\mu}_{\wedge}(f)\sigma^{\nu}_{\wedge}(g).$$

If f *or* g *belongs to the corresponding subclass of Green elements, so does the composition* fg.

Proof. The proof can easily be reduced to a corresponding composition behaviour of operator-valued symbols of the form (7). This is then a combination of a known η- wise behaviour of corresponding smoothing Mellin+Green operators on the stretched cone $X^{\wedge}$ with an evaluation of summands as classical operator-valued symbols. Details on this technique may be found, e.g., in [14, Section 3.4.2]. □

1.3 The cone algebra with parameters

Parameter-dependent operators on a (stretched) manifold $\mathbb{B}$ with conical singularities are mainly characterised by the interior symbolic structure in a neighbourhood of $\partial\mathbb{B} = X$. First we give a description in terms of local coordinates on X, with x varying in an open set $\Omega \subset \mathbb{R}^n$. A symbol $b(r, x, \rho, \xi, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+n+q}_{\rho,\xi,\eta})$ is called *edge-degenerate*, if it has the form

$$b(r, x, \rho, \xi, \eta) = \tilde{b}(r, x, \tilde{\rho}, \xi, \tilde{\eta})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta}$$

for a symbol $\tilde{b}(r, x, \tilde{\rho}, \xi, \tilde{\eta}) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+n+q}_{\tilde{\rho},\xi,\tilde{\eta}})$. Setting

$$\mathrm{op}_x(\tilde{b})(r, \tilde{\rho}, \tilde{\eta})u(x) := \int\int e^{i(x-x')\xi}\tilde{b}(r, x, \tilde{\rho}, \xi, \tilde{\eta})u(x')dx'đ\xi$$

for $u \in C_0^{\infty}(\Omega)$, we get a family

$$\mathrm{op}_x(\tilde{b})(r, \tilde{\rho}, \tilde{\eta}) \in C^{\infty}(\overline{\mathbb{R}}_+, L^{\mu}_{\mathrm{cl}}(\Omega; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})).$$

This gives rise to a family of operators

$$\mathrm{op}_x(b)(r, \rho, \eta) := \mathrm{op}_x(\tilde{b})(r, \tilde{\rho}, \tilde{\eta})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta} \in C^{\infty}(\mathbb{R}_+, L^{\mu}_{\mathrm{cl}}(\Omega; \mathbb{R}^{1+q}_{\rho,\eta})).$$

Let us now fix a system of charts $\chi_j : U_j \to \Omega_j$ on $X, j = 1, \dots, N$, for an open covering $\{U_1, \dots, U_N\}$ of X. Let $\{\varphi_1, \dots, \varphi_N\}$ be a subordinate partition of unity, and let $\{\psi_1, \dots, \psi_N\}$ be a system of functions $\psi_j \in C_0^{\infty}(U_j)$ such that $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$ for all j. Then, given a system of edge-degenerate symbols $b_j(r, x, \rho, \xi, \eta) = \tilde{b}_j(r, x, \tilde{\rho}, \xi, \tilde{\eta})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta}$ on $\mathbb{R}_+ \times \Omega_j, j = 1, \dots, N$, we form the operator push-forwards of $\mathrm{op}_x(b_j)$ (or $\mathrm{op}_x(\tilde{b}_j)$) with respect to $\chi_j^{-1} : \Omega_j \to U_j$. We then pass to a global family of operators on X by

$$\tilde{p}(r, \tilde{\rho}, \tilde{\eta}) := \sum_{j=1}^{N} \varphi_j\{(\chi_j^{-1})_* \mathrm{op}_x(\tilde{b}_j)(r, \tilde{\rho}, \tilde{\eta})\}\psi_j \tag{13}$$

and set

$$p(r, \rho, \eta) := \tilde{p}(r, \tilde{\rho}, \tilde{\eta})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta}.$$

We then obtain $\tilde{p}(r, \tilde{\rho}, \tilde{\eta}) \in C^{\infty}(\overline{\mathbb{R}}_+, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$, and $p(r, \rho, \eta) \in C^{\infty}(\mathbb{R}_+, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\rho,\eta}))$.

Definition 1.11 *Let $M^{\mu}_{O}(X; \mathbb{R}^q)$ be the set of all functions*

$$h(z, \eta) \in \mathcal{A}(\mathbb{C}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^q))$$

such that

$$h(z, \eta)|_{\Gamma_\beta \times \mathbb{R}^q} \in L^{\mu}_{\mathrm{cl}}(X; \Gamma_\beta \times \mathbb{R}^q)$$

for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$ for arbitrary $c \le c'$. For $q = 0$ we simply write $M^{\mu}_{O}(X)$. Moreover, for arbitrary $R \in \underline{\mathrm{As}}(X)$ we set

$$M^{\mu}_{R}(X) = M^{\mu}_{O}(X) + M^{-\infty}_{R}(X)$$

in the Fréchet topology of the non-direct sum.

Theorem 1.12 *For every family* (13) *there exists an element* $\tilde{h}(r, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X; \mathbb{R}^q_{\tilde{\eta}}))$ *such that*

$$\mathrm{op}_r(\tilde{p})(\tilde{\eta}) = \mathrm{op}_M^\gamma(\tilde{h})(\tilde{\eta}) \mod L^{-\infty}(X^\wedge; \mathbb{R}^q_{\tilde{\eta}})$$

for every $\gamma \in \mathbb{R}$.

Proof. This theorem is a parameter-dependent analogue of [14, Theorem 2.2.25].

Corollary 1.13 *For* $\tilde{h}(r, z, \tilde{\eta})|_{\tilde{\eta}=r\eta} =: h(r, z, \eta) \in C^\infty(\mathbb{R}_+, M_O^\mu(X; \mathbb{R}^q_\eta))$ *we have*

$$\mathrm{op}_r(p)(\eta) = \mathrm{op}_M^\gamma(h)(\eta) \mod L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta). \tag{14}$$

Moreover, setting

$$p_0(r, \rho, \eta) := \tilde{p}(0, \tilde{\rho}, \tilde{\eta})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta}$$

and

$$h_0(r, z, \eta) := \tilde{h}(0, z, \tilde{\eta})|_{\tilde{\eta}=r\eta}$$

we get

$$\mathrm{op}_r(p_0)(\eta) = \mathrm{op}_M^\gamma(h_0)(\eta) \mod L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta)$$

for every $\gamma \in \mathbb{R}$.

Definition 1.14 *The space* $\mathcal{C}^\mu(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q), \boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, *of all parameter-dependent cone operators on* $\mathbb{B}$ *of order* μ, *is defined as the set of all*

$$a(\eta) := c(\eta) + f(\eta) \tag{15}$$

for arbitrary $f(\eta) \in \mathcal{C}^\mu_{M+G}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q)$, $c(\eta) := \begin{pmatrix} c_{11}(\eta) & 0 \\ 0 & 0 \end{pmatrix}$ *for*

$$c_{11}(\eta) := \omega(r) r^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(h)(\eta) \omega_1(r) + (1 - \omega(r)) b(\eta)(1 - \omega_2(r)), \tag{16}$$

where $h(r, z, \eta)$ *is an arbitrary holomorphic parameter-dependent Mellin symbol as in Corollary* 1.13, *moreover,* $b(\eta) \in L^\mu_{\mathrm{cl}}(2\mathbb{B}; \mathbb{R}^q)$, *and* $\omega, \omega_1, \omega_2$ *are cut-off functions as in Definition* 1.9.

The definition is correct in the sense that the class is independent of the specific choice of cut-off functions.

For convenience, in expressions of the form (16) we identify a collar neighbourhood of $\partial\mathbb{B} \cong X$ with $[0, 1) \times X$ and tacitly assume that cut-off functions ω, ω_1, etc. are supported in $[0, 1)$.

Remark 1.15 *There is another equivalent definition of* $c_{11}(\eta)$ *which formally appears to be more complicated than* (16) *but gives a useful relation to edge-degenerate symbols. A result of* [4] *says that*

$$\omega(r)r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\omega_1(r) \tag{17}$$

is equal to

$$\begin{aligned}&\omega(r)\{\tilde{\omega}(r[\eta])r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}_1(r[\eta])\\ +\;&(1-\tilde{\omega}(r[\eta]))r^{-\mu}\mathrm{op}_r(p)(\eta)(1-\tilde{\omega_2}(r[\eta]))\}\omega_1(r)\end{aligned} \tag{18}$$

mod $\mathcal{R}^{\mu}_{G,O}(\mathbb{R}^q;(\gamma,\gamma-\mu,(-\infty,0]))$ (*the subspace of all, so-called flat, elements of* $\mathcal{R}^{\mu}_{G}(\mathbb{R}^q;(\gamma,\gamma-\mu,(-\infty,0]))$ *with trivial asymptotic types* P *and* Q, *i.e.,* $\pi_{\mathbb{C}}P=\pi_{\mathbb{C}}Q=\emptyset$); *here* $\tilde{\omega},\tilde{\omega}_1,\tilde{\omega}_2$ *are any cut-off functions such that* $\tilde{\omega}\tilde{\omega}_1=\tilde{\omega},\tilde{\omega}\tilde{\omega}_2=\tilde{\omega}_2$.

In the expression (18) we assume that p and h are linked to each other via relation (14). The correspondence $h\to p$ is one-to-one modulo smoothing elements. This allows us to equivalently express the interior symbolic structure of operators (17) in terms of p.

From the representation (13) we immediately get an invariantly defined parameter-dependent homogeneous principal symbol

$$\tilde{p}_{(\mu)}(r,x,\tilde{\rho},\xi,\tilde{\eta})\in C^{\infty}(T^*(\overline{\mathbb{R}}_+\times X)\times\mathbb{R}^q_{\tilde{\eta}}\setminus 0)$$

with $\tilde{\rho}$ being the covariable of r in this notation, and 0 indicating the "covector" $(\tilde{\rho},\xi,\tilde{\eta})=0$. Let $(\boldsymbol{x},\boldsymbol{\xi})$ denote points in the cotangent bundle of $2\mathbb{B}$. We then define the *principal interior symbol* $\sigma^{\mu}_{\psi}(a)(\boldsymbol{x},\boldsymbol{\xi},\eta)\in C^{\infty}(T^*(\mathrm{int}\,\mathbb{B})\times\mathbb{R}^q\setminus 0)$ of $a\in\mathcal{C}^{\mu}(\mathbb{B};\boldsymbol{g},j_-,j_+;\mathbb{R}^q)$ as the parameter-dependent homogeneous principal symbol of $c_{11}(\eta)$, regarded as an element of $L^{\mu}_{\mathrm{cl}}(\mathrm{int}\,\mathbb{B};\mathbb{R}^q)$. Then, in a collar neighbourhood of $\partial\mathbb{B}\cong X$ in the splitting of variables $(\boldsymbol{x},\boldsymbol{\xi})=(r,x,\rho,\xi),0<r<1$, we have

$$\sigma^{\mu}_{\psi}(a)(r,x,\rho,\xi,\eta):=r^{-\mu}\omega(r)\tilde{p}_{(\mu)}(r,x,r\rho,\xi,r\eta)+(1-\omega(r))b_{(\mu)}(r,x,\rho,\xi,\eta),$$

where $b_{(\mu)}(\boldsymbol{x},\boldsymbol{\xi},\eta)\in C^{\infty}(T^*(2\mathbb{B})\times\mathbb{R}^q\setminus 0)$ denotes the parameter-dependent homogeneous principal symbol of $b(\eta)$ in Definition 1.14. Moreover, the operator $c_{11}(\eta)$ has a principal edge symbol of order μ, namely

$$\sigma^{\mu}_{\wedge}(c_{11})(\eta):=r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(\eta)$$

which is the same as

$$\omega(r|\eta|)r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(\eta)\omega_1(r|\eta|)+(1-\omega(r|\eta|)r^{-\mu}\mathrm{op}_r(p_0)(\eta)(1-\omega_2(r|\eta|))$$

modulo the principal edge symbol of some flat Green element; concerning notation, cf. Corollary 1.13. We then define

$$\sigma^{\mu}_{\wedge}(a)(\eta):=\begin{pmatrix}\sigma^{\mu}_{\wedge}(c_{11})(\eta) & 0\\ 0 & 0\end{pmatrix}+\sigma^{\mu}_{\wedge}(f)(\eta),$$

$\eta\neq 0$, called the *principal edge symbol* of a of order μ.

Remark 1.16 *Similarly to* (11) *we interpret* $\sigma_\wedge^\mu(a)(\eta)$ *as a family of operators* $\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-} \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \oplus \mathbb{C}^{j_+}$, $s \in \mathbb{R}$, *and we have homogeneity of* $\sigma_\wedge^\mu(a)(\eta)$ *analogously to relation* (12).

