Edge Operators with Conditions of Toeplitz Type

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Abstract

Ellipticity of operators on a manifold with edges can be treated in the framework of a calculus of 2×2-block matrix operators with trace and potential operators on the edges. The picture is similar to the pseudodifferential analysis of boundary value problems. The extra conditions satisfy an analoque of the Shapiro-Lopatinskij condition, provided a topological obstruction for the elliptic edge-degenerate operator in the upper left corner vanishes; this is an analogue of a condition of Atiyah and Bott in boundary value problems. In general, however, we need global projection data, similarly to the situation of global boundary conditions, known for Dirac operators or other geometric operators. The present paper develops a new calculus with global projection data for operators on manifolds with edges. In particular, we show the Fredholm property in a suitable scale of spaces and construct parametrices within the calculus.

Keywords: Pseudodifferential edge-degenerate operators, operators with global projection conditions, ellipticity and parametrix constructions.

Contents

Introduction

This paper is aimed at studying ellipticity and Fredholm property of (pseudodifferential) operators on a manifold with edges. The Fredholm property in weighted

Sobolev spaces requires additional edge conditions. Similarly to boundary value problems, it may happen that there are no “Shapiro–Lopatinskij–elliptic” conditions to a given operator but conditions in terms of global projections on the edge. This gives rise to a new kind of so-called Toeplitz algebras on manifolds with edges.

Let us first have a look at the case of “standard” boundary value problems. Ellipticity of a differential operator A on a compact smooth manifold M with boundary ∂M is usually considered in connection with additional conditions at the boundary. In simplest cases we have boundary operators T that satisfy the Shapiro–Lopatinskij condition with respect to A (we then talk about SL–ellipticity). A boundary value problem of the form $Au = f$ in M, $Tu = g$ on ∂M then induces Fredholm operators $u \to (f, g)$ between standard Sobolev spaces of sufficiently large smoothness s (clearly, in the range this refers to direct sums of spaces on $\operatorname{int} M$ and ∂M, respectively). The latter property is known to be equivalent to the ellipticity of A together with SL–ellipticity of T (with respect to A). Simple examples are the Dirichlet or the Neumann problem for Laplace operators on M.

Given an elliptic (in general, pseudodifferential) operator A with the transmission property at the boundary, there is a criterion for the existence of SL–elliptic boundary conditions (in the pseudodifferential case the boundary conditions may be both of trace and potential type, cf. Boutet de Monvel [4]). To recall the condition we first consider a differential operator $A : C^\infty(M, E) \to C^\infty(M, F)$, acting between spaces of sections in $E, F \in \operatorname{Vect}(M)$ ($\operatorname{Vect}(\cdot)$ denotes the set of all smooth complex vector bundles on the manifold in the brackets). With A we associate the homogeneous principal interior symbol $\sigma_\psi(A) : \pi_M^* E \to \pi_M^* F$, with $\pi_M : T^*M \setminus 0 \to M$ being the canonical projection, and the homogeneous principal boundary symbol

$$\sigma_\partial(A) : \pi_{\partial M}^* E' \otimes H^s(\mathbb{R}_+) \to \pi_{\partial M}^* F' \otimes H^{s-\mu}(\mathbb{R}_+), \tag{0.0.1}$$

$\pi_{\partial M} : T^*(\partial M) \setminus 0 \to \partial M$; here, $\mu = \operatorname{ord} A$, $s \in \mathbb{R}$ is sufficiently large, and prime indicates restrictions of bundles to the boundary, further $H^s(\mathbb{R}_+) := H^s(\mathbb{R})|_{\mathbb{R}_+}$ for the standard Sobolev space $H^s(\mathbb{R})$ on the real axis.

Let us fix Riemannian metrics on M and ∂M such that a collar neighbourhood V of ∂M corresponds to $(\partial M) \times [0, 1)$ in the product metric, with a splitting of variables $x = (y, t)$ and covariables $\xi = (\eta, \tau)$. The family of operators (0.0.1) then has the form

$$\sigma_\partial(A)(y, \eta) = \sigma_\psi(A)(y, 0, \eta, D_t), \qquad (y, \eta) \in T^*(\partial M) \setminus 0.$$

The ellipticity of A entails the Fredholm property of

$$\sigma_\partial(A)(y, \eta) : E'_y \otimes H^s(\mathbb{R}_+) \to F'_y \otimes H^{s-\mu}(\mathbb{R}_+) \tag{0.0.2}$$

((0.0.2) is surjective in the case of a differential operator A), and we have

$$\sigma_\partial(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\partial(A)(y, \eta) \kappa_\lambda^{-1} \tag{0.0.3}$$

for all $\lambda \in \mathbb{R}_+$; here, $(\kappa_\lambda u)(t) := \lambda^{\frac{1}{2}} u(\lambda t)$. In particular, (0.0.3) shows that

$$\operatorname{ind} \sigma_\partial(A)(y, \eta) = \operatorname{ind} \sigma_\partial(A)(y, \eta/|\eta|).$$

(Analogous relations hold for pseudodifferential operators with the transmission property at the boundary; in that case $\sigma_\partial(A)(y, \eta) := \mathrm{r}^+ \sigma_\psi(A)(y, 0, \eta, D_t) \mathrm{e}^+$ where

e^+ is the operator of extension by zero from $\mathbb{R}_+$ to $\mathbb{R}$ and r^+ the restriction from $\mathbb{R}$ to $\mathbb{R}_+$.) In other words, (0.0.2) represents a family of Fredholm operators, parametrised by the points (y, η) of the unit cosphere bundle S^*Y induced by T^*Y. Then (0.0.2) gives rise to an index element $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in K(S^*Y)$, where $K(\cdot)$ denotes the K–group of the space in brackets. The condition of Atiyah and Bott [1] for differential operators A (and of Boutet de Monvel [4] for pseudodifferential operators with the transmission property) says that A admits SL–elliptic boundary conditions if and only if

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in \pi_1^* K(Y), \tag{0.0.4}$$

$\pi_1 : S^*Y \to Y$.

It is well–known that (0.0.4) may be violated, for instance, for Dirac operators in even dimensions or other interesting geometric operators. In such cases there is then an alternative concept to pose boundary conditions, namely that of global projection conditions. In particular, spectral boundary conditions or APS (Atiyah–Patodi–Singer)–conditions belong to this category. There is a long history of that kind of Fredholm boundary value problems, in particular, of a corresponding index theory, with many relations to problems of geometry and topology, see, in particular, Calderón [5], Seeley [22], Atiyah, Patodi, and Singer [2], or the papers Grubb and Seeley [11], Schulze, Sternin, and Shatalov [21], Nazaikinskij, Schulze, Sternin, and Shatalov [13].

Boundary value problems may be interpreted as particular edge problems, where the boundary is regarded as an edge and the inner normal to the boundary as the model cone of a corresponding wedge which in this case a collar neighbourhood of the boundary, cf. [16], [18]. Fredholm problems for elliptic operators with additional conditions may also be considered on a manifold with egdes in general, and the simplest case is again an analogue of SL (Shapiro–Lopatinskij)–ellipticity for the operators on the edge that substitute boundary conditions, cf. [15], [17], Egorov and Schulze [8], see also Sections 1.4 and 1.5 below.

By a manifold W with edge Y we understand a topological space such that $W \setminus Y$ and Y are C^∞ manifolds, and every $y \in Y$ has a neighbourhood V in W such that there is a homeomorphism $\chi : V \to X^\Delta \times \Omega$ for a compact closed C^∞ manifold X (with $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ being the cone with base X) and an open set $\Omega \subseteq \mathbb{R}^q$, where χ restricts to diffeomorphisms $\chi_0 : V \setminus Y \to \mathbb{R}_+ \times X \times \Omega$ and $\chi' : V \cap Y \to \Omega$. In addition, transition maps $\mathbb{R}_+ \times X \times \Omega \to \mathbb{R}_+ \times X \times \tilde{\Omega}$ are required to be restrictions of diffeomorphisms $\overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \tilde{\Omega}$ to $\mathbb{R}_+ \times X \times \Omega$. The sets $\overline{\mathbb{R}}_+ \times X \times \Omega$ may be regarded as local models of a smooth manifold $\mathbb{W}$ with boundary $\partial\mathbb{W}$, the stretched manifold associated with W, and the conditions imply that $\partial\mathbb{W}$ is an X–bundle on Y. The construction gives rise to a canonical map $\mathbb{W} \to W$ that restricts to a diffeomorphism $\mathrm{int}\, \mathbb{W} \to W \setminus Y$ and to a map $\partial\mathbb{W} \to Y$, locally induced by $X \times \Omega \to \Omega$.

Given an elliptic (edge–degenerate) differential (or pseudodifferential) operator on $\mathbb{W}$, cf. the definitions below, we may ask, whether there are SL–elliptic conditions on the edge that complete A to a block matrix

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ H^s(Y, J_-) \end{matrix} \to \begin{matrix} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \\ \oplus \\ H^{s-\mu}(Y, J_+) \end{matrix} \tag{0.0.5}$$

that is a Fredholm operator for all s. Here, $E, F \in \mathrm{Vect}(\mathbb{W})$, $J_-, J_+ \in \mathrm{Vect}(Y)$, and $\mathcal{W}^{s,\gamma}(\mathbb{W}, E)$ denote weighted Sobolev spaces on $\mathbb{W}$ of distributional sections of smoothness s and weight γ; moreover, $H^s(Y,J)$ are the standard Sobolev spaces on Y of sections in $J \in \mathrm{Vect}(Y)$.

Let us assume that the involved vector bundles E, F on $\mathbb{W}$ are pull–backs of bundles E_1, F_1 on W with respect to the canonical map $\mathbb{W} \to W$, and set $E' = E_1|_Y$, $F' = F_1|_Y$.

A differential operator A of order μ on $\mathrm{int}\,\mathbb{W}$ with smooth coefficients is said to be edge–degenerate, if it can be written in local splittings of variables $(r,x,y) \in \mathbb{R}_+ \times X \times \Omega$ near $\partial\mathbb{W}$ in the form

$$A = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y)\left(-r\frac{\partial}{\partial r}\right)^j (rD_y)^\alpha \tag{0.0.6}$$

with coefficients $a_{j\alpha}(r,y)$ that are smooth in $(r,y) \in \overline{\mathbb{R}}_+ \times \Omega$ and take values in differential operators on X of order $\mu - (j+|\alpha|)$.

In contrast to boundary value problems for differential operators on a manifold with boundary the Fredholm property of $\mathcal{A}$ may require not only trace conditions T but also potential conditions K, together with pseudodifferential operators Q on the edge Y.

