Pseudodifferential boundary value problems with global projection conditions

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Contents

Introduction

The standard form of a boundary value problem for a differential operator

$$A : C^\infty(X,E) \to C^\infty(X,F) \tag{0.0.1}$$

on a smooth manifold X with boundary is

$$Au = f \quad \text{on } X, \qquad Tu = g \quad \text{on } Y := \partial X. \tag{0.0.2}$$

Here, $E, F \in \mathrm{Vect}(X)$, where $\mathrm{Vect}(\cdot)$ denotes the set of smooth complex vector bundles on the manifold in the brackets, $T = (T_1, \ldots, T_N)$ is a vector of trace operators $T_j = \mathrm{r}'\widetilde{T}_j$, $j = 1, \ldots, N$, for differential operators $\widetilde{T}_j : C^\infty(V,E) \to C^\infty(V, \widetilde{G}_j)$ in a collar neighbourhood $V \cong Y \times [0,1)$ of Y, with bundles $\widetilde{G}_j \in$

$\mathrm{Vect}(V)$ and the restriction operator $\mathrm{r}'v := v|_Y$. Setting $G_j = \widetilde{G}_j|_Y$ and $G = \bigoplus_{j=1}^N G_j$, we can identify (0.0.2) with a column matrix operator

$$\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix} : C^\infty(X,E) \to \begin{matrix} C^\infty(X,F) \\ \oplus \\ C^\infty(Y,G) \end{matrix}.$$

For $\mu = \operatorname{ord} A$, $\mu_j = \operatorname{ord} \widetilde{T}_j$, we have continuous extensions to Sobolev spaces

$$\mathcal{A} : H^s(X,E) \to \begin{matrix} H^{s-\mu}(X,F) \\ \oplus \\ \bigoplus_{j=1}^N H^{s-\mu_j-\frac{1}{2}}(Y,G_j) \end{matrix} \tag{0.0.3}$$

for every sufficiently large real s. Here and in the sequel we assume X to be compact.

It is well–known that (0.0.3) is a Fredholm operator for any fixed (sufficiently large) $s \in \mathbb{R}$ if and only if

(i) A is elliptic, i.e., the homogeneous principal symbol

$$\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F \tag{0.0.4}$$

(with the canonical projection $\pi_X : T^*X \setminus 0 \to X$) is an isomorphism,

(ii) the trace operators T satisfy the Shapiro–Lopatinskij condition with respect to A, i.e., the boundary symbol

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ G \end{pmatrix} \tag{0.0.5}$$

(with the canonical projection $\pi_Y : T^*Y \setminus 0 \to Y$) is an isomorphism.

Here, E' denotes the restriction of any $E \in \mathrm{Vect}(X)$ to Y, and $\sigma_\partial(\mathcal{A})$ is locally given by

$$\sigma_\partial(\mathcal{A})(y,\eta) = \begin{pmatrix} \sigma_\psi(A)(y,0,\eta,D_t) \\ (\mathrm{r}'\sigma_\psi(\widetilde{T}_j)(y,0,\eta,D_t))_{j=1,\dots,N} \end{pmatrix},$$

where $(y,t) \in Y \times \overline{\mathbb{R}}_+$ are local coordinates in a collar neighbourhood of Y in X.

The Shapiro–Lopatinskij condition will also be refered to as SL–condition.

It is often convenient to restrict $\sigma_\partial(\mathcal{A})$ to the unit cosphere bundle S^*Y induced by T^*Y, where a Riemannian metric is fixed; let $\pi_1 : S^*Y \to Y$ denote the canonical projection. We simply denote $\sigma_\partial(\mathcal{A})|_{S^*Y}$ again by $\sigma_\partial(\mathcal{A})$.

If the ellipticity condition (i) is fulfilled, the boundary symbol $\sigma_\partial(A)(y,\eta) = \sigma_\psi(y,0,\eta,D_t)$ represents a family of Fredholm operators

$$\sigma_\partial(A) : \pi_1^* E' \otimes H^s(\mathbb{R}_+) \to \pi_1^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \tag{0.0.6}$$

parametrised by the compact space S^*Y. There is then an index element

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A) \in K(S^*Y).$$

In the present case the operators (0.0.6) are surjective for all $(y, \eta) \in S^*Y$, and condition (ii) entails $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) = [\pi_1^* G] \in \pi_1^* K(Y)$. In other words,

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in \pi_1^* K(Y) \tag{0.0.7}$$

is a topological obstruction to $\sigma_\psi(A)$ for the existence of SL–elliptic boundary conditions T. (0.0.7) is just the Atiyah–Bott condition from [1].

There are elliptic differential operators (0.0.1) that violate condition (0.0.7). It is well–known that Dirac operators in even dimensions and other interesting geometric operators belong to this category, and possible boundary conditions that lead to associated Fredholm operators are rather different from SL–elliptic ones. In fact, after the work of Calderón [5], Seeley [24], Atiyah, Patodi and Singer [2], another kind of conditions (here, briefly called global projection conditions) became a natural concept in the index theory of boundary value problems. There is now a stream of investigations in the literature to derive index formulas in terms of η–invariants (and their various generalisations) of elliptic operators on the boundary. Let us mention, in particular, Melrose [10], Booss-Bavnbek and Wojciechowski [3], Grubb and Seeley [8], Savin and Sternin [14], [15], as well as [13], and the references there.

General elliptic boundary value problems for differential operators and (inhomogeneous) boundary conditions in subspaces of Sobolev spaces (that are images of pseudodifferential projections on the boundary) have been studied by Seeley [24], see also joint works of the first author with Shatalov and Sternin [23]. It is natural to embed such problems into a pseudodifferential algebra, such that arbitrary elliptic operators admit either elliptic SL– or global projection conditions, where parametrices of such elliptic bondary value problems again belong to the algebra. Such a calculus for operators with the transmission property at the boundary has been introduced in [20] as a "Toeplitz extension" of Boutet de Monvels's algebra, cf. [4] or [11].

Elliptic operators in mixed, transmission or crack problems, or, more generally, on manifolds with edges, also require extra conditions along the interfaces, crack boundaries, or edges, cf. [9], [19], or Egorov and Schulze [6]. The transmission property is not a reasonable assumption in such applications. In simplest cases the additional conditions satisfy an analogue of the Shapiro–Lopatinskij condition as a direct generalisation of SL–ellipticity of boundary conditions in boundary value problems, cf. [17], [19]. However, for the existence of such conditions for a given elliptic operator in the interior we have a similar kind of topological obstruction as in boundary value problems. Thus it is again natural to ask whether there are Toeplitz extensions of the corresponding algebras that contain the original operator algebras but admit all "interior" elliptic symbols that were forbidden before by that obstruction. The present paper gives the answer for pseudodifferential boundary value problems with general interior symbols, i.e., without the condition of the transmission property at the boundary. Our algebra may also be regarded as a model for operators on manifolds with edges, though the case of boundary value problems has some properties that are not typical for edge operators in general. In a forthcoming paper [22] we will treat the case of elliptic operators on a manifold with smooth edges.

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1 Operators with the transmission property

1.1 Operators on a manifold with boundary

The study of ellipticity of (say differential) operators A on a C^∞ manifold X with C^∞ boundary Y gives rise to a number of natural questions.

First we may ask the nature of boundary conditions that (for compact X) complete A to a Fredholm operator in suitable spaces. The answer in general is that a certain type of adapted global pseudodifferential projections on Y contribute to the boundary conditions, and that Shapiro–Lopatinskij elliptic conditions are the simplest variant in this set–up.

Another question is the structure of parametrices of elliptic boundary value problems that leads to pseudodifferential operators (similarly to the case without boundary). Therefore, it is desirable to have an algebra of corresponding pseudodifferential boundary value problems. An answer is given in [20] in terms of an operator algebra $\mathcal{S}(X)$ that contains Boutet de Monvel's algebra $\mathcal{B}(X)$ as well as an algebra $\mathcal{T}(Y)$ of (generalisations of) Toeplitz operators on the boundary.

For the analytic part of this discussion it is necessary to specify the behaviour of pseudodifferential symbols (locally) on X near Y. The simplest choice consists of classical symbols $a(x,\xi)$ that are smooth in x up to the boundary Y. However, as is well–known from the classical analysis of pseudodifferential boundary value problems, cf. Vishik and Eskin [26], Eskin [7], Boutet de Monvel [4], Rempel and Schulze [12], smoothness alone is often too general.

An additional condition on symbols is the transmisssion property at the boundary. The transmission property suffices to generate an algebra that contains all differential problems together with the parametrices of elliptic elements. The transmission property has been imposed in $\mathcal{B}(X)$ as well as in $\mathcal{S}(X)$. It is a natural condition, if we prefer standard Sobolev spaces on X or scales of closed subspaces as a frame for Fredholm operators. On the other hand, to understand the structure of stable homotopies of elliptic boundary value problems, or to reach particularly interesting applications, the algebra $\mathcal{B}(X)$ appears too narrow. It is interesting to consider a larger algebra, namely, a suitable subalgebra $\mathcal{L}(X)$ of the general edge algebra on X, cf. [19] and [21]. In this interpretation X is regarded as a manifold with edge Y and $\overline{\mathbb{R}}_+$ (the inner normal from a collar neighbourhood of Y) as the model cone of the "wedge" $Y \times \overline{\mathbb{R}}_+$. The algebra $\mathcal{L}(X)$ is sufficient for studying mixed and transmission problems and consists of pseudodifferential boundary value problems not requiring the transmission property; all classical symbols on X that are smooth up to Y are admitted in $\mathcal{L}(X)$.

The operators in $\mathcal{L}(X)$ act in a certain scale $\mathcal{W}^{s,\gamma}(X)$ of weighted edge Sobolev spaces that are different from the standard Sobolev spaces $H^s(X)$, except for $s = \gamma = 0$ where we have $\mathcal{W}^{0,0}(X) = L^2(X) = H^0(X)$.

To illustrate the idea of constructing our Toeplitz extension $\mathcal{T}(X)$ of $\mathcal{L}(X)$ we first have a look at the corresponding construction for Boutet de Monvel's algebra $\mathcal{B}(X)$. The general case will be studied in Section 2 below.

Let X be a compact, smooth manifold with smooth boundary Y and $E, F \in \mathrm{Vect}(X)$, $J_-, J_+ \in \mathrm{Vect}(Y)$. Then $\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ for $\boldsymbol{b} = (E,F;J_-,J_+)$ and $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$ is defined to be the space of all block matrix operators

$$\mathcal{A} : \begin{array}{c} C^\infty(X,E) \\ \oplus \\ C^\infty(Y,J_-) \end{array} \to \begin{array}{c} C^\infty(X,F) \\ \oplus \\ C^\infty(Y,J_+) \end{array} \tag{1.1.1}$$

of the form

$$\mathcal{A} = \begin{pmatrix} \mathrm{r}^+ P \mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} + \mathcal{C}, \tag{1.1.2}$$

where the ingredients of (1.1.2) are given as follows:

(i) *P is a classical pseudodifferential operator of order μ on $2X$ (the double of X, obtained by gluing together two copies of X along the common boundary Y, such that $2X$ is a closed, compact, smooth manifold), where P has the transmission property with respect to Y, and e^+ is the operator of extension by zero from $\operatorname{int} X$ to $2X$, and r^+ is the restriction from $2X$ to $\operatorname{int} X$.*

Recall that the transmission property of an operator P in $\Omega \times \mathbb{R} \ni x = (y,t)$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, with respect to $t = 0$ is defined in terms of the homogeneous components $p_{(\mu-j)}(y,t,\eta,\tau)$, $j \in \mathbb{N}$, of a (left) symbol $p(y,t,\eta,\tau)$ of P by the condition

$$[D_t^k D_\eta^\alpha \{p_{(\mu-j)}(y,t,\eta,\tau) - (-1)^{\mu-j} p_{(\mu-j)}(y,t,-\eta,-\tau)\}]|_{t=0,\eta=0} = 0$$

for all $y \in \Omega$, $\tau \in \mathbb{R} \setminus \{0\}$, for all $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n-1}$ and all j. This is an invariant condition under coordinate changes that preserve the boundary. Thus, for $\widetilde{E}, \widetilde{F} \in \operatorname{Vect}(2X)$ we have $L^\mu_{\mathrm{cl}}(2X; \widetilde{E}, \widetilde{F})_{\mathrm{tr}}$, the space of all pseudo–differential operators on $2X$ acting between spaces of sections of $\widetilde{E}$ and $\widetilde{F}$, the symbols of which in local coordinates near Y have the transmission property. For $E = \widetilde{E}|_X$, $F = \widetilde{F}|_X$ we set $L^\mu_{\mathrm{cl}}(X; E, F)_{\mathrm{tr}} = \{\mathrm{r}^+ P \mathrm{e}^+ : P \in L^\mu_{\mathrm{cl}}(2X; \widetilde{E}, \widetilde{F})_{\mathrm{tr}}\}$. In other words, the operator in the first summand on the right of (1.1.2) belongs to $L^\mu_{\mathrm{cl}}(X; E, F)_{\mathrm{tr}}$.

We shall often set $q = n - 1$.

(ii) *The operator $\mathcal{C}$ belongs to $\mathcal{B}^{-\infty,d}(X; \boldsymbol{b})$, i.e., it is smoothing and of type d.*

Here, $\mathcal{B}^{-\infty,0}(X; \boldsymbol{b})$ is the space of all operators (1.1.1) that have C^∞ kernels up to the boundary. On X and Y we fix Riemannian metrics such that a collar neighbourhood of Y has the product metric from $[0,1) \times Y$. Then the entries of $\mathcal{C} = (C_{ij})_{i,j=1,2}$ are integral operators with C^∞ kernels over $X \times X$, $X \times Y$, $Y \times X$ and $Y \times Y$, respectively, that are sections in corresponding external tensor products of bundles on the respective Cartesian products. Now $\mathcal{B}^{-\infty,d}(X; \boldsymbol{b})$ is defined to be the space of all operators

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{d} \mathcal{C}_j \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix}$$

for arbitrary $\mathcal{C}_j \in \mathcal{B}^{-\infty,0}(X; \boldsymbol{b})$, $j = 0, \ldots, d$.

(iii) *The operator $\mathcal{G}$ in (1.1.2) is a block matrix $(G_{ij})_{i,j=1,2}$, where G_{11} on $(\operatorname{int} X) \times (\operatorname{int} X)$, G_{12} on $(\operatorname{int} X) \times Y$, G_{21} on $Y \times (\operatorname{int} X)$ have C^∞ kernels, G_{22} is a classical pseudodifferential operator of order μ on Y, while $\mathcal{G}$ near Y in local coordinates $(y,t) \in \Omega \times \overline{\mathbb{R}}_+$ has the form of a pseudo–differential operator with operator–valued symbol $g(y,\eta)$, i.e., $\mathcal{G} = \operatorname{Op}(g)$,*

$$(\operatorname{Op}(g)u)(y) = \iint e^{i(y-y')\eta} g(y,\eta) u(y') \, dy' đ\eta,$$

and

$$g(y,\eta) = g_0(y,\eta) + \sum_{j=1}^{d} g_j(y,\eta) \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix} \tag{1.1.3}$$

with $g_j(y,\eta) \in \mathcal{R}_G^{\mu-j,0}(\Omega \times \mathbb{R}^q, (k,l;j_-,j_+))$, *where* k, l, j_- *and* j_+ *are the fibre dimensions of the bundles* E, F, J_- *and* J_+, *respectively.*

Let us recall the definition of the so–called Green symbols of the class $\mathcal{R}_G^{\nu,0}(\Omega \times \mathbb{R}^q, (k,l;j_-,j_+))$ from [18] or [19]. First, if H and $\widetilde{H}$ are Hilbert spaces, and $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ are groups of isomorphisms on H and $\widetilde{H}$, respectively, strongly continuous in $\lambda \in \mathbb{R}_+$, we have the space

$$S^\nu(\Omega \times \mathbb{R}^q; H, \widetilde{H}), \qquad \nu \in \mathbb{R}, \tag{1.1.4}$$

of operator–valued symbols, defined to be the set of all $a(y,\eta) \in C^\infty(\Omega\times\mathbb{R}^q; H, \widetilde{H})$ that fulfill the following symbol estimates:

$$\|\widetilde{\kappa}_{\langle\eta\rangle}^{-1}\{D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq c\langle\eta\rangle^{\nu-|\beta|}$$

for all $y \in K$, $\eta \in \mathbb{R}^q$, for all $K \subset\subset \Omega$, $\alpha, \beta \in \mathbb{N}^{n-1}$, with constants $c = c(\alpha,\beta,K) > 0$.

If $\widetilde{H} = \varprojlim_{j\in\mathbb{N}} \widetilde{H}^j$ is a projective limit of Hilbert spaces $\widetilde{H}^j$ with continuous embeddings $\widetilde{H}^{j+1} \hookrightarrow \widetilde{H}^j$ for all j and $\widetilde{\kappa}_\lambda$ acting on $\widetilde{H}^j$ by restriction of a corresponding strongly continuous group on $\widetilde{H}^0$, we can form the symbol spaces $S^\nu(\Omega\times\mathbb{R}^q; H, \widetilde{H}^j)$ for all j and then set

$$S^\nu(\Omega \times \mathbb{R}^q; H, \widetilde{H}) = \varprojlim_{j\in\mathbb{N}} S^\nu(\Omega \times \mathbb{R}^q; H, \widetilde{H}^j).$$

An element $a(y,\eta) \in S^\nu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ is called classical, if there are functions $a_{(\nu-l)}(y,\eta) \in C^\infty(\Omega \times (\mathbb{R}^{n-1} \setminus \{0\}, \mathcal{L}(H,\widetilde{H}))$, $l \in \mathbb{N}$, such that

$$a_{(\nu-l)}(y,\lambda\eta) = \lambda^{\nu-l}\widetilde{\kappa}_\lambda a_{(\nu-l)}(y,\eta)\kappa_\lambda^{-1} \tag{1.1.5}$$

for all $\lambda \in \mathbb{R}_+$, such that $a(y,\eta) - \sum_{l=0}^N \chi(\eta)a_{(\nu-l)}(y,\eta) \in S^{\nu-(N+1)}(\Omega\times\mathbb{R}^q; H, \widetilde{H})$ for every $N \in \mathbb{N}$; here, $\chi(\eta)$ is any zero excision function in $\mathbb{R}^{n-1}$. Let $S_{\mathrm{cl}}^\nu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ denote the space of all classical symbols of order ν.

Let, in particular, $H := L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_-}$ and $\widetilde{H}^m := \langle t\rangle^{-m} H^m(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_+}$, where $H^s(\mathbb{R}_+, \mathbb{C}^l) = H^s(\mathbb{R}_+) \otimes \mathbb{C}^l$, with the standard Sobolev space $H^s(\mathbb{R}_+) = H^s(\mathbb{R})|_{\mathbb{R}_+}$ on $\mathbb{R}_+$ of smoothness $s \in \mathbb{R}$. Then we have $\widetilde{H} := \varprojlim_{m\in\mathbb{N}} \widetilde{H}^m = \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_+}$ with the Schwartz space $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$. Setting

$$\kappa_\lambda(u \oplus v) = \lambda^{\frac{1}{2}} u(\lambda t) \oplus v \qquad \text{for } u \oplus v \in H,$$

and, similarly, $\widetilde{\kappa}_\lambda := \kappa_\lambda$ on $\widetilde{H}^m$ for each m, we get the symbol spaces

$$S_{\mathrm{cl}}^\nu(\Omega \times \mathbb{R}^q; L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_-}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_+}), \tag{1.1.6}$$

and, analogously,

$$S_{\mathrm{cl}}^\nu(\Omega \times \mathbb{R}^q; L^2(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_+}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_-}). \tag{1.1.7}$$

Then $\mathcal{R}_G^{\nu,0}(\Omega\times\mathbb{R}^q;\boldsymbol{v})$ for $\boldsymbol{v}=(k,l;j_-,j_+)$ is defined to be the space of all operator–valued symbols $g(y,\eta)$ in (1.1.6), such that the pointwise adjoint $g^*(y,\eta)$ belongs to the space (1.1.7). We then define $\mathcal{R}_G^{\nu,d}(\Omega\times\mathbb{R}^q;\boldsymbol{v})$ (the space of so–called Green symbols of order ν and type d) to be the space of all symbols of the form (1.1.3) for arbitrary $g_j(y,\eta)\in\mathcal{R}_G^{\nu-j,0}(\Omega\times\mathbb{R}^q;\boldsymbol{v})$, $j=0,\dots,d$.