Note that for $a_{11}(\eta) :=$ u.l.c. $a(\eta)$ (upper left corner of $a(\eta)$) the principal edge symbol

$$\sigma_\wedge^\mu(a_{11})(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is a family of operators in the cone algebra on the infinite cone, for every $\eta \neq 0$, cf. [12] or [14]. As such there is a subordinate principal conormal symbol

$$\sigma_M\sigma_\wedge^\mu(a_{11})(z) : H^s(X) \to H^{s-\mu}(X) \tag{19}$$

with z varying on $\Gamma_{\frac{n+1}{2}-\gamma}$. Recall that (19) in this case has the form

$$\sigma_M\sigma_\wedge^\mu(a_{11})(z) = h_0(0,z,0) + m_{00}(z) \tag{20}$$

with h_0 being given by Corollary 1.13 and $m_{00}(z)$ by expression (8). Formula (20) also makes sense for $q = 0$ where we may simply write $\sigma_M(a)$ instead of (20).

Let us set

$$\sigma^\mu(a) = (\sigma_\psi^\mu(a), \sigma_\wedge^\mu(a)),$$

called the *principal symbol* of a of order μ. Here we tacitly assume $q > 0$. Otherwise, for $q = 0$ we set

$$\sigma^\mu(a) = (\sigma_\psi^\mu(a), \sigma_M(a)).$$

Let

$$\mathcal{C}^{\mu-1}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q) := \{a \in \mathcal{C}^\mu(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q) : \sigma^\mu(a) = 0\}. \tag{21}$$

Then, similarly to the above, elements of (21) have a pair of principal symbols $\sigma^{\mu-1}(a)$ of order $\mu-1$. Inductively, we get decreasing chain of subspaces $\mathcal{C}^{\mu-j}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q) \subseteqq \mathcal{C}^\mu(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q)$ for all $j \in \mathbb{N}$ with corresponding pairs $\sigma^{\mu-j}(a)$ of principal symbols. Observe that

$$\mathcal{C}^{-\infty}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q) = \bigcap_{j\in\mathbb{N}} \mathcal{C}^{\mu-j}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q).$$

Theorem 1.17 *Composition of operators induces a map*

$$\begin{aligned} \mathcal{C}^\mu(\mathbb{B}; (\gamma-\nu, \gamma-\nu-\mu, \Theta), l, j_+; \mathbb{R}^q) \quad &\times \quad \mathcal{C}^\nu(\mathbb{B}; (\gamma, \gamma-\nu, \Theta), j_-, l; \mathbb{R}^q) \\ &\to \quad \mathcal{C}^{\mu+\nu}(\mathbb{B}; (\gamma, \gamma-\nu-\mu, \Theta), j_-, j_+; \mathbb{R}^q), \end{aligned}$$

and we have

$$\sigma_\psi^{\mu+\nu}(ab) = \sigma_\psi^\mu(a)\sigma_\psi^\nu(b), \quad \sigma_\wedge^{\mu+\nu}(ab) = \sigma_\wedge^\mu(a)\sigma_\wedge^\nu(b).$$

For the subordinate conormal symbols the composition rule is

$$\sigma_M^{\mu+\nu}(ab)(z) = \sigma_M^\mu(a)(z+\nu)\sigma_M^\nu(b)(z).$$

Proof. The result is a modification of the known composition behaviour of operator-valued edge symbols, cf. Gil, Schulze and Seiler [4]. The new element here is that our operator families are edge symbols only in a collar neighbourhood of $\partial\mathbb{B} \cong X$. However, far from $\partial\mathbb{B}$ our families are (up to entries of finite-rank) parameter-dependent pseudo-differential operators in $\operatorname{int}\mathbb{B}$ which behave well under compositions. $\square$

Definition 1.18 *An element $a \in \mathcal{C}^{\mu}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q)$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ and $q \in \mathbb{N} \setminus \{0\}$ is said to be elliptic if*

(i) *$\sigma_{\psi}^{\mu}(a)$ is elliptic in the following sense: $\sigma_{\psi}^{\mu}(a)(\boldsymbol{x}, \boldsymbol{\xi}, \eta) \neq 0$ for all $(\boldsymbol{x}, \boldsymbol{\xi}, \eta) \in T^*(\operatorname{int}\mathbb{B}) \times \mathbb{R}^q \setminus 0$ and (in the splitting of variables $(r, x) \in \mathbb{R}_+ \times X$ near $\partial\mathbb{B}$) $\tilde{p}_{(\mu)}(r, x, \tilde{\rho}, \xi, \tilde{\eta}) \neq 0$ for all $(r, x, \tilde{\rho}, \xi, \tilde{\eta}) \in T^*(\overline{\mathbb{R}}_+ \times X) \times \mathbb{R}^q_{\tilde{\eta}} \setminus 0$.*

(ii) *$\sigma_{\wedge}^{\mu}(a)$ defines an isomorphism*

$$\sigma_{\wedge}^{\mu}(a)(\eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^{\wedge}) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \to \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}) \\ \oplus \\ \mathbb{C}^{j_+} \end{array} \tag{22}$$

for all $\eta \neq 0$ and some $s = s_0 \in \mathbb{R}$.

Note that when (22) is an isomorphism for $s = s_0$ then so is for all $s \in \mathbb{R}$. Moreover, (22) implies that

$$\sigma_{\wedge}^{\mu}(a_{11})(\eta) : \mathcal{K}^{s,\gamma}(X^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})$$

is a family of Fredholm operators which has the consequence that

$$\sigma_M \sigma_{\wedge}^{\mu}(a_{11})(z) : H^s(X) \to H^{s-\mu}(X) \tag{23}$$

is an isomorphism for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ and all $s \in \mathbb{R}$.

For $q = 0$ an element $a \in \mathcal{C}^{\mu}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ is called *elliptic*, if (i) holds, and if (23) is an isomorphism for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}, s \in \mathbb{R}$.

Theorem 1.19 *Let $a \in \mathcal{C}^{\mu}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q)$ be elliptic. Then there is a parametrix $b \in \mathcal{C}^{-\mu}(\mathbb{B}; (\gamma - \mu, \gamma, \Theta), j_+, j_-; \mathbb{R}^q)$ in the following sense:*

$$1 - b(\eta)a(\eta) \in \mathcal{C}^{-\infty}(\mathbb{B}; (\gamma, \gamma, \Theta), j_-, j_-; \mathbb{R}^q),$$

$$1 - a(\eta)b(\eta) \in \mathcal{C}^{-\infty}(\mathbb{B}; (\gamma - \mu, \gamma - \mu, \Theta), j_+, j_+; \mathbb{R}^q).$$

Proof. The proof of this result is analogous to the corresponding theorem on invertibility of elliptic edge symbols, cf. [14, Section 3.5.2]. $\square$

Remark 1.20 *Let $a \in \mathcal{C}^\mu(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}^q)$ be elliptic. Then there is a constant $c > 0$ such that*

$$a(\eta) : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{B}) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{B}) \\ \oplus \\ \mathbb{C}^{j_+} \end{array}$$

are isomorphisms for all $\eta \in \mathbb{R}^q, |\eta| \geq c$, and all $s \in \mathbb{R}$.

Moreover, $a_{11}(\eta)$ defines a family of Fredholm operators

$$a_{11}(\eta) : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{B})$$

for all $\eta \in \mathbb{R}^q$.

In the sequel we are mainly interested in the case $q = 1$.

Remark 1.21 *If $a_{11}(\eta) \in \mathcal{C}^\mu(\mathbb{B}; \boldsymbol{g}, 0, 0; \mathbb{R})$ is an operator which satisfies condition* (i) *of Definition* 1.18*, then*

$$\sigma^\mu_\wedge(a_{11})(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is a family of Fredholm operators if and only if

$$\sigma_M \sigma^\mu_\wedge(a_{11})(z) : H^s(X) \to H^{s-\mu}(X)$$

is a family of isomorphisms for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. The condition that $a_{11}(\eta)$ can be regarded as the upper left corner of an elliptic element $a \in \mathcal{C}^\mu(\mathbb{B}; \boldsymbol{g}; j_-, j_+; \mathbb{R})$ requires that

$$\operatorname{ind} \sigma^\mu_\wedge(a_{11})(1) = \operatorname{ind} \sigma^\mu_\wedge(a_{11})(-1)$$

is fulfilled. This is an analogue of the Atiyah-Bott obstruction from the theory of elliptic operators on manifolds with boundary for the existence of Shapiro-Lopatinskij elliptic conditions, cf. Atiyah and Bott [1] (*or Schulze* [12] *for the present case of edges that also includes the case of a q- dimensional parameter η*).

Example 1.22 *Let $a(\eta) = \sum_{j=0}^{\mu} a_j \eta^j$ be a polynomial with coefficients $a_j \in \mathcal{C}^{\mu-j}(\mathbb{B}; (\gamma, \gamma - \mu + j, \Theta), j_-, j_+)$. Then in a neighbourhood of $\partial\mathbb{B}$ the upper left corners of $a_j, j = 0, \ldots, \mu$, are Fuchs type operators*

$$r^{-\mu+j} \sum_{k=0}^{\mu-j} b_{jk}(r)(-r\frac{\partial}{\partial r})^k$$

with $b_{jk}(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j-k}(X)), j = 0, \ldots, \mu, k = 0, \ldots, \mu - j$. We then obtain

$$a_{11}(\eta) = r^{-\mu} \sum_{j=0}^{\mu} \sum_{k=0}^{\mu-j} b_{jk}(r)(-r\frac{\partial}{\partial r})^k (r\eta)^j,$$

and we have

$$\sigma_\wedge^\mu(a_{11})(\pm 1) = r^{-\mu} \sum_{j=0}^{\mu} r^j (\pm 1)^j \sum_{k=0}^{\mu-j} b_{jk}(0)(-r\frac{\partial}{\partial r})^k$$

(because of κ_λ- homogeneity of $\sigma_\wedge^\mu(a_{11})(\eta)$ it suffices to consider $|\eta| = 1$).

2 Meromorphic families of cone operators

2.1 Holomorphic families

We now consider families of operators in the cone algebra on $\mathbb{B}$ parametrised by $\eta \in \mathbb{R}$ and pass to holomorphic families by applying a kernel cut-off construction with respect to η. Let us first illustrate the idea for the case of families $b(\eta)$ with values in $L_{\mathrm{cl}}^\mu(2\mathbb{B};\mathbb{R})$ as they occur in Definition 1.14.

Set

$$k(b)(\tau) := \int e^{i\tau\eta} b(\eta) đ\eta,$$

and let $\varphi(\tau) \in C_0^\infty(\mathbb{R})$. Then

$$H(\varphi)b(\eta + i\zeta) := \int e^{-i\tau(\eta+i\zeta)} \varphi(\tau) k(b)(\tau) d\tau$$

is well-defined for all $w = \eta + i\zeta \in \mathbb{C}$ and is an element in $\mathcal{A}(\mathbb{C}, L_{\mathrm{cl}}^\mu(2\mathbb{B}))$. It has the property

$$H(\varphi)b(\eta + i\zeta) \in L_{\mathrm{cl}}^\mu(2\mathbb{B};\mathbb{R}_\eta)$$

for every $\zeta \in \mathbb{R}$, uniformly in $c \le \zeta \le c'$ for arbitrary $c \le c'$. The operator function $H(\varphi)b(w)$ belongs to an analogue $M_{\boldsymbol{O}}^\mu(2\mathbb{B})$ of the space $M_O^\mu(2\mathbb{B})$, cf. Definition 1.11 for the case $q = 0$; the only difference to the present notation is that we have interchanged the role of real and imaginary axis (to avoid confusion, holomorphy in w with $\mathrm{Re}\, w$ as parameter will be indicated by $\boldsymbol{O}$). Another important observation is the following remark.

Remark 2.1 *If $\psi \in C_0^\infty(\mathbb{R})$ is a cut-off function with respect to $\tau = 0$, i.e., $\psi \equiv 1$ in a neighbourhood of the origin, we have*

$$H(\psi)b(\eta) = b(\eta) \mod L^{-\infty}(2\mathbb{B};\mathbb{R}).$$

In a similar manner we apply the kernel cut-off operator $H(\psi)$ to $a(\eta)$ in Definition 1.14 with respect to the parameter η. We then have to consider

$$H(\psi)a(\eta + i\zeta) = H(\psi)c(\eta + i\zeta) + H(\psi)f(\eta + i\zeta),$$

cf. formula (15).

First we want to analyse $H(\psi)c(\eta+i\zeta)$. By virtue of Remark 2.1 it remains to consider

$$H(\psi)\{\omega(r)r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta+i\zeta)\omega_1(r)\} = \omega(r)r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(H(\psi)h)(\eta+i\zeta)\omega_1(r), \tag{24}$$

cf. formula (16). By assumption on the structure of h we have $h(r,z,\eta) = \tilde{h}(r,z,r\eta)$ for an operator function $\tilde{h}(r,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X;\mathbb{R}_{\tilde{\eta}}))$. The dependence of $\tilde{h}(r,z,\tilde{\eta})$ on r with smoothness up to $r=0$ does not cause additional difficulties.

