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Adiabatic limits of eta and zeta functions of elliptic operators

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ADIABATIC LIMITS OF ETA AND ZETA FUNCTIONS OF ELLIPTIC OPERATORS

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Abstract. We use the calculus of adiabatic pseudo-differential operators to study the adiabatic limit behavior of the eta and zeta functions of a differential operator δ constructed from an elliptic family of operators with base S^1. We show that for invertible families the functions $\eta(\delta_t, s)$ and $t\zeta(\delta_t, s)$ extend smoothly in the adiabatic limit. This result still holds for arbitrary elliptic families at the special value $s = 0$. Specializing to the leading terms we recover the holonomy formula of Bismut-Freed and we obtain a similar formula for the adiabatic limit of the regular value $t\zeta(\delta_t, 0)$. In the invertible case we deduce a formula for the adiabatic limit of the determinant.

1. Introduction

Let N be a closed smooth manifold, E a Hermitian vector bundle over N and δ a first order elliptic differential operator on the sections of E. The normalized zeta function of δ is defined by

$$\overline{\zeta}(\delta, s) \quad := \quad \Gamma\left(\frac{s}{2}\right)\left(\sum_{0\neq\lambda\in\mathrm{Spec}(\delta^*\delta)} \lambda^{-s/2} + \dim\ker(\delta)\right). \tag{1}$$

If δ is self-adjoint, one defines similarly its normalized eta function by

$$\overline{\eta}(\delta, s) \quad := \quad \pi^{-\frac{1}{2}}\Gamma\left(\frac{1+s}{2}\right)\left(\sum_{0\neq\lambda\in\mathrm{Spec}(\delta)} \mathrm{sign}(\lambda)|\lambda|^{-s} + \dim\ker(\delta)\right).$$

These functions are holomorphic in the half-plane $\{\mathrm{Re}(s) > \dim N\}$ and extend analytically to $\mathbb{C}$ with possible simple poles at $\dim N - 2k$, respectively $\dim N - 2k - 1$, for all $k \in \mathbb{N}$. The definitions differ from the usual zeta and eta functions by some Gamma factors which will turn out to be the key for the validity of our results outside $s = 0$. The function $\overline{\zeta}$ strikingly resembles the functional equation of the Riemann zeta function. By the result of [1], $\eta(\delta, s)$ is finite at $s = 0$, hence the regularized value of $\overline{\eta}(\delta, s)$ at $s = 0$ coincides with the refined eta-invariant of Atiyah-Patodi-Singer. We stress however that the results of this paper are independent of this regularity.

Suppose that N is the total space of a locally trivial fibration of compact manifolds $N \xrightarrow{p} M$. We are interested in the restriction of this fibration to a

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circle in the base, we can therefore assume that $M = S^1$. Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Hermitian bundle over N and D a family of elliptic differential operators of order 1 over the fibers of p. In other words, for every $x \in S^1$, the operator $D_x : C^\infty(N_x, E^+|_{N_x}) \to C^\infty(N_x, E^-|_{N_x})$ is elliptic and depends smoothly on $x \in S^1$.

We fix a connection in the fibration $N \to S^1$ and a family of metrics g on the fibers extended by 0 on the horizontal distribution. For any vector Y tangent to S^1 we denote by $\tilde{Y}$ its horizontal lift to N. We also fix differential operators $\nabla_{\tilde{Y}}$ on the sections of $E^\pm$ with the property that $(\nabla_{\tilde{Y}} e_1, e_2) + (e_1, \nabla_{\tilde{Y}} e_2) = \tilde{Y}(e_1, e_2)$. Such operators arise for instance as the restriction to horizontal vectors of metric connections in $E^\pm$. Let $\mathrm{Tr}(L_{\tilde{Y}} g)$ be the contraction by g of the Lie derivative of the tensor g. Since g is vertical and $\tilde{Y}$ is horizontal, this expression is tensorial in Y. One sees easily that the operators

$$\tilde{\nabla}_Y := \nabla_{\tilde{Y}} + \frac{1}{4}\mathrm{Tr}(L_{\tilde{Y}} g)$$

are self-adjoint.