For every $\mathcal{A}\in\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ we have a principal symbol $\sigma(\mathcal{A})=(\sigma_\psi(\mathcal{A}),\sigma_\partial(\mathcal{A}))$. Here,

$$\sigma_\psi(\mathcal{A}):\pi_X^*E\to\pi_X^*F \tag{1.1.8}$$

is the interior symbol that is the restriction of the homogeneous principal symbol of P from $T^*(2X)\setminus 0$ to $T^*X\setminus 0$, cf. formula (1.1.2). Moreover,

$$\sigma_\partial(\mathcal{A}):\pi_Y^*\begin{pmatrix}E'\otimes H^s(\mathbb{R}_+)\\ \oplus\\ J_-\end{pmatrix}\to\pi_Y^*\begin{pmatrix}F'\otimes H^{s-\mu}(\mathbb{R}_+)\\ \oplus\\ J_+\end{pmatrix} \tag{1.1.9}$$

is the boundary symbol of $\mathcal{A}$. It is defined for all $s>d-\frac{1}{2}$. It is often convenient to consider it as a family of maps

$$\sigma_\partial(\mathcal{A}):\pi_Y^*\begin{pmatrix}E'\otimes \mathcal{S}(\overline{\mathbb{R}}_+)\\ \oplus\\ J_-\end{pmatrix}\to\pi_Y^*\begin{pmatrix}F'\otimes \mathcal{S}(\overline{\mathbb{R}}_+)\\ \oplus\\ J_+\end{pmatrix}. \tag{1.1.10}$$

$\sigma_\partial(\mathcal{A})$ is defined by

$$\sigma_\partial(\mathcal{A})=\begin{pmatrix}\sigma_\partial(\mathrm{r}^+P\mathrm{e}^+) & 0\\ 0 & 0\end{pmatrix}+\sigma_\partial(\mathcal{G}),$$

with $\sigma_\partial(\mathrm{r}^+P\mathrm{e}^+)=\mathrm{r}^+\sigma_\psi(\mathcal{A})(y,0,\eta,D_t)\mathrm{e}^+$, and

$$\sigma_\partial(\mathcal{G})(y,\eta)=\sigma_\partial(g_0)(y,\eta)+\sum_{j=1}^{d}\sigma_\partial(g_j)(y,\eta)\begin{pmatrix}\partial_t^j & 0\\ 0 & 0\end{pmatrix}, \tag{1.1.11}$$

where $\sigma_\partial(g_j)$ is the homogeneous principal symbol of g_j as a classical symbol in the space (1.1.6) for $\nu=\mu-j$, $j=0,\dots,d$. Then, according to (1.1.5), we have

$$\sigma_\partial(\mathcal{A})(y,\lambda\eta)=\lambda^\mu\begin{pmatrix}\kappa_\lambda & 0\\ 0 & 1\end{pmatrix}\sigma_\partial(\mathcal{A})(y,\eta)\begin{pmatrix}\kappa_\lambda & 0\\ 0 & 1\end{pmatrix}^{-1} \tag{1.1.12}$$

for all $\lambda\in\mathbb{R}_+$. Here, $(\kappa_\lambda u)(t)=\lambda^{\frac{1}{2}}u(\lambda t)$ for $u(t)\in E'\otimes H^s(\mathbb{R}_{+,t})$, and 1 denotes the identity map in the respective fibres of $J_{\pm,y}$.

We systematically employ various standard facts about operators in $\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$. In particular, every $\mathcal{A}\in\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ induces continuous operators

$$\mathcal{A}:\begin{matrix}H^s(X,E)\\ \oplus\\ H^s(Y,J_-)\end{matrix}\to\begin{matrix}H^{s-\mu}(X,F)\\ \oplus\\ H^{s-\mu}(Y,J_+)\end{matrix} \tag{1.1.13}$$

for all real $s>d-\frac{1}{2}$, and (1.1.13) is compact for $\sigma(\mathcal{A})=0$. Moreover, $\mathcal{A}\in\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$, $\boldsymbol{b}=(E_0,F;J_0,J_+)$, and $\mathcal{B}\in\mathcal{B}^{\nu,e}(X;\boldsymbol{c})$, $\boldsymbol{c}=(E,E_0;J_-,J_0)$, implies $\mathcal{AB}\in\mathcal{B}^{\mu+\nu,h}(X;\boldsymbol{b}\circ\boldsymbol{c})$ for $\boldsymbol{b}\circ\boldsymbol{c}=(E,F;J_-,J_+)$, $h=\max(\nu+d,e)$ where

$$\sigma(\mathcal{AB})=\sigma(\mathcal{A})\sigma(\mathcal{B})$$

with componentwise multiplication.

1.2 Conditions with pseudodifferential projections

An operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ is called σ_ψ–elliptic, if the interior symbol $\sigma_\psi(\mathcal{A})$ defines an isomorphism (1.1.8). In this case,

$$\mathrm{r}^+\sigma_\psi(\mathcal{A})(y,0,\eta,D_t)\mathrm{e}^+ : E'_y \otimes H^s(\mathbb{R}_+) \to F'_y \otimes H^{s-\mu}(\mathbb{R}_+) \tag{1.2.1}$$

is known to be a family of Fredholm operators for all $(y,\eta) \in T^*Y \setminus 0$ and all $s > \max(\mu,d) - \frac{1}{2}$.

The Fredholm property of (1.2.1) is equivalent to that of

$$\mathrm{r}^+\sigma_\psi(\mathcal{A})(y,0,\eta,D_t)\mathrm{e}^+ : E'_y \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \to F'_y \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \tag{1.2.2}$$

for all $(y,\eta) \in T^*Y \setminus 0$.

An operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ is called SL–elliptic, if it is σ_ψ–elliptic and if, in addition, $\sigma_\partial(\mathcal{A})$ defines an isomorphism (1.1.9) for any $s > \max(\mu,d) - \frac{1}{2}$ (or, equivalently, an isomorphism (1.1.10)).

Let $B^{\mu,d}(X;E,F)$ denote the space of upper left corners of operator block matrices in $\mathcal{B}^{\mu,d}(X;\boldsymbol{b})$, $\boldsymbol{b} = (E,F;J_-,J_+)$. The question whether or not a σ_ψ–elliptic element $A \in B^{\mu,d}(X;E,F)$ may be interpreted as the upper left corner of an SL–elliptic operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{b})$ gives rise to an operator algebra of boundary value problems that is different from Boutet de Monvel's algebra. A general answer may be found in [20]; it consists of a new algebra with boundary conditions that we call global projection conditions. Operators in this algebra

$$\mathcal{A} : \begin{array}{c} H^s(X,E) \\ \oplus \\ P^s(Y,\boldsymbol{L}_-) \end{array} \to \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ P^{s-\mu}(Y,\boldsymbol{L}_+) \end{array} \tag{1.2.3}$$

that are characterised by the following data.

(i) *The upper left corner* u. l. c. $\mathcal{A} = A$ *belongs to* $B^{\mu,d}(X;E,F)$.

(ii) $\boldsymbol{L}_\pm$ *are triples*

$$\boldsymbol{L}_\pm = (P_\pm, J_\pm, L_\pm) \tag{1.2.4}$$

for certain bundles $J_\pm \in \mathrm{Vect}(Y)$, $L_\pm \in \mathrm{Vect}(T^*Y \setminus 0)$ *and elements* $P_\pm \in L^0_{\mathrm{cl}}(Y;J_\pm,J_\pm)$ *that are projections, i.e.,* $(P_\pm)^2 = P_\pm$, *where* $L_\pm$ *is the image of the homogeneous principal symbol*

$$p_\pm : \pi_Y^* J_\pm \to \pi_Y^* J_\pm \tag{1.2.5}$$

of $P_\pm$.

Note that then $(p_\pm)^2 = p_\pm$. Given a projection (1.2.5) there always exist such projections $P_\pm$; they are not unique, and we fix some choice, cf. Section 1.3 below.

(iii) *The spaces on the boundary* Y *in* (1.2.3) *are given by*

$$P^s(Y,\boldsymbol{L}_\pm) = P_\pm H^s(Y,J_\pm), \tag{1.2.6}$$

$s \in \mathbb{R}$. *These are closed subspaces of* $H^s(Y,J_\pm)$.

(iv) (1.2.3) *is defined to be a composition*

$$\mathcal{A} = \mathcal{P}_+ \widetilde{\mathcal{A}} \mathcal{R}_- \tag{1.2.7}$$

for an $\widetilde{\mathcal{A}} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{b})$, $\boldsymbol{b} = (E, F; J_-, J_+)$, *and*

$$\mathcal{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & P_+ \end{pmatrix}, \qquad \mathcal{R}_- = \begin{pmatrix} 1 & 0 \\ 0 & R_- \end{pmatrix},$$

where 1 are the identity operators in corresponding Sobolev spaces on X, *while* $R_- : P^s(Y, \boldsymbol{L}_-) \to H^s(Y, J_-)$ *is the canonical embedding.*

Let $\mathcal{S}^{\mu,d}(X; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$ denote the set of all operators (1.2.3) described by (i)–(iv). Continuity of (1.2.3) holds for all $s > d - \frac{1}{2}$.

Ellipticity of an operator $\mathcal{A} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{v})$ is defined by a pair of principal symbols $\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$, where $\sigma_\psi(\mathcal{A}) = \sigma_\psi(\mathrm{u.l.c.}\,\mathcal{A}) : \pi_X^* E \to \pi_X^* F$ is the interior symbol and $\sigma_\partial(\mathcal{A})$ the boundary symbol which is a bundle homomorphism

$$\sigma_\partial(\mathcal{A}) : \begin{matrix} \pi_Y^* E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ L_- \end{matrix} \to \begin{matrix} \pi_Y^* F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ L_+ \end{matrix} \tag{1.2.8}$$

where

$$\sigma_\partial(\mathcal{A})(y, \lambda\eta) = \lambda^\mu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \sigma_\partial(\mathcal{A})(y, \eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1}; \tag{1.2.9}$$

here, 1 are the identity maps in L_- and L_+, respectively. $\mathcal{A}$ is called elliptic if both $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ are isomorphisms. Instead of $\mathcal{S}(\overline{\mathbb{R}}_+)$ in (1.2.8) we could equivalently consider Sobolev spaces $H^s(\mathbb{R}_+)$ for arbitrary $s > \max(\mu, d) - \frac{1}{2}$.

Given a $\varrho \in \mathbb{R}$ we set $\varrho^+ = \max(\varrho, 0)$. Let us recall from [20] that if $\mathcal{A} \in \mathcal{S}^{\mu,d}(X; \boldsymbol{v})$ is elliptic the operator (1.2.3) is Fredholm for any (and then all) real $s > \max(\mu, d) - \frac{1}{2}$. There is then a parametrix $\mathcal{B} \in \mathcal{S}^{-\mu,e}(X; \boldsymbol{v}^{-1})$ for $e = (d - \mu)^+$ and $\boldsymbol{v}^{-1} = (F, E; \boldsymbol{L}_+, \boldsymbol{L}_-)$ in the sense that

$$\mathcal{B}\mathcal{A} - \mathcal{I} \in \mathcal{S}^{-\infty, d_l}(X; \boldsymbol{v}_l), \qquad \mathcal{A}\mathcal{B} - \mathcal{I} \in \mathcal{S}^{-\infty, d_r}(X; \boldsymbol{v}_r) \tag{1.2.10}$$

for $d_l = \max(\mu, d)$, $d_r = (d - \mu)^+$, $\boldsymbol{v}_l = (E, E; \boldsymbol{L}_-, \boldsymbol{L}_-)$, $\boldsymbol{v}_r = (F, F; \boldsymbol{L}_+, \boldsymbol{L}_+)$. Clearly, the remainders in (1.2.10) are compact in the respective spaces in (1.2.3).

Notice that $\mathrm{ind}\,\mathcal{A}$ depends on the choice of the global pseudodifferential projections $P_\pm$. However, the freedom in the choice of the projections does not affect the Fredholm property. This is a general fact on operators in Hilbert spaces, as we shall discuss now. To this end, let H^+ and H^- be Hilbert spaces, $P_\pm, \widetilde{P}_\pm \in \mathcal{L}(H_\pm)$ be projections such that $P_\pm - \widetilde{P}_\pm$ is compact. If we now set $PH_\pm := \mathrm{im}\, P_\pm = P_\pm(H_\pm)$, and, similarly, $\widetilde{P}H_\pm$, the following result holds:

Proposition 1.2.1 *Let* $A \in \mathcal{L}(H_-, H_+)$ *such that*

$$\widetilde{\mathcal{A}} = \widetilde{P}_- A : \widetilde{P}H_+ \to \widetilde{P}H_-,$$

is a Fredholm operator. Then this is also true for

$$\mathcal{A} = P_- A : PH_+ \to PH_-$$

and we have the relative index formula

$$\mathrm{ind}\,\mathcal{A} - \mathrm{ind}\,\widetilde{\mathcal{A}} = \mathrm{ind}(\widetilde{P}_+ : PH_+ \to \widetilde{P}H_+) + \mathrm{ind}(P_- : \widetilde{P}H_- \to PH_-). \tag{1.2.11}$$

Proof. First let us show that the operators on the righ–hand side of (1.2.11) are indeed Fredholm. Since P_+ acts as the identity on PH_+,

$$P_+\widetilde{P}_+ - 1 = P_+\widetilde{P}_+ - (P_+)^2 = P_+(\widetilde{P}_+ - P_+) : PH_+ \to PH_+$$

is compact. Hence P_+ is the Fredholm inverse of $\widetilde{P}_+$, and $P_+\widetilde{P}_+ : PH_+ \to PH_+$ is Fredholm with index 0. The analogue holds for the projections P_-, $\widetilde{P}_-$. Therefore, the operator

$$\mathcal{B} : PH_+ \xrightarrow{\widetilde{P}_+} \widetilde{P}H_+ \xrightarrow{\widetilde{\mathcal{A}}} \widetilde{P}H_- \xrightarrow{P_-} PH_-$$

is Fredholm with index

$$\operatorname{ind}\mathcal{B} = \operatorname{ind}\widetilde{\mathcal{A}} + \operatorname{ind}(\widetilde{P}_+ : PH_+ \to \widetilde{P}H_+) + \operatorname{ind}(P_- : \widetilde{P}H_- \to PH_-).$$

On the other hand,

$$\mathcal{B} = (P_-\widetilde{P}_-)\mathcal{A}(P_+\widetilde{P}_+) - P_-[\widetilde{P}_-, P_-]AP_+\widetilde{P}_+ + P_-\widetilde{P}_-A(1 - P_+)\widetilde{P}_+$$

where $[\widetilde{P}_-, P_-]$ is the commutator, which is a compact operator $H_- \to H_-$, since

$$[\widetilde{P}_-, P_-] = \widetilde{P}_-P_- - P_-\widetilde{P}_- = (\widetilde{P}_- - P_-)(1 - P_- - \widetilde{P}_-).$$

Moreover, $(1 - P_+)\widetilde{P}_+ = (\widetilde{P}_+ - P_+)\widetilde{P}_+ : H_+ \to H_+$ is also compact. Hence $(P_-\widetilde{P}_-)\mathcal{A}(P_+\widetilde{P}_+)$ differs from $\mathcal{B}$ by a compact remainder and thus is itself Fredholm with the same index

$$\operatorname{ind}\mathcal{B} = \operatorname{ind}(P_-\widetilde{P}_-)\mathcal{A}(P_+\widetilde{P}_+).$$

Above we have seen that $P_-\widetilde{P}_-$ and $P_+\widetilde{P}_+$ are Fredholm operators of index 0. Hence, $\mathcal{A}$ itself is Fredholm and $\operatorname{ind}\mathcal{B} = \operatorname{ind}\mathcal{A}$. □

1.3 Projections and Fredholm families

In this section we recall the result on the existence of a pseudodifferential projection to a given homogeneous principal symbol that is a projection. Moreover, we give some construction on families of Fredholm operators that will be used below in boundary value problems.

Let M be a closed compact C^∞ manifold with the space $L^\mu_{\mathrm{cl}}(M; E, F)$ of classical pseudodifferential operators of order μ, acting between distributional sections of vector bundles E and F on M. Recall that the homogeneous principal symbol of order μ of an operator $A \in L^\mu_{\mathrm{cl}}(M; E, F)$ is a bundle homomorphism $\sigma_\psi(A) : \pi^*E \to \pi^*F$ where $\pi : T^*M \setminus 0 \to 0$.

Theorem 1.3.1 *Let $p : \pi^*E \to \pi^*E$, $E \in \mathrm{Vect}(M)$, be a projection, i.e., $p^2 = p$, with $p(x, \lambda\xi) = p(x, \xi)$ for all $(x, \xi) \in T^*M \setminus 0$, $\lambda \in \mathbb{R}_+$. Then there exists an element $P \in L^0_{\mathrm{cl}}(M; E, E)$ with $P^2 = P$ and $\sigma_\psi(P) = p$.*

Moreover, if $p = p^2$ satisfies the condition $p = p^*$, there is a choice of $P = P^2 \in L^0_{\mathrm{cl}}(M; E, E)$ with $\sigma_\psi(P) = p$ and $P = P^*$.

The adjoint of p refers to a given Hermitian metric in E and the adjoint of P to a fixed scalar product in the space $L^2(M, E)$, with respect to a Riemannian metric on M and the Hermitian metric in E.

Let H be a (complex) Hilbert space, $\mathcal{L}(H)$ the space of linear continuous operators, $\mathcal{K}(H)$ the subspace of compact operators in H, $\mathcal{L}(H)/\mathcal{K}(H)$ the Calkin algebra, and $\pi : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{K}(H)$ the canonical map.

Lemma 1.3.2 *Let $p \in \mathcal{L}(H)/\mathcal{K}(H)$ be an element with $p^2 = p$ and choose any $Q \in \mathcal{L}(H)$ with $\pi Q = p$. Then the spectrum $\sigma_{\mathcal{L}(H)}(Q)$ of Q has the property that $\sigma_{\mathcal{L}(H)}(Q) \cap (\mathbb{C} \setminus (\{0\} \cup \{1\}))$ is discrete.*

Proof. First observe that $p^2 = p$ implies $\sigma_{\mathcal{L}(H)/\mathcal{K}(H)}(p) \subseteq \{0\} \cup \{1\}$. In fact, for $\lambda \in \mathbb{C} \setminus (\{0\} \cup \{1\}) =: U$ there exists $(\lambda e - p)^{-1} = \frac{1}{\lambda-1}p + \frac{1}{\lambda}(e-p)$, where $e \in \mathcal{L}(H)/\mathcal{K}(H)$ is the identity, $e = \pi I$ for the identity $I \in \mathcal{L}(H)$. Now $U \ni \lambda \to \lambda I - Q \in \mathcal{L}(H)$ is a holomorphic Fredholm family in U, and $\lambda I - Q$ is invertible in $\mathcal{L}(H)$ for $|\lambda| > \|Q\|_{\mathcal{L}(H)}$. A well–known invertibility result on holomorphic Fredholm families, cf. [], implies that $\lambda I - Q$ is invertible for all $\lambda \in U \setminus D$ for a certain discrete subset D (i.e., D is countable and $D \cap K$ finite for every compact subset $K \subset U$). □

Proof of Theorem 1.3.1**.** Lemma 1.3.2 implies that there exists a $0 < \delta < 1$ such that the circle $C_\delta := \{\lambda : \ |\lambda - 1| = \delta\}$ does not intersect $\sigma_{\mathcal{L}(H)}(Q)$. We set

$$P := \frac{1}{2\pi i} \int\limits_{C_\delta} (\lambda I - Q)^{-1} \, d\lambda. \tag{1.3.1}$$

Then $P^2 = P$, and we have $P \in L^0_{\mathrm{cl}}(M; E, E)$ as a consequence of the holomorphic functional calculus for $L^0_{\mathrm{cl}}(M; E, E)$. Moreover, we have

$$\begin{aligned} \sigma_\psi(P) &= \frac{1}{2\pi i} \int\limits_{C_\delta} (\lambda e - p)^{-1} \, d\lambda \\ &= \left\{ \frac{1}{2\pi i} \int\limits_{C_\delta} \frac{1}{\lambda - 1} \, d\lambda \right\} p + \left\{ \frac{1}{2\pi i} \int\limits_{C_\delta} \frac{1}{\lambda} \, d\lambda \right\} (e - p). \end{aligned}$$

The second summand on the right hand side vanishes, while the first one equals p by the Residue theorem.

To prove the second part of Theorem 1.3.1 we suppose $p = p^*$. Then, if $P_1 = P_1^2 \in L^0_{\mathrm{cl}}(M; E, E)$ is any choice with $\sigma_\psi(P_1) = p$, also $Q := P_1^* P_1 \in L^0_{\mathrm{cl}}(M; E, E)$ satisfies $\sigma_\psi(Q) = p^* p = p^2 = p$. For Q we have $Q = Q^* \geq 0$. Let η be the spectral measure of Q. Then the projection $P \in L^0_{\mathrm{cl}}(M; E, E)$ defined by formula (1.3.1) equals the spectral projection

$$\eta(B_\delta(1) \cap \sigma_{\mathcal{L}(L^2(M,E))}(Q)) \qquad \text{for } B_\delta = \{\lambda \in \mathbb{C} : \ |\lambda - 1| < \delta\}.$$

In particular, we have $P = P^* = P^2$, and $\sigma_\psi(P) = p$ as above. □

As noted in the beginning, the boundary symbols to elliptic symbols with the transmission property on a manifold X with boundary Y are families of Fredholm operators, acting in spaces normal to the boundary, parametrised by points in S^*Y. The situation for symbols without the transmission property will be similar. To analyse the nature of associated boundary conditions, we need some observations on Fredholm families in general. Let H_1, H_2 be Hilbert spaces and M a compact topological space (for simplicity assumed to be arcwise connected). For every operator family $a \in C(M, \mathcal{L}(H_1, H_2))$ such $a(m) : H_1 \to H_2$ is Fredholm for every $m \in M$ we have an index element $\mathrm{ind}_M \, a \in K(M)$ in the K–group of M. If the dimension of $\ker a(m)$ (and then also of $\mathrm{coker}\, a(m)$) is independent of $m \in M$, there are subbundles $\widetilde{L}_+ \subset M \times H_1$ and $\widetilde{L}_- \subset M \times H_2$ such that $\widetilde{L}_{+,m} = \ker a(m)$,

$\widetilde{L}_{-,m} + \operatorname{im} a(m) = H_2$ and $\widetilde{L}_{-,m} \cap \operatorname{im} a(m) = \{0\}$ for all $m \in M$. In general, this is not the case. To simplify notation the trivial bundle $M \times F$ with fibre F is often denoted by F.