Setting

$$k_\eta(h)(r,z,\tau) := \int e^{i\tau\eta}\tilde{h}(r,z,r\eta)đ\eta, \;\; k_{\tilde{\eta}}(\tilde{h})(r,z,\tilde{\tau}) := \int e^{i\tilde{\tau}\tilde{\eta}}\tilde{h}(r,z,\tilde{\eta})đ\tilde{\eta},$$

yields

$$k_\eta(h)(r,z,\tau) = r^{-1}\int e^{i\frac{\tau}{r}\tilde{\eta}}\tilde{h}(r,z,\tilde{\eta})đ\tilde{\eta} = r^{-1}k_{\tilde{\eta}}(\tilde{h})(r,z,\frac{\tau}{r}).$$

Then, if $\psi(\tau)$ is any cut-off function, we get

$$\begin{aligned} H(\psi)h(r,z,\eta+i\zeta) &= \int e^{-i\tau(\eta+i\zeta)}\psi(\tau)k_\eta(h)(r,z,\tau)d\tau \\ &= r^{-1}\int e^{-i\tau(\eta+i\zeta)}\psi(\tau)k_{\tilde{\eta}}(\tilde{h})(r,z,\frac{\tau}{r})d\tau = \int e^{-i\tilde{\tau}r(\eta+i\zeta)}\psi(r\tilde{\tau})k_{\tilde{\eta}}(\tilde{h})(r,z,\tilde{\tau})d\tilde{\tau} \\ &= (H(\psi_r)\tilde{h})(r,z,r(\eta+i\zeta)) \end{aligned} \tag{25}$$

for $\psi_r(\tilde{\tau}) := \psi(r\tilde{\tau})$ which is an r- dependent cut-off function.

Definition 2.2 *Let $M_{\boldsymbol{O}}^\mu(X;\overline{\mathbb{R}}_+\times\mathbb{C}_w)$ denote the space of all operator functions $f(r,z,w) = \tilde{f}(r,z,rw)$ for $\tilde{f}(r,z,\tilde{w}) \in \mathcal{A}(\mathbb{C}_{\tilde{w}}, C^\infty(\mathbb{R}_+, M_O^\mu(X))$ such that*

$$\tilde{f}(r,z,rw) \in \mathcal{A}(\mathbb{C}_w, C^\infty(\mathbb{R}_+, M_O^\mu(X)) \tag{26}$$

and

$$\tilde{f}(r,z,\tilde{\eta}+ir\zeta) \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X;\mathbb{R}_{\tilde{\eta}})) \tag{27}$$

for every $\zeta\in\mathbb{R}$, uniformly in $c\le\zeta\le c'$ for every $c\le c'$.

As above we set

$$h(r,z,\eta) = \tilde{h}(r,z,r\eta) \tag{28}$$

for an $\tilde{h}(r,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X;\mathbb{R}_{\tilde{\eta}}))$ and set

$$\sigma_M^\mu(h)(z) := \tilde{h}(0,z,0).$$

In particular, for any $f(r,z,w) \in M_{\boldsymbol{O}}^\mu(X;\overline{\mathbb{R}}_+\times\mathbb{C}_w)$ we can form $\sigma_M(f)(z) = \tilde{f}(0,z,0)$, using property (27).

Proposition 2.3 *Let $\psi \in C_0^\infty(\mathbb{R})$ be a cut-off function. Then, if $h(r,z,\eta)$ is defined by* (28) *we have $(H(\psi))h(r,z,w) \in M^\mu_{\boldsymbol{O}}(X;\overline{\mathbb{R}}_+ \times \mathbb{C}_w)$, and*

$$\sigma^\mu_M(h)(z) = \sigma^\mu_M(H(\psi)h)(z). \tag{29}$$

Proof. The property (26) for $f(r,z,w) := H(\psi)h(r,z,w)$ is an immediate consequence of kernel cut-off operators which produce holomorphic functions in $w \in \mathbb{C}$, see [14, Section 2.2.2]. For (27) we first consider

$$(H(\psi_{r_1})\tilde{h})(r,z,\tilde{\eta}+ir_2\zeta), \tag{30}$$

where r_1 in the first argument indicates dependence on the half-axis variable coming from ψ_{r_1}, while r_2 comes from the r- factor at ζ. To verify the smoothness of (27) in r up to $r=0$ it suffices to show that (30) is C^∞ in $(r_1,r_2) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$. The smoothness in $r_2 \in \overline{\mathbb{R}}_+$ is evident for every $\zeta \in \mathbb{R}$, uniformly in finite intervals. The crucial point is the smoothness in r_1 which is a parameter in the family of cut-off functions $\psi(r_1\tilde{\tau})$ that tends to 1 for $r_1 \to 0$. Here we can apply [14, Section 1.1.3, Remark 1.1.51] which shows the desired smooth dependence in the parameter up to 0 in the topology of symbols. At the same time we get relation (29). □

We now introduce a notion of holomorphic dependence of families of operators $a(w), w=\eta+i\zeta$, with values in $\mathcal{C}^\mu(\mathbb{B};\boldsymbol{g},j_-,j_+)$ (the latter space is included in Definition 1.14 for $q=0$). We separately consider the ingredients of Definition 1.14. First, as a variant of Definition 1.11 we have the space $M^\mu_{\boldsymbol{O}}(2\mathbb{B})$ of all functions $b(w) \in \mathcal{A}(\mathbb{C}_w, L^\mu_{\mathrm{cl}}(2\mathbb{B}))$ such that $b(\eta+i\zeta) \in L^\mu_{\mathrm{cl}}(2\mathbb{B};\mathbb{R}_\eta)$ for every $\zeta \in \mathbb{R}$, uniformly in $c \le \zeta \le c'$ for every $c \le c'$.

Analogously to (16) we now form

$$c_{11}(w) := \omega(r)r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(w)\omega_1(r) + (1-\omega_1(r))b(w)(1-\omega_2(r)), \tag{31}$$

for any $f(r,z,w) \in M^\mu_{\boldsymbol{O}}(X;\overline{\mathbb{R}}_+ \times \mathbb{C})$ and $b(w) \in M^\mu_{\boldsymbol{O}}(2\mathbb{B})$; (without loss of generality, we may set $f = H(\psi)h$ as in Proposition 2.3).

For every fixed $\zeta \in \mathbb{R}$ the operator family

$$\omega(r)r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta+i\zeta)\omega_1(r) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

has the structure of an edge symbol with covariable η (and constant coefficients) in the representation from [4]. This is particularly convenient under the aspect of holomorphy in w. The operators (31) are a holomorphic family

$$c_{11}(w) : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{B})$$

for every $s \in \mathbb{R}$.

We now perform an analogous construction for the second term $f(\eta)$ in formula (15). In this case in the representation (9) we set $f_1 \equiv 0$ (because holomorphic smoothing families acting outside a neighbourhood of $\partial\mathbb{B}$ are already contained in $b(w)$), and it suffices to look at $f_0(\eta)$.

Definition 2.4 *Let $E = \varprojlim E^j, \tilde{E} = \varprojlim \tilde{E}^k$ be Fréchet spaces, written as projective limits of Hilbert spaces, with group action, cf. the notation in Section* 1.1*, and set $\mathcal{L}(E, \tilde{E})_{\mathrm{r}} := \bigcap_{j\in\mathbb{N}} \mathcal{L}(E^{\mathrm{r}(j)}, \tilde{E}^j)$ for some map $\mathrm{r} : \mathbb{N} \to \mathbb{N}$. Then $S^\mu_{\mathrm{cl}}(\mathbb{C}; E, \tilde{E})_{\mathrm{r}}$ denotes the space of all $f(w) \in \mathcal{A}(\mathbb{C}, \mathcal{L}(E, \tilde{E})_{\mathrm{r}})$ such that $f(\eta + i\zeta) \in \bigcap_{j\in\mathbb{N}} S^\mu_{\mathrm{cl}}(\mathbb{R}_\eta; E^{\mathrm{r}(j)}, \tilde{E}^j)$ for every $\zeta \in \mathbb{R}$, uniformly in $c \le \zeta \le c'$ for every $c \le c'$. Set $S^\mu_{\mathrm{cl}}(\mathbb{C}; E, \tilde{E}) = \bigcup_{\mathrm{r}} S^\mu_{\mathrm{cl}}(\mathbb{C}; E, \tilde{E})_{\mathrm{r}}$.*

Setting $k(a)(\tau) := \int e^{i\tau\eta} a(\eta) d\eta$ for an $a(\eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}; E, \tilde{E})$ for every $\psi \in C_0^\infty(\mathbb{R})$ we have a kernel cut-off map $H(\psi)a(\eta+i\zeta) := \int e^{-i\tau(\eta+i\zeta)} \psi(\tau) k(a)(\tau) d\tau$,

$$H(\psi) : S^\mu_{\mathrm{cl}}(\mathbb{R}; E, \tilde{E}) \to S^\mu_{\mathrm{cl}}(\mathbb{C}; E, \tilde{E}) \tag{32}$$

which is continuous between the relevant subspaces with supscript r. We shall apply this, in particular, to cut-off functions ψ and obtain, similarly to Remark 2.1,

$$H(\psi)a(\eta) = a(\eta) \mod S^{-\infty}(\mathbb{R}^q; E, \tilde{E}).$$

The kernel cut-off (32) can be applied to symbols $a(\eta) \in \mathcal{R}^\mu_{M+G}(\mathbb{R}^q; \boldsymbol{g}, j_-, j_+)$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, as operator-valued symbols in $S^\mu_{\mathrm{cl}}(\mathbb{R}^q; E, \tilde{E})$ for spaces $E = \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}, \tilde{E} = \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge) \oplus \mathbb{C}^{j_+}$ for arbitrary $s \in \mathbb{R}$, as well as for (Fréchet) subspaces with asymptotics. In particular, Definition 2.4 can be specified for $\mathcal{R}^\mu_G(\mathbb{R}; \boldsymbol{g}, j_-, j_+)_{P,Q}$. This gives us a space of holomorphic Green symbols $\mathcal{R}^\mu_G(\mathbb{C}; \boldsymbol{g}, j_-, j_+)_{P,Q}$ and a corresponding kernel cut-off map

$$H(\psi) : \mathcal{R}^\mu_G(\mathbb{R}; \boldsymbol{g}, j_-, j_+)_{P,Q} \to \mathcal{R}^\mu_G(\mathbb{C}; \boldsymbol{g}, j_-, j_+)_{P,Q}$$

for any cut-off function ψ. Set

$$\mathcal{R}^\mu_G(\mathbb{C}; \boldsymbol{g}, j_-, j_+) := \bigcup_{P,Q} \mathcal{R}^\mu_G(\mathbb{C}; \boldsymbol{g}, j_-, j_+)_{P,Q}.$$

Next, starting from an operator family $m_0(\eta)$ of the form (8) we construct

$$m(w) := H(\psi) m_0(w), \tag{33}$$

where ψ is any cut-off function. Then it can easily be checked that the function $m(\eta + i\zeta)$ of η belongs to $\mathcal{R}^\mu_{M+G}(\mathbb{R}; \boldsymbol{g}, j_-, j_+)$ for every fixed $\zeta \in \mathbb{R}$, with a uniformity condition in intervals $c \le \zeta \le c'$ for arbitrary $c \le c'$. We now define

$$\mathcal{R}^\mu_{M+G}(\mathbb{C}; \boldsymbol{g}, j_-, j_+) := \{m(w) + g(w)\}, \tag{34}$$

where m is of the form (33) for any m_0 and $g \in \mathcal{R}^\mu_G(\mathbb{C}; \boldsymbol{g}, j_-, j_+)$.

Finally, by $\mathcal{M}^{-\infty}_{\boldsymbol{O}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ we denote the union over $P \in \mathrm{As}(X; (\gamma - \mu, \Theta)), Q \in \mathrm{As}(X; (-\gamma, \Theta))$ of all spaces

$$\mathcal{A}(\mathbb{C}_w, \mathcal{C}_G(\mathbb{B}; \boldsymbol{g}, j_-, j_+)_{P,Q}) \ni k(w)$$

such that

$$k(\eta + i\zeta) \in \mathcal{C}^{-\infty}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}_\eta)_{P,Q}$$

for every $\zeta \in \mathbb{R}$, uniformly in finite intervals $c \leq \zeta \leq c'$ for arbitrary $c \leq c'$.