The homogeneous principal symbol of A of order μ which is a bundle homomorphism

$$\sigma_\psi(A) : \pi^*E \to \pi^*F, \qquad \pi : T^*(\mathrm{int}\,\mathbb{W}) \setminus 0 \to \mathrm{int}\,\mathbb{W},$$

has a special degeneracy near $\partial\mathbb{W}$.

Writing $\sigma_\psi(A)$ locally near $\partial\mathbb{W}$ as $\sigma_\psi(A)(r,x,y,\varrho,\xi,\eta)$ with variables (r,x,y) and covariables (ϱ,ξ,η), the function $\widetilde{\sigma}_\psi(A)(r,x,y,\varrho,\xi,\eta) := r^\mu\sigma_\psi(A)(r,x,y,r^{-1}\varrho,\xi,r^{-1}\eta)$ is smooth up to $r=0$.

There is a simple invariant interpretation of the latter modified symbol; for the moment we content ourselves with local descriptions.

The operator A is called elliptic (in the edge–degenerate sense), if A is elliptic on $\mathrm{int}\,\mathbb{W}$ as usual, and if $\widetilde{\sigma}_\psi(A)$ is bijective for $(\varrho,\xi,\eta) \neq 0$ up to $r=0$.

An analogue of the above–mentioned boundary symbol of A in the present case is the operator family

$$\sigma_\wedge(A)(y,\eta) := r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(0,y)\left(-r\frac{\partial}{\partial r}\right)^j (r\eta)^\alpha$$

acting between weighted Sobolev spaces on the infinite open stretched cone $X^\wedge := \mathbb{R}_+ \times X$

$$\sigma_\wedge(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E'_y \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F'_y \tag{0.0.7}$$

for all $(y,\eta) \in T^*\Omega \setminus 0$. Here, $\mathcal{K}^{s,\gamma}(X^\wedge)$ are weighted Sobolev spaces on $X^\wedge$, cf. [15] (we shall recall the definition below).

The space $\mathcal{K}^{s,\gamma}(X^\wedge)$ is endowed with a strongly continuous group of isomorphisms

$$\kappa_\lambda : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge), \qquad \lambda \in \mathbb{R}_+,$$

$(\kappa_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$, $n = \dim X$. We then have

$$\sigma_\wedge(A)(y,\lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\wedge(A)(y,\eta)\kappa_\lambda^{-1} \tag{0.0.8}$$

for all $\lambda \in \mathbb{R}_+$. It is well–known, cf. [17], that when A is elliptic, the operator (0.0.7) for any fixed $(y,\eta) \in T^*\Omega \setminus 0$ is Fredholm if and only if $\sigma_M(A)(y,z) := \sum_{j=0}^{\mu} a_{j0}(0,y)z^j$ defines a family of isomorphisms

$$\sigma_M(A)(y,z) : H^s(X) \otimes E'_y \to H^{s-\mu}(X) \otimes F'_y \tag{0.0.9}$$

for all $z \in \mathbb{C}$, $\operatorname{Re} z = \frac{n+1}{2} - \gamma$. The choice of s is unessential in these considerations, while γ plays a more specific role and is kept fixed. Relation (0.0.8) gives us $\operatorname{ind} \sigma_\wedge(A)(y,z) = \operatorname{ind} \sigma_\wedge(A)(y,\eta/|\eta|)$, and hence (0.0.7) can be regarded as a Fredholm family parametrised by S^*Y (where we assume the bijectivity of (0.0.9) for all $y \in Y$ and $\operatorname{Re} z = \frac{n+1}{2} - \gamma$.) We thus have an index element

$$\operatorname{ind} \sigma_\wedge(A) \in K(S^*Y). \tag{0.0.10}$$

Analogous relations hold for the case of edge–degenerate pseudodifferential operators on $\mathbb{W}$. In particular, with an operator $\mathcal{A}$ of the form (0.0.5) in the calculus of edge–pseudodifferential operators (here referred to as the edge algebra, cf. [15], [17]) we can associate a homogeneous principal edge symbol

$$\sigma_\wedge(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}, \tag{0.0.11}$$

$\pi_Y : T^*Y \setminus 0 \to Y$, homogeneous of order μ in the sense

$$\sigma_\wedge(\mathcal{A})(y,\lambda\eta) = \lambda^\mu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \sigma_\wedge(\mathcal{A})(y,\eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tag{0.0.12}$$

for all $\lambda \in \mathbb{R}_+$. If (0.0.11) is an isomorphism, we talk about SL–ellipticity of the edge conditions to A that are represented by the operators T, K and Q. The following conditions are equivalent (see [15], [8], [7]):

(i) the operator (0.0.5) is Fredholm,

(ii) A is elliptic, and $\sigma_\wedge(\mathcal{A})$ is SL–elliptic.

SL–ellipticity implies

$$\operatorname{ind}_{S^*Y} \sigma_\wedge(A) \in \pi_1^* K(Y), \tag{0.0.13}$$

since $\operatorname{ind}_{S^*Y} \sigma_\wedge(A) = [\pi_1^* J_+] - [\pi_1^* J_-]$ in this case.

Relation (0.0.13) is an analogue of the Atiyah–Bott condition for the existence of SL–elliptic edge conditions for A. Condition (0.0.13) can be violated; explicit examples are given in Nazaikinskij, Savin, Schulze, and Sternin [12].

1 Shapiro–Lopatinskij ellipticity in the edge algebra

1.1 Edge–degenerate symbols

Local representations of pseudodifferential operators on a manifold with edges will refer to variables $(r,x,y) \in \mathbb{R}_+ \times B \times \Omega$ for open sets $B \subseteq \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^q$, where

$x \in B$ and $y \in \Omega$ are local coordinates on X and Y, respectively. Let $S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^M)$ denote the space of (classical) symbols in variables in an open set $U \subseteq \mathbb{R}^m$ and covariables in $\mathbb{R}^M$. An element $p(r, x, y, \varrho, \xi, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times B \times \Omega \times \mathbb{R}^{1+n+q})$ is said to be edge–degenerate, if it has the form

$$p(r, x, y, \varrho, \xi, \eta) = \widetilde{p}(r, x, y, \widetilde{\varrho}, \xi, \widetilde{\eta})|_{\widetilde{\varrho}=r\varrho, \widetilde{\eta}=r\eta} \tag{1.1.1}$$

for some $\widetilde{p}(r, x, y, \widetilde{\varrho}, \xi, \widetilde{\eta}) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times B \times \Omega \times \mathbb{R}^{1+n+q})$ (the latter notation means that symbol estimates hold uniformly up to $r = 0$). All occurring spaces of symbols will be taken in their natural Fréchet topologies. Let $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times B \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ defined to be the space of all $h(r, x, y, z, \xi, \eta) \in \mathcal{A}(\mathbb{C}, S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times B \times \Omega \times \mathbb{R}^{n+q}))$ (where $\mathcal{A}(D, E)$ denotes the space of all holomorphic functions in an open set $D \subseteq \mathbb{C}$ with values in a Fréchet space E) such that

$$h(r, x, y, \beta + i\varrho, \xi, \eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times B \times \Omega \times \mathbb{R}^{1+n+q}_{\varrho,\xi,\eta})$$

for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for every $c \leq c'$. Set $\Gamma_\beta := \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$.

Let M denote the Mellin transform, $Mu(z) = \int_0^\infty r^{z-1}u(r)\, dr$, first for $u \in C_0^\infty(\mathbb{R}_+)$ where $z \in \mathbb{C}$, and then extended to suitable spaces of distributions, also vector–valued ones. Set $(M_\gamma u)(z) = M(r^{-\gamma}u)(z + \gamma)$, $\gamma \in \mathbb{R}$, and form pseudodifferential operators with respect to the Mellin transform

$$\mathrm{op}^{\gamma}_{M}(f)u(r) := \iint \left(\frac{r}{r'}\right)^{-(\frac{1}{2}\gamma+i\varrho)} f\left(r, r', \frac{1}{2} - \gamma + i\varrho\right) \frac{dr'}{r} d\!\!{}^-\!\varrho, \tag{1.1.2}$$

$d\!\!{}^-\!\varrho = (2\pi)^{-1}\, d\varrho$, with symbols $f(r, r', z) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$ (with $\operatorname{Im} z$ for $z \in \Gamma_{\frac{1}{2}-\gamma}$ being the covariable). More generally, Mellin pseudodifferential operators (1.1.2) will also be considered for operator–valued symbols f. The standard pseudodifferential action (based on the Fourier transform) will be denoted by $\mathrm{op}(\cdot)$ (when it refers to a part of variables and covariables we indicate variables as subscript). For instance, with a symbol $p(r, x, y, \varrho, \xi, \eta)$ we associate (y, η)–dependent families of pseudodifferential operators on $\mathbb{R}_+ \times B$

$$\mathrm{op}_{r,x}(p)(y, \eta)u(r, x) = \iint e^{i(r-r')\varrho + i(x-x')\xi} p(r, x, y, \varrho, \xi, \eta)u(r', x')\, dr' dx' d\!\!{}^-\!\varrho d\!\!{}^-\!\xi,$$

$d\!\!{}^-\!\xi = (2\pi)^{-n}\, d\xi$. If G is a C^∞ manifold, $L^{\mu}_{(\mathrm{cl})}(G; \mathbb{R}^l)$ denotes the space of (classical) parameter–dependent pseudodifferential operators on G, with $\mathbb{R}^l \ni \lambda$ being the space of parameters (that is, local symbols contain λ as a component of covariables, and $L^{-\infty}(G; \mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(G))$ with respect to the canonical Fréchet topology of the space $L^{-\infty}(G)$ of smoothing operators on G).

Theorem 1.1.1 *For every edge–degenerate symbol* $p(r, x, y, \varrho, \xi, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times B \times \Omega \times \mathbb{R}^{1+n+q})$ *there is an* $\widetilde{h}(r, x, y, z, \xi, \widetilde{\eta}) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times B \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ *such that* $h(r, x, y, , z, \xi, \eta) := \widetilde{h}(r, x, y, z, \xi, r\eta)$ *satisfies the relation*

$$\mathrm{op}_{r,x}(p)(y, \eta) = \mathrm{op}^{\gamma}_{M}\, \mathrm{op}_x(h)(y, \eta) \bmod C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+ \times B; \mathbb{R}^q)) \tag{1.1.3}$$

for all $\gamma \in \mathbb{R}$ *(both sides are regarded as operators on* $C_0^\infty(\mathbb{R}_+ \times B)$*). The symbol* $\widetilde{h}$ *is uniquely determined* $\bmod S^{-\infty}(\overline{\mathbb{R}}_+ \times B \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$.