Lemma 1.3.3 *Let $L_\pm \in \operatorname{Vect}(M)$ be any fixed choice such that $\operatorname{ind}_M a = [L_+] - [L_-]$. Then there exists an operator function $c \in C(M, \mathcal{L}(H_1, H_2))$, where $c(m)$ is of finite rank for every $m \in M$, such that $\widetilde{a} := a + c$ has the following properties:*

(i) $\ker \widetilde{a} \cong L_+$, $\operatorname{coker} \widetilde{a} \cong L_-$, *i.e., there are subbundles $\widetilde{L}_+ \subset M \times H_1$, $\widetilde{L}_+ \cong L_+$, and $\widetilde{L}_- \subset M \times H_2$, $\widetilde{L}_- \cong L_-$, such that $\widetilde{L}_{+,m} = \ker \widetilde{a}(m)$, $\widetilde{L}_{-,m} + \operatorname{im} \widetilde{a}(m) = H_2$ and $\widetilde{L}_{-,m} \cap \operatorname{im} a(m) = \{0\}$ for all $m \in M$.*

(ii) *There are (continuous) homomorphisms*

$$\widetilde{k} : L_- \to H_2, \qquad \widetilde{t} : H_1 \to L_+ \tag{1.3.2}$$

such that

$$\begin{pmatrix} \widetilde{a} & \widetilde{k} \\ \widetilde{t} & 0 \end{pmatrix} : \begin{array}{c} H_1 \\ \oplus \\ L_- \end{array} \to \begin{array}{c} H_2 \\ \oplus \\ L_+ \end{array} \tag{1.3.3}$$

is an isomorphism.

Proof. We first apply the well–known fact that there is an N_- and an injective m–independent map $k_{N_-} : \mathbb{C}^{N_-} \to H_2$ such that

$$\begin{pmatrix} a(m) & k_{N_-} \end{pmatrix} : \begin{array}{c} H_1 \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to H_2$$

is surjective for all $m \in M$. Recall that k_{N_-} can be found via a finite–dimensional subspace $V \subset H_2$ such that $\operatorname{im} a(m) + V = H_2$ for every $m \in M$; such a V always exists, and then k_{N_-} may be taken as any isomorphism $\mathbb{C}^{N_-} \to V$ for $N_- = \dim V$. Set $\widetilde{J}_+ := \ker \begin{pmatrix} a & k_{N_-} \end{pmatrix}$ which is a subbundle of H_1, and identify $\widetilde{J}_+$ via an isomorphism with an element $J_+ \in \operatorname{Vect}(M)$. We then have

$$\operatorname{ind}_M a = [J_+] - [\mathbb{C}^{N_-}] = [L_+] - [L_-]$$

in $K(M)$. We can choose N_- as large as we want, and we now replace N_- by $N_- + N$ for some N and form the corresponding injective operator $k_{N_-+N} : \mathbb{C}^{N_-+N} \to H_2$. Let $p : H_2 \to \operatorname{im} k_{N_-+N}$ denote the orthogonal projection and write $a^0 := (1-p)a$. Then, applying the above construction to a^0 instead of a with k_{N_-+N} in place of k_N, we get a bundle $\widetilde{J}^0_+ \subset H_1$ and a corresponding $J^0_+ \cong J_+ \oplus \mathbb{C}^N \in \operatorname{Vect}(M)$ such that $\widetilde{J}^0_+ = \ker a^0$,

$$\operatorname{ind}_M a^0 = [J^0_+] - [\mathbb{C}^{N_-+N}] = [L_+] - [L_-]. \tag{1.3.4}$$

We may assume that the given bundles $L_\pm$ are both subbundles of $\mathbb{C}^N$ for a sufficiently large choice of N. Taking that N in our construction we find complementary subbundles $L^\perp_\pm$ of $L_\pm$ in J^0_+ and $\mathbb{C}^{N_-+N}$, respectively, i.e.,

$$L_+ \oplus L^\perp_+ = J^0_+, \qquad L_- \oplus L^\perp_- = \mathbb{C}^{N_-+N}.$$

Then (1.3.4) implies $[L_+^\perp] = [L_-^\perp]$ in $K(M)$, i.e., there is an R such that $L_+^\perp \oplus \mathbb{C}^R \cong L_-^\perp \oplus \mathbb{C}^R$. By replacing the number N by $N + R =: N_R$ and carrying out the above construction with N_R we arrive at an operator family a_R^0 instead of a^0 where $\mathrm{ind}_M\, a_R^0 = [J_{+,R}^0] - [\mathbb{C}^{N_- + N_R}]$, $J_{+,R}^0 = J_+^0 \oplus \mathbb{C}^R$, and we get bundles $L_{\pm,R}^\perp$ instead of $L_\pm^\perp$. Since $L_{\pm,R}^\perp = L_\pm^\perp \oplus \mathbb{C}^R$, we therefore obtain $L_{+,R}^\perp \cong L_{-,R}^\perp$. In other words, choosing N sufficiently large in the construction of a^0 we get

$$L_-^\perp \cong L_+^\perp.$$

By construction there are subbundles $\widetilde{L}_-$, $\widetilde{L}_-^\perp \subset H_2$ such that $\widetilde{L}_- \cong L_-$, $\widetilde{L}_-^\perp \cong L_-^\perp$ where $\widetilde{L}_- \oplus \widetilde{L}_-^\perp = \mathrm{Im}\, k_{N_- + N}$, and there are subbundles $\widetilde{L}_+$, $\widetilde{L}_+^\perp \subset H_1$ such that $\widetilde{L}_+ \cong L_+$, $\widetilde{L}_+^\perp \cong L_+^\perp$ where $\widetilde{L}_+ \oplus \widetilde{L}_+^\perp = \ker a_0$. Choose any isomorphism $\lambda : \widetilde{L}_+^\perp \to \widetilde{L}_-^\perp$, and let $\pi^\perp : H_1 \to \widetilde{L}_+^\perp$ denote the orthogonal projection, $\iota^\perp : \widetilde{L}_-^\perp \to H_2$ the canonical embedding. Then $q := \iota^\perp \circ \lambda \circ \pi^\perp : H_1 \to H_2$ is a continuous family of operators of finite rank, and we can form $\widetilde{a} := a + c$ for $c := -pa + q$. Then $\widetilde{a}$ satisfies the relations of (1.3.3). To construct the isomorphism (1.3.2) it suffices to choose isomorphisms $k : L_- \to \widetilde{L}_-$, $t : \widetilde{L}_+ \to L_+$ and to set $\widetilde{k} := \iota k$, $\widetilde{t} = t\pi$, where $\iota : \widetilde{L}_- \to H_2$ is the canonical embedding, $\pi : H_1 \to \widetilde{L}_+$ the orthogonal projection. □

Corollary 1.3.4 *Let $p_{(\mu)} : \pi_X^* E \to \pi_X^* F$ be a homogeneous elliptic (of order $\mu \in \mathbb{Z}$) principal symbol with the transmission property at the boundary (that is, locally belonging to a classical symbol with the transmission property), and form the associated boundary symbol $\sigma_\partial(p_{(\mu)})(y,\eta) = \mathrm{r}^+ p_{(\mu)}(y,0,\eta,D_t)\mathrm{e}^+ : E' \otimes H^s(\mathbb{R}_+) \to F' \otimes H^{s-\mu}(\mathbb{R}_+)$, $s > \max(\mu,0) - \frac{1}{2}$, that is a family of Fredholm operators, $(y,\eta) \in S^*Y$, and let $L_\pm \in \mathrm{Vect}(S^*Y)$ such that $[L_+] - [L_-] = \mathrm{ind}_{S^*Y}\, \sigma_\partial(p_{(\mu)})$. Then there exists a homogeneous principal Green symbol (of order μ) $g_{(\mu)}(y,\eta)$ such that $\mathrm{coker}(\sigma_\partial(p_{(\mu)}) + g_{(\mu)}) \cong L_-$ and $\ker(\sigma_\partial(p_{(\mu)}) + g_{(\mu)}) \cong L_+$.*

2 Boundary value problems not requiring the transmission property

2.1 Interior operators

Let X be a smooth, compact manifold with smooth boundary Y, let $\widetilde{E}, \widetilde{F} \in \mathrm{Vect}(2X)$, and set $E : \widetilde{E}|_X$, $F = \widetilde{F}|_X$. We then define the space

$$L_{\mathrm{cl}}^\mu(X;E,F)_{\mathrm{smooth}} := \{\mathrm{r}^+ P \mathrm{e}^+ + C : \; P \in L_{\mathrm{cl}}^\mu(2X;\widetilde{E},\widetilde{F}),\; C \in L^{-\infty}(\mathrm{int}\, X;E,F)\}. \tag{2.1.1}$$

Note that (2.1.1) can also be defined mod $L^{-\infty}(\mathrm{int}\, X;E,F)$ in terms of charts on X and local symbols in the half–space that are smooth up to the boundary. We admit arbitrary such symbols, and, as is well–known, cf. [21], the calculus of operators in (2.1.1) is far from that of the subspace $L_{\mathrm{cl}}^\mu(X;E,F)_{\mathrm{tr}}$. Let $S^{(\mu)}(T^*X \setminus 0;E,F)$ for $\mu \in \mathbb{R}$, $E,F \in \mathrm{Vect}(X)$ denote the set of all bundle homomorphisms $a_{(\mu)} : \pi_X^* E \to \pi_X^* F$ such that $a_{(\mu)}(x,\lambda\xi) = \lambda^\mu a_{(\mu)}(x,\xi)$ for all $(x,\xi) \in T^*X \setminus 0$. Every $A \in L_{\mathrm{cl}}^\mu(X;E,F)_{\mathrm{smooth}}$ has a well–defined homogeneous principal symbol $\sigma_\psi^\mu(A) \in S^{(\mu)}(T^*X \setminus 0;E,F)$, namely $\sigma_\psi^\mu(A) := \sigma_\psi^\mu(\widetilde{A})|_{T^*X\setminus 0}$ for any $\widetilde{A} \in L^{\mu}_{\mathrm{cl}}(2X;\widetilde{E},\widetilde{F})$ such that $A = \mathrm{r}^{+}\widetilde{A}\mathrm{e}^{+} + C$. Moreover, there is a (non–canonical) linear map

$$\mathrm{op} : S^{(\mu)}(T^{*}X \setminus 0; E, F) \to L^{\mu}_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}} \tag{2.1.2}$$

that can be generated by a standard procedure in terms of charts and local representatives of operators with given principal symbols where $\sigma^{\mu}_{\psi}(\mathrm{op}(p_{(\mu)})) = a_{(\mu)}$. To introduce a convenient scale of weighted Sobolev spaces we first look at $\mathbb{R}_{+}$. Let M denote the standard Mellin transform $Mu(z) = \int_0^\infty t^{z-1}u(t)\,dt$, first defined for $u \in C_0^\infty(\mathbb{R}_{+})$ (then $Mu(z)$ is holomorphic in z) and then extended to various distribution spaces (also vector–valued ones) where z is often assumed to vary on some weight line $\Gamma_\beta := \{z:\ \mathrm{Re}\, z = \beta\}$. Let $\mathcal{H}^{s,\gamma}(\mathbb{R}_{+})$ for $s,\gamma \in \mathbb{R}$ denote the completion of $C_0^\infty(\mathbb{R}_{+})$ with respect to the norm $\{(2\pi i)^{-1}\int(1+|z|^2)^s|Mu(z)|^2\,dz\}^{\frac{1}{2}}$. Moreover, let

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_{+}) := \{\omega f + (1-\omega)g :\ f \in \mathcal{H}^{s,\gamma}(\mathbb{R}_{+}), g \in H^{s}(\mathbb{R}_{+})\} \tag{2.1.3}$$

where $\omega(t)$ is any fixed cut–off function (that is, $\omega \in C_0^\infty(\overline{\mathbb{R}}_{+})$ and $\omega \equiv 1$ near $t=0$). The space (2.1.3) will be considered in a natural norm (corresponding to $\|u\|_{\mathcal{K}^{s,\gamma}(\mathbb{R}_{+})} = \inf\{\|\omega f\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_{+})} + \|(1-\omega)g\|^2_{H^{s}(\mathbb{R}_{+})}\}^{\frac{1}{2}}$, where the infimum is taken over all f and g in the respective spaces such that $u = \omega f + (1-\omega)g$. In particular, we have $\mathcal{K}^{0,0}(\mathbb{R}_{+}) = L^2(\mathbb{R}_{+})$ that we take with the standard scalar product.

Let H be a Hilbert space with a strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}}$. Then the "abstract" edge Sobolev space $\mathcal{W}^{s}(\mathbb{R}^{q}, H)$, $s \in \mathbb{R}$, is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, H)$ (the Schwartz space of H–valued functions on $\mathbb{R}^q$) with respect to the norm

$$\Bigg\{\int\limits_{\mathbb{R}^q} \langle\eta\rangle^{2s}\|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H\,d\eta\Bigg\}^{\frac{1}{2}},$$

where $\hat{u}(\eta)$ is the Fourier transform of u, cf. [17]. There are also "comp" and "loc" variants of such spaces $\mathcal{W}^{s}_{\mathrm{comp}}(\Omega, H)$ and $\mathcal{W}^{s}_{\mathrm{loc}}(\Omega, H)$, respectively, on any open set $\Omega \subseteq \mathbb{R}^q$; the scheme of the definition is analogous to that for scalar Sobolev spaces. More generally, we may insert Fréchet spaces H, written as projective limits of Hilbert spaces.

Given a symbol $a(y,\eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; H, \tilde{H})$ we can form the associated pseudodifferential operator $\mathrm{Op}(a)u(y) = \iint e^{i(y-y')\eta}a(y,\eta)u(y')\,dy' đ\eta$. We then have continuity

$$\mathrm{Op}(a) : \mathcal{W}^{s}_{\mathrm{comp}}(\Omega, H) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{H}) \tag{2.1.4}$$

for all $s \in \mathbb{R}$.

Setting $(\kappa_\lambda u)(t) = \lambda^{\frac{1}{2}}u(\lambda t)$, $\lambda \in \mathbb{R}_{+}$, we get a strongly continuous group of isomorphisms on $\mathcal{K}^{0,0}(\mathbb{R}_{+})$ for every $s,\gamma \in \mathbb{R}$. According to notation in Section 1.1 we then obtain spaces

$$\mathcal{W}^{s}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathbb{R}_{+})) =: \mathcal{W}^{s,\gamma}(\mathbb{R}^q \times \mathbb{R}_{+}).$$

Note that $\mathcal{W}^{s,\gamma}(\mathbb{R}^q \times \mathbb{R}_{+}) \subset H^{s}_{\mathrm{loc}}(\mathbb{R}^q \times \mathbb{R}_{+})$ for every $s,\gamma \in \mathbb{R}$. There is now a staightforward definition of spaces $\mathcal{W}^{s,\gamma}(X, E)$ on X for any $E \in \mathrm{Vect}(X)$, using an atlas on X with transition maps that we assume to be independent of the normal variable in a collar neighbourhood of the boundary.

Theorem 2.1.1 [18] *For every $A \in L^{\mu}_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}}$ and every $\gamma \in \mathbb{R}$ there is an operator $C_\gamma \in L^{-\infty}(\mathrm{int}\, X; E, F)$ such that $A_\gamma := A - C_\gamma$ induces continuous operators*

$$A_\gamma : \mathcal{W}^{s,\gamma}(X, E) \to \mathcal{W}^{s-\mu,\gamma-\mu}(X, F) \tag{2.1.5}$$

for all $s \in \mathbb{R}$.

There are many ways to find suitable operators C_γ. We shall see more details later. Any choice of a correspondence $A \to A_\gamma$ may be regarded as an operator convention that maps a complete symbol of A (i.e., a corresponding system of local symbols $(a_j)_{j=1,\ldots,N}$) to a continuous operator (2.1.5). Setting $\mathrm{op}_\gamma(a_{(\mu)}) := \mathrm{op}(a_{(\mu)})_\gamma$, cf. (2.1.2), we also get a map

$$\mathrm{op}_\gamma : S^{(\mu)}(T^*X \setminus 0; E, F) \to \bigcap_{s \in \mathbb{R}} \mathcal{L}(\mathcal{W}^{s,\gamma}(X, E), \mathcal{W}^{s-\mu,\gamma-\mu}(X, F)).$$

In this paper we solve the question of how to construct an operator algebra $\mathcal{T}(X)$ of boundary value problems

$$\mathcal{A} = \begin{pmatrix} A_\gamma & K \\ T & Q \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(X, E) \\ \oplus \\ P^s(Y, \boldsymbol{L}_-) \end{matrix} \to \begin{matrix} \mathcal{W}^{s-\mu,\gamma-\mu}(X, F) \\ \oplus \\ P^{s-\mu}(Y, \boldsymbol{L}_+) \end{matrix} \tag{2.1.6}$$

for arbitrary $A \in L^{\mu}_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}}$ and operators T, K and Q, where

(i) $P^s(Y, \boldsymbol{L}^{\pm})$ are spaces of the kind (1.2.6),

(ii) every σ_ψ–elliptic operator $A \in L^{\mu}_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}}$ occurs (up to a stabilisation) as an upper left corner of an elliptic (and then Fredholm) operator (2.1.6) for a suitable choice of data $\boldsymbol{L}^{\pm}$ and T, K and Q,

(iii) $\mathcal{T}(X)$ contains the parametrices of elliptic elements.

The construction will be given in such a way that $\mathcal{T}(X)$ is the Toeplitz extension of another algebra $\mathcal{L}(X)$ of operators

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A_\gamma & \widetilde{K} \\ \widetilde{T} & \widetilde{Q} \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(X, E) \\ \oplus \\ H^s(Y, J_-) \end{matrix} \to \begin{matrix} \mathcal{W}^{s-\mu,\gamma-\mu}(X, F) \\ \oplus \\ H^{s-\mu}(Y, J_+) \end{matrix} \tag{2.1.7}$$

that plays a similar role as $\mathcal{B}(X)$ in connection with its Toeplitz extension $\mathcal{S}(X)$.

2.2 Edge amplitude functions

The algebra $\mathcal{L}(X)$ will locally be defined by a specific kind of operator–valued amplitude functions. Let $\Omega \subseteq \mathbb{R}^q$ be an open set corresponding to a chart on Y, and consider the symbol space $S^{\mu}_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ $(= S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n})$, $\mu \in \mathbb{R}$. We use the fact that for every $p(y, t, \eta, \tau) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n_{\eta,\tau})$ there exists an element $\widetilde{p}(y, t, \widetilde{\eta}, \widetilde{\tau}) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n_{\widetilde{\eta},\widetilde{\tau}})$ such that

$$p(y, t, \eta, \tau) = t^{-\mu}\widetilde{p}(y, t, t\eta, t\tau) \bmod S^{-\infty}(\Omega \times \mathbb{R}_+ \times \mathbb{R}^n_{\eta,\tau}). \tag{2.2.1}$$

With $p(y, t, \eta, \tau) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n_{\widetilde{\eta},\widetilde{\tau}})$ we want to associate an operator–valued symbol

$$a(y, \eta) := \sigma(t)t^{-\mu}\{a_M(y, \eta) + a_F(y, \eta)\}\widetilde{\sigma}(t) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \tag{2.2.2}$$

with cut–off functions $\sigma(t)$, $\widetilde{\sigma}(t)$ and operator families $a_M(y,\eta)$ and $a_F(y,\eta)$ that will be constructed as follows.

First, let $M^\mu_{\mathcal{O}}(\mathbb{R}^q\times\mathbb{C})$ denote the space of all functions $f(\eta,z)\in\mathcal{A}(\mathbb{C},S^\mu_{\mathrm{cl}}(\mathbb{R}^q))$ such that $f(\eta,\beta+i\tau)\in S^\mu_{\mathrm{cl}}(\mathbb{R}^{q+1}_{\eta,\tau})$ for every $\beta\in\mathbb{R}$, uniformly in compact β–intervals. The space $M^\mu_{\mathcal{O}}(\mathbb{R}^q\times\mathbb{C})$ in Fréchet in a natural way. Given an element $\widetilde{h}(y,t,\widetilde{\eta},z)\in C^\infty(\Omega\times\overline{\mathbb{R}}_+,M^\mu_{\mathcal{O}}(\mathbb{R}^q_{\widetilde{\eta}}\times\mathbb{C}))$ we set

$$h(y,t,\eta,z):=\widetilde{h}(y,t,t\eta,z). \tag{2.2.3}$$

Functions of the form (2.2.3) will play the role of symbols of Mellin pseudodifferential operators. Set $\Gamma_\beta:=\{z\in\mathbb{C}:\ \mathrm{Re}\,z=\beta\}$ for any real β and consider $\mathrm{Im}\,z$ as the Mellin covariable. Let $S^\mu_{\mathrm{cl}}(\Gamma_\beta)$ denote the space of classical symbols of order μ with respect to z on the line Γ_β. For every $f(t,t',z)\in C^\infty(\mathbb{R}_+\times\mathbb{R}_+,S^\mu_{\mathrm{cl}}(\Gamma_{\frac12-\gamma}))$, $\gamma\in\mathbb{R}$, we set

$$\mathrm{op}^\gamma_M(f)u(t):=\frac{1}{2\pi}\int\left(\frac{t}{t'}\right)^{-\left(\frac12-\gamma+i\tau\right)}f(t,t',z)u(t')\,\frac{dt'}{t'}d\tau,$$

first regarded as an operator $C^\infty_0(\mathbb{R}_+)\to C^\infty(\mathbb{R}_+)$. Inserting the function (2.2.3) as the Mellin symbol we get a family of operators $\mathrm{op}^\gamma_M(h)(y,\eta)$, where (by Cauchy's theorem)

$$\mathrm{op}^\gamma_M(h)(y,\eta)u=\mathrm{op}^{\widetilde{\gamma}}_M(h)(y,\eta)u\qquad\text{for every }\gamma,\widetilde{\gamma}\in\mathbb{R}.$$

Theorem 2.2.1 [16] *For every $\widetilde{p}(y,t,\widetilde{\eta},\widetilde{\tau})\in S^\mu_{\mathrm{cl}}(\Omega\times\overline{\mathbb{R}}_+\times\mathbb{R}^n)$ there exists an element $\widetilde{h}(y,t,\widetilde{\eta},z)\in C^\infty(\Omega\times\overline{\mathbb{R}}_+,M^\mu_{\mathcal{O}}(\mathbb{R}^q\times\mathbb{C}))$ such that for*

$$b(y,t,\eta,\tau):=\widetilde{p}(y,t,t\eta,t\tau),\qquad h(y,t,\eta,z):=\widetilde{h}(y,t,t\eta,z)$$

we have

$$\mathrm{op}_t(b)(y,\eta)=\mathrm{op}^\gamma_M(h)(y,\eta)\ \mathrm{mod}\ C^\infty(\Omega,L^{-\infty}(\mathbb{R}_+;\mathbb{R}^q)) \tag{2.2.4}$$

for all $\gamma\in\mathbb{R}$, and $\widetilde{h}$ is unique $\mathrm{mod}\,C^\infty(\Omega\times\overline{\mathbb{R}}_+,M^{-\infty}_{\mathcal{O}}(\mathbb{R}^q\times\mathbb{C}))$.