Definition 2.5 *Let $\mathcal{M}^\mu_{\boldsymbol{O}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ for $\Theta := (-(k+1), 0], k \in \mathbb{N}$, denote the space of all operator families $a(w), w = \eta + i\zeta \in \mathbb{C}$, with values in $\mathcal{C}^\mu(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ such that in the representation*

$$a(w) = c(w) + f(w) + k(w), \quad c(w) = \begin{pmatrix} c_{11}(w) & 0 \\ 0 & 0 \end{pmatrix},$$

$c_{11}(w)$ is as in (31), *further $f(w) = \mathrm{diag}(\omega, 1) f_0(w) \mathrm{diag}(\omega_1, 1)$ for an element $f_0(w) \in \mathcal{R}^\mu_{M+G}(\mathbb{C}; \boldsymbol{g}, j_-, j_+)$, and $k(w) \in \mathcal{M}^{-\infty}_{\boldsymbol{O}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$.*

From the constructions of this section we obtain altogether the following result:

Theorem 2.6 *For every $a(\eta) \in \mathcal{C}^\mu(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R})$ there exists an $h(w) \in \mathcal{M}^\mu_{\boldsymbol{O}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+), w = \eta + i\zeta \in \mathbb{C}$, such that*

(i) $h(\eta) - a(\eta) \in \mathcal{C}^{-\infty}(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R})$,

(ii) *$h(\eta + i\zeta) \in C^\mu(\mathbb{B}; \boldsymbol{g}, j_-, j_+; \mathbb{R}_\eta)$ for all $\zeta \in \mathbb{R}$, uniformly in $c \leq \zeta \leq c'$ for every $c \leq c'$.*

Remark 2.7 *If $a(\eta)$ is elliptic, cf. Definition* 1.18, *then $h(w), w = \eta + i\zeta$, associated with $a(\eta)$ via Theorem* 2.6 *is elliptic for every $\zeta \in \mathbb{R}$. Then we say that $h(w)$ is elliptic.*

2.2 The algebra of meromorphic families

Next let $\boldsymbol{R}$ denote a sequence

$$\boldsymbol{R} = \{(p_j, m_j, \boldsymbol{N}_j)\}_{j \in \mathbb{Z}}$$

with $p_j \in \mathbb{C}, m_j \in \mathbb{N}$ and $\boldsymbol{N}_j \subset \mathcal{C}^{-\infty}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ which are finite-dimensional spaces of operators of finite rank, such that $\pi_\mathbb{C} \boldsymbol{R} := \{p_j\}_{j \in \mathbb{Z}}$ has the property that $\pi_\mathbb{C} \boldsymbol{R} \cap \{w : c \leq \mathrm{Im}\, w \leq c'\}$ is a finite set for every $c \leq c'$. Let $\boldsymbol{As}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ denote the set of all such sequences. Moreover, ${}^{\mathrm{t}}\boldsymbol{R}$ will denote the transposed of $\boldsymbol{R}$, i.e., ${}^{\mathrm{t}}\boldsymbol{R} = \{(p_j, m_j, {}^{\mathrm{t}}\boldsymbol{N}_j)\}_{j \in \mathbb{Z}}$ when $\boldsymbol{R} = \{(p_j, m_j, \boldsymbol{N}_j)\}_{j \in \mathbb{Z}}$.

Definition 2.8 *Define $\mathcal{M}^{-\infty}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ for $\boldsymbol{R} \in \boldsymbol{As}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ to be the space of all operator families $a(w) \in \mathcal{A}(\mathbb{C} \setminus \pi_\mathbb{C} \boldsymbol{R}, \mathcal{C}^{-\infty}(\mathbb{B}; \boldsymbol{g}, j_-, j_+))$ such that*

(i) *for any $\pi_\mathbb{C} \boldsymbol{R}$ - excision function χ we have $(\chi a)(\eta + i\zeta) \in \mathcal{S}(\mathbb{R}_\eta, \mathcal{C}^{-\infty}(\mathbb{B}; \boldsymbol{g}, j_-, j_+))$ for every $\zeta \in \mathbb{R}$, uniformly in $c \leq \zeta \leq c'$ for every $c \leq c'$,*

(ii) *close to every* $p_j \in \pi_{\mathbb{C}} \boldsymbol{R}$ *the function* a *has a representation*

$$a(w) = \sum_{k=0}^{m_j} a_{jk}(w - p_j)^{-(k+1)} + h(w) \tag{35}$$

with coefficients $a_{jk} \in \boldsymbol{N}_j, 0 \le k \le m_j$, *where* $h(w)$ *is holomorphic near* p_j *with values in* $\mathcal{C}^{-\infty}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$.

Remark 2.9 *The spaces* $\mathcal{M}^{\mu}_{\boldsymbol{O}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ *and* $\mathcal{M}^{-\infty}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ *are Fréchet (corresponding systems of semi-norms are immediate from the definition), and for every* $\boldsymbol{R} \in \boldsymbol{As}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ *we define*

$$\begin{aligned} \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+) &= \mathcal{M}^{\mu}_{\boldsymbol{O}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+) \\ &+ \mathcal{M}^{-\infty}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+) \end{aligned} \tag{36}$$

in the sense of a non-direct sum.

It follows that close to every $p_j \in \pi_{\mathbb{C}} \boldsymbol{R}$ *an element* $a \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ *can be written in the form* (35) *with coefficients* $a_{jk} \in \boldsymbol{N}_j, 0 \le k \le m_j$, *and* $h(w)$ *holomorphic near* p_j *with values in* $\mathcal{C}^{\mu}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$.

Note that differentiation with respect to w induces continuous maps

$$\frac{d}{dw} : \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+) \to \mathcal{M}^{\mu-1}_{\boldsymbol{R}'}(\mathbb{B}; (\gamma, \gamma - \mu + 1, \Theta), j_-, j_+)$$

for every $\boldsymbol{R} \in \boldsymbol{As}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ with a resulting asymptotic type $\boldsymbol{R}' \in \boldsymbol{As}(\mathbb{B}; (\gamma, \gamma - \mu + 1, \Theta), j_-, j_+)$. Below we simply set $a'(w) := \frac{d}{dw} a(w)$.

Theorem 2.10 *If* $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; (\gamma - \nu, \gamma - \nu - \mu, \Theta), l, j_+)$ *for* $\boldsymbol{R} \in \boldsymbol{As}(\mathbb{B}; (\gamma - \nu, \gamma - \nu - \mu, \Theta), l, j_+)$ *and* $b(w) \in \mathcal{M}^{\nu}_{\boldsymbol{Q}}(\mathbb{B}; (\gamma, \gamma - \nu, \Theta), j_-, l)$ *for* $\boldsymbol{Q} \in \boldsymbol{As}(\mathbb{B}; (\gamma, \gamma - \nu, \Theta), j_-, l)$, *then* $a(w)b(w) \in \mathcal{M}^{\mu+\nu}_{\boldsymbol{P}}(\mathbb{B}; (\gamma, \gamma - \mu - \nu, \Theta), j_-, j_+)$, *with a resulting asymptotic type* $\boldsymbol{P} \in \boldsymbol{As}(\mathbb{B}; (\gamma, \gamma - \mu - \nu, \Theta), j_-, j_+)$ *determined by* a *and* b.

Proof. The composition for every fixed $w \in \mathbb{C}$ is stated in Theorem 1.17. Locally, in finite regions of $\mathbb{C}$, we have compositions of meromorphic operator functions, and we easily see that the Laurent coefficients of the product are generated by the factors in a similar way as for scalar meromorphic functions; in the present case we employ that the coefficient spaces $\boldsymbol{N}_j$ in the asymptotic types are of finite rank and smoothing; those are an ideal in the cone algebra on $\mathbb{B}$, which yields analogous Laurent coefficient spaces for the resulting asymptotic type $\boldsymbol{P}$. $\square$

Let $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$. Then (36) gives us a decomposition

$$a(w) = a_{\boldsymbol{o}}(w) + a_{\boldsymbol{1}}(w), \tag{37}$$

where $a_{\boldsymbol{o}}(w) \in \mathcal{M}^{\mu}_{\boldsymbol{O}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ and $a_{\boldsymbol{1}}(w) \in \mathcal{M}^{-\infty}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$.

Definition 2.11 *An element* $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ *is said to be elliptic if so is* $a_{\boldsymbol{o}}(w)$ *for any decomposition of* $a(w)$ *of the form* (37).

This definition is correct, i.e., independent of the particular choice of the splitting (37) of a.

Theorem 2.12 *Let* $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+), \boldsymbol{R} \in \boldsymbol{As}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$, *be elliptic. Then there is a countable subset* $D \subset \mathbb{C}$, *with finite intersection* $D \cap \{w : c \leq \operatorname{Im} w \leq c'\}$ *for every* $c \leq c'$, *such that*

$$a(w) : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{B}) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \rightarrow \begin{array}{c} \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{B}) \\ \oplus \\ \mathbb{C}^{j_+} \end{array}$$

is invertible for all $w \in \mathbb{C} \setminus D$. *Moreover, there is an inverse* $a^{-1}(w)$ (*in the sense of the composition of Theorem* 2.10) *belonging to* $\mathcal{M}^{-\mu}_{\boldsymbol{Q}}(\mathbb{B}; (\gamma - \mu, \gamma, \Theta), j_+, j_-)$ *for a resulting* $\boldsymbol{Q} \in \boldsymbol{As}(\mathbb{B}; (\gamma - \mu, \gamma, \Theta), j_+, j_-)$.

Proof. Write $a(w)$ in the form (37). Because of Remark 1.20 we can apply results from [14, Section 1.2.4] and obtain an $h(w) \in \mathcal{M}^{-\mu}_{\boldsymbol{S}}(\mathbb{B}; (\gamma - \mu, \gamma, \Theta), j_+, j_-)$ for some $\boldsymbol{S} \in \boldsymbol{As}(\mathbb{B}; (\gamma-\mu, \gamma, \Theta), j_+, j_-)$ such that $a_{\boldsymbol{o}}(w)h(w) = h(w)a_{\boldsymbol{o}}(w) = 1$ for all $w \in \mathbb{C}$.

Proposition 2.10 gives us $h(w)a_{\boldsymbol{1}}(w) \in \mathcal{M}^{-\infty}_{\boldsymbol{P}}(\mathbb{B}; (\gamma, \gamma, \Theta), j_-, j_-)$ for some $\boldsymbol{P} \in \boldsymbol{As}(\mathbb{B}; (\gamma, \gamma, \Theta), j_-, j_-)$. There is now an analogue of Lemma 4.3.13 of [11] that can be applied in our case, i.e., there is an element $g(w) \in \mathcal{M}^{-\infty}_{\boldsymbol{Q}}(\mathbb{B}; (\gamma, \gamma, \Theta), j_-, j_-)$ for a $\boldsymbol{Q} \in \boldsymbol{As}(\mathbb{B}; (\gamma, \gamma, \Theta), j_-, j_-)$ such that $(1 + h(w)a_{\boldsymbol{1}}(w))^{-1} = 1 + g(w)$, cf. Theorem 2.10. Now it is easy to see that $(1 + g(w))h(w)$ is a left inverse of $a(w)$. Analogously, starting from $a_{\boldsymbol{1}}(w)h(w)$, we get a right inverse for $a(w)$. $\square$

The operator family $a^{-1}(w)$ can be interpreted as the *resolvent* of $a(w)$.

2.3 Characteristic values and a factorisation of meromorphic families

Let $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ be elliptic. For a fixed $w \in \mathbb{C}$ away from the set of poles, $a(w)$ defines an operator $\mathcal{H}^{s,\gamma}(\mathbb{B}) \oplus \mathbb{C}^{j_-} \rightarrow \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{B}) \oplus \mathbb{C}^{j_+}$. The particular choice of s is not important, because the kernel and cokernel of $a(w)$ consist of functions in $\mathcal{H}^{\infty,\gamma}(\mathbb{B}) \oplus \mathbb{C}^{j_-}$ and $\mathcal{H}^{\infty,\gamma-\mu}(\mathbb{B}) \oplus \mathbb{C}^{j_+}$, respectively.

A point $w_0 \in \mathbb{C}$ is called a *characteristic value* of $a(w)$, if there exists a vector-valued function $u(w)$ with values in $\mathcal{H}^{s,\gamma}(\mathbb{B}) \oplus \mathbb{C}^{j_-}$, holomorphic in a neighbourhood of w_0 with $u(w_0) \neq 0$, such that the vector-valued function $a(w)u(w)$ is holomorphic at w_0 and vanishes at this point. We call $u(w)$ a *root function* of $a(w)$ at w_0. Suppose that w_0 is a characteristic value of $a(w)$ und $u(w)$ a corresponding root function. The order of w_0 as a zero of $a(w)u(w)$ is called the *multiplicity* of $u(w)$, and the vector $u_0 = u(w_0)$ an *eigenvector* of $a(w)$ at w_0. The eigenvectors of $a(w)$ at w_0 (together with the zero function) form a vector space. This space is called the *kernel* of $a(w)$ at w_0, and is denoted by $\ker a(w_0)$. By the *rank* of an eigenvector u_0 we mean the supremum of the multiplicities of all root functions $u(w)$ such that $u(w_0) = u_0$.

The elements $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ represent meromorphic operator functions in $\mathbb{C}$ with values in $\mathcal{L}(\mathcal{H}^{s,\gamma}(\mathbb{B}) \oplus \mathbb{C}^{j_-}, \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{B}) \oplus \mathbb{C}^{j_+}), s \in \mathbb{R}$, taking values in $\mathcal{C}^{\mu}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ for every $w \in \mathbb{C} \backslash \pi_{\mathbb{C}} \boldsymbol{R}$. Meromorphy or holomorphy of operator functions with such properties also makes sense when w varies in an arbitrary open subset of $\mathbb{C}$. This will be the interpretation of locally given operator functions in the following consideration.

Proposition 2.13 *Let w_0 be a characteristic value of $a(w)$. Then*

(i) *the space* $\ker a(w_0)$ *is a finite-dimensional subspace of* $\mathcal{H}^{\infty,\gamma}(\mathbb{B}) \oplus \mathbb{C}^{j_-}$,

(ii) *the rank of each eigenfunction of $a(w)$ at w_0 is finite.*

Proof. (i) We have

$$a(w) = \sum_{j=-m}^{-1} a_j (w - w_0)^j + h(w) \tag{38}$$

in some neighbourhood U of w_0, where $a_j \in \mathcal{C}^{-\infty}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ are of finite rank for $j = -m, \ldots, -1$, and $h(w)$ is a holomorphic function near w_0 with values in $\mathcal{C}^{\mu}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$.