Theorem 1.1.1 may be interpreted as a Mellin quantisation result. Proofs (based on different methods) may be found in [14] or [9], [10]. Theorem 1.1.1 entails a corresponding global variant, with B being replaced by a smooth, closed manifold X.

Definition 1.1.2 *Let X be a smooth, closed manifold. Then $M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^q)$ is defined as the space of all $h(z,\eta) \in \mathcal{A}(\mathbb{C}, L^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^q))$ such that $h|_{\Gamma_\beta \times \mathbb{R}^l} \in L^{\mu}_{\mathrm{cl}}(X;\Gamma_\beta \times \mathbb{R}^q)$ for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for every $c \leq c'$.*

The space $M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^q)$ is Fréchet in a natural way.

Let $\{U_1,\ldots,U_N\}$ be an open covering of X by coordinate neighbourhoods, let $\{\varphi_1,\ldots,\varphi_N\}$ be a subordinate partition of unity and $\{\psi_1,\ldots,\psi_N\}$ another system of elements $\psi_j \in C^\infty_0(U_j)$ such that $\varphi_j\psi_j = \varphi_j$ for all j. Fix charts $\kappa_j : U_j \to B_j$ for open sets $B_j \subseteq \mathbb{R}^n_x$, and consider a system of symbols $p(r,x,y,\varrho,\xi,\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times B \times \Omega \times \mathbb{R}^{1+n+q})$, $j = 1,\ldots,N$, that are edge–degenerate. Then the operator push–forwards of $\mathrm{op}_{r,x}(p_j)(y,\eta)$ with respect to $(\mathrm{id} \times \kappa_j^{-1}) : \mathbb{R}_+ \times B_j \to \mathbb{R}_+ \times U_j$ gives us elements $P_j(y,\eta) \in C^\infty(\Omega, L^{\mu}_{\mathrm{cl}}(U_j;\mathbb{R}^q))$ and we can pass to the family of operators

$$P(y,\eta) := \sum_{j=1}^{N} \varphi_j P_j(y,\eta)\psi_j \in C^\infty(\Omega, L^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^q)). \tag{1.1.4}$$

As a corollary of Theorem 1.1.1 we then obtain the following result. There exists an $\tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^q))$ such that $h(r,y,z,\eta) := \tilde{h}(r,y,z,r\eta)$ satisfies the relation

$$P(y,\eta) = \mathrm{op}^{\gamma}_{M}(h)(y,\eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q)) \tag{1.1.5}$$

for all $\gamma \in \mathbb{R}$, and $\tilde{h}$ is unique mod $C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^{-\infty}_{\mathcal{O}}(X;\mathbb{R}^q))$.

By assumption, we have $p_j(r,x,y,\varrho,\xi,\eta) = \tilde{p}_j(r,x,y,r\varrho,\xi,r\eta)$ for certain elements $\tilde{p}_j(r,x,y,\tilde{\varrho},\xi,\tilde{\eta}) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times B_j \times \Omega \times \mathbb{R}^{1+n+q})$.

Let us set $p_{j,0}(r,x,y,\varrho,\xi,\eta) := \tilde{p}_j(0,x,y,r\varrho,\xi,r\eta)$, and

$$P_0(y,\eta) := \sum_{j=1}^{N} \varphi_j P_{j,0}(y,\eta)\psi_j, \tag{1.1.6}$$

where $P_{j,0}(y,\eta)$ denotes the corresponding operator push–forward of $\mathrm{op}_{r,x}(p_{j,0})(y,\eta)$. Then, for $h_0(r,y,z,\eta) := \tilde{h}(0,y,z,r\eta)$, we also have

$$P_0(y,\eta) = \mathrm{op}^{\gamma}_{M}(h_0)(y,\eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q)). \tag{1.1.7}$$

1.2 Smoothing Mellin pseudodifferential operators

In this paper by a cut-off function $\omega(r)$ we understand any $\omega \in C^\infty_0(\overline{\mathbb{R}}_+)$ that equals 1 in a neighbourhood of $r = 0$.

Let $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s,\gamma \in \mathbb{R}$ denote the completion of $C^\infty_0(X^\wedge)$ with respect to the norm $\{(2\pi i)^{-1}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\mathrm{Im}\, z)Mu(z)\|^2_{L^2(X)}dz\}^{1/2}$, where $R^s(\rho) \in L^s_{\mathrm{cl}}(X;\mathbb{R})$ is a parameter-dependent elliptic operator of order s that defines isomorphisms $H^s(X) \to H^{r-s}(X)$ for all reals r and ρ. Moreover, let $\mathcal{K}^{s,\gamma}(X^\wedge)$ be the subspace of all $u(r,x) \in H^s_{\mathrm{loc}}(X^\wedge)$ such that

$$wu \in \mathcal{H}^{s,\gamma}(X^\wedge), (1-\omega)u \in H^s(X^\wedge)$$

for any cut–off function $\omega(r)$; here $H^s(X^\wedge)$ is the Sobolev space on the cylinder $X^\wedge = \mathbb{R}_+ \times X$ with the cone metric at infinity (for $X^\wedge = \mathbb{R}_+ \times S^n$ this j ist corresponds to $H^s(\mathbb{R}^{n+1})$ in polar coordinates, for general X the definition reduces to the case of the sphere by a decomposition of X into conical subsets, for details, cf. [15]). The spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge)$ can be endowed with Hilbert space scalar products. For $s = \gamma = 0$ we have an indification $\mathcal{H}^{0,0}(X^\wedge) = \mathcal{K}^{0,0}(X^\wedge) = r^{-\frac{n}{2}}L^2(\mathbb{R}_+ \times X)$, where $L^2(\mathbb{R}_+ \times X)$ refers to the measure $drdx$, $n = \dim X$. On $\mathcal{K}^{s,\gamma}(X^\wedge) \ni u(r,x)$ we consider a strongly continuous group of isomorphisms

$$\kappa_\lambda : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s,\gamma}(X^\wedge),$$

$\lambda \in \mathbb{R}_+$, defined by $(\kappa_\lambda u)(r,x) := \lambda^{\frac{n+1}{2}} u(\lambda x)$. For any $\varepsilon > 0$ we set

$$\mathcal{S}^\gamma(X^\wedge)_\varepsilon := \bigcap_{k=0}^{\infty} r^{\frac{\varepsilon k}{k+1}} \langle r\rangle^{-k} \mathcal{K}^{k,\gamma}(X^\wedge)$$

which is a Fréchet space in the topology of the projective limit. Also $E_k := r^{\frac{\varepsilon k}{k+1}} \langle r\rangle^{-k} \mathcal{K}^{s,\gamma}(X^\wedge)$ is a Hilbert space, where $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ acts as a strongly continuous group of isomorphisms (in such a case we say that the Hilbert space in consideration is endowed with a group action.)

Definition 1.2.1 (i) *Given a Hilbert space E endowed with a strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, the space $\mathcal{W}^s(\mathbb{R}^q, E)$ for $s \in \mathbb{R}$ is defined as the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm*

$$\Big\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \widehat{u}(\eta)\|^2_E d\eta \Big\}^{1/2}$$

(ii) *If E and $\widetilde{E}$ are Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, then $S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ for $U \subseteq \mathbb{R}^q$ open, $\mu \in \mathbb{R}$, is defined to be the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that*

$$\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle} \{D^\alpha_y D^\beta_\eta a(y,\eta)\} \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,\widetilde{E})} \le c\langle\eta\rangle^{\mu-\beta}$$

for all $(y,\eta) \in K \times \mathbb{R}^q$ for arbitrary $K \subset\subset U$ and $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, with constants $c = c(\alpha,\beta,k) > 0$.

An operator function $f(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E,\widetilde{E}))$ is said to belong to $S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E})$, if

$$f(y, \lambda\eta) = \lambda^\mu \widetilde{\kappa}_\lambda f(y,\eta) \kappa_\lambda$$

for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$. A symbol $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ is called classical, if there are elements $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E})$, $j = 0, 1, 2, \dots$, such that

$$a(y,\eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y,\eta) \in S^{(\mu-(N+1)}(U \times \mathbb{R}^q; E, \widetilde{E})$$

for all $N \in \mathbb{N}$; here $\chi(\eta)$ is any excision function.

Let $M^{-\infty}(X;\beta)_\varepsilon$ for $\beta \in \mathbb{R}$ and $\varepsilon > 0$ denote the subspace of all $f(z) \in \mathcal{A}(\{\beta-\varepsilon < \operatorname{Re} z < \beta+\varepsilon\}, L^{-\infty}(X))$ such that $f(\delta+i\varrho) \in \mathcal{S}(\mathbb{R}_\varrho, L^{-\infty}(X))$ for every $\beta-\varepsilon<\delta<\beta+\varepsilon$, uniformly in $c \le \delta \le c'$ for every $\beta-\varepsilon<c,\, c'<\beta+\varepsilon$. The space $M^{-\infty}(X;\beta)_\varepsilon$ is Fréchet in a natural way. Given an element $f(y,z) \in C^\infty(\Omega, M^{-\infty}(X;\gamma-\frac{n}{2})_\varepsilon)$ we form the family of operators

$$m(y,\eta) := \omega(r[\eta])r^{-\mu}\operatorname{op}_M^{\gamma-\frac{n}{2}}(f)(y)\widetilde{\omega}(r[\eta]) \tag{1.2.1}$$

for some cut–off functions ω and $\widetilde{\omega}$. Such a family represents an element

$$m(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)). \tag{1.2.2}$$

1.3 Green, trace, and potential operators

Compositions of operators of the form (0.0.5) that are possible in our class of edge problems (when bundles and weights in the middle fit together) give rise to more general upper left corners than assumed in (0.0.5). The same is true of parametrices for elliptic operators. Instead of A we obtain operators of the form $A+M+G$, where M is a smoothing Mellin operator, cf. Section 1.2, and G a so–called Green operator. While Green operators as well as trace and potential operators in [15] or [17] are given terms of symbolic structures with asymptotics, we consider here a larger class of operators which is, in fact, simpler and only controls weight improvements in an ε–strip around given weight lines $\Gamma_{\frac{n+1}{2}-\gamma}$ and $\Gamma_{\frac{n+1}{2}-\delta}$, respectively, for some $\gamma,\delta\in\mathbb{R}$ and $\varepsilon>0$.