Then, if we set

$$b_0(y,t,\eta,\tau):=\widetilde{p}(y,0,t\eta,t\tau),\qquad h_0(y,t,\eta,z):=\widetilde{h}(y,0,t\eta,z), \tag{2.2.5}$$

it follows that

$$\mathrm{op}_t(b_0)=\mathrm{op}^\gamma_M(h_0)(y,\eta)\ \mathrm{mod}\ C^\infty(\Omega,L^{-\infty}(\mathbb{R}_+;\mathbb{R}^q)).$$

In this paper a cut–off function $\omega(t)$ is an element of $C^\infty_0(\overline{\mathbb{R}}_+)$ that equals 1 in a neighbourhood of $t=0$. If $\omega,\widetilde{\omega}$ are cut–off functions, we write $\omega\prec\widetilde{\omega}$ if $\widetilde{\omega}$ equals 1 on $\mathrm{supp}\,\omega$. Let us fix a function $\eta\to[\eta]$ in $C^\infty(\mathbb{R}^q)$ where $[\eta]=|\eta|$ for $|\eta|\geq c$ for some $c>0$.

Remark 2.2.2 *Let $\omega_1,\omega_2,\omega_3$ be arbitrary cut–off functions such that $\omega_3\prec\omega_1\prec\omega_2$. Then, starting from an arbitrary $p(y,t,\eta,\tau)\in S^\mu_{\mathrm{cl}}(\Omega\times\overline{\mathbb{R}}_+\times\mathbb{R}^n)$, we form a symbol $\widetilde{p}(y,t,\widetilde{\eta},\tau\eta)$ via relation* (2.2.1) *and set*

$$a_M(y,\eta):=\omega_1(t[\eta])\,\mathrm{op}^\gamma_M(h)(y,\eta)\omega_2(t[\eta]), \tag{2.2.6}$$

$$a_F(y,\eta):=(1-\omega_1(t[\eta]))\,\mathrm{op}_t(b)(y,\eta)(1-\omega_3(t[\eta])) \tag{2.2.7}$$

with $b(y,t,\eta,\tau)$ and $h(y,t,\eta,\tau)$ being related to each other as in Theorem 2.2.1*. We then have*

$$\mathrm{op}_t(b)(y,\eta)=a_M(y,\eta)+a_F(y,\eta)\ \mathrm{mod}\ C^\infty(\Omega,L^{-\infty}(\mathbb{R}_+;\mathbb{R}^q)).$$

Set $S(c,c') := \{z \in \mathbb{C} : \ c < \operatorname{Re} z < c'\}$, and let $M^{-\infty}(\Gamma_\beta)_\varepsilon$ for $\beta \in \mathbb{R}$, $\varepsilon > 0$, denote the subspace of all $f(z) \in \mathcal{A}(S(\beta-\varepsilon, \beta+\varepsilon))$ such that $f(z)|_{\Gamma_\delta} \in \mathcal{S}(\Gamma_\delta)$ for every $\beta - \varepsilon < \delta < \beta + \varepsilon$, uniformly in compact subintervals. The space $M^{-\infty}(\Gamma_\beta)_\varepsilon$ is Fréchet in a natural way.

Then, if $\omega_1(t)$ and $\omega_2(t)$ are arbitrary cut–off functions and $f(y,z) \in C^\infty(\Omega, M^{-\infty}(\Gamma_{\frac{1}{2}-\gamma})_\varepsilon)$, the operator family

$$m(y,\eta) := \omega_1(t[\eta]) r^{-\mu} \operatorname{op}_M^\gamma(f)(y) \omega_2(t[\eta]) \tag{2.2.8}$$

is an element of $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{K}^{\infty,\gamma-\mu}(\mathbb{R}_+))$ for every $s \in \mathbb{R}$. The spaces $\mathcal{K}^{s,\beta}(\mathbb{R}_+)$ are endowed with the group action $(\kappa_\lambda u)(t) := \lambda^{\frac{1}{2}} u(\lambda t)$, $\lambda \in \mathbb{R}_+$.

More generally, we also may employ spaces $\mathcal{K}^{s,\gamma;\varrho}(\mathbb{R}_+) := \langle t \rangle^{-\varrho} \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ for $s, \gamma, \varrho \in \mathbb{R}$, with $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ being given as before; then $\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = \mathcal{K}^{s,\gamma;0}(\mathbb{R}_+)$. The following definition will refer to spaces of the form $\mathcal{K}^{s,\gamma;\varrho}(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^j$ with the group action $\{\kappa_\lambda \oplus \mathrm{id}_{\mathbb{C}^j}\}_{\lambda \in \mathbb{R}_+}$.

Definition 2.2.3 $\mathcal{R}^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ *for* $\mu \in \mathbb{R}$, $\varepsilon > 0$, $\boldsymbol{g} = (\gamma, \delta)$, $\boldsymbol{v} = (k, l; j_-, j_+)$, *is defined to be the space of all*

$$g(y,\eta) \in \bigcap_{\substack{s,r,\varrho \in \mathbb{R} \\ 0 \le \alpha < \varepsilon}} S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_-}, \mathcal{K}^{r,\delta+\alpha;\varrho}(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_+})$$

such that

$$g^*(y,\eta) \in \bigcap_{\substack{s,r,\varrho \in \mathbb{R} \\ 0 \le \alpha < \varepsilon}} S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\delta}(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_+}, \mathcal{K}^{r,-\gamma+\alpha;\varrho}(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_-}).$$

The (pointwise) formal adjoint g^* *is defined by*

$$(gu, v)_{L^2(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_+}} = (u, g^* v)_{L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_-}}$$

for all $u \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_-}$, $v \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^l) \oplus \mathbb{C}^{j_+}$.

For purposes below we set

$$\mathcal{S}^\gamma(\mathbb{R}_+)_\varepsilon := \varprojlim_{\substack{r,\varrho \in \mathbb{R} \\ 0 \le \alpha < \varepsilon}} \mathcal{K}^{r,\gamma+\alpha;\varrho}(\mathbb{R}_+). \tag{2.2.9}$$

The elements $g(y,\eta) \in \mathcal{R}^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ are called Green symbols of flatness $\varepsilon > 0$ relative to the weights $\gamma, \delta \in \mathbb{R}$.

Moreover, let $\mathcal{R}^\mu_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ denote the space of all operator families of the form

$$c(y,\eta) := \begin{pmatrix} m(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + g(y,\eta) \tag{2.2.10}$$

for arbitrary $g(y,\eta) \in R^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ (regarded as a block matrix with an $l \times k$–block matrix upper left corner) and an $m(y,\eta)$ of the form (2.2.8) for $f(y,z) \in C^\infty(\Omega, M^{-\infty}(\Gamma_{\frac{1}{2}-\gamma})_\varepsilon) \otimes \mathbb{C}^l \otimes \mathbb{C}^k$.

Finally, we introduce $\mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ as the space of all operator families of the form

$$a(y,\eta) = \begin{pmatrix} a_\psi(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + c(y,\eta) \tag{2.2.11}$$

for arbitrary $c(y,\eta) \in R^{\mu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ and $l \times k$–block matrix family $a_\psi(y,\eta)$ where the entries have the form

$$\sigma(t)t^{-\mu}(a_M(y,\eta) + a_F(y,\eta))\widetilde{\sigma}(t)$$

with arbitrary $a_M(y,\eta)$, $a_F(y,\eta)$ as in Remark 2.2.2 and cut–off functions $\sigma(t), \widetilde{\sigma}(t)$.

2.3 Boundary value problems

We now establish pseudodifferential boundary value problems as a subalgebra of an algebra of edge operators. Concerning general material on pseudodifferential operators near edge singularities, cf. Schulze [19] or Seiler [25].

For $H = \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ we set

$$\mathcal{W}^{s,\gamma}(\mathbb{R}^n_+) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathbb{R}_+))$$

where $n = q+1$, further $\mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+) = \mathcal{W}^s_{\mathrm{comp}}(\Omega, \mathcal{K}^{s,\gamma}(\mathbb{R}_+))$, and similarly for "loc". By virtue of $\mathcal{W}^{s,\gamma}(\mathbb{R}^n_+) \subset H^s_{\mathrm{loc}}(\mathbb{R}^n_+)$ we define the spaces $\mathcal{W}^{s,\gamma}(X)$ on a compact smooth manifold X with smooth boundary Y as the subspace of all $u \in H^s_{\mathrm{loc}}(\mathrm{int}\, X)$ such that the restriction to neighbourhood of a boundary point in local coordinates $(y,t) \in \Omega \times \mathbb{R}_+$ belongs to $\mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+)$. More generally, for a bundle $E \in \mathrm{Vect}(X)$ we define spaces $\mathcal{W}^{s,\gamma}(X,E)$ of distributional sections by a similar scheme as for standard Sobolev spaces, using the invariance of the local spaces under transition diffeomorphisms (we suppose the coordinate diffeomorphisms to be independent of the normal variable t in a collar neighbourhood of the boundary).

Note that there are continuous embeddings

$$\mathcal{W}^{s',\gamma'}(X,E) \hookrightarrow \mathcal{W}^{s,\gamma}(X,E) \tag{2.3.1}$$

for all $s' \geq s$, $\gamma' \geq \gamma$ which are compact for $s' > s$, $\gamma' > \gamma$.

Let $R^{\mu}(\Omega \times \mathbb{R}^q; \boldsymbol{g}, (k,l))_\varepsilon = R^{\mu}(\Omega \times \mathbb{R}^q; \boldsymbol{g})_\varepsilon \otimes \mathbb{C}^l \otimes \mathbb{C}^k$ for $\boldsymbol{g} = (\gamma, \gamma - \mu)$, and let

$$\mathcal{R}^{\mu}(\Omega \times \mathbb{R}^q; \boldsymbol{g}, \boldsymbol{v})_\varepsilon \qquad \text{for } \boldsymbol{v} = (k,l; j_-, j_+) \tag{2.3.2}$$

defined to be the set of all operator families

$$a(y,\eta) = \begin{pmatrix} p(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + g(y,\eta) : \begin{matrix} \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \otimes \mathbb{C}^k \\ \oplus \\ \mathbb{C}^{j_-} \end{matrix} \to \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \otimes \mathbb{C}^l \\ \oplus \\ \mathbb{C}^{j_+} \end{matrix} \tag{2.3.3}$$

for arbitrary $p(y,\eta) \in R^{\mu}(\Omega \times \mathbb{R}^q; \boldsymbol{g}, (k,l))_\varepsilon$ and $g(y,\eta) \in R^{\mu}_G(\Omega \times \mathbb{R}^q; \boldsymbol{g}, \boldsymbol{v})_\varepsilon$, cf. Section 2.2.

Denoting the space in (2.3.3) for a moment by H and $\widetilde{H}$, respectively, endowed with group actions $\mathrm{diag}(\kappa_\lambda, \mathrm{id})_{\lambda \in \mathbb{R}_+}$, where id means the identity in $\mathbb{C}^{j_-}$ or $\mathbb{C}^{j_+}$, we have $a(y,\eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ for every $s, \gamma \in \mathbb{R}$. If $U \subset X$ is an open neighbourhood, $U' := U \cap Y \neq \emptyset$, where U is diffeomorphic to $\Omega \times \overline{\mathbb{R}}_+$, and if $E|_U \to (\Omega \times \overline{\mathbb{R}}_+) \times \mathbb{C}^k$, $F|_U \to (\Omega \times \overline{\mathbb{R}}_+) \times \mathbb{C}^l$, $J_\pm|_{U'} \to \Omega \times \mathbb{C}^{j_\pm}$ are trivialisations of bundles $E, F \in \mathrm{Vect}(X)$, $J_\pm \in \mathrm{Vect}(Y)$, we can pull–back $\mathrm{Op}_y(a)$ to U as an operator

$$A_U : \begin{matrix} C_0^\infty(U, E|_U) \\ \oplus \\ C_0^\infty(U', J_-|_{U'}) \end{matrix} \to \begin{matrix} C^\infty(U, F|_U) \\ \oplus \\ C_0^\infty(U', J_+|_{U'}) \end{matrix},$$

using a corresponding invariance under transmission maps.

Now let $\sigma, \widetilde{\sigma} \in C^\infty(X)$ be elements supported by a tubular neighbourhood $\cong Y \times [0,1) \ni (y,t)$ of the boundary Y where $\sigma(t), \widetilde{\sigma}(t)$ equal 1 for $0 \leq t \leq \frac{1}{2}$, choose an open covering $\{U_1', \ldots, U_M'\}$ of Y by coordinate neighbourhoods, and let $\{\varphi_1, \ldots, \varphi_M\}$ denote a subordinate partition of unity and $\{\psi_1, \ldots, \psi_M\}$ another system of functions $\psi_j \in C_0^\infty(U_j')$ such that $\varphi_j \psi_j = \varphi_j$ for all j.

Let $U_j := U_j' \times [0,1)$, and form operators

$$\sum_{j=0}^{M} \begin{pmatrix} \varphi_j \sigma & 0 \\ 0 & \varphi_j \end{pmatrix} \mathcal{A}_{U_j} \begin{pmatrix} \psi_j \widetilde{\sigma} & 0 \\ 0 & \psi_j \end{pmatrix} + \begin{pmatrix} (1-\sigma) A_{\mathrm{int}} (1-\widetilde{\widetilde{\sigma}}) & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.3.4}$$

where $\mathcal{A}_{U_j}$ is associated with local symbols $a_j(y,\eta)$ in the above–mentioned sense, $A_{\mathrm{int}} \in L^\mu_{\mathrm{cl}}(\mathrm{int}\, X; E, F)$, and $\widetilde{\widetilde{\sigma}} \in C^\infty(X)$ is a function supported in $Y \times [0,1)$ such that $\sigma \widetilde{\widetilde{\sigma}} = \widetilde{\widetilde{\sigma}}$. To get a calculus we also define global smoothing operators, that are defined by the mapping properties

$$\mathcal{G} : \begin{array}{c} \mathcal{W}^{s,\gamma}(X,E) \\ \oplus \\ H^s(Y,J_-) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,\gamma-\mu+\alpha}(X,F) \\ \oplus \\ H^\infty(Y,J_+) \end{array} \tag{2.3.5}$$

where

$$\mathcal{G}^* : \begin{array}{c} \mathcal{W}^{s,-\gamma+\mu}(X,F) \\ \oplus \\ H^s(Y,J_+) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,-\gamma+\alpha}(X,E) \\ \oplus \\ H^\infty(Y,J_-) \end{array} \tag{2.3.6}$$

for all $s \in \mathbb{R}$ and all $0 \leq \alpha < \varepsilon$ for some $\varepsilon > 0$. Here, $\mathcal{G}^*$ is the formal adjoint of $\mathcal{G}$ in the sense

$$(\mathcal{G}u, v)_{\mathcal{W}^{0,0}(X,F) \oplus H^0(Y,J_+)} = (u, \mathcal{G}^* v)_{\mathcal{W}^{0,0}(X,E) \oplus H^0(Y,J_-)}$$

for all $u \in C_0^\infty(\mathrm{int}\, X, E) \oplus C^\infty(Y, J_-)$, $v \in C_0^\infty(\mathrm{int}\, X, F) \oplus C^\infty(Y, J_+)$, with respect to a choice of scalar products in the spaces $\mathcal{W}^{0,0}(X,E)$, $\mathcal{W}^{0,0}(X,F)$ and $H^0(Y,J_-)$, $H^0(Y,J_+)$, referring to the Riemannian metrics and bundles E, F and J_-, J_+. Let $\mathcal{L}_G^{-\infty}(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ denote the space of all operators $\mathcal{G}$ with these mapping properties, where $\boldsymbol{g} = (\gamma, \gamma - \mu)$ are given weight data, $\gamma, \mu \in \mathbb{R}$, and $\boldsymbol{v} = (E, F; J_-, J_+)$ the tuple of bundles.

Definition 2.3.1 *$\mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ for any fixed $\varepsilon > 0$ is defined to be the space of all operators $\mathcal{A} = \mathcal{A}_1 + \mathcal{G}$, where $\mathcal{A}_1$ is of the form* (2.3.4) *and $\mathcal{G} \in \mathcal{L}_G^{-\infty}(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$.*

Note that

$$\mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_{\varepsilon'} \subseteq \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$$

for every $\varepsilon' \geq \varepsilon$.

Let $\mathcal{L}^\mu_{M+G}(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ ($\mathcal{L}^\mu_G(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$) denote the subset of all elements in $\mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ where the local amplitude functions $a_j(y,\eta)$ belong to $\mathcal{R}^\mu_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ ($\mathcal{R}^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$) for all j. Moreover, let $L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$, $L^\mu_{M+G}(X, \boldsymbol{g}; E, F)_\varepsilon$, and $L^\mu_G(X, \boldsymbol{g}; E, F)_\varepsilon$ be the space of upper left corners of the corresponding spaces $\mathcal{L}^\mu(\ldots)_\varepsilon$, $\mathcal{L}^\mu_{M+G}(\ldots)_\varepsilon$, and $\mathcal{L}^\mu_G(\ldots)_\varepsilon$, respectively, where $\boldsymbol{v} = (E, F; J_-, J_+)$.

Remark 2.3.2 *We have*

$$L^\mu_{M+G}(X, \boldsymbol{g}; E, F)_\varepsilon = L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon \cap L^{-\infty}(\mathrm{int}\, X; E, F).$$

Theorem 2.3.3 *Every $\mathcal{A} \in \mathcal{L}^{\mu}(X, \boldsymbol{g}; \boldsymbol{v})_{\varepsilon}$ induces continuous operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(X, E) \\ \oplus \\ H^{s}(Y, J_{-}) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(X, F) \\ \oplus \\ H^{s-\mu}(Y, J_{+}) \end{array} \tag{2.3.7}$$

for all $s \in \mathbb{R}$. Elements $\mathcal{A} \in \mathcal{L}_{G}^{-\infty}(X, \boldsymbol{g}; \boldsymbol{v})_{\varepsilon}$ induce compact operators (2.3.7) *for all $s, \mu \in \mathbb{R}$.*

Proof. It suffices to observe that the summands in the definition of $\mathcal{A}$ have the asserted mapping property. For $\mathcal{G}$ this is the case by definition; the interior part in (2.3.4) is standard. Concerning $\operatorname{diag}(\varphi_j \sigma, \varphi_j)\mathcal{A}_{U_j} \operatorname{diag}(\psi_j \widetilde{\sigma}, \psi_j)$ in (2.3.4) we may apply (2.1.4) to the underlying local symbols $a_j(y, \eta)$.

For the second assertion it suffices to combine the mapping property with corresponding compact embeddings of our Sobolev spaces, cf. formula (2.3.1). $\square$

Let us now establish the principal symbolic structure of $\mathcal{L}^{\mu}(X, \boldsymbol{g}; \boldsymbol{v})_{\varepsilon}$. First, the upper left corner A of an element $\mathcal{A} \in \mathcal{L}^{\mu}(X, \boldsymbol{g}; \boldsymbol{v})_{\varepsilon}$ belongs to $L_{\mathrm{cl}}^{\mu}(X; E, F)_{\mathrm{smooth}}$, cf. notation (2.1.1), and there is then a homogeneous principal symbol of order μ

$$\sigma_{\psi}(\mathcal{A}) := \sigma_{\psi}(A) : \pi_X^* E \to \pi_X^* F, \tag{2.3.8}$$

$\pi_X : T^*X \setminus 0 \to X$. In addition, we have a homogeneous principal boundary symbol of order μ

$$\sigma_{\partial}(\mathcal{A}) : \pi_Y^* \begin{pmatrix} E' \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ J_{-} \end{pmatrix} \to \pi_Y^* \begin{pmatrix} F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \\ \oplus \\ J_{+} \end{pmatrix}, \tag{2.3.9}$$

$\pi_Y : T^*Y \setminus 0 \to Y$, that is invariantly defined by local terms, i.e., for the amplitude functions (2.2.11).

Locally, we set

$$\sigma_{\partial}(\mathcal{A})(y, \eta) = \begin{pmatrix} \sigma_{\partial}(a_{\psi})(y, \eta) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_{\partial}(c)(y, \eta), \tag{2.3.10}$$

where $\sigma_{\partial}(c)$ is the homogeneous principal component of the classical operator–valued symbol $c(y, \eta)$, cf. formula (2.2.10), in particular,

$$\sigma_{\partial}(m)(y, \eta) = \omega_1(t|\eta|)t^{-\mu} \operatorname{op}_M^{\gamma}(f)(y)\omega_2(t|\eta|) \tag{2.3.11}$$

in the notation of (2.2.4). Moreover, we set

$$\begin{aligned} \sigma_{\partial}(a_{\psi})(y, \eta) = {} & \omega_1(t|\eta|)t^{-\mu} \operatorname{op}_M^{\gamma}(h_0)(y, \eta)\omega_2(t|\eta|) \\ & + (1 - \omega_1(t|\eta|))t^{-\mu} \operatorname{op}_t(b_0)(y, \eta)(1 - \omega_3(t|\eta|)), \end{aligned} \tag{2.3.12}$$

cf. formulas (2.2.5) and (2.2.6), (2.2.7).