Relations (38) and (37) imply that $h(w_0) \in \mathcal{C}^{\mu}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ is elliptic.

Now the vector-valued function $u(w)$ is a root function of $a(w)$ at w_0 if and only if

$$h(w_0)u(w_0) = -\sum_{k=1}^{m} \frac{1}{k!} a_{-k} u^{(k)}(w_0), \quad \sum_{k=0}^{m+\nu} \frac{1}{k!} a_{\nu-k} u^{(k)}(w_0) = 0 \tag{39}$$

for all $\nu = -m, \ldots, -1$. The first equation of (39) yields that $h(w_0)u(w_0)$ belongs to a finite-dimensional subspace of $\mathcal{H}^{\infty,\gamma-\mu}(\mathbb{B}) \oplus \mathbb{C}^{j_+}$. Hence the ellipticity of $h(w_0)$ shows that $u(w_0)$ lies in a finite-dimensional subspace of $\mathcal{H}^{\infty,\gamma} \oplus \mathbb{C}^{j_-}$.

(ii) Let $u(w)$ be a root function of $a(w)$ at w_0. Then $g(w) := a(w)u(w)$ is holomorphic near w_0 and $g(w_0) = 0$. Choose a neighbourhood U of w_0 so that $a(w)$ is invertible for all $w \in U \setminus \{w_0\}$ (cf. Theorem 2.12). We have

$$u(w) = a^{-1}(w) g(w) \quad \text{for} \quad w \in U \setminus \{w_0\}.$$

Because of $u(w_0) \neq 0$ the order of w_0 as a zero of $g(w)$ does not exceed the order of w_0 as a pole of $a^{-1}(w)$. $\square$

By a *canonical system* of eigenvectors of $a(w)$ at w_0 we understand a system of eigenvectors $u_0^{(1)}, \ldots, u_0^{(N)}$, $N = \dim\ker a(w_0)$, with the following property: $\operatorname{rank} u_0^{(1)}$ is the maximum of the ranks of all eigenvectors of $a(w)$ at w_0; $\operatorname{rank} u_0^{(i)}, i = 2, \ldots, N$, is the maximum of the ranks of all eigenvectors in a direct complement in $\ker a(w_0)$ of the linear span of the vectors $u_0^{(1)}, \ldots, u_0^{(i-1)}$. Let $r_i = \operatorname{rank} u_0^{(i)}$, $i = 1, \ldots, N$. The rank of any eigenvector of $a(w)$ at w_0 is always equal to one of the r_i. Hence the numbers r_i are uniquely determined by the function $a(w)$. Note that, in general, a canonical system of eigenvectors is not uniquely determined. The numbers r_i are called the *partial null-multiplicities*, and $\boldsymbol{n}(a(w_0)) = r_1 + \ldots + r_N$ the *null-multiplicity* of the characteristic value w_0 of $a(w)$. If $a(w)$ has no root function at w_0, we set $\boldsymbol{n}(a(w_0)) = 0$. We call both the characteristic values of $a(w)$ and $a^{-1}(w)$ the *singular values* of $a(w)$. Suppose that w_0 is a characteristic value of $a^{-1}(w)$. Let $P = \dim\ker a^{-1}(w_0)$, and $\rho_1, \ldots, \rho_P$ are partial null multiplicities of characteristic value w_0 of $a^{-1}(w)$. Then we call $\rho_1, \ldots, \rho_P$ the *partial polar-multiplicities* of the singular value w_0 of $a(w)$, and $\boldsymbol{p}(a(w_0)) = \rho_1 + \ldots \rho_P$ the *polar-multiplicity* of the singular value w_0 of $a(w)$. We then call $\boldsymbol{m}(a(w_0)) = \boldsymbol{n}(a(w_0)) - \boldsymbol{p}(a(w_0))$ the *multiplicity* of a singular value w_0 of $a(w)$. If $a(w)$ is holomorphic at a point $w_0 \in \mathbb{C}$ and the operator $a(w_0)$ is invertible, then w_0 is said to be a *regular point* of $a(w)$.

Remark 2.14 *According to Theorem* 2.12 *for every* $a \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$, $\boldsymbol{R} \in \boldsymbol{As}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$, *the singular values of* a *form a countable set* $D \subset \mathbb{C}$ *with finite intersections* $D \cap \{w : c \leq \operatorname{Im} w \leq c'\}$ *for every* $c \leq c'$.

Remark 2.15 *Let* $\pi_j, j = 1, \ldots, n$ *be a system of mutually orthogonal projections with* $\sum_{j=1}^{n} \pi_j = 1$ *and* $a_j(w)$ *holomorphic in some punctured neighbourhood of* $w_0 \in \mathbb{C}$ *acting in* $\pi_j(\mathcal{H}^{s,\gamma}(\mathbb{B}) \oplus \mathbb{C}^{j_-})$. *Then if*

$$a(w) = \sum_{j=1}^{n} a_j(w)\pi_j,$$

and if w_0 *is a characteristic value of* $a(w)$, *a vector-valued function* $u(w)$ *is a root function of* $a(w)$ *at* w_0 *if and only if for each* $j = 1, \ldots, n$ *the vector-valued function* $\pi_j u(w)$ *is either a root function of* $a_j(w)$ *at* w_0 *or is identically zero, and not all of the* $\pi_j u(w)$ *are identically zero. The multiplicity of* $u(w)$ *is equal to the minimum of the multiplicities of the root functions* $\pi_j u(w)$ *of the operator-valued functions* $a_j(w)$. *On the other hand, each root function of* $a_j(w)$ *at* w_0 *is a root function of* $a(w)$ *at* w_0, *and the corresponding multiplicities are equal.*

The nature of projections π_j *will be specified below in Theorem* 2.17 *together with Remark* 2.16.

Clearly, in the scalar case the multiplicity of a singular value w_0 of $a(w)$ is equal to the order of the pole, if w_0 is a pole of $a(w)$, and the multiplicity of the zero, if w_0 is a zero of $a(w)$.

Remark 2.16 *Let w_0 be a characteristic value of $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$. If $b_1(w), b_2(w)$ are invertible holomorphic functions near w_0 with values in $\mathcal{C}^{\mu_1}(\mathbb{B}; (\gamma_0, \gamma, \Theta), j_0, j_-)$ and $\mathcal{C}^{\mu_2}(\mathbb{B}; (\gamma - \mu, \gamma_1, \Theta), j_+, j_1)$, respectively, then w_0 is a characteristic value of $c(w) := b_2(w)a(w)b_1(w)$ and the partial null and polar multiplicities of w_0 for $a(w)$ and $c(w)$ coincide.*

Theorem 2.17 *Let $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ be elliptic and $w_0 \in \mathbb{C}$ a singular value of $a(w)$. Then, in a neighbourhood of w_0 there are invertible holomorphic functions $b_1(w)$ and $b_2(w)$ with values in $\mathcal{C}^0(\mathbb{B}; (\gamma, \gamma, \Theta), j_-, j_-)$ and $\mathcal{C}^{\mu}(\mathbb{B}; \boldsymbol{g}; j_-, j_+)$, repectively, satisfying the representation*

$$a(w) = b_2(w)\{\pi_0 + \sum_{j=1}^{n} \pi_j (w - w_0)^{m_j}\} b_1(w), \tag{40}$$

where $\pi_j, j = 0, \dots, n$, are mutually orthogonal projections such that $\pi_j \in \mathcal{C}^{-\infty}(\mathbb{B}; (\gamma, \gamma, \Theta), j_-, j_-), j = 1, \dots, n$, are of rank 1 with $\pi_0 + \sum_{j=1}^{n} \pi_j = 1$, and $m_1 \le m_2 \le \dots \le m_n$ are integers.

Proof. This theorem is an analogue of [16, Proposition 3.1]. □

Corollary 2.18 *Let $a(w), b_1(w), b_2(w)$ and w_0 be as in Theorem 2.17. Using (40) we get*

$$a^{-1}(w) = b_1^{-1}(w)\{\pi_0 + \sum_{j=1}^{n} \pi_j (w - w_0)^{-m_j}\} b_2^{-1}(w) \tag{41}$$

in a punctured neighbourhood $U \setminus \{w_0\}$ of w_0.

Suppose the numbers $m_j, j = 1, \dots, n$, from (40) satisfy the conditions $m_1 \le \dots \le m_r < 0, m_{r+1} = \dots = m_{r+p} = 0$ and $0 < m_{r+p+1} \le \dots \le m_n$. Then, using Remark 2.15 and (40), (41), we obtain that the partial null multiplicities of the singular value w_0 of $a(w)$ are equal to $m_{r+p+1}, \dots, m_n$; the partial polar multiplicities are equal to $m_1, \dots, m_r$. In particular, it follows that $\boldsymbol{m}(a(w_0)) = \sum_{j=1}^{n} m_j$.

For $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ let ${}^{\mathrm{t}}a(w)$ denote the transposed pseudodifferential operator of $a(w)$ for any $w \in \mathbb{C}$. It is easy to see that ${}^{\mathrm{t}}a(w) \in \mathcal{M}^{\mu}_{{}^{\mathrm{t}}\boldsymbol{R}}(\mathbb{B}; (-\gamma + \mu, -\gamma), j_+, j_-)$ and is elliptic if so is $a(w)$.

Corollary 2.19 *Let $a(w)$ be elliptic. Then $a(w)$ and ${}^{\mathrm{t}}a(w)$ have the same singular values with the same partial null and polar multiplicities. In particular, $\boldsymbol{m}({}^{\mathrm{t}}a(w_0)) = \boldsymbol{m}(a(w_0))$ for any singular value w_0.*

Let $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ be elliptic, and let w_0 be a singular value of $a(w)$. Then in a punctured neighbourhood of w_0 the principal part (p.p.) of the Laurent expansion (38) of $a(w)$ is an operator in $\mathcal{C}^{-\infty}(\mathbb{B}; (\gamma, \gamma - \mu, \Theta), j_-, j_+)$ of finite rank. Hence the trace (tr) of p.p. $a'(z)a^{-1}(z)$ is well-defined.

Lemma 2.20 *Let $a(w)$ and w_0 be as above. Then we have*

$$\operatorname{tr} \text{p.p.}\, a'(w)a^{-1}(w) = \frac{\boldsymbol{m}(a(w_0))}{w - w_0}. \tag{42}$$

Proof. Cf. [16, Corollary 3.2]. $\square$

Let $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ be elliptic. Suppose w_0 is a characteristic value of $a(w)$ and $u(w)$ a root function of $a(w)$ at w_0. Denote by r the multiplicity of $u(w)$.

The vector-valued functions

$$\frac{1}{k!}\Big(\frac{\partial}{\partial w}\Big)^k u(w_0),\ k = 1, \dots, r-1,$$

are said to be associated vectors for the eigenvector $u_0 = u(w_0)$.

Remark 2.21 *For each characteristic value w_0 of $a(w)$ the associated vectors of $a(w)$ at w_0 lie in a finite dimensional subspace of $\mathcal{H}^{\infty,\gamma}(\mathbb{B}) \oplus \mathbb{C}^{j_-}$.*

Let $u_0^{(1)}, \dots, u_0^{(N)}$ be a canonical system of eigenvectors of $a(w)$ at w_0, and as above, let r_i denotes the rank of $u_0^{(i)}$. Moreover, let $u_1^{(i)}, \dots, u_{r_i-1}^{(i)}$ be associated vectors for the eigenvector $u_0^{(i)}$. Then the system

$$(u_0^{(i)}, u_1^{(i)}, \dots, u_{r_i-1}^{(i)})_{i=1,\dots,N}$$

is called a *canonical system* of eigenvectors and associated vectors of $a(w)$ at w_0.

Example 2.22 *Let $a(w) = \sum_{j=0}^{\mu} a_j w^j$, $a_j \in \mathcal{C}^{\mu-j}(\mathbb{B}; \boldsymbol{g}, j_-, j_+), j = 0, \dots, \mu$, and let w_0 be a characteristic value of $a(w)$. For convenience we assume that $\dim\ker a(w_0) = 1$. Furthermore, let u_0 be an eigenvector of rank r, and $u_1, \dots, u_{r-1}$ be associated vectors for u_0. Then we have the following relations*

$$\sum_{m=0}^{k} \frac{1}{(k-m)!} \sum_{j=k-m}^{\mu} a_j(u_m) \frac{j!}{(j-k+m)!} w_0^{j-k+m} = 0$$

for $k = 0, 1, \dots, r-1$.

Proposition 2.23 *For each characteristic value w_0 of $a(w)$, there are canonical systems*

$$(u_0^{(i)}, u_1^{(i)}, \ldots, u_{r_i-1}^{(i)})_{i=1,\ldots,N} \quad \text{and} \quad (v_0^{(i)}, v_1^{(i)}, \ldots, v_{r_i-1}^{(i)})_{i=1,\ldots,N}$$

of eigenvectors and associated vectors of $a(w)$ and ${}^{\mathrm{t}}(a(w))$ at w_0, respectively, such that

$$\text{p.p.}\, a^{-1}(w) = \sum_{i=1}^{N} \sum_{j=-r_i}^{-1} \left\{ \sum_{k=0}^{r_i+j} \langle v_k^{(i)}, \cdot \rangle u_{r_i+j-k}^{(i)} \right\} (w - w_0)^j$$

in a neighbourhood of w_0.