For the following definition we set

$$\mathcal{E}^{s,\gamma} := \mathcal{K}^{s,\gamma}(X^\wedge,\mathbb{C}^k)\oplus\mathbb{C}^{j_-}, \tag{1.3.1}$$

$$\mathcal{S}^\delta_\varepsilon := \varprojlim_{N\in\mathbb{N}} \langle r\rangle^{-N}\mathcal{K}^{N,\delta+\varepsilon}(X^\wedge,\mathbb{C}^l)\oplus\mathbb{C}^{j_+}, \tag{1.3.2}$$

where $\mathcal{E}^{s,\gamma}$ is equipped with a strongly continuous group of isomorphisms

$$\boldsymbol{\kappa}_\lambda : u(r,x)\oplus c \to (\lambda^{\frac{n+1}{2}}u(\lambda r,x))\oplus c, \qquad \lambda\in\mathbb{R}_+.$$

An analogous action, also denoted by $\boldsymbol{\kappa}_\lambda$, is considered on the spaces in (1.3.2) for every N. This allows us to talk about operator–valued symbols, acting from $\mathcal{E}^{s,\gamma}$ to $\mathcal{S}^\delta_\varepsilon$. Given an operator $g:\mathcal{E}^{s,\gamma}\to\mathcal{S}^\delta_\varepsilon$ that is continuous for all s, the formal adjoint g^*, is defined by

$$(gu,v)_{\mathcal{K}^{0,0}(X^\wedge,\mathbb{C}^l)\oplus\mathbb{C}^{j_+}} = (u,g^*v)_{\mathcal{K}^{0,0}(X^\wedge,\mathbb{C}^k)\oplus\mathbb{C}^{j_-}}$$

for all $u\in C_0^\infty(X^\wedge,\mathbb{C}^k)\oplus\mathbb{C}^{j_-}$, $v\in C_0^\infty(X^\wedge,\mathbb{C}^l)\oplus\mathbb{C}^{j_+}$.

Similarly to (1.3.1) and (1.3.2) we set

$$\widetilde{\mathcal{E}}^{s,-\delta} := \mathcal{K}^{s,-\delta}(X^\wedge,\mathbb{C}^l)\oplus\mathbb{C}^{j_+}, \tag{1.3.3}$$

$$\widetilde{\mathcal{S}}^{-\gamma}_{\varepsilon'} := \varprojlim_{N\in\mathbb{N}} \langle r\rangle^{-N}\mathcal{K}^{N,-\gamma+\varepsilon}(X^\wedge,\mathbb{C}^k)\oplus\mathbb{C}^{j_-}, \tag{1.3.4}$$

also endowed with $\{\boldsymbol{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$.

Definition 1.3.1 *The space $\mathcal{R}^{\mu}_{G}(U \times \mathbb{R}^q, (\gamma, \delta); \boldsymbol{w})_{\varepsilon}$ for an open set $U \subseteq \mathbb{R}^p$ and a tuple $\boldsymbol{w} = (k, l; j_-, j_+)$ of fibre dimensions (below belonging to vector bundles on $\mathbb{W}$ and Y, respectively) is defined to be the set of all operator–valued symbols*

$$g(y,\eta) \in \bigcap_{\substack{s\in\mathbb{R}\\ 0\le\varepsilon'<\varepsilon}} S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^q; \mathcal{E}^{s,\gamma}, \mathcal{S}^{\delta}_{\varepsilon'})$$

such that the pointwise formal adjoint $g^*(y,\eta)$ has the property

$$g^*(y,\eta) \in \bigcap_{\substack{s\in\mathbb{R}\\ 0\le\varepsilon'<\varepsilon}} S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^q; \widetilde{\mathcal{E}}^{s,-\delta}, \widetilde{\mathcal{S}}^{-\gamma}_{\varepsilon'}).$$

1.4 The edge algebra

The edge algebra that will be formulated in this section is based on local interior and operator–valued edge symbols. Let us first have a look at a special class of scalar operator–valued symbols $a(y,\eta)$ for $(y,\eta) \in \Omega \times \mathbb{R}^q$, $\Omega \subseteq \mathbb{R}^q$ open.

Let us fix cut–off functions $\omega_1, \omega_2, \omega_3$ such that $\omega_2 \equiv 1$ on $\operatorname{supp}\omega_1$ and $\omega_1 \equiv 1$ on $\operatorname{supp}\omega_3$, and set

$$a_M(y,\eta) := \omega_1(r[\eta])\, \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\omega_2(r[\eta]), \tag{1.4.1}$$
$$a_F(y,\eta) := (1-\omega_1(r[\eta]))P(y,\eta)(1-\omega_3(r[\eta])), \tag{1.4.2}$$

where $P(y,\eta)$ is an operator family of the form (1.1.4) and $h(r,y,z,\eta) = \widetilde{h}(r,y,z,r\eta)$ a Mellin symbol, $\widetilde{h}(r,y,z,\eta) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega, M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^q))$, such that relation

$$P(y,\eta) = \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta) \bmod C^{\infty}(\Omega, L^{-\infty}(X^\wedge, \mathbb{R}^q)) \tag{1.4.3}$$

holds. Then, if $\sigma, \widetilde{\sigma}$ are arbitrary cut–off functions,

$$p(y,\eta) := \sigma(r)r^{-\mu}\{a_M(y,\eta) + a_F(y,\eta)\}\widetilde{\sigma}(r) \tag{1.4.4}$$

belongs to $S^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ for every $s \in \mathbb{R}$.

Let $R^{\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g})_{\varepsilon}$ for $\boldsymbol{g} := (\gamma, \gamma-\mu)$, $\varepsilon > 0$, denote the set of all families of operators

$$p(y,\eta) + (m+g_1)(y,\eta) \tag{1.4.5}$$

for arbitrary $a(y,\eta)$ of the form (1.4.4) and $(m+g_1)(y,\eta) \in R^{\mu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})_{\varepsilon}$, cf. Section 1.3. More generally, we set $R^{\mu}(\Omega\times\mathbb{R}^q, \boldsymbol{g}; (k,l))_{\varepsilon} := R^{\mu}(\Omega\times\mathbb{R}^q, \boldsymbol{g})_{\varepsilon} \otimes \mathbb{C}^l \otimes \mathbb{C}^k$.

For $\mu \in \mathbb{R}$, $\boldsymbol{g} = (\gamma, \gamma-\mu)$, $\varepsilon > 0$, and $\boldsymbol{w} := (k, l; j_-, j_+)$ we now denote by $\mathcal{R}^{\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_{\varepsilon}$ the space of all families of operators

$$a(y,\eta) := \begin{pmatrix} p(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + g(y,\eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, \mathbb{C}^k) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \to \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge, \mathbb{C}^l) \\ \oplus \\ \mathbb{C}^{j_+} \end{array} \tag{1.4.6}$$

for arbitrary $p(y,\eta) \in R^{\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; (k,l))_{\varepsilon}$ and $g(y,\eta) \in \mathcal{R}^{\mu}_{G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_{\varepsilon}$.

Denoting the spaces in (1.4.6) for a moment by H and $\widetilde{H}$, respectively, endowed with group actions $\mathrm{diag}(\kappa_\lambda, \mathrm{id})_{\lambda\in\mathbb{R}_+}$, where id means the identity in $\mathbb{C}^{j-}$ or $\mathbb{C}^{j+}$, we have $a(y,\eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ for every $s, \gamma \in \mathbb{R}$. If $U \subset Y$ is a coordinate neighbourhood on the edge Y, diffeomorphic to Ω, and if $E'|_U \to \Omega \times \mathbb{C}^k$, $F'|_U \to \Omega \times \mathbb{C}^l$, $J_\pm|_U \to \Omega \times \mathbb{C}^{j_\pm}$ are trivialisations of bundles $E', F', J_\pm \in \mathrm{Vect}(Y)$, we can pull-back $\mathrm{Op}_y(a)$ to U as an operator

$$\mathcal{A}_U : \begin{array}{c} C_0^\infty(X^\wedge \times U, E|_{X^\wedge \times G}) \\ \oplus \\ C_0^\infty(U, J_-|_U) \end{array} \to \begin{array}{c} C^\infty(X^\wedge \times U, F|_{X^\wedge \times G}) \\ \oplus \\ C^\infty(U, J_+|_U) \end{array} \tag{1.4.7}$$

where E and F are the pull-backs of E' and F' under the cononical projection $X^\wedge \times Y \to Y$. The global interpretation of (1.4.7) is a consequence of a corresponding invariance under transition maps. Now let $\sigma, \widetilde{\sigma} \in C^\infty(\mathbb{W})$ be elements supported by a collar neighbourhood $\cong [0,1) \times X \times Y$ of $\partial\mathbb{W} \cong X \times Y$ where σ and $\widetilde{\sigma}$ equal 1 in a neighbourhood of $\partial\mathbb{W}$ and $\widetilde{\sigma} \equiv 1$ on $\mathrm{supp}\,\sigma$. Choose an open covering $\{U_1, \ldots, U_M\}$ of Y by coordinate neighbourhoods, let $\{\varphi_1, \ldots, \varphi_M\}$ be a subordinate partition of unity, and let $\{\psi_1, \ldots, \psi_M\}$ be another system of functions $\psi_j \in C_0^\infty(U_j)$ such that $\psi_j \equiv 1$ on $\mathrm{supp}\,\varphi_j$ for all j.

For any system of local symbols $a_j(y,\eta) \in \mathcal{R}^\mu(\Omega_j \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_\varepsilon$ with open $\Omega_j \subseteq \mathbb{R}^q$ diffeomorphic to U_j, $j = 1, \ldots, M$, and an operator $A_{\mathrm{int}} \in L^\mu_{\mathrm{cl}}(\mathrm{int}\,\mathbb{W}; E, F)$ (for $E, F \in \mathrm{Vect}(\mathbb{W})$ that restrict on $[0,1) \times X \times Y$ to the corresponding bundles E, F occurring in the definition of (1.4.7)) we now form an operator

$$\sum_{j=1}^M \begin{pmatrix} \varphi_j\sigma & 0 \\ 0 & \varphi_j \end{pmatrix} \mathcal{A}_{U_j} \begin{pmatrix} \psi_j\widetilde{\sigma} & 0 \\ 0 & \psi_j \end{pmatrix} + \begin{pmatrix} (1-\sigma)A_{\mathrm{int}}(1-\widetilde{\widetilde{\sigma}}) & 0 \\ 0 & 0 \end{pmatrix} \tag{1.4.8}$$

where $\widetilde{\widetilde{\sigma}}$ is another function in $C^\infty(\mathbb{W})$ supported in a collar neighbourhood of $\partial\mathbb{W}$ such that $1 - \widetilde{\widetilde{\sigma}}$ equals 1 on $\mathrm{supp}(1-\sigma)$ and $1 - \widetilde{\widetilde{\sigma}}$ vanishes near $\partial\mathbb{W}$.