The homogeneity of $\sigma_{\partial}(\mathcal{A})(y, \eta)$ is formally analogous to (1.2.9).

Let us set

$$\sigma(\mathcal{A}) = (\sigma_{\psi}(\mathcal{A}), \sigma_{\partial}(\mathcal{A})). \tag{2.3.13}$$

Note that every $\mathcal{A} \in \mathcal{L}^{\mu}(X, \boldsymbol{g}; \boldsymbol{b})_{\varepsilon}$ has a subordinate principal conormal symbol

$$\sigma_M \sigma_{\partial}(\mathcal{A})(y, z) : E'_y \to F'_y \tag{2.3.14}$$

for every $y \in Y$, $z \in \mathbb{C}$, $\operatorname{Re} z = \frac{1}{2} - \gamma$, that only depends on $A = \text{u.l.c.}\,\mathcal{A}$, namely

$$\sigma_M \sigma_\partial(A)(y,z) := \widetilde{h}(y,0,0,z) + f(y,z), \tag{2.3.15}$$

cf. formulas (2.2.5) and (2.2.8).

Theorem 2.3.4 *$\mathcal{A} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{b})_\varepsilon$ for $\boldsymbol{g} = (\gamma, \nu, \gamma - (\mu + \nu))$, $\boldsymbol{b} = (E_0, F; J_0, J_+)$, and $\mathcal{B} \in \mathcal{L}^\nu(X, \boldsymbol{h}; \boldsymbol{c})_{\widetilde{\varepsilon}}$ for $\boldsymbol{h} = (\gamma, \gamma - \nu)$, $\boldsymbol{c} = (E, E_0; J_-, J_0)$ implies $\mathcal{AB} \in \mathcal{L}^{\mu+\nu}(X, \boldsymbol{g} \circ \boldsymbol{h}; \boldsymbol{b} \circ \boldsymbol{c})_{\min(\varepsilon, \widetilde{\varepsilon})}$ for $\boldsymbol{g} \circ \boldsymbol{h} = (\gamma, \gamma - (\mu + \nu))$, $\boldsymbol{b} \circ \boldsymbol{c} = (E, F; J_-, J_+)$, and we have*

$$\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

with componentwise multiplication. If $\mathcal{A}$ or $\mathcal{B}$ belongs to the subspace with subscript $M + G$ (G), then the same is true of the composition.

Definition 2.3.5 *An operator $\mathcal{A} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ for $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\boldsymbol{v} = (E, F; J_-, J_+)$, is called SL–elliptic if both* (2.3.8) *and* (2.3.9) *are isomorphisms.*

The bijectivity of (2.3.9) is an analogue of the Shapiro–Lopatinskij condition. If this holds, we also call $\sigma_\partial(\mathcal{A})$ SL–elliptic. The bijectivity of (2.3.3) for an $s = s_0 \in \mathbb{R}$ is equivalent to that for all $s \in \mathbb{R}$.

Theorem 2.3.6 *Let $\mathcal{A} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ be SL–elliptic. Then there is a parametrix $\mathcal{P} \in \mathcal{L}^{-\mu}(X, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})_{\widetilde{\varepsilon}}$ for a certain $\widetilde{\varepsilon} > 0$, where $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma)$, $\boldsymbol{v}^{-1} = (F, E; J_+, J_-)$, i.e., the remainders*

$$\mathcal{C}_l := 1 - \mathcal{PA} \qquad \text{and} \qquad \mathcal{C}_r := 1 - \mathcal{AP} \tag{2.3.16}$$

belong to $\mathcal{L}_G^{-\infty}(X, \boldsymbol{g}_l; \boldsymbol{v}_l)_{\min(\varepsilon, \widetilde{\varepsilon})}$ and $\mathcal{L}_G^{-\infty}(X, \boldsymbol{g}_r; \boldsymbol{v}_r)_{\min(\varepsilon, \widetilde{\varepsilon})}$, respectively, for $\boldsymbol{g}_l = (\gamma, \gamma)$, $\boldsymbol{v}_l = (E, E; J_-, J_-)$ and $\boldsymbol{g}_r = (\gamma - \mu, \gamma - \mu)$, $\boldsymbol{v}_r = (F, F; J_+, J_+)$.

Theorem 2.3.4 entails $\sigma(\mathcal{P}) = \sigma^{-1}(\mathcal{A})$ with componentwise inversion.

Remark 2.3.7 *If $\mathcal{A} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ is SL–elliptic, the operator* (2.3.7) *is Fredholm for every $s \in \mathbb{R}$, and the parametrix $\mathcal{P}$ can be chosen in such a way that the smoothing remainders are projections of finite rank, where $\mathcal{C}_l$ projects to* $\ker \mathcal{A}$ *and $\mathcal{C}_r$ to a complement of* $\operatorname{im} \mathcal{A}$ *for every fixed s;* $\ker \mathcal{A}$ *is independent of s as well as* $\dim \operatorname{coker} \mathcal{A}$*, i.e.,* $\operatorname{ind} \mathcal{A}$ *is independent of s.*

For references below we want to formulate the following result:

Theorem 2.3.8 *For every $\gamma, \mu \in \mathbb{R}$ and $E \in \operatorname{Vect}(X)$ there exists an elliptic operator $R_E^\mu \in L^\mu(X, (\gamma, \gamma - \mu); E, E)_\varepsilon$ for some $\varepsilon > 0$ such that $\sigma_\psi(R_E^\mu)(\xi) = |\xi|^\mu \operatorname{id}_E$, where*

$$R_E^\mu : \mathcal{W}^{s,\gamma}(X, E) \to \mathcal{W}^{s-\mu, \gamma-\mu}(X, E)$$

induces isomorphisms for all $s \in \mathbb{R}$ and $(R_E^\mu)^{-1} \in L^{-\mu}(X, (\gamma - \mu, \gamma); E, E)_\varepsilon$.

Theorem 2.3.8 corresponds to [21, Theorem 4.2.8]; the version in [21] is more precise insofar the operators R_E^μ even belong to operator spaces with constant discrete asymptotics. These spaces are contained in the present ones for a sufficiently small $\varepsilon > 0$.

3 Operators with global projection conditions

3.1 Constructions for boundary symbols

The results of Section 2.3 show that for every $a_{(\mu)}(x,\xi) \in S^{(\mu)}(T^*X \setminus 0; E, F)$ there exists an element $A_\gamma \in L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$ for arbitrary $\gamma \in \mathbb{R}$, $\varepsilon > 0$, such that $\sigma_\psi(A) = a_{(\mu)}$. In fact, in formula (2.3.4) it suffices to assume $A_{\text{int}} \in L^\mu_{\text{cl}}(\text{int}\, X; E, F)$ to be of the form $A_{\text{int}} = \widetilde{A}|_{\text{int}\, X}$ for some $\widetilde{A} \in L^\mu_{\text{cl}}(X; E, F)_{\text{smooth}}$ where $\sigma_\psi(\widetilde{A}) = a_{(\mu)}$, and to take local symbols $p(y,t,\eta,\tau) \in S^\mu_{\text{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n) \otimes \mathbb{C}^l \otimes \mathbb{C}^k$ having $a_{(\mu)}$ as homogeneous principal part (cf. Remark 2.2.2 and the construction of $a_\psi(y,\eta)$).

Let

$$\text{op}_\gamma : S^{(\mu)}(T^*X \setminus 0; E, F) \to L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon \tag{3.1.1}$$

denote the map that follows from such a construction. Clearly, (3.1.1) is not canonical (and not necessarily linear), but it is a right inverse of the principal symbolic map

$$\sigma_\psi : L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon \to S^{(\mu)}(T^*X \setminus 0; E, F). \tag{3.1.2}$$

Let $S^{(\mu)}(T^*Y \setminus 0; E', F')_\varepsilon$ for $\boldsymbol{g} = (\gamma, \gamma - \mu)$ denote the space of all homogeneous principal boundary symbols

$$\sigma_\partial(A)(y,\eta) : \pi_Y^* E' \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \pi_Y^* F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+), \tag{3.1.3}$$

$\pi_Y : T^*Y \setminus 0 \to Y$, belonging to elements $A \in L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$.

Moreover, set

$$S^{(\mu)}_{M+G}(T^*Y \setminus 0; E', F')_\varepsilon = \{\sigma_\partial(A) : \; A \in L^\mu_{M+G}(X, \boldsymbol{g}; E, F)_\varepsilon\},$$

and define $S^{(\mu)}_G(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$ in a similar manner in terms of $L^\mu_G(X, \boldsymbol{g}; E, F)_\varepsilon$.

Observe that (y,η)–wise the operators (3.1.3) are elements of the cone algebra on $\mathbb{R}_+$ with weight control of breadth ε relative to the weights γ and $\gamma - \mu$, respectively. From the cone theory we have an interior symbolic structure in $(t,\tau) \in T^*\mathbb{R}_+ \setminus 0$ that is the standard one of classical pseudodifferential operators on $\mathbb{R}_+ \ni t$, the exit symbolic structure that is responsible for $t \to \infty$ and the principal conormal symbolic structure for $t \to 0$

$$\sigma_M \sigma_\partial(A)(y,z) : E'_y \to F'_y, \tag{3.1.4}$$

$y \in Y$, $z \in \mathbb{C}$, $\text{Re}\, z = \frac{1}{2} - \gamma$.

Let us set $T^*_Y X := T^*X|_Y$, and let $S^{(\mu)}(T^*_Y X \setminus 0; E', F')$ denote the space of all restrictions of elements of $S^{(\mu)}(T^*X \setminus 0; E, F)$ to $T^*_Y X \setminus 0$. Setting $A = \text{op}_\gamma(a_{(\mu)})$ for an $a_{(\mu)}(x,\xi) \in S^{(\mu)}(T^*X \setminus 0; E, F)$, the operator family (3.1.3) admits to recover $a_{(\mu)}|_{T^*_Y X \setminus 0} \in S^{(\mu)}(T^*_Y X \setminus 0; E, F)$ in a unique way which gives us a linear map

$$\sigma'_\psi : S^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon \to S^{(\mu)}(T^*_Y X \setminus 0; E', F')$$

where

$$\ker \sigma'_\psi = S^{(\mu)}_{M+G}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon. \tag{3.1.5}$$

Remark 3.1.1 *A pair $(p_\psi, p_\partial) \in S^{(\mu)}(T^*X \setminus 0; E, F) \times S^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$ equals the symbol $\sigma(A) = (\sigma_\psi(A), \sigma_\partial(A))$ of some $A \in L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$ if and only if $p_\psi = \sigma_\psi(A)$ and $p_\psi|_{T^*_Y X \setminus 0} = \sigma'_\psi(p_\partial)$.*

Remark 3.1.2 *For every choice of* op_γ *the composition* $\sigma_\partial\,\mathrm{op}_\gamma$ *induces a linear map*

$$[\sigma_\partial\,\mathrm{op}_\gamma] : S^{(\mu)}(T^*X \setminus 0; E, F) \\ \to S^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon / S^{(\mu)}_{M+G}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon.$$

An element of $S^{(\mu)}(T^*X \setminus 0; E, F)$ is called elliptic, if it defines an isomorphism $\pi_X^* E \to \pi_X^* F$.

Theorem 3.1.3 *Let* $Y = \partial X$ *satisfy the following condition. There exists a vector field* v *on* Y *(i.e., a section in* T^*Y*) such that* $v(y) \neq 0$ *for every* $y \in Y$*. Then for every* $\gamma \in \mathbb{R}$ *the map* op_γ *can be chosen in such a way that the ellipticity of* $a_{(\mu)}(x,\xi) \in S^{(\mu)}(T^*X \setminus 0; E, F)$ *entails the Fredholm property of*

$$b_{(\mu)}(y,\eta) := \sigma_\partial\,\mathrm{op}_\gamma(a_{(\mu)})(y,\eta) : E'_y \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to F'_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \qquad (3.1.6)$$

for every $(y,\eta) \in T^*Y \setminus 0$*.*

For general X a similar result holds up to stabilisation. By that we mean an elliptic symbol

$$\widetilde{a}_{(\mu)}(x,\xi) \in S^{(\mu)}(T^*X \setminus 0; E \oplus \widetilde{E}, F \oplus \widetilde{E})$$

for some $\widetilde{E} \in \mathrm{Vect}(X)$ such that

$$\widetilde{a}_{(\mu)}(x,\xi)|_{S^*X} = a_{(\mu)}(x,\xi)|_{S^*X} \oplus \mathrm{id}_{\pi_X^*}\,\widetilde{E}.$$

Here S^*X is the unit sphere bundle induced by T^*X.

Theorem 3.1.4 *For a given elliptic* $a_{(\mu)}(x,\xi) \in S^{(\mu)}(T^*X \setminus 0; E, F)$ *there is an* $\widetilde{E} \in \mathrm{Vect}(X)$ *such that for a suitable choice of the map* op_γ*,* $\gamma \in \mathbb{R}$*,*

$$\widetilde{b}_{(\mu)}(y,\eta) := \sigma_\partial(\mathrm{op}_\gamma(\widetilde{a}_{(\mu)}))(y,\eta) : (E \oplus \widetilde{E})_y \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to (F \oplus \widetilde{E})_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$$

is Fredholm for every $(y,\eta) \in T^*Y \setminus 0$*.*

A proof of Theorem 3.1.3 and Theorem 3.1.4 will be given in Section 3.3 below.

Remark 3.1.5 *If* $a_{(\mu)}$ *is elliptic, the operator* (3.1.6) *is Fredholm for any* $s = s_0 \in \mathbb{R}$ *and* $\eta \neq 0$ *if and only if the principal conormal symbol*

$$\sigma_M \sigma_\partial\,\mathrm{op}_\gamma(a_{(\mu)})(y,z) : E'_y \to F'_y$$

is a family of isomorphisms for all $y \in Y$*,* $\mathrm{Re}\, z = \frac{1}{2} - \gamma$*. In that case* $b_{(\mu)}(y,\eta)$ *is Fredholm for all* $s \in \mathbb{R}$*,* $\ker b_{(\mu)}(y,\eta)$ *is independent of* s *and a finite–dimensional subspace of* $E'_y \otimes \mathcal{S}^\gamma(\mathbb{R}_+)_\varepsilon$*.*

Moreover, there is a finite–dimensional subspace of $F'_y \otimes \mathcal{S}^{\gamma-\mu}(\mathbb{R}_+)_\varepsilon$ *that is direct to* $\mathrm{im}\, b_{(\mu)}(y,\eta)$ *and spans together with* $\mathrm{im}\, b_{(\mu)}(y,\eta)$ *the space* $F'_y \otimes \mathcal{K}^{-\mu,\gamma-\mu}(\mathbb{R}_+)$ *for all* $s \in \mathbb{R}$*. This is true for all* $y \in Y$*,* $\mathrm{Re}\, z = \frac{1}{2} - \gamma$*.*

3.2 Ellipticity of boundary value problems with projection data

Let $b_{(\mu)}(y,\eta) \in S^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$ be an element such that

$$b_{(\mu)}(y,\eta) : E'_y \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to F'_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \tag{3.2.1}$$

is Fredholm for every $s \in \mathbb{R}$ and $(y,\eta) \in T^*Y \setminus 0$, cf. Theorem 3.1.3. Because of the homogeneity $b_{(\mu)}(y,\lambda\eta) = \lambda^\mu \kappa_\lambda b_{(\mu)}(y,\eta)\kappa_\lambda^{-1}$ for all $\lambda \in \mathbb{R}_+$ it is often sufficient to consider $b_{(\mu)}$ on the unit cosphere bundle S^*Y. Let us denote that restriction simply again by $b_{(\mu)}$. We then have an index element

$$\mathrm{ind}_{S^*Y}\, b_{(\mu)} \in K(S^*Y).$$

If $\widetilde{b}_{(\mu)} \in S^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$ is another choice such that $\sigma'_\psi(\widetilde{b}_{(\mu)}) = a_{(\mu)}|_{T^*_Y X \setminus 0}$, relation (3.1.5) gives us

$$b_{(\mu)} - \widetilde{b}_{(\mu)} \in S^{(\mu)}_{M+G}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon.$$

$\widetilde{b}_{(\mu)}(y,\eta)$ is not necessarily a Fredholm family in the sense of (3.2.1). Moreover, if this is the case, it may happen that

$$\mathrm{ind}_{S^*Y}\, b_{(\mu)} \neq \mathrm{ind}_{S^*Y}\, \widetilde{b}_{(\mu)}.$$

Now let $\mathcal{A} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ be an SL–elliptic operator, $\boldsymbol{v} = (E, F; J_-, J_+)$, and let $A \in L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$ be the upper left corner of $\mathcal{A}$. Setting $\sigma(A) = (\sigma_\psi(A), \sigma_\partial(A)) =: (a_{(\mu)}, b_{(\mu)})$ we then have a Fredholm family (3.2.1) where

$$\mathrm{ind}\, \sigma_\partial(A) = [\pi_1^* J_+] - [\pi_1^* J_-], \tag{3.2.2}$$

$\pi_1 : S^*Y \to Y$. Thus, like in the calculus of boundary value problems with the transmission property, we have

$$\mathrm{ind}\, \sigma_\partial(A) \in \pi_1^* K(Y), \tag{3.2.3}$$

cf. relation (0.0.7).

Given an elliptic symbol $a_{(\mu)} \in S^{(\mu)}(T^*X \setminus 0; E, F)$ we may ask, whether to a given weight $\gamma \in \mathbb{R}$ there is an SL–elliptic $\mathcal{A} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ for a suitable choice of bundles $J_\pm \in \mathrm{Vect}(Y)$ such that $a_{(\mu)} = \sigma_\psi(\mathcal{A})$.

Theorem 3.2.1 *Let $a_{(\mu)} \in S^{(\mu)}(T^*X \setminus 0; E, F)$ be elliptic, let $\gamma \in \mathbb{R}$, and let $A := \mathrm{op}_\gamma(a_{(\mu)})$ be chosen in such a way that* (3.1.6) *is a family of Fredholm operators. Then the following conditions are equivalent:*

(i) *There exists an SL–elliptic $\mathcal{A} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ for $\boldsymbol{v} = (E, F; J_-, J_+)$ for suitable $J_-, J_+ \in \mathrm{Vect}(Y)$ such that $a_{(\mu)} = \sigma_\psi(\mathcal{A})$.*

(ii) *$\sigma_\partial(A)$ has the property* (3.2.3).

Proof. After the discussion before it remains to show (ii)⇒(i). Relation (3.2.3) implies the existence of elements $J_\pm \in \mathrm{Vect}(Y)$ such that (3.2.2) holds. Similarly to Corollary 1.3.4 there exists a $g_{(\mu)}(y,\eta) \in S_G^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$ such that (in the notation of (3.1.6))

$$\ker(b_{(\mu)} + g_{(\mu)})(y,\eta) \cong J_{+,y}, \qquad \mathrm{coker}(b_{(\mu)} + g_{(\mu)})(y,\eta) \cong J_{-,y}$$

for all $(y,\eta) \in T^*Y \setminus 0$, independent of the specific s. Now the construction of the isomorphism (1.3.3) in Lemma 1.3.3 allows us to fill up the family of Fredholm operators $(b_{(\mu)} + g_{(\mu)})(y,\eta)$ to a family of isomorphisms

$$\begin{pmatrix} b_{(\mu)} + g_{(\mu)} & k_{(\mu)} \\ c_{(\mu)} & 0 \end{pmatrix} (y,\eta) : \begin{matrix} E'_y \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ J_{-,y} \end{matrix} \to \begin{matrix} F'_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \\ \oplus \\ J_{+,y} \end{matrix},$$

first for all $(y,\eta) \in S^*Y$ and then, by homogeneity of order μ (according to a relation of the kind (1.2.9)) for all $(y,\eta) \in T^*Y \setminus 0$. In this construction we may easily achieve smoothness in the variables (y,η) when we repeat the arguments in the proof of Lemma 1.3.3 for C^∞ operator functions and a parameter space M that is a C^∞ manifold. In addition, since $C_0^\infty(\mathbb{R}_+)$ is dense in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ for every $s, \gamma \in \mathbb{R}$, the potential part $k_{(\mu)}(y,\eta)$ can be chosen as a map $k_{(\mu)} : \pi_Y^* J_- \to \pi_Y^* F' \otimes C_0^\infty(\mathbb{R}_+)$, while $c_{(\mu)}(y,\eta)$ may be represented by an element in $\pi_Y^*(J_+ \oplus (E')^* \otimes C_0^\infty(\mathbb{R}_+))$ such that the map $c_{(\mu)}(y,\eta)$ is defined by an integration $\int_0^\infty (c_{(\mu)}(y,\eta)(t), u(t))_{E'_y}\, dt$, where $(\cdot,\cdot)_{E'_y}$ denotes the pairing between E'_y and its dual $(E'_y)^*$. Let us now restrict $g_{(\mu)}$, $k_{(\mu)}$, $c_{(\mu)}$ to a coordinate neighbourhood U'_j on Y and interpret the variables y as local coordinates in $\Omega \subseteq \mathbb{R}^q$ with respect to a chart $U'_j \to \Omega$. Then, if $\chi(\eta)$ is an excision function, we get operator–valued symbols

$$\begin{aligned}
g(y,\eta) &= \chi(\eta) g_{(\mu)}(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^k), \mathcal{K}^{\infty,\gamma-\mu}(\mathbb{R}_+, \mathbb{C}^k)),\\
k(y,\eta) &= \chi(\eta) k_{(\mu)}(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathbb{C}^{j_-}, \mathcal{K}^{\infty,\gamma-\mu}(\mathbb{R}_+, \mathbb{C}^k)),\\
c(y,\eta) &= \chi(\eta) c_{(\mu)}(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^k), \mathbb{C}^{j_+}),
\end{aligned}$$

for all $s \in \mathbb{R}$, where k and $j_\pm$ are the fibre dimensions of the bundles E, F and $J_\pm$, respectively. Let G_{U_j}, K_{U_j} and C_{U_j} denote the pull–backs of $\mathrm{Op}(g)$, $\mathrm{Op}(k)$ and $\mathrm{Op}(c)$ from Ω to U'_j with respect to the charts and the trivialisations of the involved bundles. Then, similarly to (2.3.4) we can pass to an operator

$$\begin{pmatrix} G & K \\ C & 0 \end{pmatrix} := \sum_{j=0}^{M} \begin{pmatrix} \varphi_j \sigma & 0 \\ 0 & \varphi_j \end{pmatrix} \begin{pmatrix} G_{U_j} & K_{U_j} \\ C_{U_j} & 0 \end{pmatrix} \begin{pmatrix} \psi_j \widetilde{\sigma} & 0 \\ 0 & \psi_j \end{pmatrix} \tag{3.2.4}$$

and set $\mathcal{A} := \begin{pmatrix} \mathrm{op}_\gamma(a_{(\mu)}) + G & K \\ C & 0 \end{pmatrix}$ which belongs to the space $\mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ for $\boldsymbol{v} = (E, F; J_-, J_+)$ where (3.2.4) equals $\sigma_\partial(\mathcal{A})$ and $a_{(\mu)} = \sigma_\psi(\mathrm{op}_\gamma(a_{(\mu)}) + G)$. □

Remark 3.2.2 *It can also be proved that there is an SL–elliptic $\mathcal{A}_1 \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{v}_1)_\varepsilon$ such that $A = \mathrm{op}_\gamma(a_{(\mu)})$ equals the upper left corner of $\mathcal{A}_1$, where $\boldsymbol{v}_1 = (E; F; J_{-,1}, J_{+,1})$ for a suitable choice of $J_{\pm,1} \in \mathrm{Vect}(Y)$. It suffices to set $J_{-,1} = \mathbb{C}^{N_-}$ for a sufficiently large N_- and to choose some homogeneous potential symbol $k_{(\mu),1} : \pi_Y^* \mathbb{C}^{N_-} \to \pi_Y^* F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$ such that*

$$\begin{pmatrix} b_{(\mu)} & k_{(\mu),1} \end{pmatrix} : \pi_Y^* \begin{pmatrix} E' \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{pmatrix} \to \pi_Y^* F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \tag{3.2.5}$$

is surjective. For sufficiently large N_- this is possible, and then $\ker \begin{pmatrix} b_{(\mu)} & k_{(\mu),1} \end{pmatrix} \in \mathrm{Vect}(Y)$ *can be taken as a copy of $J_{+,1}$. Finally, (3.2.5) can be filled up by a second row $\begin{pmatrix} c_{(\mu),1} & q_{(\mu),1} \end{pmatrix}$ to a block matrix isomorphism that plays the role of $\sigma_\partial(\mathcal{A}_1)$. Then we can pass to a corresponding operator $\mathcal{A}_1$ by a similar scheme as in the proof of Theorem* 3.2.1.