Proof. The meromorphic operator functions in our context may be regarded as a special case of the ones in the paper [6] of Gohberg and Sigal. In other words, we can directly apply [6, Theorem 7.1] in the present situation. □

3 Operators in the infinite cylinder

3.1 Weighted edge spaces

On the infinite cylinder with conical cross section we consider specific so-called *edge Sobolev* spaces. First, on the cylinder $\mathbb{R} \times 2\mathbb{B}$ we have the spaces $H^{s,\delta}(\mathbb{R} \times 2\mathbb{B})$, $s, \delta \in \mathbb{R}$, defined as the completion of $C_0^\infty(\mathbb{R} \times 2\mathbb{B})$ with respect to the norm

$$\left\{ \int_{I_\delta} ||R^s(\operatorname{Re} w)(Fu)(w)||^2_{L^2(2\mathbb{B})} dw \right\}^{\frac{1}{2}}.$$

Here $I_\delta := \{w \in \mathbb{C} : \operatorname{Im} w = \delta\}$ and F is the Fourier transform on $\mathbb{R}_t$ with covariable η, extended to complex arguments $w = \eta + i\zeta$ (for functions with compact support). Moreover, $R^s(\eta) \in L^s_{\mathrm{cl}}(2\mathbb{B}; \mathbb{R}_\eta)$ is a parameter-dependent elliptic family of classical pseudo-differential operators on $2\mathbb{B}$ which induce isomorphisms $R^s(\eta) : H^r(2\mathbb{B}) \to H^{r-s}(2\mathbb{B})$ for all $r, s \in \mathbb{R}, \eta \in \mathbb{R}$.

Note that for $H^s(\mathbb{R} \times 2\mathbb{B}) := H^{s,0}(\mathbb{R} \times 2\mathbb{B})$ we have

$$H^{s,\delta}(\mathbb{R} \times 2\mathbb{B}) = e^{-\delta t} H^s(\mathbb{R} \times 2\mathbb{B}).$$

Let E be a Hilbert space with group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, and let $\mathcal{W}^s(\mathbb{R}, E)$, $s \in \mathbb{R}$, denote the completion of $\mathcal{S}(\mathbb{R}, E)$ with respect to the norm

$$||u||_{\mathcal{W}^s(\mathbb{R},E)} = \left(\int \langle \eta \rangle^{2s} ||\kappa^{-1}_{\langle \eta \rangle} Fu(\eta)||^2_E d\eta \right)^{\frac{1}{2}}$$

Let $\varphi \in C_0^\infty(\mathbb{R}_+)$ be any element supported in $(0,1) \ni r$. As is known from [13], we have

$$\varphi \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma}(X^\wedge)) = \varphi H^s(\mathbb{R} \times 2\mathbb{B}). \tag{43}$$

For $s, \delta \in \mathbb{R}$ we define

$$\mathcal{W}^{s,\delta}(\mathbb{R}, \mathcal{K}^{s,\gamma}(X^\wedge)) = e^{-\delta t}\mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma}(X^\wedge)).$$

Definition 3.1 *For arbitrary $s, \delta, \gamma \in \mathbb{R}$ we set*

$$\mathcal{W}^{s,\delta;\gamma}(\mathbb{R} \times \mathbb{B}) := [\omega]\mathcal{W}^{s,\delta}(\mathbb{R}, \mathcal{K}^{s,\gamma}(X^\wedge)) + [1-\omega]H^{s,\delta}(\mathbb{R} \times 2\mathbb{B})$$

as a non-direct sum of Hilbert spaces, where ω is any cut-off function, supported in $[0,1)$.

Because of (43) this is a correct definition, i.e., independent of the specific choice of ω.

The norm on $\mathcal{W}^{s,\delta;\gamma}(\mathbb{R} \times \mathbb{B})$ is defined by

$$||u||_{\mathcal{W}^{s,\delta;\gamma}(\mathbb{R}\times\mathbb{B})} = (||\omega u||^2_{\mathcal{W}^{s,\delta}(\mathbb{R},\mathcal{K}^{s,\gamma}(X^\wedge))} + ||(1-\omega)u||^2_{H^{s,\delta}(\mathbb{R}\times 2\mathbb{B})})^{\frac{1}{2}}.$$

Remark 3.2 *The space $C_0^\infty(\mathbb{R} \times \operatorname{int}\mathbb{B})$ is dense in $\mathcal{W}^{s,\delta;\gamma}(\mathbb{R} \times \mathbb{B})$.*

Indeed, this follows from the fact that $C_0^\infty(\mathbb{R} \times X^\wedge)$ and $C_0^\infty(\mathbb{R} \times X)$ are dense in the spaces $\mathcal{W}^{s,\delta}(\mathbb{R}, \mathcal{K}^{s,\gamma}(X^\wedge))$ and $H^{s,\delta}(\mathbb{R} \times X)$, respectively.

Let $\boldsymbol{\delta} = (\delta_-, \delta_+)$ be a pair of real numbers. For $s, \gamma \in \mathbb{R}$, set

$$\mathcal{W}^{s,\boldsymbol{\delta};\gamma}(\mathbb{R} \times \mathbb{B}) = [\sigma]\mathcal{W}^{s,\delta_-;\gamma}(\mathbb{R} \times \mathbb{B}) + [1-\sigma]\mathcal{W}^{s,\delta_+;\gamma}(\mathbb{R} \times \mathbb{B})$$

and

$$H^{s,\boldsymbol{\delta}}(\mathbb{R}, \mathbb{C}^j) = [\sigma]H^{s,\delta_-}(\mathbb{R}, \mathbb{C}^j) + [1-\sigma]H^{s,\delta_+}(\mathbb{R}, \mathbb{C}^j)$$

in the sense of non-direct sums of Hilbert spaces, where $\sigma(t)$ is a fixed cut-off function for the point $t = -\infty$ on the real axis, i.e., σ is a C^∞ function on $\mathbb{R}$ equal to 1 near $t = -\infty$ and vanishing near $t = +\infty$.

A norm on the space $\mathcal{W}^{s,\boldsymbol{\delta};\gamma}(\mathbb{R} \times \mathbb{B})$ is defined by

$$||u||_{\mathcal{W}^{s,\boldsymbol{\delta};\gamma}(\mathbb{R}\times\mathbb{B})} = (||\sigma u||^2_{\mathcal{W}^{s,\delta_-;\gamma}(\mathbb{R}\times\mathbb{B})} + ||(1-\sigma)u||^2_{\mathcal{W}^{s,\delta_+;\gamma}(\mathbb{R}\times\mathbb{B})})^{\frac{1}{2}}. \qquad (44)$$

In an analogous manner we define a norm on the space $H^{s,\boldsymbol{\delta}}(\mathbb{R}, \mathbb{C}^{j_\pm})$.

Let us set

$$E^\pm_{s,\delta;\gamma} = \mathcal{W}^{s,\delta;\gamma}(\mathbb{R} \times \mathbb{B}) \oplus H^{s,\delta}(\mathbb{R}, \mathbb{C}^{j_\pm})$$

for any $s, \delta, \gamma \in \mathbb{R}$ and

$$E^\pm_{s,\boldsymbol{\delta};\gamma} = \mathcal{W}^{s,\boldsymbol{\delta};\gamma}(\mathbb{R} \times \mathbb{B}) \oplus H^{s,\boldsymbol{\delta}}(\mathbb{R}, \mathbb{C}^{j_\pm})$$

for any $s, \gamma \in \mathbb{R}$ and any pair of real numbers $\boldsymbol{\delta} = (\delta_-, \delta_+)$.

3.2 Inhomogeneous equation

Let $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ be elliptic, and assume that a has no poles on I_δ. Consider the operator

$$Au(t) := \frac{1}{2\pi} \int_{I_\delta} e^{itw} a(w) Fu(w) dw \tag{45}$$

first for $u \in C_0^\infty(\mathbb{R} \times \operatorname{int} \mathbb{B}) \oplus C_0^\infty(\mathbb{R}, \mathbb{C}^{j_-})$. We then have $F(Au)(w) = a(w)Fu(w)$ for all $w \in I_\delta$.

Proposition 3.3 *The operator A, defined as in* (45)*, induces continuous map*

$$A : E^-_{s,\delta;\gamma} \to E^+_{s-\mu,\delta;\gamma-\mu} \tag{46}$$

for each $s, \gamma \in \mathbb{R}$.

Proof. We prove the continuity of the upper left corner operator

$$A : \mathcal{W}^{s,\delta;\gamma}(\mathbb{R} \times \mathbb{B}) \to \mathcal{W}^{s-\mu,\delta;\gamma-\mu}(\mathbb{R} \times \mathbb{B})$$

(for simplicity, we denote the new operator and its symbol also by A and a, respectively); the continuity of the full block matrix operator A can be proved in an analogous way.

Since the space $C_0^\infty(\mathbb{R} \times \operatorname{int} \mathbb{B})$ is dense in $\mathcal{W}^{s,\delta;\gamma}(\mathbb{R} \times \mathbb{B})$, it suffices to prove the assertion for $u \in C_0^\infty(\mathbb{R} \times \operatorname{int} \mathbb{B})$. We have

$$\begin{aligned}
&||\omega A u||^2_{\mathcal{W}^{s-\mu,\delta}(\mathbb{R}, \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))} \\
= \; & \int \langle\eta\rangle^{2(s-\mu)} ||\kappa^{-1}_{\langle\eta\rangle} a(\eta + i\delta) F(\omega u)(\eta + i\delta)||^2_{\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)} d\eta \\
= \; & \int \langle\eta\rangle^{2(s-\mu)} e^{2\delta t} ||\kappa^{-1}_{\langle\eta\rangle} a(\eta + i\delta) F(\omega u)(\eta)||^2_{\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)} d\eta \\
\leq \; & \sup_{\eta\in\mathbb{R}} \{\langle\eta\rangle^{-2\mu} ||a(\eta + i\delta)||^2_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))}\} ||\omega u||^2_{\mathcal{W}^{s,\delta;\gamma}(\mathbb{R}\times\mathbb{B})}
\end{aligned}$$

and

$$\begin{aligned}
&||(1-\omega)Au||^2_{H^{s-\mu,\delta}(\mathbb{R}\times 2\mathbb{B})} = \int_{I_\delta} ||R^{s-\mu}(\eta) a(w) F((1-\omega)u)(w)||^2_{L^2(2\mathbb{B})} dw \\
\leq \; & \sup_{\eta\in\mathbb{R}} \{||R^{s-\mu}(\eta) a(\eta + i\delta) R^{-s}(\eta)||^2_{\mathcal{L}(L^2(2\mathbb{B}))}\} ||(1-\omega)u||^2_{H^{s,\delta}(\mathbb{R}\times 2\mathbb{B})}.
\end{aligned}$$

Since $a(w)$ is an operator-valued symbol of order μ with respect to η, and because of Theorem 2.12 concerning continuity of $a(w)$ in the respective spaces, these estimates yield the assertion. □

Proposition 3.4 *Let $a(w)$ have no singular values on I_δ. Then* (46) *extends to an isomorphism.*

Proof. This is a consequence of Propositions 2.12 and 3.3. □

3.2.1 The case $\delta_- \leq \delta_+$

Lemma 3.5 *Let $\boldsymbol{\delta} = (\delta_-, \delta_+)$ satisfy $\delta_- \leq \delta_+$. Then, a function u belongs to $E^{\pm}_{s,\boldsymbol{\delta};\gamma}$ if and only if $u \in E^{\pm}_{s,\delta;\gamma}$ for each $\delta \in [\delta_-, \delta_+]$. More precisely, $u \in E^{\pm}_{s,\delta_+;\gamma}$ and $u \in E^{\pm}_{s,\delta_-;\gamma}$ entails $u \in E^{\pm}_{s,\boldsymbol{\delta};\gamma}$.*

Proof. We give the proof for the spaces $\mathcal{W}^{s,\boldsymbol{\delta};\gamma}(\mathbb{R} \times \mathbb{B})$ and $\mathcal{W}^{s,\delta;\gamma}(\mathbb{R} \times \mathbb{B})$. The case of spaces $E^{\pm}_{s,\boldsymbol{\delta};\gamma}$ and $E^{\pm}_{s,\delta;\gamma}$ is analogous and will be omitted. For $\delta \in [\delta_-, \delta_+]$ we have

$$\begin{aligned} ||u||^2_{\mathcal{W}^{s,\delta;\gamma}(\mathbb{R}\times\mathbb{B})} &\leq 2(||\sigma u||^2_{\mathcal{W}^{s,\delta;\gamma}(\mathbb{R}\times\mathbb{B})} + ||(1-\sigma)u||^2_{\mathcal{W}^{s,\delta;\gamma}(\mathbb{R}\times\mathbb{B})}) \\ &\leq c||u||^2_{\mathcal{W}^{s,\boldsymbol{\delta};\gamma}(\mathbb{R}\times\mathbb{B})}, \end{aligned}$$

since

$$\begin{aligned} ||\sigma u||^2_{\mathcal{W}^{s,\delta;\gamma}(\mathbb{R}\times\mathbb{B})} &= ||\omega\sigma u||^2_{\mathcal{W}^{s,\delta}(\mathbb{R},\mathcal{K}^{s,\gamma}(X^\wedge))} + ||(1-\omega)\sigma u||^2_{H^{s,\delta}(\mathbb{R}\times 2\mathbb{B})} \\ &\leq c_1(||\omega\sigma u||^2_{\mathcal{W}^{s,\delta_-}(\mathbb{R},\mathcal{K}^{s,\gamma}(X^\wedge))} + ||(1-\omega)\sigma u||^2_{H^{s,\delta_-}(\mathbb{R}\times 2\mathbb{B})}) \\ &= c_2||\sigma u||^2_{\mathcal{W}^{s,\delta_-;\gamma}(\mathbb{R}\times\mathbb{B})}. \end{aligned}$$