For the calculus of operators on the stretched manifold $\mathbb{W}$ with edges we also need global smoothing operators $\mathcal{G}$, characterised by the continuity of

$$\mathcal{G} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ H^s(Y, J_-) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,\gamma-\mu+\alpha}(\mathbb{W}, F) \\ \oplus \\ H^\infty(Y, J_+) \end{array} \tag{1.4.9}$$

and

$$\mathcal{G}^* : \begin{array}{c} \mathcal{W}^{s,-\gamma+\mu}(\mathbb{W}, F) \\ \oplus \\ H^s(Y, J_+) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,-\gamma+\alpha}(\mathbb{W}, E) \\ \oplus \\ H^\infty(Y, J_-) \end{array} \tag{1.4.10}$$

for all $s \in \mathbb{R}$ and all $0 \le \alpha < \varepsilon$ for some $\varepsilon > 0$. Here, $\mathcal{G}^*$ is the formal adjoint of $\mathcal{G}$ in the sense

$$(\mathcal{G}u, v)_{\mathcal{W}^{0,0}(\mathbb{W},F)\oplus H^0(Y,J_+)} = (u, \mathcal{G}^*v)_{\mathcal{W}^{0,0}(\mathbb{W},E)\oplus H^0(Y,J_-)}$$

for all $u \in C_0^\infty(\mathrm{int}\,\mathbb{W}, E) \oplus C^\infty(Y, J_-)$, $v \in C_0^\infty(\mathrm{int}\,\mathbb{W}, F) \oplus C^\infty(Y, J_+)$ with respect to a choice of scalar products in the spaces $\mathcal{W}^{0,0}(\mathbb{W}, E)$, $\mathcal{W}^{0,0}(\mathbb{W}, F)$, $H^0(Y, J_\pm)$, referring to the Riemanian metrics on $\mathbb{W}$ and Y and Hermitian metrics in the bundles E, F and J_-, J_+. Let $\mathcal{Y}_G^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ denote the space of all operators $\mathcal{G}$ with these mapping properties, where $\boldsymbol{g} = (\gamma, \gamma - \mu)$ are the given weight data, $\gamma, \mu \in \mathbb{R}$, and $\boldsymbol{v} = (E, F; J_-, J_+)$ the tuple of bundles, and set

$$\mathcal{Y}_G^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) := \bigcup_{\varepsilon > 0} \mathcal{Y}_G^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})_\varepsilon .$$

Definition 1.4.1 *Let $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} := (\gamma, \gamma - \mu)$, $\boldsymbol{v} := (E, F; J_-, J_+)$ denote the set of all operators $\mathcal{A}_0 + \mathcal{G}$, where $\mathcal{A}_0$ is of the form* (1.4.8) *for arbitrary $a_j \in \mathcal{R}^\mu(\Omega_j \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_\varepsilon$ and arbitrary $\varepsilon > 0$, $j = 1, \dots, M$, $A_{\mathrm{int}} \in L^\mu_{\mathrm{cl}}(\mathrm{int}\,\mathbb{W}; E, F)$, and $\mathcal{G} \in \mathcal{Y}^{-\infty}_G(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$.*

By definition, the elements $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{v} := (E, F; J_-, J_+)$, are 2×2-block matrices. The space of upper left cornes u.l.c.$\,\mathcal{A}$ will be denoted by $Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ (which is, of course, the same as $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; (E, F; 0, 0))$ with 0 being interpreted as corresponding bundles of fibre dimension 0). Similarly, we employ the notation $Y^\mu_G(\mathbb{W}, \boldsymbol{g}; E, F)$. Moreover, let $A \in Y^\mu_{M+G}(\mathbb{W}, \boldsymbol{g}; E, F)$ denote the subspace of all $A \in Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ such that $A \in L^{-\infty}(\mathrm{int}\,\mathbb{W}; E, F)$ (these operators are characterised by $A_{\mathrm{int}} \equiv 0$ in the representation (1.4.8) and $a_M(y, \eta) = a_F(y, \eta) = 0$ in the local amplitude functions, cf. (1.4.1) and (1.4.2)).

Theorem 1.4.2 *Every $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ represents continuous operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ H^s(Y, J_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \\ \oplus \\ H^{s-\mu}(Y, J_+) \end{array} \tag{1.4.11}$$

for all $s \in \mathbb{R}$.

Let us now formulate the principal symbolic structure of elements $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$. It consists of two components

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\wedge(\mathcal{A})).$$

The so-called (principal) interior symbol $\sigma_\psi(\mathcal{A})$ is the standard homogeneous principal symbol order μ of the upper left corner $A :=$ u.l.c.$\,\mathcal{A}$ of $\mathcal{A}$

$$\sigma_\psi(\mathcal{A}) : \pi^*_{\mathrm{int}\,\mathbb{W}} E \to \pi^*_{\mathrm{int}\,\mathbb{W}} F, \tag{1.4.12}$$

$\pi_{\mathrm{int}\,\mathbb{W}} : T^*(\mathrm{int}\,\mathbb{W}) \setminus 0 \to \mathrm{int}\,\mathbb{W}$. Because of the degenerate behaviour of the underlying symbols they have in local coordinates $(r, x, y) \in \mathbb{R}_+ \times B \times \Omega$ near $\partial\mathbb{W}$ the form

$$r^{-\mu} \widetilde{p}_{(\mu)}(r, x, y, \widetilde{\rho}, \xi, \widetilde{\eta})|_{\widetilde{\rho} = r\rho, \widetilde{\eta} = r\eta} \tag{1.4.13}$$

for an $l \times k$-matrix $\widetilde{p}_{(\mu)}(r, x, y, \widetilde{\rho}, \xi, \widetilde{\eta})$ of functions, homogeneous in $(\widetilde{\rho}, \xi, \widetilde{\eta}) \neq 0$ of order μ and smooth in (r, x, y) up to $r = 0$. In addition, we have the so-called (principal) edge symbol

$$\sigma_\wedge(\mathcal{A}) : \pi^*_Y \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \to \pi^*_Y \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}, \tag{1.4.14}$$

$\pi_Y : T^*Y \setminus 0 \to Y$.

Locally, $\sigma_\wedge(\mathcal{A})$ is defined by the expression

$$\begin{pmatrix} \sigma_\wedge(p)(y, \eta) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\wedge(g)(y, \eta), \tag{1.4.15}$$

where $\sigma_\wedge(g)(y, \eta)$ is the homogeneous principal component of $g(y, \eta)$ in the representation (1.4.5) of local amplitude functions $a_j(y, \eta)$, cf. Definition 1.3.1, while

$$\begin{aligned} \sigma_\wedge(p)(y, \eta) \quad := \quad & r^{-\mu} \omega_1(r|\eta|)\, \mathrm{op}_M^{\gamma - \frac{n}{2}}(h_0)(y, \eta) \omega_2(r|\eta|) \\ & + r^{-\mu}(1 - \omega_1(r|\eta|)) P_0(y, \eta)(1 - \omega_3(r|\eta|)) \\ & + r^{-\mu} \omega(r|\eta|)\, \mathrm{op}_M^{\gamma - \frac{n}{2}}(f)(y) \widetilde{\omega}(r|\eta|) \end{aligned} \tag{1.4.16}$$

when we represent $p(y,\eta)$ in the form (1.4.5), now without g_1, because this Green term is already contained in the second summand in (1.4.6); notation in (1.4.16) refers to (1.1.6) (1.1.7), (1.2.1) and (1.4.1), (1.4.2).

Remark 1.4.3 *$\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ and $\sigma(\mathcal{A}) = 0$ implies that* (1.4.11) *is a compact operator for every $s \in \mathbb{R}$.*

For purposes below we also define subspaces

$$\mathcal{Y}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) \quad \text{for} \quad \boldsymbol{g} := (\gamma, \gamma - \mu), \tag{1.4.17}$$

$j = 0, 1, 2, \ldots$, where local interior edge-degenerate symbols (1.1.1) are of order $\mu - j$ and multiplied by $r^{-\mu+j}$ (instead of $r^{-\mu}$ before) while local Green symbols are of order $\mu - j$, cf. Definition 1.3.1.

Similarly to principal interior and edge symbols we have pair $\sigma^{\mu-j}(\mathcal{A}) := (\sigma_\psi^{\mu-j}(\mathcal{A}), \sigma_\wedge^{\mu-j}(\mathcal{A}))$ of principal symbols for $\mathcal{A} \in \mathcal{Y}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, in particular, $\sigma(\mathcal{A}) = \sigma_\psi^\mu(\mathcal{A}), \sigma_\wedge^\mu(\mathcal{A}))$. It can easily be proved that then

$$\mathcal{Y}^{\mu-(j+1)}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) = \{\mathcal{A} \in \mathcal{Y}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) : \sigma^{\mu-j}(\mathcal{A}) = 0\}. \tag{1.4.18}$$

The following result is the motivation of our notation “edge algebra”. Let us set $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\boldsymbol{h} := (\gamma + \nu, \gamma)$, $\boldsymbol{g} \circ \boldsymbol{h} := (\gamma + \nu, \gamma - \mu)$ and $\boldsymbol{v} := (E_0, F; J_0, J_+)$, $\boldsymbol{w} := (E, E_0; J, J_0)$, $\boldsymbol{v} \circ \boldsymbol{w} = (E, F; J_-, J_+)$.

Theorem 1.4.4 *$\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, $\mathcal{B} \in \mathcal{Y}^\nu(\mathbb{W}, \boldsymbol{h}; \boldsymbol{w})$ implies $\mathcal{AB} \in \mathcal{Y}^{\mu+\nu}(\mathbb{W}, \boldsymbol{g} \circ \boldsymbol{h}; \boldsymbol{v} \circ \boldsymbol{w})$, and we have*

$$\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

(*with componentwise multiplication*).

1.5 SL–ellipticity

Definition 1.5.1 *Let $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{g} := (\gamma, \gamma - \mu)$, $\boldsymbol{v} := (E, F; J_-, J_+)$. Then $\mathcal{A}$ is called SL-elliptic, if*

(i) (1.4.12) *is an isomorphism and if, in addition the local $k \times k$-matrices $\widetilde{p}_{(\mu)}$ in representation* (1.4.13) (*with k being the fibre dimension of E and F*) *are non-singular up to $r = 0$.*

(ii) (1.4.14) *is an isomorphism for an $s = s_0 \in \mathbb{R}$.*

Remark 1.5.2 (i) *The homogeneous principal symbol* (1.4.12) *may be interpreted as a homomorphism between bundle pull-backs with respect to another representation of $T^*(\operatorname{int}\mathbb{W}) \setminus 0$, an analogue of the compressed cotangent bundle. We want to avoid explaining this in detail here, but content ourselves observing the “compressed” symbol $\widetilde{p}_{(\mu)}$ in local form. An operator $A \in Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ will be called σ_ψ-elliptic Definition* 1.5.1 (i).