Proposition 3.2.3 *Let* $\widetilde{\mathrm{op}}_\gamma : S^{(\mu)}(T^*X \setminus 0; E, F) \to L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$ *be another choice of an operator convention in the sense of* (3.1.1), *let* $a_{(\mu)} \in S^{(\mu)}(T^*X \setminus 0; E, F)$ *be elliptic, set* $A = \mathrm{op}_\gamma(a_{(\mu)})$, $\widetilde{A} = \widetilde{\mathrm{op}}_\gamma(a_{(\mu)})$, *and assume that both* $\sigma_\partial(A)$ *and* $\sigma_\partial(\widetilde{A})$ *are families of Fredholm operators* $E'_y \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to F'_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$, $(y,\eta) \in T^*Y \setminus 0$. *Then we have* $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in \pi_1^* K(Y)$ *if and only if* $\mathrm{ind}_{S^*Y}\, \sigma_\partial(\widetilde{A}) \in \pi_1^* K(Y)$.

Proof. $\sigma_\partial(A)(y,\eta)$ can be written

$$\sigma_\partial(A)(y,\eta) = \sigma_\partial(a_\psi)(y,\eta) + \sigma_\partial(m+g)(y,\eta),$$

where $\sigma_\partial(a_\psi)$ is of the form (2.3.12), $\sigma_\partial(m)$ of the form (2.3.11), and $\sigma_\partial(g) \in S_G^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$. Similarly, we have

$$\sigma_\partial(\widetilde{A})(y,\eta) = \sigma_\partial(\widetilde{a}_\psi)(y,\eta) + \sigma_\partial(\widetilde{m} + \widetilde{g})(y,\eta).$$

By virtue of $\sigma_\partial(a_\psi) - \sigma_\partial(\widetilde{a}_\psi) \in S_{M+G}^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$, without loss of generality we may assume $\sigma_\partial(a_\psi) = \sigma_\partial(\widetilde{a}_\psi)$. In addition, since elements of $S_G^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$ represent families of compact operators, the property of $\mathrm{ind}\, \sigma_\partial(A)$ or $\mathrm{ind}\, \sigma_\partial(\widetilde{A})$ to belong to $\pi_1^* K(Y)$ is not affected by a Green summand. Therefore, $\sigma_\partial(g)$ and $\sigma_\partial(\widetilde{g})$ may be ignored.

There exists an $N_- \in \mathbb{N}$ and an injective homomorphism

$$k_{(\mu)} : \pi_1^* \mathbb{C}^{N_-} \to \pi_1^* F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+),$$

$(y,\eta) \in S^*Y$ (point–wise mapping to $F'_y \otimes C_0^\infty(\mathbb{R}_+)$) such that both

$$\begin{pmatrix} \sigma_\partial(A) & k_{(\mu)} \end{pmatrix} : \pi_1^* \begin{pmatrix} E' \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{pmatrix} \to \pi_1^* F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \tag{3.2.6}$$

and

$$\begin{pmatrix} \sigma_\partial(\widetilde{A}) & k_{(\mu)} \end{pmatrix} : \pi_1^* \begin{pmatrix} E' \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{pmatrix} \to \pi_1^* F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \tag{3.2.7}$$

are surjective (as usual, the choice of s is unessential).

Set $b_{(\mu)} := \begin{pmatrix} \sigma_\partial(A) & k_{(\mu)} \end{pmatrix}$ and $\widetilde{b}_{(\mu)} := \begin{pmatrix} \sigma_\partial(\widetilde{A}) & k_{(\mu)} \end{pmatrix}$. Observe that the property $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in \pi_1^* K(Y)$ is equivalent to the fact that for sufficiently large N_- the bundle $L_+ := \ker b_{(\mu)} \in \mathrm{Vect}(S^*Y)$ may be represented by a system of trivialisations where the transition isomorphisms only depend on y, not on the covariable η. Clearly, we have $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in \pi_1^* K(Y) \Leftrightarrow \mathrm{ind}_{S^*Y}\, b_{(\mu)} \in \pi_1^* K(Y)$, and the same for the operator families with tilde.

Let $\widetilde{b}_{(\mu)}^{-1}(y,\eta)$ denote a right inverse of $\sigma_\partial(\widetilde{A})(y,\eta)$; it can be calculated within our class of boundary symbols. Then we have $b_{(\mu)}(y,\eta)\widetilde{b}_{(\mu)}^{-1}(y,\eta) - 1 =: \sigma_\partial(n)(y,\eta) + l_{(0)}(y,\eta) \in S_{M+G}^{(0)}(T^*Y \setminus 0, (\gamma-\mu, \gamma-\mu); F', F')_\varepsilon|_{S^*Y}$, where $n(y,\eta)$ is a smoothing Mellin family of a similar form as (2.3.11) (for $\mu = 0$) and $l_{(0)}(y,\eta) \in S_G^{(0)}(T^*Y \setminus 0, (\gamma-\mu, \gamma-\mu); F', F')_\varepsilon|_{S^*Y}$. We have

$$\mathrm{ind}_{S^*Y}(1 + \sigma_\partial(n) + l_{(0)}) = \mathrm{ind}_{S^*Y}(1 + \sigma_\partial(n)) \in \pi_1^* K(Y),$$

because $L_{(0)}$ takes values in compact operators and $\sigma_\partial(n)(y,\eta)$ is independent of η on S^*Y. From

$$\mathrm{ind}_{S^*Y}(\widetilde{b}_{(\mu)}) = \mathrm{ind}_{S^*Y}(b_{(\mu)}) - \mathrm{ind}_{S^*Y}(1 + l_{(0)})$$

we then immediately get the assertion. □

Proposition 3.2.4 *Let $\gamma, \widetilde{\gamma} \in \mathbb{R}$, and let $\mathrm{op}_{\widetilde{\gamma}} : S^{(\mu)}(T^*X\backslash 0; E, F) \to L^\mu(X, \widetilde{\boldsymbol{g}}; E, F)_{\widetilde{\varepsilon}}$ for $\widetilde{\boldsymbol{g}} = (\widetilde{\gamma}, \widetilde{\gamma} - \mu)$, be similarly defined as* (3.1.1) *for γ (with arbitrary $\varepsilon, \widetilde{\varepsilon} > 0$). Let $a_{(\mu)} \in S^{(\mu)}(T^*X \setminus 0; E, F)$ be elliptic, set $A_\gamma := \mathrm{op}_\gamma(a_{(\mu)})$, $A_{\widetilde{\gamma}} := \mathrm{op}_{\widetilde{\gamma}}(a_{(\mu)})$, and assume that*

$$\sigma_\partial(A_\gamma)(y,\eta) : E'_y \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to F'_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$$

and

$$\sigma_\partial(A_{\widetilde{\gamma}})(y,\eta) : E'_y \otimes \mathcal{K}^{s,\widetilde{\gamma}}(\mathbb{R}_+) \to F'_y \otimes \mathcal{K}^{s-\mu,\widetilde{\gamma}-\mu}(\mathbb{R}_+)$$

are families of Fredholm operators, $(y,\eta) \in T^*Y\backslash 0$. Then we have $\mathrm{ind}_{S^*Y}\,\sigma_\partial(A_\gamma) \in \pi_1^*K(Y)$ if and only if $\mathrm{ind}_{S^*Y}\,\sigma_\partial(A_{\widetilde{\gamma}}) \in \pi_1^*K(Y)$.

Proof. Starting from our operators

$$A_\gamma : \mathcal{W}^{s,\gamma}(X,E) \to \mathcal{W}^{s-\mu,\gamma-\mu}(X,F), \quad A_{\widetilde{\gamma}} : \mathcal{W}^{s-\gamma+\widetilde{\gamma},\widetilde{\gamma}}(X,E) \to \mathcal{W}^{s-\gamma+\widetilde{\gamma},\widetilde{\gamma}-\mu}(X,F)$$

that are continuous for all s we pass to $\widetilde{A}_\gamma := (R_F^{\gamma-\widetilde{\gamma}})^{-1} A_{\widetilde{\gamma}} R_E^{\gamma-\widetilde{\gamma}} \in L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$, using the operators from Theorem 2.3.8. We then have $\sigma_\psi(A_\gamma) = \sigma_\psi(\widetilde{A}_\gamma) = a_{(\mu)}$, and hence, setting $A := A_\gamma$, $\widetilde{A} := \widetilde{A}_\gamma$, the boundary symbols $\sigma_\partial(A)$ and $\sigma_\partial(\widetilde{A})$ satisfy the assumptions of Proposition 3.2.3. To complete the proof it suffices to note that relation $\mathrm{ind}_{S^*Y}\,\sigma_\partial(\widetilde{A}_\gamma) \in \pi_1^*K(Y)$ is equivalent to $\mathrm{ind}_{S^*Y}\,\sigma_\partial(A_{\widetilde{\gamma}}) \in \pi_1^*K(Y)$. □

We now pass to boundary value problems with global projection conditions.

Let us fix $\boldsymbol{v} := (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$ for $E, F \in \mathrm{Vect}(X)$ and $\boldsymbol{L}_\pm := (P_\pm, J_\pm, L_\pm)$, cf. Section 1.2, (ii).

Definition 3.2.5 *The space $\mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ for $\boldsymbol{g} = (\gamma, \gamma-\mu)$, $\gamma, \mu \in \mathbb{R}$, $\varepsilon > 0$, is defined to be the space of all operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(X,E) \\ \oplus \\ P^s(Y, \boldsymbol{L}_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(X,F) \\ \oplus \\ P^{s-\mu}(Y, \boldsymbol{L}_+) \end{array}, \tag{3.2.8}$$

$s \in \mathbb{R}$, such that

(i) *$A := \mathrm{u.l.c.}\,\mathcal{A} \in L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$, cf. Section 3.2,*

(ii) *(3.2.8) can be written as a composition*

$$\mathcal{A} = \mathcal{P}_+ \widetilde{\mathcal{A}} \mathcal{R}_- \tag{3.2.9}$$

for an $\widetilde{\mathcal{A}} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{b})_\varepsilon$, $\boldsymbol{b} := (E, F; J_-, J_+)$,

where the operators $\mathcal{P}_+$ and $\mathcal{R}_-$ have the same meaning as in Section 1.2 above.

Let $\mathcal{T}^{\mu}_{M+G}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$ ($\mathcal{T}^{\mu}_{G}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$) denote the subspace of all $\mathcal{A} \in \mathcal{T}^{\mu}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$ auch that $\widetilde{\mathcal{A}} \in \mathcal{L}^{\mu}_{M+G}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$ ($\mathcal{L}^{\mu}_{G}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$) in the representation (3.2.9). Moreover, let $T^{\mu}(X,\boldsymbol{g};E,F)_{\varepsilon}$, $T^{\mu}_{M+G}(X,\boldsymbol{g};E,F)_{\varepsilon}$ and $T^{\mu}_{G}(X,\boldsymbol{g};E,F)_{\varepsilon}$ be the spaces of upper left corners in the respective classes of 2×2–block matrices. Finally, $\mathcal{T}^{-\infty}_{G}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$ will denote the set of all operators (3.2.9) where $\widetilde{A} \in \mathcal{L}^{-\infty}_{G}(X,\boldsymbol{g};\boldsymbol{b})_{\varepsilon}$, and we write $T^{-\infty}_{G}(X,\boldsymbol{g};E,F)_{\varepsilon}$ for the corresponding space of upper left corners.

The principal symbolic structure of $\mathcal{T}^{\mu}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$ consists of pairs

$$\sigma(\mathcal{A}) := (\sigma_{\psi}(\mathcal{A}), \sigma_{\partial}(\mathcal{A})),$$

where $\sigma_{\psi}(\mathcal{A}) := \sigma_{\psi}(A)$ for $A := \mathrm{u.l.c.}\,\mathcal{A}$ is the standard homogeneous principal symbol of the upper left corner

$$\sigma_{\psi}(\mathcal{A}) : \pi^*_X E \to \pi^*_X F, \tag{3.2.10}$$

$\pi_X : T^*X \setminus 0 \to X$, while $\sigma_{\partial}(\mathcal{A})$ is the homogeneous principal boundary symbol

$$\sigma_{\partial}(\mathcal{A}) : \begin{array}{c} \pi^*_Y E' \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ L_- \end{array} \to \begin{array}{c} \pi^*_Y F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \\ \oplus \\ L_+ \end{array} \tag{3.2.11}$$

$\pi_Y : T^*Y \setminus 0 \to Y$, given as the composition

$$\sigma_{\partial}(\mathcal{A})(y,\eta) := \begin{pmatrix} 1 & 0 \\ 0 & p_+(y,\eta) \end{pmatrix} \sigma_{\partial}(\widetilde{\mathcal{A}})(y,\eta) \begin{pmatrix} 1 & 0 \\ 0 & r_-(y,\eta) \end{pmatrix},$$

where $\sigma_{\partial}(\widetilde{\mathcal{A}})$ is the boundary symbol of $\widetilde{\mathcal{A}}$, cf. formula (3.2.9), $p_+(y,\eta)$ is the homogeneous principal symbol of P_+ and $r_-(y,\eta) : L_{-,(y,\eta)} \to (\pi^*_Y J_-)_{(y,\eta)}$ the canonical embedding.

Proposition 3.2.6 *Let $\mathcal{A} \in \mathcal{T}^{\mu}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$ and $\sigma(\mathcal{A}) = 0$. Then we have $\mathcal{A} \in \mathcal{T}^{\mu-1}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$, and $\mathcal{A}$ as an operator* (3.2.8) *is compact for every $s \in \mathbb{R}$.*

Proof. Let us write $\mathcal{A}$ in the form (3.2.9) for an $\widetilde{\mathcal{A}} \in \mathcal{L}^{\mu}(X,\boldsymbol{g};\boldsymbol{b})_{\varepsilon}$. If we set

$$\widetilde{\mathcal{A}}_0 := \begin{pmatrix} 1 & 0 \\ 0 & p_+ \end{pmatrix} \widetilde{\mathcal{A}} \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix}$$

we also have $\mathcal{A} = \mathcal{P}_+ \widetilde{\mathcal{A}}_0 \mathcal{R}_-$, and $\sigma(\mathcal{A}) = 0$ implies $\sigma(\widetilde{\mathcal{A}}_0) = 0$; the latter symbol refers to $\mathcal{L}^{\mu}(X,\boldsymbol{g};\boldsymbol{b})_{\varepsilon}$. This gives us $\widetilde{\mathcal{A}}_0 \in \mathcal{L}^{\mu-1}(X,\boldsymbol{g};\boldsymbol{b})_{\varepsilon}$, and hence $\mathcal{A} \in \mathcal{T}^{\mu-1}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon}$. The compactness of (3.2.8) follows from the compactness of $\widetilde{\mathcal{A}}_0$ in the usual Sobolev spaces. $\square$

For $\boldsymbol{g} = (\gamma, \gamma - \mu)$ we define inductively

$$\mathcal{T}^{\mu-j}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon} := \{\mathcal{A} \in \mathcal{T}^{\mu-(j-1)}(X,\boldsymbol{g};\boldsymbol{v})_{\varepsilon} : \sigma(\mathcal{A}) = 0\},$$

$j = 1, 2, \cdots$, where $\sigma(\cdot)$ in the latter notation indicates the pair of principal symbols of order $\mu - (j-1)$.

Proposition 3.2.7 *Let* $\boldsymbol{g} = (\gamma, \gamma - \mu)$, *and let* $\mathcal{A}_j \in \mathcal{T}^{\mu-j}(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$, $j \in \mathbb{N}$, *be an arbitrary sequence. Then there exists an* $\mathcal{A} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ *unique* $\mathrm{mod}\mathcal{T}_G^{-\infty}(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$, *such that* $\mathcal{A} \sim \sum_{j=0}^\infty \mathcal{A}_j$, *i.e.*,

$$\mathcal{A} - \sum_{j=0}^{\mathcal{N}} \mathcal{A}_j \in \mathcal{T}^{\mu-(\mathcal{N}+1)}(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$$

for every $\mathcal{N} \in \mathbb{N}$.

The proof is an easy consequence of a corresponding result for the space $\mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{b})_\varepsilon$.

Consider any $\mathcal{A} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ for $\boldsymbol{v} = (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$ in the notation of Definition 3.2.5 and an operator $\mathcal{B} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{w})_\varepsilon$ for $\boldsymbol{w} := (V, W; \boldsymbol{M}_-, \boldsymbol{M}_+)$, $V, W \in \mathrm{Vect}(X)$, triples $\boldsymbol{M}_\pm = (Q_\pm, G_\pm, M_\pm)$ with $G_\pm \in \mathrm{Vect}(Y)$, $M_\pm \in \mathrm{Vect}(T^*Y \setminus 0)$, and projections $Q_\pm \in L^0_{\mathrm{cl}}(Y; G_\pm, G_\pm)$ that have corresponding principal symbols. There is then a direct sum

$$\mathcal{A} \oplus \mathcal{B} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v} \oplus \boldsymbol{w})_\varepsilon,$$

defined in a canonical way, where $\boldsymbol{v} \oplus \boldsymbol{w} := (E \oplus V, F \oplus W; \boldsymbol{L}_- \oplus \boldsymbol{M}_-, \boldsymbol{L}_+ \oplus \boldsymbol{M}_+)$ for

$$\boldsymbol{L}_- \oplus \boldsymbol{M}_- := (P_- \oplus Q_-, J_- \oplus G_-, L_- \oplus M_-)$$

and, similarly, $\boldsymbol{L}_+ \oplus \boldsymbol{M}_+$. Then

$$\mathcal{A} \oplus \mathcal{B} : \begin{array}{c} \mathcal{W}^{s,\gamma}(X, E \oplus V) \\ \oplus \\ P^s(Y, \boldsymbol{L}_- \oplus \boldsymbol{M}_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(X, F \oplus W) \\ \oplus \\ P^{s-\mu}(Y, \boldsymbol{L}_+ \oplus \boldsymbol{M}_+) \end{array}$$

is continuous for all $s \in \mathbb{R}$.