In this way we obtain

$$||(1-\sigma)u||^2_{\mathcal{W}^{s,\delta;\gamma}(\mathbb{R}\times\mathbb{B})} \leq c_3||(1-\sigma)u||^2_{\mathcal{W}^{s,\delta_+;\gamma}(\mathbb{R}\times\mathbb{B})}.$$

Conversely, for $u \in \mathcal{W}^{s,\delta;\gamma}(\mathbb{R} \times \mathbb{B}), \delta \in [\delta_-, \delta_+]$, relation (44) gives us

$$||u||^2_{\mathcal{W}^{s,\boldsymbol{\delta};\gamma}(\mathbb{R}\times\mathbb{B})} \leq c(||u||^2_{\mathcal{W}^{s,\delta_-;\gamma}(\mathbb{R}\times\mathbb{B})} + ||u||^2_{\mathcal{W}^{s,\delta_+;\gamma}(\mathbb{R}\times\mathbb{B})}). \tag{47}$$

The latter estimate shows that when u belongs both to $\mathcal{W}^{s,\delta_-;\gamma}(\mathbb{R} \times \mathbb{B})$ and $\mathcal{W}^{s,\delta_+;\gamma}(\mathbb{R} \times \mathbb{B})$ it follows that $u \in \mathcal{W}^{s,\boldsymbol{\delta};\gamma}(\mathbb{R} \times \mathbb{B})$. □

Remark 3.6 *For any $u \in E^{-}_{s,\boldsymbol{\delta};\gamma}$ the Fourier transform Fu of u is holomorphic in the strip $\delta_- < \operatorname{Im} w < \delta_+$.*

Indeed, this is an immediate consequence of the representation of u, i.e., u has the factor $e^{-\delta_- t}$ in a neighbourhood of $-\infty$ and $e^{-\delta_+ t}$ in a neighbourhood of $+\infty$.

Let us write

$$\mathrm{op}^{\delta}(h)u(t) := \int_{\mathbb{R}} e^{it(\eta+i\delta)} h(\eta+i\delta) Fu(\eta+i\delta)d\eta.$$

Proposition 3.7 *Let $a(w)$ have no poles on the lines I_{δ_-} and I_{δ_+}, where $\delta_- \leq \delta_+$. Then, for each $u \in E^{-}_{s,\boldsymbol{\delta};\gamma}$, we have*

$$\mathrm{op}^{\delta_-}(a)u(t) - \mathrm{op}^{\delta_+}(a)u(t) = 2\pi i \sum_{\operatorname{Im} p \in (\delta_-, \delta_+)} \mathrm{res}_p e^{itw} a(w) Fu(w).$$

Proof. According to Remark 2.9 the operator function $a(w)$ has a representation $a(w) = a_{\boldsymbol{O}}(w) + a_{\boldsymbol{1}}(w)$ for certain $a_{\boldsymbol{O}}(w) \in \mathcal{M}^{\mu}_{\boldsymbol{O}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ and $a_{\boldsymbol{1}}(w) \in \mathcal{M}^{-\infty}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$. It suffices to show that

$$\mathrm{op}^{\delta_-}(a_{\boldsymbol{1}})u(t) - \mathrm{op}^{\delta_+}(a_{\boldsymbol{1}})u(t) = 2\pi i \sum_{\mathrm{Im}\, p \in (\delta_-, \delta_+)} \mathrm{res}_p e^{itw} a_{\boldsymbol{1}}(w) Fu(w) \tag{48}$$

($a_{\boldsymbol{1}}(w)$ on the right hand side of (48) may be replaced by $a(w)$) and

$$\mathrm{op}^{\delta_-}(a_{\boldsymbol{O}})u(t) = \mathrm{op}^{\delta_+}(a_{\boldsymbol{O}})u(t). \tag{49}$$

The relation (48) is an easy consequence of Cauchy's integral formula and the Residue Theorem.

Concerning (49) we use the fact that for $u \in C_0^\infty(\mathbb{R} \times \mathrm{int}\, \mathbb{B}) \oplus C_0^\infty(\mathbb{R}, \mathbb{C}^{j_-})$ the desired relation holds by Cauchy's theorem. On the other hand, since $C_0^\infty(\mathbb{R} \times \mathrm{int}\, \mathbb{B}) \oplus C_0^\infty(\mathbb{R}, \mathbb{C}^{j_-})$ is dense in $E^-_{s,\delta,\gamma}$ for every $\delta \in \mathbb{R}$, cf. Remark 3.2, using Proposition 3.3, we also have

$$\mathrm{op}^{\delta}(a_{\boldsymbol{O}})u(t) = \lim_{k \to \infty} \mathrm{op}^{\delta}(a_{\boldsymbol{O}})u_k(t)$$

in $E^+_{s-\mu,\delta;\gamma-\mu}$ whenever $u = \lim_{k\to\infty} u_k$ in the space $E^-_{s,\delta;\gamma}$. This implies relation (49) for all $u \in E^-_{s,\boldsymbol{\delta};\gamma}$. $\square$

Fix $s, \gamma \in \mathbb{R}$ and weight data $\boldsymbol{\delta} = (\delta_-, \delta_+)$ satisfying $\delta_- \leq \delta_+$. Set

$$\mathrm{dom}\, A = \{u \in E^-_{s,\boldsymbol{\delta};\gamma} : \mathrm{res}_p e^{itw} a(w) Fu(w) = 0 \text{ for } \mathrm{Im}\, p \in (\delta_-, \delta_+)\}.$$

Proposition 3.7 shows that Au is independent of the particular choice of $\delta \in (\delta_-, \delta_+)$ (if $a(w)$ has no poles on the line I_δ) for any $u \in \mathrm{dom}\, A$.

Lemma 3.8 *Let $a(w)$ be elliptic. Then* $\mathrm{dom}\, A$ *is a closed subspace of* $E^-_{s,\boldsymbol{\delta};\gamma}$ *of finite codimension*

$$\mathrm{codim}\,\mathrm{dom}\, A = \sum_{\mathrm{Im}\, p \in (\delta_-, \delta_+)} \boldsymbol{p}(a(p)).$$

Proof. Let p be a pole of $a(w)$ in the strip $\delta_- < \mathrm{Im}\, w < \delta_+$, then p is a characteristic value of the inverse $a^{-1}(w)$. Because of Proposition 2.23, there are canonical systems

$$(f_0^{(i)}, f_1^{(i)}, \ldots, f_{\rho_i - 1}^{(i)})_{i=1,\ldots,P} \quad \text{and} \quad (g_0^{(i)}, g_1^{(i)}, \ldots, g_{\rho_i - 1}^{(i)})_{i=1,\ldots,P}$$

of eigenvectors and associated vectors of $a^{-1}(w)$ and ${}^{\mathrm{t}}(a^{-1}(w))$ at p, respectively, such that

$$\mathrm{p.p.}\, a(w) = \mathrm{p.p.}\, (a^{-1})^{-1}(w) = \sum_{i=1}^{P} \sum_{j=-\rho_i}^{-1} \Big\{ \sum_{k=0}^{\rho_i + j} \langle g_k^{(i)}, \cdot \rangle f^{(i)}_{\rho_i + j - k} \Big\} (w - p)^j$$

in a neighbourhood of p. We have

$$\begin{aligned}
&\operatorname{res}_p e^{itw} a(w) Fu(w) = \operatorname{res}_p (\text{p.p.}\, a(w)) e^{itw} Fu(w)\\
= &\sum_{i=1}^{P} \sum_{j=-\rho_i}^{-1} \sum_{k=0}^{\rho_i+j} \frac{1}{(-j-1)!} \left(\frac{\partial}{\partial w}\right)^{-j-1} \left\{ e^{itw} \int_{\mathbb{R}} e^{-it'w} \langle g_k^{(i)}, u(t') \rangle dt' \right\}\Big|_{w=p} f^{(i)}_{\rho_i+j-k}\\
= &\; e^{itp} \sum_{i=1}^{P} \sum_{j=-\rho_i}^{-1} \sum_{k=0}^{\rho_i+j} \left\{ \frac{1}{(-j-1)!} \int_{\mathbb{R}} e^{-it'p} (it - it')^{-j-1} \langle g_k^{(i)}, u(t') \rangle dt' \right\} f^{(i)}_{\rho_i+j-k}\\
= &\; e^{itw} \sum_{i=1}^{P} \sum_{\alpha=0}^{\rho_i-1} \left\{ \sum_{\beta=1}^{\rho_i-\alpha} \frac{1}{(\beta-1)!} \int_{\mathbb{R}} e^{-it'p} (it - it')^{\beta-1} \langle g^{(i)}_{\rho_i-\alpha-\beta}, u(t') \rangle dt' \right\} f^{(i)}_{\alpha},
\end{aligned}$$

where $\alpha := \rho_i + j - k, \beta := -j$. Hence, we see that $u \in \operatorname{dom} A$ if and only if u satisfies a system of $\sum_{i=1}^{P} \rho_i = \boldsymbol{p}(a(p))$ linearly independent conditions. □

Proposition 3.9 *Let $a(w)$ be elliptic, and let $a(w)$ have no poles on the lines I_{δ_-} and I_{δ_+}. Then we have $Au \in E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$, and the operator*

$$A : \operatorname{dom} A \to E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$$

is continuous.

Proof. Because of Proposition 3.7 for $u \in \operatorname{dom} A$ we have

$$Au(t) = \frac{1}{2\pi} \int_{I_{\delta_-}} e^{itw} a(w) Fu(w) dw = \frac{1}{2\pi} \int_{I_{\delta_+}} e^{itw} a(w) Fu(w) dw.$$

Hence, using Proposition 3.3 and Lemma 3.5 we get $Au \in E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$, and Au satisfies the following estimates $||Au||_{E^+_{s-\mu,\delta_\pm;\gamma-\mu}} \le c_\pm ||u||_{E^-_{s,\delta_\pm;\gamma}}$, where the constants $c_\pm$ are independent of u. This completes the proof. □

The following proposition describes the set of all elements $f \in E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ for which the inhomogeneous equation $Au = f$ has solution in $\operatorname{dom} A$.

Proposition 3.10 *Let $a(w)$ have no singular values on the lines I_{δ_-} and I_{δ_+}. Then, for $f \in E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ there exists a solution $u \in \operatorname{dom} A$ of the equation $Au = f$ if and only if*

$$\operatorname{res}_p e^{itw} a^{-1}(w) Ff(w) = 0 \;\text{ for }\; \operatorname{Im} p \in (\delta_-, \delta_+).$$

(By $\operatorname{im} A$ *we denote the set of all such functions $f \in E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$).*

Proof. Let $f = Au$ for some $u \in E^-_{s,\boldsymbol{\delta};\gamma}$. Then $Ff(w) = a(w)Fu(w)$ or $a^{-1}(w)Ff(w) = Fu(w)$. As $Fu(w)$ is holomorphic in the strip $\delta_- < \operatorname{Im} w < \delta_+$, we get that $f \in \operatorname{im} A$.

Conversely, let $f \in \operatorname{im} A$. Then, because of Proposition 3.7 the integral

$$u(t) = \frac{1}{2\pi} \int_{I_\delta} e^{itw} a^{-1}(w) Ff(w) dw$$

is independent of $\delta \in [\delta_-, \delta_+]$, if $a^{-1}(w)$ has no poles on I_δ. In particular, taking $\delta = \delta_\pm$ we conclude from Lemma 3.5 that $u \in E^-_{s,\boldsymbol{\delta};\gamma}$. As $Ff(w)$ is holomorphic in the strip $\delta_- < \operatorname{Im} w < \delta_+$, the relation $a(w)Fu(w) = Ff(w)$ implies that $u \in \operatorname{dom} A$. Finally, a simple calculation shows that $Au = f$. □

Corollary 3.11 *The operator $A : \operatorname{dom} A \to E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ is injective,* $\operatorname{im} A$ *is a closed subspace of $E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ of finite codimension*

$$\operatorname{codim} \operatorname{im} A = \sum_{\operatorname{Im} p \in (\delta_-, \delta_+)} \boldsymbol{n}(a(p)).$$

In fact, this is a consequence of Proposition 3.10 and Lemma 3.8.