(ii) *The ellipticity condition* (1.4.14) *is an analogue of the Shapiro-Lopatinskij condition from the analysis of boundary value problems, here with the edge in place of the boundary. If it is satisfied for an $s = s_0$ then so is for arbitrary $s \in \mathbb{R}$. Moreover, it may be replaced by the bijectivity condition of*

$$\sigma_\wedge(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{S}_\varepsilon^\gamma(X^\wedge) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{S}_\varepsilon^{\gamma-\mu}(X^\wedge) \otimes F' \\ \oplus \\ J_+ \end{pmatrix} \tag{1.5.1}$$

for any $\varepsilon > 0$*, where* $\mathcal{S}^{\gamma}_{\varepsilon}(X^{\wedge}) := \varprojlim_{N \in \mathbb{N}} \langle r \rangle^{-N} \mathcal{K}^{N,\gamma+\varepsilon}(X^{\wedge})$.

Theorem 1.5.3 *For an operator* $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{g} := (\gamma, \gamma-\mu)$, $\boldsymbol{v} := (E, F; J_-, J_+)$, *the following conditions are equivalent:*

(i) $\mathcal{A}$ *is elliptic.*

(ii) *The operator* (1.4.11) *is Fredholm for an* $s = s_0 \in \mathbb{R}$.

Theorem 1.5.4 *If* $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *is elliptic, the operator* (1.4.11) *is Fredholm for every* $s \in \mathbb{R}$*, and there is a parametrix* $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ *for* $\boldsymbol{g}^{-1} := (\gamma-\mu, \gamma)$, $\boldsymbol{v}^{-1} := (F, E; J_+, J_-)$ *in the sense that*

$$\mathcal{C}_l := \mathcal{I} - \mathcal{P}\mathcal{A} \in \mathcal{Y}^{-\infty}_{G}(\mathbb{W}, \boldsymbol{g}_l; \boldsymbol{v}_l), \tag{1.5.2}$$

$$\mathcal{C}_r := \mathcal{I} - \mathcal{A}\mathcal{P} \in \mathcal{Y}^{-\infty}_{G}(\mathbb{W}, \boldsymbol{g}_r; \boldsymbol{v}_r) \tag{1.5.3}$$

for $\boldsymbol{g}_l = (\gamma, \gamma)$, $\boldsymbol{v}_l = (E, E; J_-, J_-)$*, and* $\boldsymbol{g}_r = (\gamma-\mu, \gamma-\mu)$, $\boldsymbol{v}_r = (F, F; J_+, J_+)$.

Remark 1.5.5 *The parametrix* $\mathcal{P}$ *in Theorem* 1.5.4 *can be chosen in such a way that the operators* (1.5.2) *and* (1.5.3) *define continuous projections*

$$\mathcal{C}_l : \begin{array}{c} \mathcal{W}^{s,\gamma}(X,E) \\ \oplus \\ H^{(}Y, J_-) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,\gamma}(X,E) \\ \oplus \\ C^{\infty}(Y, J_-) \end{array}, \qquad \mathcal{C}_r : \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(X,F) \\ \oplus \\ H^{s-\mu}(Y, J_+) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,\gamma-\mu}(X,F) \\ \oplus \\ C^{\infty}(Y, J_+) \end{array}$$

such that $V := \operatorname{im} \mathcal{C}_l$ *and* $W := \operatorname{im} \mathcal{C}_r$ *are of finite dimension and independent of* s*, and* $V = \ker \mathcal{A}$, $W \cap \operatorname{im} \mathcal{A} = \emptyset$, $W + \operatorname{im} \mathcal{A} = \mathcal{W}^{s-\mu,\gamma-\mu}(X,F) \oplus H^{s-\mu}(Y, J_+)$. *In the latter relations* $\mathcal{A}$ *is interpreted as an operator* (1.4.11) *for any fixed* $s \in \mathbb{R}$*, and the assertions are true of all* $s \in \mathbb{R}$.

2 Edge operators with global projection conditions

2.1 Fredholm families in the edge symbolic calculus

Given an operator $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, cf. Definition 1.4.1, and $A :=$ u.l.c.$\mathcal{A}$, the condition of bijectivity of the operator family (1.4.14) implies that

$$\sigma_{\wedge}(A) : \pi^{*}_{Y}(E' \otimes \mathcal{K}^{s,\gamma}(X^{\wedge})) \to \pi^{*}_{Y}(F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})) \tag{2.1.1}$$

$\pi_Y : T^{*}Y \setminus 0 \to Y$, is a family of Fredholm operators, where $\ker \sigma_{\wedge}(A)(y, \eta)$ and $\operatorname{coker} \sigma_{\wedge}(A)(y, \eta)$ are independent of s. By virtue of the homogeneity

$$\sigma_{\wedge}(A)(y, \lambda\eta) = \lambda^{\mu} \kappa_{\lambda} \sigma_{\wedge}(A)(y, \eta) \kappa^{-1}_{\lambda}$$

for all $\lambda \in \mathbb{R}_+$, the index $\operatorname{ind} \sigma_{\wedge}(A)(y, \eta)$ only depends on $\eta/|\eta|$. Therefore, we interpret $\sigma_{\wedge}(A)$ as a family of Fredholm operators parametrised by $(y, \eta) \in S^{*}Y$. This gives us an index element

$$\operatorname{ind}_{S^{*}Y} \sigma_{\wedge}(A) \in K(S^{*}Y). \tag{2.1.2}$$

As noted in the beginning, similarly to boundary value problems with the transmission property, cf. Atiyah and Bott [1], Boutet de Monvel [4], in the edge algebra there is the a topological criterion for the existence of SL-elliptic conditions.

Theorem 2.1.1 *Let $A \in Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ be σ_ψ-elliptic, and let* (2.1.1) *be a family of Fredholm operators. There exists an SL-elliptic element $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ with $A := \mathrm{u.l.c.}\, \mathcal{A}$, $\boldsymbol{v} := (E, F; J_-, J_+)$ for certain $J_\pm \in \mathrm{Vect}(Y)$, if and only if condition* (2.1.2) *is fulfilled.*

In fact, it is evident that condition (2.1.2) is necessary for the existence of an SL-elliptic element $\mathcal{A}$. Conversely, if (2.1.2) is fulfilled, we can choose vector bundles $J_\pm \in \mathrm{Vect}(Y)$ and block matrix of the form (1.4.14) with $\sigma_\wedge(A)$ as upper left corner such that (1.4.14) is an isomorphism. The idea of constructing corresponding extra entries is very close to the case of boundary value problem, see [20, Theorem 3.2.1]. The generalisation to the present case is simple, so we omit details. Recall that from the (1,2)-, (2,1) -, and (2,2) - entries of (1.4.14) we get classical operator-valued symbols when we multiply them by an excision function in η. Associated operators then complete A to a Fredholm operator (1.4.11), cf. Theorem 1.5.3.

Remark 2.1.2 *It may happen that there are different weights $\gamma, \widetilde{\gamma} \in \mathbb{R}$ such that $A \in Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ for $\boldsymbol{g} := (\gamma, \gamma - \mu)$ and $\widetilde{A} \in Y^\mu(\mathbb{W}, \widetilde{\boldsymbol{g}}; E, F)$ for $\widetilde{\boldsymbol{g}} := (\widetilde{\gamma}, \widetilde{\gamma} - \mu)$ restrict to the same element in $L^\mu_{\mathrm{cl}}(\mathrm{int}\,\mathbb{W}; E, F) \mod L^{-\infty}(\mathrm{int}\,\mathbb{W}; E, F)$. This is the case, for instance, when we set $\gamma = \widetilde{\gamma}$ and $\widetilde{A} = A + C$ for any $C \in Y^\mu_{M+G}(\mathbb{W}, \boldsymbol{g}; E, F)$. Then, if both* (2.1.1) *and*

$$\sigma_\wedge(\widetilde{A}) : \pi_Y^*(\mathcal{K}^{s,\widetilde{\gamma}}(X^\wedge) \otimes E') \to \pi_Y^*(\mathcal{K}^{s-\mu,\widetilde{\gamma}-\mu}(X^\wedge) \otimes F')$$

are Fredholm families, we have

$$\mathrm{ind}_{S^*Y}\, \sigma_\wedge(A) \in \pi_1^* K(Y) \iff \mathrm{ind}_{S^*Y}\, \sigma_\wedge(\widetilde{A}) \in \pi_1^* K(Y).$$

Theorem 2.1.3 *Let $A \in Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ be elliptic in the sense of Definition* 1.5.1 (i)*, and let* (2.1.1) *be a Fredholm family. If $L_\pm \in \mathrm{Vect}(T^*Y \setminus 0)$ are bundles with the property*

$$\mathrm{ind}_{S^*Y}\, \sigma_\wedge(A) = [L_+|_{S^*Y}] - [L_-|_{S^*Y}], \tag{2.1.3}$$

there exists an element $G \in Y^\mu_G(\mathbb{W}, \boldsymbol{g}; E, F)$ such that

$$\ker_{S^*Y}(A+G) \cong L_+|_{S^*Y}, \qquad \mathrm{coker}_{S^*Y}(A+G) \cong L_-|_{S^*Y}.$$

If $J_\pm \in \mathrm{Vect}(Y)$ is a bundles such that $L_\pm$ are subbundles of $\pi_Y^* J_\pm$, there exists an $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; J_-, J_+)$ with $A + G = \mathrm{u.l.c.}\, \mathcal{A}$ such that $\sigma_\wedge(\mathcal{A})$ induces an isomorphism

$$\sigma_\wedge(\mathcal{A}) : \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge) \otimes \pi_Y^* E' \\ \oplus \\ L_- \end{pmatrix} \to \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes \pi_Y^* F' \\ \oplus \\ L_+ \end{pmatrix}. \tag{2.1.4}$$

Proof. $Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ is a direct generalization of the space $L^\mu(M, \boldsymbol{g}; E, F)$ from [20] of boundary value problems on a (smooth, compact) manifold M with boundary. The proof of Theorem 2.1.3 is formally the same as that of [20, Theorem 5.2.1]. $\square$

2.2 A Toeplitz extension of the edge algebra

In the following we use the fact that for every $L \in \mathrm{Vect}(T^*Y \setminus 0)$ and every $J \in \mathrm{Vect}(Y)$ such that L is a subbundle of $\pi_Y^* J$ there exists an element $P \in L^0_{\mathrm{cl}}(Y; J, J)$ such that $P^2 = P$, and the homogeneous principal symbols symbol of order zero

$$\sigma_\psi(P) : \pi_Y^* J \to L$$

is a projection. Clearly, there are many such choices of P. Let us write $\mathbb{L} := (P, J, L)$ for a fixed triple of data of that kind, and set

$$P^s(Y, \mathbb{L}) := PH^s(Y, J), \tag{2.2.1}$$

$s \in \mathbb{R}$. There is then a canonical embedding $R : P^s(Y, \mathbb{L}) \to H^s(Y, J)$. Given two such triples

$$\mathbb{L}_\pm := (P_\pm, J_\pm, L_\pm), \tag{2.2.2}$$

for every operator $\widetilde{\mathcal{A}} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{v} := (E, F; J_-, J_+)$ we can form

$$\mathcal{A} := \operatorname{diag}(1, P_+)\widetilde{\mathcal{A}} \operatorname{diag}(1, R_-) \tag{2.2.3}$$

which is a continuous operator

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(W, E) \\ \oplus \\ P^s(Y, \mathbb{L}_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(W, F) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{L}_+) \end{array}; \tag{2.2.4}$$

R_- is the above-mentioned embedding belonging to $\mathbb{L}_-$, cf. formula (1.4.11).