Theorem 3.2.8 $\mathcal{A} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ *for* $\boldsymbol{g} = (\gamma - \nu, \gamma - (\mu + \nu))$, $\boldsymbol{v} = (E_0, F; \boldsymbol{L}_0, \boldsymbol{L}_+)$, *and* $\mathcal{B} \in \mathcal{T}^\nu(X, \boldsymbol{h}; \boldsymbol{w})_{\tilde{\varepsilon}}$ *for* $\boldsymbol{h} = (\gamma, \gamma - \nu)$, $\boldsymbol{w} = (E, E_0; \boldsymbol{L}_-, \boldsymbol{L}_0)$ *implies* $\mathcal{A}\mathcal{B} \in \mathcal{T}^{\mu+\nu}(X, \boldsymbol{g} \circ \boldsymbol{h}; \boldsymbol{v} \circ \boldsymbol{w})_{\min(\varepsilon, \tilde{\varepsilon})}$ *for* $\boldsymbol{g} \circ \boldsymbol{h} = (\gamma, \gamma - (\mu + \nu))$, $\boldsymbol{v} \circ \boldsymbol{w} = (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$, *and we have*

$$\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

with componentwise multiplication. If $\mathcal{A}$ *or* $\mathcal{B}$ *belongs to the subspace with subscript* $M + G$ (G), *then the same is true of the composition.*

Proof. The assertion is an immediate consequence of Theorem 2.3.4 and of Definition 3.2.5. □

Definition 3.2.9 *An operator* $\mathcal{A} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ *for* $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\boldsymbol{v} = (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$, $\varepsilon > 0$, *is called elliptic, if both* (3.2.10) *and* (3.2.11) *are isomorphisms.*

Remark 3.2.10 *The condition that* (3.2.11) *is an isomorphism is independent of* s. *If it is satisfied for an* $s = s_0 \in \mathbb{R}$ *then so is for all* $s \in \mathbb{R}$.

Theorem 3.2.11 *Let* $a_{(\mu)} \in S^{(\mu)}(T^*X \setminus 0; E, F)$ *be an arbitrary elliptic element. Then for every* $\gamma \in \mathbb{R}$ *there exist triples* $\boldsymbol{L}_\pm = (P_\pm, J_\pm, L_\pm)$, $\boldsymbol{L}_\pm = \boldsymbol{L}_\pm(\gamma)$, *and an elliptic operator* $\mathcal{A} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ *for some* $\varepsilon > 0$, $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\boldsymbol{v} = (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$, *such that* $\sigma_\psi(\mathcal{A}) = a_{(\mu)}$.

Proof. According to Theorem 3.1.3 we choose an operator $A_\gamma = \mathrm{op}_\gamma(a_{(\mu)}) \in L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$ for some $\varepsilon > 0$ such that $b_{(\mu)}(y,\eta) := \sigma_\partial(A_\gamma)(y,\eta)$ represents a family of Fredholm operators (3.1.6).

Choose elements $L_\pm \in \mathrm{Vect}(S^*Y)$ such that $[L_+] - [L_-] = \mathrm{ind}_{S^*Y}(b_{(\mu)})$. Then, similarly to the proof of Theorem 3.2.1 we find a family of isomorphisms

$$\begin{pmatrix} b_{(\mu)} + g_{(\mu)} & k_{(\mu)} \\ c_{(\mu)} & 0 \end{pmatrix}(y,\eta) : \begin{array}{c} E'_y \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ L_{-,(y,\eta)} \end{array} \to \begin{array}{c} F'_y \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \\ \oplus \\ L_{+,(y,\eta)} \end{array} \tag{3.2.12}$$

where $g_{(\mu)} \in S_G^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; E', F')_\varepsilon$ is a suitably chosen Green boundary symbol such that

$$L_+ \cong \ker(b_{(\mu)} + g_{(\mu)}), \qquad L_- \cong \mathrm{coker}(b_{(\mu)} + g_{(\mu)}),$$

while $k_{(\mu)}$ and $c_{(\mu)}$ are smooth families that are (y,η)–wise of the type of potential and trace symbols, where

$$k_{(\mu)}(y, \lambda\eta) = \lambda^\mu \kappa_\lambda k_{(\mu)}(y,\eta), \qquad c_{(\mu)}(y,\lambda\eta) = \lambda^\mu c_{(\mu)}(y,\eta)\kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in T^*Y \setminus 0$. For convenience, bundles $L_\pm$ on S^*Y will be identified with their pull–backs to $T^*Y \setminus 0$ under the canonical projection $(y,\eta) \to (y, \eta/|\eta|)$; we hope that this does not cause confusions. Choose arbitrary bundles $J_\pm \in \mathrm{Vect}(Y)$ such that $L_\pm$ are subbundles of $\pi_1^* J_\pm$. From (3.2.12) we can pass to a homomorphism

$$\begin{pmatrix} b_{(\mu)} + g_{(\mu)} & \widetilde{k}_{(\mu)} \\ \widetilde{c}_{(\mu)} & 0 \end{pmatrix} : \pi_Y^* \begin{pmatrix} E' \otimes \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} F' \otimes \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix} \tag{3.2.13}$$

by extending $k_{(\mu)}$ to $\widetilde{k}_{(\mu)}$ by zero on a complementary bundle $L_-^\perp$ to L_- in $\pi_Y^* J_-$, while $\widetilde{c}_{(\mu)}$ is defined by composing $c_{(\mu)}$ with the embedding $L_+ \to \pi_Y^* J_+$.

As in the proof of Theorem 3.2.1 we can form an operator $\widetilde{\mathcal{A}} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{b})_\varepsilon$ for $\boldsymbol{b} = (E, F; J_-, J_+)$ that has (3.2.13) as its principal boundary symbol. In addition, the projections $\pi_Y^* J_\pm \to L_\pm$ along complementary bundles $L_\pm^\perp$ of $L_\pm$ in $\pi_Y^* J_\pm$ can be interpreted as principal symbols of projections $P_\pm \in L^0_{\mathrm{cl}}(Y; J_\pm, J_\pm)$. Then, forming $\mathcal{A}$ by formula (3.2.12), where the operators $\mathcal{P}_+, \mathcal{R}_-$ are as Section 1.2, we get an elliptic element $\mathcal{A} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$ for $\boldsymbol{v} = (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$ $\boldsymbol{L}_\pm = (P_\pm, J_\pm, L_\pm)$, where, in particular, $\sigma_\psi(\mathcal{A}) = a_{(\mu)}$. □

Theorem 3.2.12 *For every elliptic operator $\mathcal{A} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{v})_\varepsilon$, $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\boldsymbol{v} = (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$, there exists an elliptic operator $\mathcal{B} \in \mathcal{T}^\mu(X, \boldsymbol{g}; \boldsymbol{w})_\varepsilon$ for $\boldsymbol{w} = (F, E; \boldsymbol{M}_-, \boldsymbol{M}_+)$ with suitable triples $\boldsymbol{M}_\pm = (Q_\pm, \mathbb{C}^N, M_\pm)$ with projections $Q_\pm \in L^0_{\mathrm{cl}}(Y; \mathbb{C}^N, \mathbb{C}^N)$ and $M_\pm \in \mathrm{Vect}(S^*Y)$, such that $\mathcal{A} \oplus \mathcal{B} \in \mathcal{L}^\mu(X, \boldsymbol{g}; \boldsymbol{b})_\varepsilon$ for $\boldsymbol{b} = (E \oplus F, F \oplus E; \mathbb{C}^N, \mathbb{C}^N)$ is SL–elliptic.*

Proof. Let $A := \mathrm{u.l.c.}\,\mathcal{A}$ which belongs to $L^\mu(X, \boldsymbol{g}; E, F)_\varepsilon$ and form the formal adjoint A^* that is an element in $L^\mu(X, \boldsymbol{g}^*; F, E)_\varepsilon$ for $\boldsymbol{g}^* = (-\gamma + \mu, -\gamma)$. The definition of A^* is based on the relation

$$(Au, v)_{\mathcal{W}^{0,0}(X,F)} = (u, A^* v)_{\mathcal{W}^{0,0}(X,E)}$$

for all $u \in C_0^\infty(\operatorname{int} X, E)$, $v \in C_0^\infty(\operatorname{int} X, F)$ which is compatible with the (y,η)–wise formal adjoint on the level of principal boundary symbols

$$(\sigma_\partial(A)(y,\eta)f, g)_{\mathcal{K}^{0,0}(\mathbb{R}_+,\mathbb{C}^k)} = (f, \sigma_\partial(A^*)(y,\eta)g)_{\mathcal{K}^{0,0}(\mathbb{R}_+,\mathbb{C}^k)},$$

where $\sigma_\partial(A^*)(y,\eta)$ defines a bundle homomorphism

$$\sigma_\partial(A^*) : \pi_Y^* F' \otimes \mathcal{K}^{s,-\gamma+\mu}(\mathbb{R}_+) \to \pi_Y^* E' \otimes \mathcal{K}^{s-\mu,-\gamma}(\mathbb{R}_+)$$

that is Fredholm for every $s \in \mathbb{R}$ where

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A^*) = -\operatorname{ind}_{S^*Y} \sigma_\partial(A).$$

Choosing any N such that $L_\pm \in \operatorname{Vect}(S^*Y)$ have complementary bundles $M_\pm \in \operatorname{Vect}(S^*Y)$ in $\mathbb{C}^N$, i.e., $L_\pm \oplus M_\pm = \mathbb{C}^N$, we have

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A^*) = [M_+] - [M_-].$$

From Theorem 2.3.8 we have order and weight reducing isomorphisms

$$R_E^{\mu-2\gamma} : \mathcal{W}^{s,-\gamma}(X,E) \to \mathcal{W}^{s-\mu+2\gamma,\gamma-\mu}(X,E)$$

and

$$R_F^{\mu-2\gamma} : \mathcal{W}^{s,-\gamma+\mu}(X,F) \to \mathcal{W}^{s-\mu+2\gamma,\gamma}(X,F).$$

Let us pass from A^* to the operator $B := R_E^{\mu-2\gamma} A^* (R_F^{\mu-2\gamma})^{-1} \in L^\mu(X,\boldsymbol{g};F,E)_\varepsilon$ that has the property

$$\operatorname{ind}_{S^*Y} \sigma_\partial(B) = \operatorname{ind}_{S^*Y} \sigma_\partial(A^*).$$

As in the proof of Theorem 3.2.11 we find an element $g_{(\mu)} \in S_G^{(\mu)}(T^*Y \setminus 0, \boldsymbol{g}; F, E)_\varepsilon$ such that

$$M_+ \cong \ker(\sigma_\partial(B) + g_{(\mu)}), \qquad M_- \cong \operatorname{coker}(\sigma_\partial(B) + g_{(\mu)}),$$

and we can form triples $\boldsymbol{M}_\pm = (Q_\pm, \mathbb{C}^N, M_\pm)$ with a choice of projections $Q_\pm \in L^0_{\mathrm{cl}}(Y;\mathbb{C}^N, M_\pm)$ having the projections $\mathbb{C}^N \to M_\pm$ along $L_\pm$ as principal symbols. The constructions of the proof of Theorem 3.2.11 then give us an elliptic operator $\mathcal{B} \in \mathcal{T}^\mu(X,\boldsymbol{g};\boldsymbol{w})_\varepsilon$ with the desired properties. □

Remark 3.2.13 *Setting* $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{B}$ *we have*

$$\mathcal{A} = \mathcal{P}_+ \widetilde{\mathcal{A}} \mathcal{R}_- \qquad \text{and} \qquad \mathcal{B} = \mathcal{P}_+^\perp \widetilde{\mathcal{A}} \mathcal{R}_-^\perp$$

where $\mathcal{P}_+ = \operatorname{diag}(1, P_+)$ *and* $\mathcal{R}_- = \operatorname{diag}(1, R_-)$ *are as in* (3.2.7), *while* $\mathcal{P}_+^\perp = \operatorname{diag}(1, Q_+)$, $\mathcal{R}_-^\perp = \operatorname{diag}(1, R_-^\perp)$, *with* $R_-^\perp$ *being the canonical embedding* $P^s(Y, \boldsymbol{M}_-) \to H^s(Y, \mathbb{C}^N)$.

Definition 3.2.14 *Let* $\mathcal{A} \in \mathcal{T}^\mu(X,\boldsymbol{g};\boldsymbol{v})_\varepsilon$, $\boldsymbol{g} = (\gamma, \gamma-\mu)$, $\boldsymbol{v} = (E, F; \boldsymbol{L}_-, \boldsymbol{L}_+)$, $\varepsilon > 0$. *An operator* $\mathcal{P} \in \mathcal{T}^{-\mu}(X, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})_{\widetilde{\varepsilon}}$ *for* $\boldsymbol{g}^{-1} = (\gamma-\mu, \gamma)$, $\boldsymbol{v}^{-1} = (F, E; \boldsymbol{L}_+, \boldsymbol{L}_-)$, $\widetilde{\varepsilon} > 0$, *is called a parametrix of* $\mathcal{A}$, *if the operators*

$$\mathcal{C}_l := \mathcal{I} - \mathcal{P}\mathcal{A} \qquad \text{and} \qquad \mathcal{C}_r := \mathcal{I} - \mathcal{A}\mathcal{P} \tag{3.2.14}$$

belong to $\mathcal{T}_G^{-\infty}(X, \boldsymbol{g}_l; \boldsymbol{v}_l)_{\min(\varepsilon,\widetilde{\varepsilon})}$ *and* $\mathcal{T}_G^{-\infty}(X, \boldsymbol{g}_r; \boldsymbol{v}_r)_{\min(\varepsilon,\widetilde{\varepsilon})}$, *respectively, for* $\boldsymbol{g}_l = (\gamma,\gamma)$, $\boldsymbol{v}_l = (E, E; \boldsymbol{L}_-, \boldsymbol{L}_-)$ *and* $\boldsymbol{g}_r = (\gamma-\mu, \gamma-\mu)$, $\boldsymbol{v}_r = (F, F; \boldsymbol{L}_+, \boldsymbol{L}_+)$.

Theorem 3.2.15 *Every elliptic operator* $\mathcal{A} \in \mathcal{T}^{\mu}(X, \boldsymbol{g}; \boldsymbol{v})_{\varepsilon}$ *has a parametrix* $\mathcal{P} \in \mathcal{T}^{-\mu}(X, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})_{\tilde{\varepsilon}}$, *(cf. the notation in Definition* 3.2.9 *and* 3.2.14.)

Proof. Let us apply Theorem 3.2.12 to $\mathcal{A}$ and form $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{B} \in \mathcal{L}^{\mu}(X, \boldsymbol{g}; \boldsymbol{b})_{\varepsilon}$ with some complementary elliptic operator $\mathcal{B}$. By virtue of Theorem 2.3.6 we have a parametrix $\widetilde{\mathcal{P}} \in \mathcal{L}^{-\mu}(X, \boldsymbol{g}^{-1}; \boldsymbol{b}^{-1})_{\tilde{\varepsilon}}$ where $\sigma(\widetilde{\mathcal{P}}) = \sigma^{-1}(\widetilde{\mathcal{A}})$. Let us set

$$\mathcal{P}_0 = \operatorname{diag}(1, P_-)\widetilde{\mathcal{P}} \operatorname{diag}(1, R_+),$$

where $R_+ : P^{s-\mu}(Y, \boldsymbol{L}_+) \to H^{s-\mu}(Y, J_+)$ is the canonical embedding and $P_- \in L^0_{\mathrm{cl}}(Y; J_-, L_-)$ the projection involved in $\boldsymbol{L}_-$. Then we have

$$\mathcal{P}_0\mathcal{A} = \operatorname{diag}(1, P_-)\widetilde{\mathcal{P}} \operatorname{diag}(1, \mathcal{P}_+)\widetilde{\mathcal{A}} \operatorname{diag}(1, R_-).$$

It follows that for $\mathcal{E}_l := \mathcal{I} - \mathcal{P}_0\mathcal{A} \in \mathcal{T}^0(X, \boldsymbol{g}_l; \boldsymbol{v}_l)_{\tilde{\varepsilon}}$ we have $\sigma(\mathcal{E}_l) = 0$, i.e., $\mathcal{E}_l \in \mathcal{T}^{-1}(X, \boldsymbol{g}_l; \boldsymbol{v}_l)_{\tilde{\varepsilon}}$, cf. Proposition 3.2.6. Applying Proposition 3.2.7 we find an operator $\mathcal{D}_l \in \mathcal{T}^{-1}(X, \boldsymbol{g}_l; \boldsymbol{v}_l)_{\tilde{\varepsilon}}$, such that $(\mathcal{I}+\mathcal{D}_l)(\mathcal{I}-\mathcal{E}_l) = \mathcal{I} \bmod \mathcal{T}_G^{-\infty}(X, \boldsymbol{g}_l; \boldsymbol{v}_l)_{\tilde{\varepsilon}}$; it suffices to form the asymptotic sum $\mathcal{D}_l := \sum_{j=1}^{\infty} \mathcal{E}_l^j$. This yields $(\mathcal{I} + \mathcal{D}_l)\mathcal{P}_0\mathcal{A} = 1 \bmod \mathcal{T}_G^{-\infty}(X, \boldsymbol{g}_l; \boldsymbol{v}_l)_{\tilde{\varepsilon}}$, and hence $\mathcal{P}_l := \mathcal{I} + \mathcal{D}_l\mathcal{P}_0 \in \mathcal{T}^{-\mu}(X, \boldsymbol{g}^{-}1; \boldsymbol{v}^{-}1)_{\tilde{\varepsilon}}$ is a left parametrix of $\mathcal{A}$. In a similar manner we find a right parametrix; thus we may take $\mathcal{P} := \mathcal{P}_l$. □

Theorem 3.2.16 *Let* $\mathcal{A} \in \mathcal{T}^{\mu}(X, \boldsymbol{g}; \boldsymbol{v})_{\varepsilon}$ *be elliptic. Then*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(X, E) \\ \oplus \\ P^s(Y, \boldsymbol{L}_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(X, F) \\ \oplus \\ P^{s-\mu}(Y, \boldsymbol{L}_+) \end{array} \tag{3.2.15}$$

is a Fredholm operator for every $s \in \mathbb{R}$. *The parametrix* $\mathcal{P}$ *in Theorem* 3.2.15 *can be chosen in such a way that the smoothing remainders are projections of finite rank, where* $\mathcal{C}_l$ *projects to* $\ker \mathcal{A}$ *and* $\mathcal{C}_r$ *to a complement of* $\operatorname{im} \mathcal{A}$ *for every fixed* s*;* $\ker \mathcal{A}$ *is independent of* s *as well as* $\dim \operatorname{coker} \mathcal{A}$, *i.e.,* $\operatorname{ind} \mathcal{A}$ *is independent of* s.

Proof. The Fredholm property of 3.2.15 is a direct consequence of the fact that the remainders $\mathcal{E}_l$ and $\mathcal{E}_r$ in relation 3.2.14 are compact operators, cf. also Proposition 3.2.7. The second part of Theorem 3.2.16 is a consequence on generalities on elliptic operators that are always fulfilled when we have elliptic regularity in the respective scales of spaces.

□

3.3 Operators of order zero

In this section we study operators $A \in L^0(X, (0, 0); E, F)_{\varepsilon}$ and associated boundary symbols in more detail. Setting $A_0 = R_F^{-\gamma+\mu} A R_E^{-\gamma}$ we get an isomorphism

$$L^{\mu}(X, (\gamma, \gamma - \mu); E, F)_{\varepsilon} \to L^0(X, (0, 0); E, F)_{\varepsilon},$$

cf. Theorem 2.3.8.

Set $S_Y^*X := S^*X|_Y$ and

$$S^{(0)}(S_Y^*X; E', F') := \{a|_{S_Y^*X} : \ a \in S^{(0)}(T^*X \setminus 0; E, F)\}.$$

Given any $A \in L^0(X,(0,0);E,F)_\varepsilon$ such that $\sigma_\psi(A) \in S^{(0)}(T^*X\setminus 0;E,F)$ is elliptic, we form $a := \sigma_\psi(A)|_{S^*_Y X}$ and ask the Fredholm property of

$$\mathrm{op}^+(a)(y,\eta) = \mathrm{r}^+ a(y,\eta,D_t)\mathrm{e}^+ : E'_y \otimes L^2(\mathbb{R}_+) \to F'_y \otimes L^2(\mathbb{R}_+) \tag{3.3.1}$$

for $(y,\eta) \in S^*Y$. In contrast to (3.1.3) we now prefer L^2-spaces, because in the case of (possibly) violated transmission property the standard Sobolev spaces or Schwartz spaces with smoothness up to the boundary are not always respected under the operation. Define $\Xi := S^*_Y X \cup N$, where N is the trivial $[-1,+1]$–bundle on Y induced by the conormal bundle to Y, i.e., the fibres are intervals $\{\tau: \ -1 \le \tau \le 1\}$ connecting the south pole $((\eta,\tau)=(0,-1))$ with the north pole $((\eta,\tau)=(0,1))$ of $S^*_y X$, $y \in Y$.

Let us recall a criterion for the Fredholm property of (3.3.1) in terms of

$$g^\pm(z) := (1-e^{\mp 2\pi i z})^{-1}, \tag{3.3.2}$$

cf. Eskin [7]. The functions $g^\pm(z)$ are meromorphic in $z \in \mathbb{C}$ with simple poles at the real integers. Thus the lines $\Gamma_\beta = \{z \in \mathbb{C}: \ \mathrm{Re}\, z = \beta\}$ do not contain poles for $\beta \notin \mathbb{Z}$. Choose a diffeomorphism $\zeta : (-1,1) \to \Gamma_{\frac{1}{2}}$ with $\zeta(\tau) \to \pm\infty$ for $\tau \to \mp 1$. Then, setting

$$a^\pm(y) := a(y,0,\pm 1),$$

the family of homomorphisms

$$\widetilde{a}(y,\tau) := a^+(y)g^+(\zeta(\tau)) + a^-(y)g^-(\zeta(\tau)) : E'_y \to F'_y \tag{3.3.3}$$

is well–defined for $-1 \le \tau \le 1$, since $g^+(z)+g^-(z)=1$, and $g^\pm|_{\Gamma_{\frac{1}{2}}}$ strongly tends to 1 for $\mathrm{Im}\, z \to \mp\infty$. More precisely, (3.3.3) is the convex combination of the homomorphisms $a^\pm(y) : E'_y \to F'_y$.