3.2.2 The case $\delta_- > \delta_+$

In the case $\delta_- > \delta_+$ we cannot define the operator A in the form (45), since $E^-_{s,\boldsymbol{\delta};\gamma} \hookrightarrow E^-_{s,\delta;\gamma}$ for a $\delta \in \mathbb{R}$ implies $\delta_- \le \delta_+$. Moreover, the Fourier transform Fu of $u \in E^-_{s,\boldsymbol{\delta};\gamma}$ is not holomorphic in the strip $\delta_+ < \operatorname{Im} w < \delta_-$. Only in the case that $a(w)$ is a polynomial in w, i.e., A is a differential operator, we have no additional problems.

To investigate the case $\delta_- > \delta_+$ we need the transposed operator ${}^{\mathrm{t}}A$ of the operator A.

Proposition 3.12 *Let $a(w)$ have no pole on a line I_δ. Then the transposed operator ${}^{\mathrm{t}}A : E^+_{-s+\mu,-\delta;-\gamma+\mu} \to E^-_{-s,-\delta;-\gamma}$ of the operator* (46) *is given by*

$${}^{\mathrm{t}}Av(t) = \frac{1}{2\pi} \int_{I_{-\delta}} e^{itw}\, {}^{\mathrm{t}}a(-w) Fv(w) dw, \quad t \in \mathbb{R}, \tag{50}$$

for any $v \in E^+_{-s+\mu,-\delta;-\gamma+\mu}$.

Proof. It suffices to verify relation (50) for elements $v \in C_0^\infty(\mathbb{R} \times \operatorname{int} \mathbb{B}) \oplus C_0^\infty(\mathbb{R}, \mathbb{C}^{j_+})$. For $u \in C_0^\infty(\mathbb{R} \times \operatorname{int} \mathbb{B}) \oplus C_0^\infty(\mathbb{R}, \mathbb{C}^{j_-})$ we have

$$\begin{aligned}
\langle v, Au \rangle &= \int_{\mathbb{R}} \langle v(t), \frac{1}{2\pi} \int_{I_\delta} e^{itw} a(w) Fu(w) dw \rangle dt \\
&= \int_{\mathbb{R}} \langle \frac{1}{2\pi} \int_{I_\delta} e^{-it'w}\, {}^{\mathrm{t}}a(w) Fv(-w) dw, u(t') \rangle dt' \\
&= \int_{\mathbb{R}} \langle \frac{1}{2\pi} \int_{I_{-\delta}} e^{it'w}\, {}^{\mathrm{t}}a(-w) Fv(w) dw, u(t') \rangle dt' = \langle {}^{\mathrm{t}}Av, u \rangle.
\end{aligned}$$

□

We see that the formulas (45) and (50) coincide with $a(w)$ and δ replaced by ${}^{\mathrm{t}}a(-w)$ and $-\delta$, respectively. For $\delta_- > \delta_+$ the weight data $-\boldsymbol{\delta} = (-\delta_-, -\delta_+)$ satisfy the condition of the previous section. Hence, we may study the operator ${}^{\mathrm{t}}A$ in a similar manner as the operator A in Section 3.2.1. It follows that

$$\begin{aligned}\operatorname{dom}{}^{\mathrm{t}}A &= \{v \in E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu} : \operatorname{res}_p e^{itw}\ {}^{\mathrm{t}}a(-w)Fv(w) = 0 \\ &\quad\ \text{for } \operatorname{Im} p \in (-\delta_-, -\delta_+)\}\end{aligned}$$

is a closed subspace of $E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu}$ of finite codimension

$$\operatorname{codim}\operatorname{dom}{}^{\mathrm{t}}A = \sum_{\operatorname{Im} p\in(-\delta_-,-\delta_+)} \boldsymbol{p}({}^{\mathrm{t}}a(-p)) = \sum_{\operatorname{Im} p\in(\delta_+,\delta_-)} \boldsymbol{p}(a(p)).$$

The latter equality is a consequence of Corollary 2.19. Moreover, Corollary 3.11 shows that if $a(w)$ has no singular values on the lines I_{δ_-} and I_{δ_+}, the operator

$${}^{\mathrm{t}}A : \operatorname{dom}{}^{\mathrm{t}}A \to E^-_{-s,-\boldsymbol{\delta};-\gamma}$$

is injective, and $\operatorname{im}{}^{\mathrm{t}}A$ is a closed subspace in $E^-_{-s,-\boldsymbol{\delta};-\gamma}$ of finite codimension

$$\operatorname{codim}\operatorname{im}{}^{\mathrm{t}}A = \sum_{\operatorname{Im} p\in(-\delta_-,-\delta_+)} \boldsymbol{n}({}^{\mathrm{t}}a(-p)) = \sum_{\operatorname{Im} p\in(\delta_+,\delta_-)} \boldsymbol{n}(a(p)).$$

The latter equality is again a consequence of Corollary 2.19.

We now define the operator $A : E^-_{s,\boldsymbol{\delta};\gamma} \to E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ for the case $\delta_- > \delta_+$. To this end we need the following result.

Lemma 3.13 *For each $u \in E^-_{s,\boldsymbol{\delta};\gamma}$ there is a unique $f \in E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ such that*

$$\begin{aligned}\langle v, f\rangle &= \langle {}^{\mathrm{t}}Av, u\rangle \quad \text{for all } v \in \operatorname{dom}{}^{\mathrm{t}}A,\\ \langle v, f\rangle &= 0 \qquad\qquad\ \text{for all } v \in E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu} \ominus \operatorname{dom}{}^{\mathrm{t}}A.\end{aligned} \tag{51}$$

Proof. Let π be the projection operator of $E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu}$ to $\operatorname{dom}{}^{\mathrm{t}}A$. Then for $u \in E^-_{s,\boldsymbol{\delta};\gamma}$ we define

$$\langle v, f\rangle = \langle {}^{\mathrm{t}}A\pi v, u\rangle$$

for $v \in E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu}$. Obviously, f is a continuous linear functional on $E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu}$ (therefore, it can be identified with an element of $E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$) and satisfies (51). Furthermore, if $f_1, f_2 \in E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ satisfy the relations (51), we have

$$\langle v, f_1 - f_2\rangle = \langle \pi v, f_1 - f_2\rangle + \langle (1-\pi)v, f_1 - f_2\rangle = 0$$

for all $v \in E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu}$, i.e., $f_1 = f_2$. $\square$

For $u \in E^-_{s,\boldsymbol{\delta};\gamma}$ we set
$$Au := f \tag{52}$$
for f associated with u via Lemma 3.13. Then A is a linear continuous operator $E^-_{s,\boldsymbol{\delta};\gamma} \to E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$.

Theorem 3.14 *Let $a(w)$ have no singular values on the lines I_{δ_-} and I_{δ_+}. The operator $A : E^-_{s,\boldsymbol{\delta};\gamma} \to E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ defined as in* (51) *is a Fredholm operator. More precisely, we have*
$$\dim \ker A = \sum_{\operatorname{Im} p \in (\delta_+,\delta_-)} \boldsymbol{n}(a(p))$$
and
$$\dim \operatorname{coker} A = \sum_{\operatorname{Im} p \in (\delta_+,\delta_-)} \boldsymbol{p}(a(p)).$$

Proof. The relation (51) shows that $u \in \ker A$ is equivalent to $\langle {}^{\mathrm{t}}Av, u\rangle = 0$ for all $v \in E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu}$, and $Au = f$ has a solution for $f \in E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ if and only if $\langle v, f\rangle = 0$ for all $v \in E^+_{-s+\mu,-\boldsymbol{\delta};-\gamma+\mu} \ominus \operatorname{dom} {}^{\mathrm{t}}A$. Hence, the dimension of $\ker A$ is equal to the codimension of $\operatorname{im} {}^{\mathrm{t}}A$, and the codimension of $\operatorname{im} A$ is equal to the codimension of $\operatorname{dom} {}^{\mathrm{t}}A$. $\square$

Example 3.15 *Let $a(w)$ be as in Example* 2.22, *and let p be a characteristic value of $a(w)$ with $\operatorname{Im} p \in (\delta_+,\delta_-)$. Then from Theorem* 3.14 *there correspond $\boldsymbol{n}(a(p))$ linearly independent solutions of the differential operator $Au = 0$. We show that the solutions are*
$$\sum_{s=1}^{r_i-k} e^{ipt}\frac{(it)^{s-1}}{(s-1)!}u^{(i)}_{r_i-k-s}, \quad i = 1,\dots,N,\ k = 0,1\dots,r_i-1,$$
where $(u^{(i)}_k), i = 1,\dots,N, k = 0,1,\dots,r_i-1$, is a canonical system of eigenvectors and associated vectors of $a(w)$ at p.

In fact, by Leibniz formula we see that
$$\sum_{s=1}^{r_i-k} e^{ipt}\frac{(it)^{s-1}}{(s-1)!}u^{i}_{r_i-k-s} = \frac{1}{(r_i-k-1)!}\Big(\frac{\partial}{\partial w}\Big)^{r_i-k-1}(e^{iwt}u^{(i)}(w))|_{w=p},$$
where $\frac{1}{k!}\Big(\frac{\partial}{\partial w}\Big)^{k}u^{(i)}(p) = u^{(i)}_k, k = 0,1,\dots,r_i-1$. Now a direct calculation gives us
$$A\Big(\sum_{s=1}^{r_i-k} e^{ipt}\frac{(it)^{s-1}}{(s-1)!}u^{(i)}_{r_i-k-s}\Big) = \frac{1}{(r_i-k-1)!}\Big(\frac{\partial}{\partial w}\Big)^{r_i-k-1}(e^{iwt}a(w)u^{(i)}(w))|_{w=p},$$
which is equal to zero since for $a(w)u^{(i)}(w)$ the point p is a zero of order r_i-1 for all $i = 1,\dots,N$.

3.2.3 Index formula

Let $a(w) \in \mathcal{M}^{\mu}_{\boldsymbol{R}}(\mathbb{B}; \boldsymbol{g}, j_-, j_+)$ be elliptic, and fix weight data $\boldsymbol{\delta} = (\delta_-, \delta_+)$.

If $\delta_- \leq \delta_+$ (Section 3.2.1) we define the operator $A : E^-_{s,\boldsymbol{\delta};\gamma} \to E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ on a subset $\operatorname{dom} A$ of $E^-_{s,\boldsymbol{\delta};\gamma}$. To define A on the space $E^-_{s,\boldsymbol{\delta};\gamma}$ itself we compose A with the projection π of $E^-_{s,\boldsymbol{\delta};\gamma}$ onto $\operatorname{dom} A$. For notational convenience we denote the resulting operator again by A. Then A is a linear continuous operator with null-space $E^-_{s,\boldsymbol{\delta};\gamma} \ominus \operatorname{dom} A$. If $a(w)$ has no singular values on the lines I_{δ_-} and I_{δ_+}, Lemma 3.8 and Corollary 3.11 give that A is a Fredholm operator of index

$$\begin{aligned}
\operatorname{ind} A &= \sum_{\operatorname{Im} p \in (\delta_-, \delta_+)} \boldsymbol{p}(a(p)) - \sum_{\operatorname{Im} p \in (\delta_-, \delta_+)} \boldsymbol{n}(a(p)) \\
&= - \sum_{\operatorname{Im} p \in (\delta_-, \delta_+)} \boldsymbol{m}(a(p)).
\end{aligned}$$

If $\delta_- > \delta_+$ (Section 3.2.2) we define the operator $A : E^-_{s,\boldsymbol{\delta};\gamma} \to E^+_{s-\mu,\boldsymbol{\delta};\gamma-\mu}$ as in (52) which is a linear continuous operator. If $a(w)$ has no singular values on the lines I_{δ_-} and I_{δ_+}, using Theorem 3.14, it follows that A is a Fredholm operator of index

$$\begin{aligned}
\operatorname{ind} A &= \sum_{\operatorname{Im} p \in (\delta_+, \delta_-)} \boldsymbol{n}(a(p)) - \sum_{\operatorname{Im} p \in (\delta_+, \delta_-)} \boldsymbol{p}(a(p)) \\
&= \sum_{\operatorname{Im} p \in (\delta_+, \delta_-)} \boldsymbol{m}(a(p)).
\end{aligned}$$

The following theorem gives us an explicit formula for the index.

Theorem 3.16 *Let $a(w)$ have no singular values on the lines I_{δ_-} and I_{δ_+}. Then*

$$\operatorname{ind} A = \operatorname{tr} \left(\frac{1}{2\pi i} \int_{I_{\delta_+}} a^{-1}(w) a'(w) dw - \frac{1}{2\pi i} \int_{I_{\delta_-}} a^{-1}(w) a'(w) dw \right). \quad (53)$$

Proof. Let $\delta_- \leq \delta_-$ and Q_T be a rectangle with vertices $\pm T + i\delta_\pm$, which contains all singular values of $a(w)$ in the strip $\operatorname{Im} w \in (\delta_-, \delta_+)$. Using Lemma 2.20 and the residue formula, we get

$$\operatorname{tr} \left(\frac{1}{2\pi i} \int_{Q_T} a^{-1}(w) a'(w) dw \right) = \sum_{\operatorname{Im} p \in (\delta_-, \delta_+)} m(a(p)) = -\operatorname{ind} A. \quad (54)$$

Now, for $T \to \infty$ on the left of (54), we get the assertion. Analogously, we argue for the case $\delta_+ < \delta_-$. □

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