Definition 2.2.1 *Let* $\mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ *for* $\boldsymbol{g} := (\gamma, \gamma - \mu)$, $\boldsymbol{l} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$ *denote the space of all operators* (2.2.3).

For $\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ of the form (2.2.3) we define $\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\wedge(\mathcal{A}))$, $\sigma_\psi(\mathcal{A}) := \sigma_\psi(\widetilde{\mathcal{A}})$, and $\sigma_\wedge(\mathcal{A})$ is a bundle homomorphism (2.1.4), given as

$$\sigma_\psi(\mathcal{A}) := \operatorname{diag}(1, \sigma_\psi(P_+))\sigma_\wedge(\mathcal{A}) \operatorname{diag}(1, r_-)$$

with $r_- : L_- \to \pi_Y^* J_-$ being the embedding.

Remark 2.2.2 *Let us choose projection data* $\mathbb{L}_\pm := (P_\pm, J_\pm, L_\pm)$ *and* $\widetilde{\mathbb{L}}_\pm := (\widetilde{P}_\pm, \widetilde{J}_\pm, L_\pm)$ *such that* $J_\pm$ *are subbundles of* $\widetilde{J}_\pm$ *and*

$$\widetilde{P}_\pm|_{H^s(Y, J_\pm)} = P_\pm. \tag{2.2.5}$$

Then we have a canonical isomorphism

$$\mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \widetilde{\boldsymbol{l}}) \cong \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$$

for $\widetilde{\boldsymbol{l}} := (E, F; \widetilde{\mathbb{L}}_-, \widetilde{\mathbb{L}}_+)$, $\boldsymbol{l} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$. *Moreover, if* $\mathbb{L}_\pm$ *are given, and* $\widetilde{J}_\pm$ *are bundles containing* $J_\pm$ *as subbundles, we find projections* $\widetilde{P}_\pm$ *with the property* (2.2.5).

Remark 2.2.3 *For* $L_\pm = J_\pm$ *and* $P_\pm = \mathrm{id}$ *we recover edge operators in the sense of* $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *for* $\boldsymbol{v} = (E, F; J_-, J_+)$. *For that reason we call* $\mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *the Toeplitz extension of* $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$.

Similarly to (1.4.17) we can also form $\mathcal{T}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, $j \in \mathbb{N}$, as the space of all operators (2.2.3) such that $\widetilde{\mathcal{A}} \in \mathcal{Y}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$. Every $\mathcal{A} \in \mathcal{T}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ then has a pair of principal symbols $\sigma^{\mu-j}(\mathcal{A}) = (\sigma_\psi^{\mu-j}(\mathcal{A}), \sigma_\wedge^{\mu-j}(\mathcal{A}))$, and we have an analogue of relation (1.4.18) for $\mathcal{T}^{\mu-(j+1)}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ instead of $\mathcal{Y}^{\mu-(j+1)}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$. Let us set $\mathcal{T}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) := \bigcap_{j \in N} \mathcal{T}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$.

A simple consequence of Remark 1.4.3 in the following observation.

Remark 2.2.4 *Let $\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} := (\gamma, \gamma - \mu)$. Then, if $\sigma(\mathcal{A}) = 0$, the operator* (2.2.4) *is compact for every $s \in \mathbb{R}$.*

Another useful property is the existence of asymptotic sums, i.e., for every sequence $\mathcal{A}_j \in \mathcal{T}^{\mu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, $j \in \mathbb{N}$, there is an $\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ such that $\mathcal{A} - \sum_{j=0}^N \in \mathcal{T}^{\mu-(N+1)}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for every N, and $\mathcal{A}$ is unique $\mod \mathcal{T}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$.

Theorem 2.2.5 *$\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, $\mathcal{B} \in \mathcal{T}^\nu(\mathbb{W}, \boldsymbol{h}; \boldsymbol{w})$ implies $\mathcal{A}\mathcal{B} \in \mathcal{T}^{\mu+\nu}(\mathbb{W}, \boldsymbol{g} \circ \boldsymbol{h}; \boldsymbol{v} \circ \boldsymbol{w})$, and we have*

$$\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B}).$$

2.3 Ellipticity with global projection data

We now turn to ellipticity of problems in the Toeplitz extension of the edge algebra.

Definition 2.3.1 *An operator $\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ for $\boldsymbol{g} := (\gamma, \gamma - \mu)$, $\boldsymbol{l} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$ with global projection data ($\mathbb{L}_\pm$) is called elliptic, if $A :=$ u.l.c. $\mathcal{A}$ is elliptic in the sense of Definition* 1.5.1 (i)*, and if the principal edge symbol* (2.1.4) *is an isomorphism for an $s = s_0$.*

Similarly to SL-ellipticity, the second condition implies the bijectivity of (2.1.4) for all $s \in \mathbb{R}$, and we can equivalently pass to subspaces $\mathcal{S}_\varepsilon^\gamma(X^\wedge)$ and $S_\varepsilon^{\gamma-\mu}(X^\wedge)$ in the upper left corners, for a certain $\varepsilon > 0$.

Remark 2.3.2 *Let $\mathcal{A} \in Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ be elliptic in the sense of Definition* 1.5.1 (i)*, and let* (2.1.1) *be a Fredholm family.*

(i) *For every $L_\pm \in \mathrm{Vect}(T^*Y \setminus 0)$ with the property* (2.1.3) *there exists a $G \in Y_G^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ and an elliptic operator $\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ such that $A + G =$ u.l.c.$\mathcal{A}$.*

(ii) *For the operator A and chosen global projection data $\mathbb{L}_\pm = (P_\pm, J_\pm, L_\pm)$ and an elliptic operator $\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ such that $A =$ u.l.c.$\mathcal{A}$.*

In fact, (i) is a consequence of Theorem 2.1.3. To show (ii) it suffices to choose bundles $J_\pm \in \mathrm{Vect}(Y)$ of sufficiently large fibre dimension and a potential edge symbol

$$\sigma_\wedge(K) : \pi_Y^* J_- \to \pi_Y^* \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F'$$

of κ_λ-homogeneity μ such that

$$(\sigma_\wedge(A) \quad \sigma_\wedge(K)) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F' \tag{2.3.1}$$

is surective; this is always possible (even for a trivial bundle J_- of sufficiently large fibre dimension.) Then

$$\ker_{S^*Y}(\sigma_\wedge(A) \quad \sigma_\wedge(K)) =: L_+$$

has the property $\mathrm{ind}_{S*Y}\, \sigma_\wedge(A) = [L_+|_{S^*Y}] - [\pi_1^* J_-]$. This allows us to apply (i) for $\mathbb{L}_- := (\mathrm{id}, J_-, J_-)$.

Theorem 2.3.3 *For every elliptic operator $\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$, $\boldsymbol{g} := (\gamma, \gamma - \mu)$, $\boldsymbol{l} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$ there exists an elliptic operator $\mathcal{B} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{m})$ for $\boldsymbol{m} := (F, E; \mathbb{M}_-, \mathbb{M}_+)$ for suitable global projection data $\mathbb{M}_\pm$ such that $\mathcal{A} \oplus \mathcal{B} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{v} := (E \oplus F, F \oplus E; \mathbb{C}^N, \mathbb{C}^N)$ for a certain $N \in \mathbb{N}$.*

Proof. The operator $A := \text{u.l.c.}\,\mathcal{A} \in Y^\mu(\mathbb{W}, \boldsymbol{g}; E, F)$ induces continuous maps

$$A : \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \to \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \tag{2.3.2}$$

for all $s \in \mathbb{R}$. Let us apply (2.3.2) for $s = \gamma$. There are order reducing isomorphisms

$$R_E^\gamma : \mathcal{W}^{\gamma,\gamma}(\mathbb{W}, E) \to \mathcal{W}^{0,0}(\mathbb{W}, E) \quad R_F^{\gamma-\mu} : \mathcal{W}^{\gamma-\mu,\gamma-\mu}(\mathbb{W}, F) \to \mathcal{W}^{0,0}(\mathbb{W}, F)$$

belonging to $Y^\gamma(\mathbb{W}, (\gamma, 0); E, E)$ and $Y^{\gamma-\mu}(\mathbb{W}, (\gamma - \mu, 0); F, F)$ respectively, cf. Behm [3], or Dorschfeldt [6]. The operators R_E^γ, $R_F^{\gamma-\mu}$ are elliptic, i.e., their principal symbols are isomorphisms. Set

$$A_0 := R_F^{\gamma-\mu} A (R_E^\gamma)^{-1}$$

which belongs to $Y^0(\mathbb{W}, (0, 0); E, F)$, cf. Theorem 1.4.4. We then have $\sigma(A_0) = \sigma(R_F^{\gamma-\mu})\sigma(A)\sigma(R_E^\gamma)^{-1}$. The latter relation for the edge symbolic components $\sigma_\wedge$ gives us

$$\operatorname{ind}_{S^*Y} \sigma_\wedge(A_0) = \operatorname{ind}_{S^*Y} \sigma_\wedge(A). \tag{2.3.3}$$

There is a sufficiently large N such that the bundles $J_\pm$ in $\mathbb{L}_\pm$ are subbundles of $\mathbb{C}^N$. Because of Remark 2.2.2 without loss of generality we assume $\mathbb{L}_+ = (P_\pm, \mathbb{C}^N, L_\pm)$. Let $L_\pm^\perp$ denote complementary bundles of $L_\pm$ in $\mathbb{C}^N$. Then we have

$$[L_+^\perp|_{S^*Y}] - [L_-^\perp|_{S^*Y}] = [L_+|_{S^*Y}] - [L_-|_{S^*Y}]. \tag{2.3.4}$$