Proposition 3.3.1 *The operators* (3.3.1) *are Fredholm for all* $(y,\eta) \in S^*Y$ *if and only if*

$$\widetilde{a}(y,\eta,\tau) = \begin{cases} (3.3.3) & \text{for } \eta = 0, \ -1 \le \tau \le 1, \\ \sigma_\psi(A)|_{S^*_Y X} & \text{for } |\eta,\tau| = 1. \end{cases} \tag{3.3.4}$$

is a family of isomorphisms

$$\widetilde{a}(y,\eta,\tau) : E'_y \to F'_y \tag{3.3.5}$$

for all $(y,\eta,\tau) \in \Xi$.

Proposition 3.3.1 is known from the theory of singular integral operators, cf. the framework of Eskin [7]. An explicit proof of the necessity of the isomorphism (3.3.5) for the Fredholm property of (3.3.3) may be found in Rempel and Schulze [12].

Recall that when $\mathrm{op}^+(a)$ stems from a symbol $\sigma_\psi(A)$ with the transmission property, we have $a^+(y) = a^-(y)$, and hence the criterion of Proposition 3.3.1 is automatically fulfilled as soon as $\sigma_\psi(A)$ is elliptic.

In general, each family of isomorphisms (3.3.5) represents an element $\sigma(\widetilde{a}) \in K(B,\Xi)$ in the relative K–group of the pair (B,Ξ) for $B := B^*X|_Y$, where B^*X is the unit ball bundle induced by T^*X.

By virtue of $K(B,\Xi) \cong K(\mathbb{R}^2 \times S^*Y)$ there is an isomorphism

$$\iota : K(B,\Xi) \to K(S^*Y)$$

via Bott periodicity.

Proposition 3.3.2 *Let $\sigma_\psi(A)$ be elliptic (of order zero), assume that $\sigma_\psi(A)|_{S_Y^*X}$ extends to Ξ as a family of isomorphisms (3.3.5), and let $\sigma(\widetilde{a}) \in K(B,\Xi)$ be the associated element. Then we have*

$$\operatorname{ind}_{S^*Y} \operatorname{op}^+(a) = \iota(\sigma(\widetilde{a}))$$

for $a(y,\xi) = \sigma_\psi(A)|_{T_Y^*X\setminus 0}$.

For symbols with the transmission property Proposition 3.3.2 is known by Boutet de Monvel [4]; a related statement for symbols of elliptic differential operators is due to Atiyah and Bott [1]. The general case (not requiring the transmission property) is treated in Rempel and Schulze [12].

Clearly, any other extension $\widetilde{\widetilde{a}}$ of $\sigma_\psi(A)|_{S_Y^*X}$ to Ξ as a family of isomorphisms

$$\widetilde{\widetilde{a}} : E'_y \to F'_y, \tag{3.3.6}$$

$(y,\eta,\tau) \in \Xi$, also represents an element $\sigma(\widetilde{\widetilde{a}}) \in K(B,\Xi)$ and hence a certain $\iota(\sigma(\widetilde{\widetilde{a}})) \in K(S^*Y)$.

It is not obvious at first glance how $\iota(\sigma(\widetilde{\widetilde{a}}))$ can be interpreted as $\operatorname{ind}_{S^*Y} b$ for some family

$$b(y,\eta) : E'_y \otimes L^2(\mathbb{R}_+) \to F'_y \otimes L^2(\mathbb{R}_+)$$

of Fredholm operators, parametrised by $(y,\eta) \in S^*Y$. But the pointwise analytic information from Eskin [7] combined with that on the structure of pseudodifferential boundary value problems not requiring the transmission property from Rempel and Schulze [12] and [18] gives us the following scenario: Let $\theta(E',F')$ denote the set of all families of homomorphisms $E'_y \to F'_y$, continuously parametrised by $(y,\eta,\tau) \in \Xi$, that vanish on S_Y^*X. Every element of $\theta(E',F')$ can be canonically identified with a continuous family of homomorphisms parametrised by $(y,\tau) \in N = Y \times [-1,1]$, vanishing on $Y \cup \{-1\} \cup \{+1\}$. We then have $(\widetilde{a}^{-1}\widetilde{\widetilde{a}})(y,\eta,\tau) = 1 + f_0(y,\tau)$ for some $f_0(y,\tau) \in \theta(E',E')$, or

$$\begin{aligned}\widetilde{\widetilde{a}}(y,\eta,\tau) &= \widetilde{a}(y,\eta,\tau)(1+f_0(y,\tau)) \\ &= \widetilde{a}(y,\eta,\tau) + f_1(y,\tau)\end{aligned}$$

for an $f_1(y,\tau) \in \theta(E',F')$. It suffices to consider elements $\widetilde{\widetilde{a}}(y,\eta,\tau)$ of the above kind such that $f_1(y,\zeta^{-1}(z))$ is a Schwartz function with respect to $z \in \Gamma_{\frac{1}{2}}$. In fact, we can obviously construct such $\widetilde{\widetilde{a}}$ from an arbitrary family $\widetilde{\widetilde{a}}_1$ of isomorphisms satisfying $\widetilde{a} - \widetilde{\widetilde{a}}_1 \in \theta(E,F')$ by a small change of $\widetilde{\widetilde{a}}_1|_N$ near $Y \cup \{-1\} \cup \{+1\}$ within the homotopy class of families of isomorphisms represented by $\widetilde{\widetilde{a}}_1$. We then have $\sigma(\widetilde{\widetilde{a}}_1) = \sigma(\widetilde{\widetilde{a}})$ and hence $\iota\sigma(\widetilde{\widetilde{a}}_1) = \iota\sigma(\widetilde{\widetilde{a}})$.

For our purposes it is even sufficient to assume a smaller class of functions on $\Gamma_{\frac{1}{2}}$. For every $\mu \in \mathbb{R}$ we define $M_{\mathcal{O}}^\mu(\Gamma_\beta)_\varepsilon$ to be the subspace of all $h(z) \in \mathcal{A}(\{z : \beta - \varepsilon < \operatorname{Re} z < \beta + \varepsilon\})$ with $h|_{\Gamma_\delta} \in S_{\mathrm{cl}}^\mu(\Gamma_\delta)$ for all $\delta \in (\beta-\varepsilon, \beta+\varepsilon)$, uniformly in $\delta \in [\beta - \varepsilon', \beta + \varepsilon']$ for arbitrary $0 < \varepsilon' < \varepsilon$. Also the space $M_{\mathcal{O}}^\mu(\Gamma_\beta)_\varepsilon$ is Fréchet, and we set $M^\mu(\Gamma_\beta) = \bigcup_{\varepsilon>0} M_{\mathcal{O}}^\mu(\Gamma_\beta)_\varepsilon$. An analogous definition makes sense for $\operatorname{Hom}(E',F')$–valued functions, $E',F' \in \operatorname{Vect}(Y)$. The corresponding spaces will be denoted by $M_{\mathcal{O}}^\mu(Y \times \Gamma_\beta; E',F')_\varepsilon$ and $M^\mu(Y \times \Gamma_\beta; E',F')$, defined to be the union of these spaces over $\varepsilon > 0$.

In the sequel $\omega(t), \widetilde{\omega}(t), \ldots$ are cut–off functions on $\mathbb{R}_+$. The following assertion is a generalisation of Propositions 3.3.1 and 3.3.2.

Proposition 3.3.3 *Let $\sigma_\psi(A)$ be elliptic (of order zero), $a(y,\xi) = \sigma_\psi(A)|_{T_Y^*X\setminus 0}$ (cf. the notation of Proposition 3.3.1), and let $l(y,z) \in M^{-\infty}(Y \times \Gamma_{\frac{1}{2}}; E', F')$ be an element such that*

$$\tilde{\tilde{a}}(y,\eta,\tau) := \begin{cases} f(y,\zeta(\tau)) & \text{for } \eta = 0,\ -1 \le \tau \le 1, \\ \sigma_\psi(A)|_{S_Y^*X} & \text{for } |\eta,\tau| = 1 \end{cases} \tag{3.3.7}$$

for $f(y,z) := a^+(y)g^+(z) + a^-(y)g^-(z) + l(y,z)$ defines a family of isomorphisms (3.3.6) *for all $(y,\eta,\tau) \in \Xi$. Then, if $\omega(t)$, $\tilde{\omega}(t)$ are arbitrary cut–off functions,*

$$r(y,\eta) := \mathrm{op}^+(a)(y,\eta) + \omega(t|\eta|)\,\mathrm{op}_M(l)\tilde{\omega}(t|\eta|) : E'_y \otimes L^2(\mathbb{R}_+) \to F'_y \otimes L^2(\mathbb{R}_+) \tag{3.3.8}$$

is a family of Fredholm operators parametrised by $(y,\eta) \in T^*Y \setminus 0$, and for its restriction to S^*Y we have

$$\mathrm{ind}_{S^*Y}(r) = \iota\sigma(\tilde{\tilde{a}}). \tag{3.3.9}$$

The Fredholm property (3.3.8) is shown in [7] (in a slightly modified form without $\tilde{\omega}$; the present formulation is given in [18] and relation (3.3.9) in [12], [11]).

Remark 3.3.4 *We have $r(y,\lambda\eta) = \kappa_\lambda r(y,\eta)\kappa_\lambda^{-1}$ for all $\lambda > 0$ for $(\kappa_\lambda u)(t) = \lambda^{\frac{1}{2}}u(\lambda t)$.*

Let

$$S_G^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); \boldsymbol{g}, \boldsymbol{v}) \qquad \text{for } \boldsymbol{g} = (\gamma,\delta),\ \boldsymbol{v} = (k,l;j_-,j_+) \tag{3.3.10}$$

denote the space of all $g_{(\mu)}(y,\eta)$ induced by Green symbols $g(y,\eta)$. This space can equivalently be defined by similar mapping properties of $g_{(\mu)}(y,\eta)$ and $g^*_{(\mu)}(y,\eta)$ as of $g(y,\eta)$ itself, now with C^∞ dependence on $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$ and homogeneity in $\eta \neq 0$. Let, in particular, $\Omega \subseteq \mathbb{R}^q$ be open, regarded as a local patch of Y, and interpret k, l and j_-, j_+ as the fibre dimensions of bundles E', F' and J_-, J_+ on Y. Then the spaces (3.3.10) behave invariantly under substitution of the transition maps to the pull–backs of the bundles under $\pi_Y : T^*Y \setminus 0 \to Y$, and we get global spaces

$$S_G^{(\mu)}(T^*Y \setminus 0; \boldsymbol{g}, \boldsymbol{b}') \qquad \text{for } \boldsymbol{g} = (\gamma,\delta),\ \boldsymbol{b}' = (E',F';J_-,J_+). \tag{3.3.11}$$

Elements $g_{(\mu)}(y,\eta)$ in the latter space can be written as 2×2–block matrices. Define the space

$$\begin{aligned} &S^{(0)}(T^*Y \setminus 0; \boldsymbol{b}') \\ &\quad = \Big\{ \left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right) + g(y,\eta) :\ r(y,\eta) \text{ is an operator family of the form (3.3.8) for} \\ &\qquad \text{some } a(y,\xi) \in S^{(0)}(T^*X \setminus 0; E,F)|_{T_Y^*X\setminus 0},\ l(y,z) \in \\ &\qquad M^{-\infty}(Y \times \Gamma_{\frac{1}{2}}; E',F'), \text{ and } g_{(0)}(y,\eta) \in S_G^{(0)}(T^*Y \setminus 0; (0,0), \boldsymbol{b}') \Big\}. \end{aligned} \tag{3.3.12}$$

Here, we suppose the bundles $E', F' \in \mathrm{Vect}(Y)$ to be restrictions of corresponding $E, F \in \mathrm{Vect}(X)$ to Y.

Proposition 3.3.5 *We have*

$$S^{(0)}(T^*Y \setminus 0; \boldsymbol{b}') = \{\sigma_\partial(\mathcal{A}) :\ \mathcal{A} \in \mathcal{L}^0(X,(0,0);\boldsymbol{b})_\varepsilon \text{ for some } \varepsilon > 0\},$$

for $\boldsymbol{b} = (E,F;J_-,J_+)$.

For $p \in S^{(0)}(T^*Y \setminus 0; \boldsymbol{b}')$, given by an expression on the right of (3.3.12), we set

$$\sigma'_\psi(p)(y,\xi) = a(y,\xi), \qquad \sigma_M(p)(y,z) = m(y,z).$$

Proof. If we are given an element $\mathcal{A} \in \mathcal{L}^0(X;(0,0);\boldsymbol{b})_\varepsilon$, the boundary symbol $\sigma_\partial(\mathcal{A})$ belongs to (3.3.12) as a consequence of the definitions. Conversely, given an element in $S^{(0)}(T^*Y \setminus 0; b')$ we want to show to that there is an associated operator. The part for the Green symbols is trivial, because Green operators are classical pseudo–differential operators with the respective homogeneous principal symbols. Thus it suffices to consider upper left corners. In other words, we consider an operator family $r(y,\eta)+g(y,\eta)$ where $r(y,\eta)$ has the form (3.3.8), and $a(y,\eta)$ is the restriction of an element $a(x,\xi) \in S^{(0)}(T^*X \setminus 0; E, F)$ to the boundary. As we know from [21] the operator family $\mathrm{op}^+(a)(y,\eta)$ equals the boundary symbol of an element $A_0 \in L^0(X,(0,0);E,F)_\varepsilon$ modulo an operator family $m(y,\eta) \in R^0_{M+G}(\Omega \times \mathbb{R}^q,(0,0);E',F')_\varepsilon$ (in local coordinates (y,t) near the boundary). The operator A_0 may be defined as $A_0 = \mathrm{r}^+\widetilde{A}\mathrm{e}^+$ for an $\widetilde{\mathcal{A}} \in L^0_{\mathrm{cl}}(2X;2E,2F)$, where $2X$ is the double of X, and $2E, 2F \in \mathrm{Vect}(2X)$ are bundles such that $2E|_X = E$, $2F|_X = F$. Thus it remains to observe that the elements of $S^{(0)}(T^*Y \setminus 0; E', F')$ that are locally in $R^0_{M+G}(\Omega \times \mathbb{R}^q,(0,0);E',F')_\varepsilon$ are boundary symbols of associated operators. For the Green part this has been discussed before, while for the smoothing Mellin part we replace in the cut–off functions $|\eta|$ by $[\eta]$ and thus obtain local operator–valued symbols. The operators themselves are then obtained by the standard operator convention with the Fourier transform in y–variables and passing to a sum, using a system of charts near the boundary and a partition of unity. $\square$

Remark 3.3.6 *$p(y,\eta) \in S^{(0)}(T^*Y \setminus 0; \boldsymbol{b}')$, $\boldsymbol{b}' = (E_0,F;J_0,J_+)$, and $\widetilde{p}(y,\eta) \in S^{(0)}(T^*Y \setminus 0; \boldsymbol{c}')$, $\boldsymbol{c}' = (E,E_0;J_-,J_0)$, implies $(p\widetilde{p})(y,\eta) \in S^{(0)}(T^*Y \setminus 0; \boldsymbol{b}' \circ \boldsymbol{c}')$ for $\boldsymbol{c}' \circ \boldsymbol{c}' = (E,F;J_-,J_+)$, and we have*

$$\sigma'_\psi(p\widetilde{p})(y,\xi) = \sigma'_\psi(p)(y,\xi)\sigma'_\psi(\widetilde{p})(y,\xi),$$
$$\sigma_M(p\widetilde{p})(y,z) = \sigma_M(p)(y,z)\sigma_M(\widetilde{p})(y,z).$$

Proof of Theorem 3.1.3. It suffices to consider the case $\mu = \gamma = 0$. In fact, the reduction to order and weight zero as in the beginning of Section 3.3 can also be done on the level of interior and boundary symbols. In other words, we can first pass to a symbol of order zero by setting $a_{(0)} = \sigma_\psi(R^{-\gamma+\mu})a_{(\mu)}\sigma_\psi(R_E^{-\gamma})$, carry out our construction, that yields an element $b_{(0)}(y,\eta)$ that is a Fredholm family as asserted in (3.1.6), where it suffices to consider

$$b_{(0)}(y,\eta) : E'_y \otimes L^2(\mathbb{R}_+) \to F'_y \otimes L^2(\mathbb{R}_+).$$

Then we may set

$$b_{(\mu)}(y,\eta) := \sigma_\partial(R_F^{\gamma-\mu})(y,\eta)b_{(0)}(y,\eta)\sigma_\partial(R_E^\gamma).$$

By virtue of Proposition 3.3.5 for the case of upper left corners (i.e., when the fibre dimensions of $J_\pm$ are zero) it suffices to show that $a_{(0)}(x,\xi)|_{S^*_Y X}$ for an elliptic principal symbol $a_{(0)}(x,\xi) : \pi_X^* E \to \pi_X^* F$ admits an extension to an isomorphism

$$\widetilde{a} : \pi_\Xi^* E' \to \pi_\Xi^* F', \qquad \pi_\Xi : \Xi \to Y. \tag{3.3.13}$$

In fact, knowing this, an approximation argument as explained before yields an element $l(y,z) \in M^{-\infty}(Y \times \Gamma_{\frac{1}{2}}; E', F')$ such that $\widetilde{\widetilde{a}}$ given by (3.3.7) for $a_{(0)}(x,\xi)|_{S^*_Y X}$

instead of $\sigma_\psi(A)|_{S^*_Y X}$ is also an extension of $a_{(0)}(x,\xi)|_{S^*_Y X}$ to an isomorphism to Ξ (that is homotopic to $\widetilde{a}$ through isomorphisms).

By hypothesis there is a non–vanishing vector field $v(y)$ on Y. Without loss of generality we assume $|v(y)| = 1$ for all y. Applying an isomorphism $TY \to T^*Y$ that induces a diffeomorphism $\alpha : SY \to S^*Y$ between the respective unit cosphere bundles we get a map $\alpha \circ v : Y \to S^*Y$. For every $y \in Y$ there is a unique half–circle $\widetilde{N}_y$ on $S^*_y X$ containing the points $\alpha \circ v(y)$ and $(y, 0, 0, \pm 1)$ (north and south pole of the sphere.) This yields a trivial bundle $\widetilde{N}$ on Y with fibre $\widetilde{N}_y$ over y. There is a projection of $S^*_Y X$ to the conormal bundle N, given by $(y, 0, \eta, \tau) \to (y, \tau)$, which induces an isomorphism $\nu : \widetilde{N} \to N$ (as fibre bundles in the set–up of homeomorphisms between fibres).

To get an extension of $a_{(0)}|_{S^*_Y X}$ to an isomorphism (3.3.13) it suffices to set $\widetilde{a}(y, \tau) := a_{(0)}(y, 0, \widetilde{\eta}, \widetilde{\tau})$, for $\nu_y(\widetilde{\eta}, \widetilde{\tau}) = \tau$, $\nu_y : \widetilde{N}_y \to N_y$, $y \in Y$. □

Proof of Theorem 3.1.4. Similarly to the preceding proof it suffices to consider $\gamma = 0$ and any fixed order $\mu \in \mathbb{R}$. In the present case it is convenient to set $\mu = 1$. Let $a_{(1)} \in S^{(1)}(T^*X \setminus 0; E, F)$ be elliptic, and let $a'_{(1)} := a_{(1)}|_Y$ which belongs to $S^{(1)}(T^*_Y X \setminus 0; E', F')$. By the standard difference construction we have an element $[a'_{(1)}] \in K(T^*_Y X) = K(T^*Y \times \mathbb{R})$. Every element in $K(T^*Y \times \mathbb{R})$ can be represented by a homomorphism

$$b(y, \eta) + i\tau : V \to V \tag{3.3.14}$$

for a vector bundle V on $Y \times \mathbb{R}$, where $b : \pi^*_Y V' \to \pi^*_Y V'$ is a self–adjoint elliptic symbol of order 1 on Y, cf. Atiyah, Patodi and Singer [2, III]. Since $b(y, \eta)$ is elliptic, (3.3.14) is an isomorphism between corresponding fibres for $\tau = 0$. Moreover, since b is self–adjoint, there are only real eigenvalues. Thus (3.3.14) is an isomorphism for all $\tau \in \mathbb{R}$. Passing to stabilisations both of $a'_{(1)}$ and (3.3.14) we see that for a suitable $M \in \mathbb{N}$ the homomorphism $a'_{(1)} \oplus \mathrm{id}_{\mathbb{C}^M}$ between pull–backs of $E' \oplus \mathbb{C}^M$ and $F' \oplus \mathbb{C}^M$ to $S^*_Y X$ has an extension to an isomorphism $\widetilde{a} : \pi^*_\Xi(E' \oplus \mathbb{C}^M) \to \pi^*_\Xi(F' \oplus \mathbb{C}^M)$. Similarly to the preceding proof we find an element $l(y, z) \in M^{-\infty}(Y \times \Gamma_{\frac{1}{2}}; E' \oplus \mathbb{C}^M, F' \oplus \mathbb{C}^M)$ such that (3.3.7) for $a'_{(1)} \oplus \mathrm{id}_{\mathbb{C}^M}|_{S^*_Y X}$ instead of $\sigma_\psi(A)|_{S^*_Y X}$ defines an extension of $a'_{(1)} \oplus \mathrm{id}_{\mathbb{C}^M}|_{S^*_Y X}$ to an isomorphism to Ξ, homotopic to $\widetilde{a}$ through isomorphisms. In analogy to (3.3.8) we now form

$$\begin{aligned} r(y, \eta) &:= \mathrm{op}^+(a_{(1)})(y, \eta) + t\omega(t|\eta|)\,\mathrm{op}_M(l)\widetilde{\omega}(t|\eta|) \\ &: (E'_y \oplus \mathbb{C}^M) \otimes \mathcal{K}^{1,0}(\mathbb{R}_+) \to (F'_y \oplus \mathbb{C}^M) \otimes L^2(\mathbb{R}_+). \end{aligned} \tag{3.3.15}$$

Then to complete the proof, we apply a reduction of order and weight by a similar scheme as in the proof of Theorem 3.1.3. □

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