Let us form the adjoint A_0^* of A_0 which belongs to $Y^0(\mathbb{W}; (0, 0); F, E)$. Then, relations (2.3.3) and (2.3.4) imply

$$\operatorname{ind}_{S^*Y} \sigma_\wedge(A_0^*) = [L_+^\perp|_{S^*Y}] - [L_-^\perp|_{S^*Y}].$$

Because of Theorem 2.1.3 we find a Green operator $G_0 \in Y^0(\mathbb{W}, (0, 0); F, E)$ such that

$$\ker_{S^*Y} \sigma_\wedge(A_0^* + G_0) \cong L_+^\perp|_{S^*Y}, \qquad \operatorname{coker} \sigma_\wedge(A_0^* + G_0) \cong L_-^\perp|_{S^*Y}.$$

Then, setting $B := (R_E^{\gamma-\mu})^{-1}(A_0^* + G_0)R_F^\gamma \in Y^\mu(\mathbb{W}, (\gamma, \gamma - \mu); F, E)$, we also get

$$\ker_{S^*Y} \sigma_\wedge(B) \cong L_+^\perp|_{S^*Y}, \qquad \operatorname{coker} \sigma_\wedge(B) \cong L_-^\perp|_{S^*Y}.$$

Let us set $\mathbb{M}_\pm := (P_\pm^\perp, \mathbb{C}^N, L_\pm^\perp)$, where $P_\pm^\perp$ are complementary projections to $P_\pm$.

Using Remark 2.3.2 we can pass to an operator $\mathcal{B} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{m})$ for $\boldsymbol{m} := (F, E; \mathbb{M}_-, \mathbb{M}_+)$ which is elliptic and has $\mathcal{B}$ as upper left corner. From the construction it is now clear that $\mathcal{A} \oplus \mathcal{B}$ has the desired property. □

2.4 Parametrices and Fredholm property

Let $\mathcal{A} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ for $\boldsymbol{g} := (\gamma, \gamma - \mu)$ and $\boldsymbol{l} := (E, F; \mathbb{L}_-, \mathbb{L}_+)$. An operator $\mathcal{P} \in \mathcal{T}^\mu(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{l}^{-1})$ for $\boldsymbol{g}^{-1} := (\gamma - \mu, \gamma)$ and $\boldsymbol{l}^{-1} := (F, E; \mathbb{L}_+, \mathbb{L}_-)$ is called a parametrix of $\mathcal{A}$, if the operators

$$\mathcal{C}_l := \mathcal{I} - \mathcal{P}\mathcal{A} \qquad \text{and} \qquad \mathcal{C}_r := \mathcal{I} - \mathcal{A}\mathcal{P} \tag{2.4.1}$$

belong to $\mathcal{T}_G^{-\infty}(\mathbb{W}, \boldsymbol{g}_l; \boldsymbol{l}_l)$ and $\mathcal{T}_G^{-\infty}(\mathbb{W}, \boldsymbol{g}_r; \boldsymbol{l}_r)$, respectively, where $\boldsymbol{g}_l := (\gamma, \gamma)$, $\boldsymbol{l}_l := (E, E; \mathbb{L}_-, \mathbb{L}_-)$, $\boldsymbol{g}_r := (\gamma - \mu, \gamma - \mu)$, $\boldsymbol{l}_r := (F, F; \mathbb{L}_+, \mathbb{L}_+)$.

Theorem 2.4.1 *Let $\mathcal{A} \in \mathcal{T}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ be elliptic. Then $\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{T}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{l}^{-1})$.*

Proof. Applying Theorem 2.3.3 to the operator $\mathcal{A}$ we find a complementary operator $\mathcal{B}$ such that $\widetilde{\mathcal{A}} := \mathcal{A} \oplus \mathcal{B}$ is elliptic in the space $\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$. Theorem 1.5.4 gives us a parametrix $\mathcal{P} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$. Similarly to (2.2.3) we pass to an operator

$$\mathcal{P}_0 = \operatorname{diag}(1, P_-)\widetilde{P} \operatorname{diag}(1, R_+),$$

where $R_+ : P^{s-\mu}(Y, \mathbb{L}_+) \to H^{s-\mu}(Y, J_+)$ is the canonical embedding and $P_- \in L^0_{\mathrm{cl}}(Y; J_-, J_+)$ the projection from $\mathbb{L}_-$. We then have

$$\mathcal{P}_0\mathcal{A} = \operatorname{diag}(1 - P_-)\widetilde{\mathcal{P}} \operatorname{diag}(1, P_+)\widetilde{\mathcal{A}} \operatorname{diag}(1, R_-).$$

Setting $\mathcal{C} := \mathcal{I} - \mathcal{P}_0\mathcal{A}$ which belongs to $\mathcal{T}^0(\mathbb{W}, \boldsymbol{g}_l; \boldsymbol{v}_l)$ (cf. the notation of Theorem 1.5.4) we have $\sigma(\mathcal{C}) = 0$ which implies $\mathcal{C} \in \mathcal{T}^{-1}(\mathbb{W}, \boldsymbol{g}_l; \boldsymbol{v}_l)$. We then find an operator $\mathcal{D} \in \mathcal{T}^{-1}(\mathbb{W}, \boldsymbol{g}_l; \boldsymbol{v}_l)$ such that $(\mathcal{I} + \mathcal{D})(\mathcal{I} - \mathcal{C}) = \mathcal{I} \mod \mathcal{T}^{-\infty}(\mathbb{W}, \boldsymbol{g}_l; \boldsymbol{v}_l)$, namely $\mathcal{D} := \sum_{j=1}^{\infty} \mathcal{C}^j$ as an asymptotic sum. This yields $(\mathcal{I} + \mathcal{D})\mathcal{P}_0\mathcal{A} = 1 \mod \mathcal{T}^{-\infty}(\mathbb{W}, \boldsymbol{g}_l; \boldsymbol{v}_l)$, is a left parametrix of $\mathcal{A}$. In a similar manner we find a right parametrix, and hence we may set $\mathcal{P} := \mathcal{P}_l$. $\square$

Theorem 2.4.2 *If $\mathcal{A} \in \mathcal{T}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ is elliptic, then*

(i) *The operator*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ P^s(Y, \mathbb{L}_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{L}_+) \end{array} \tag{2.4.2}$$

is Fredholm for every $s \in \mathbb{R}$.

(ii) *$\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{T}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{l}^{-1})$ that can be chosen in such a way that*

$$\mathcal{C}_l : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ P^s(Y, \mathbb{L}_-) \end{array} \to \ker \mathcal{A} =: V$$

and

$$\mathcal{C}_r : \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{L}_+) \end{array} \to W$$

are projections, where $W \subset \mathcal{W}^{\infty,\gamma-\mu}(\mathbb{W}, F) \oplus \mathcal{P}^{\infty}(Y, \mathbb{L}_+)$ is a subspace of finite dimension. Here, the operator $\mathcal{A}$ is interpreted in the sense of (2.4.2) *and the assertion holds for all $s \in \mathbb{R}$.*

Proof. (i) By virtue of Theorem 2.4.1 the ellipticity of $\mathcal{A}$ entails the existence of a parametrix $\mathcal{P}$ of $\mathcal{A}$. Since the remainders (2.4.1) are compact operators in the respective spaces, cf. Remark 2.2.4, the operator (2.4.2) is Fredholm for every s.

(ii) The existence of a parametrix $\mathcal{P}$ of a Fredholm operator $\mathcal{A}$ between Hilbert spaces such that the remainders (2.4.1) are projections of finite rank is standard. In the present case we have more information, namely that $\mathcal{A}$ is Fredholm between spaces of a scale indexed by $s \in \mathbb{R}$. The choice of any parametrix $\mathcal{P}_1$ of $\mathcal{A}$ in our case already gives us elliptic regularity, in particular, that

$\ker \mathcal{A}$ and $\ker \mathcal{P}_1$ are finite-dimensional subspaces in corresponding spaces for $s = \infty$. Then a simple functional analytic arguments allow us to pass to another parametrix $\mathcal{P} = \mathcal{P}_1 + \mathcal{C}$ for an operator $\mathcal{C}$ of order $-\infty$ that has the desired properties. In our case, we have $\mathcal{C} \in \mathcal{T}^{-\infty}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{l}^{-1})$.

$\square$

Remark 2.4.3 *Let* $\mathcal{A} \in \mathcal{T}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{l})$ *for* $\boldsymbol{l} = (E, F; \mathbb{L}_-, \mathbb{L}_+)$, $\mathbb{L}_\pm := (P_\pm, J_\pm, L_\pm)$ *be an elliptic operator, written as* (2.2.3) *with an* $\widetilde{\mathcal{A}} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *for* $\boldsymbol{v} = (E, F; J_-, J_+)$. *Moreover, let* $\mathcal{B} \in \mathcal{T}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{m})$ *for* $\boldsymbol{m} = (E, F; \mathbb{M}_-, \mathbb{M}_+)$, $\mathbb{M}_\pm := (Q_\pm, J_\pm, L_\pm)$, *be another operator*

$$\mathcal{B} := \operatorname{diag}(1, Q_+)\widetilde{\mathcal{A}} \operatorname{diag}(1, S_-)$$

with the same $\widetilde{\mathcal{A}}$, *projections* $Q_\pm : H^{s-\mu}(Y, J_\pm) \to P^{s-\mu}(Y, \mathbb{M}_\pm)$ *and the canonical embedding* $S_- : P^s(Y, \mathbb{M}_-) \to H^s(Y, J_-)$. *Then* $\mathcal{B}$ *is also elliptic and hence defines Fredholm operators*

$$\mathcal{B} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ P^s(Y, \mathbb{M}_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \\ \oplus \\ P^{s-\mu}(Y, \mathbb{M}_+) \end{array}$$

for all $s \in \mathbb{R}$, *and we have*

$$\begin{array}{rcl} \operatorname{ind} \mathcal{A} - \operatorname{ind} \mathcal{B} & = & \operatorname{ind}(Q_- : P^s(Y, \mathbb{L}_-) \to P^s(Y, \mathbb{M}_-)) \\ & + & \operatorname{ind}(P_+ : P^s(Y, \mathbb{M}_+) \to P^s(Y, \mathbb{L}_+)), \end{array}$$

where the expressions on the right are independent of s.

This is a consequence of [19, Proposition 1.2.2].

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