On S^1 we fix the canonical metric $d\sigma^2$ (any other metric is isometric to a multiple of $d\sigma^2$). Let ∂_σ be the positively oriented unit vector field on S^1. For $t \in [0, \infty)$ consider the self-adjoint operator

$$\delta_t := \begin{bmatrix} -ti\tilde{\nabla}_{\partial_\sigma} & D^* \\ D & ti\tilde{\nabla}_{\partial_\sigma} \end{bmatrix}. \tag{2}$$

acting on $C^\infty(N, E)$. Such an operator arises for instance in [3], where the fibers of $N \to S^1$ are even-dimensional and carry a continuous spin structure, $E^\pm$ are the spinor bundles of the fibers and ∇ is the Levi-Civita connection. In that case δ_t is the Dirac operator on N associated to the metric

$$g_t^N := g + \frac{d\sigma^2}{t^2}. \tag{3}$$

This example is presented in detail in Section 2. Note that δ_t is elliptic if and only if $t > 0$; nonetheless, δ_t is an elliptic *adiabatic* family of operators including at the limiting value $t = 0$, in the sense of Section 5.

The main goal of this paper is to relate spectral invariants of the family $\{D_x\}_{x \in S^1}$ and of the family $\{\delta_t\}_{t \in (0,\infty)}$.

In the case when D is a family of twisted Dirac operators, a conjecture of Witten [9] proved by Bismut-Freed [3] states that

$$\lim_{t \to 0} e^{-i\pi\eta(\delta_t)}$$

exists, and equals the holonomy of the determinant line bundle $\det(D)$ over S^1. The determinant bundle is reviewed in Section 4. We generalize this statement to any family D of first order elliptic differential operators, and moreover we prove that $\overline{\eta}(\delta_t)$ is smooth in t (modulo $2\mathbb{Z}$) for $t \in [0, \infty)$. If additionally one assumes that the family D is invertible, we prove that the function $\overline{\eta}(\delta_t, s)$ admits a Taylor expansion at $t = 0$ with coefficients meromorphic functions on $\mathbb{C}$, and we compute its first two coefficients.

Similar results hold for the zeta function. In the invertible case, we show that the family $t\overline{\zeta}(\delta_t, s)$ admits a Taylor expansion in powers of t with coefficients meromorphic functions on $\mathbb{C}$. We show that

$$\lim_{t\to 0} t\overline{\zeta}(\delta_t, s) = \frac{1}{\sqrt{\pi}} \int_{S^1} \overline{\zeta}(D, s-1) d\sigma.$$

As a corollary, we obtain formulas (24), (21) for the adiabatic limit of $t\zeta(\delta_t, 0)$ and of $t \log\det(\delta_t)$. The later formula takes a simpler form when the manifold N is odd-dimensional. The former is valid for any elliptic not necessarily invertible family D.

The proof of these results uses the algebra $\Psi_a(N)$ of *adiabatic pseudo-differential operators* [4], [7] that we review in Section 5.

A remark about the Gamma functions: the adiabatic limit process involves passing from a $(n+1)$-dimensional manifold to a family of n-dimensional manifolds. From zeta-type functions of operators on N one obtains in the limit a factor

$$f(s) := \int_{-\infty}^{\infty} (1+\tau^2)^{-s} d\tau$$

where τ encodes the missing dimension. It is straightforward to check that

$$f(s) = \frac{\sqrt{\pi}\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}. \tag{4}$$

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2. The lifted Dirac operator

This section computes an important example which motivates the definition (2). Let $p : X \to M$ be a fibration of closed manifolds, whose fibers are oriented Spin Riemannian manifolds of dimension $2n$, and assume that there exists a continuous spin structure on the fibers, meaning that TX/M is associated to a principal $Spin_{2n}$ bundle which is a double cover of the orthonormal frame bundle of the fibers. We denote by $\{D_x\}_{x\in M}$ the associated family of Dirac operators. We fix a connection in the fibration p, i.e. a choice of a horizontal distribution in X. This allows us to extend the family of fiber-wise metrics to a 2-tensor g on the total space of X. Any metric g^M on M induces by pull-back, together with g, a Riemannian metric on X:

$$g^X = g + p^* g^M.$$

We define a connection ∇ on the vertical sub-bundle TX/M of TX as the vertical projection of the Levi-Civita connection of the metric g^X on X. This depends on the tensor g, but not on the metric g^M on the base. We also denote by ∇ the induced connection on the spinors over the fibers. For all vectors fields Y on M, we define a first-order differential operator $\tilde{\nabla}_Y$ on sections of the spinor bundles by

$$\tilde{\nabla}_Y := \nabla_{\tilde{Y}} + \frac{1}{4}\mathrm{Tr}(L_{\tilde{Y}}g). \tag{5}$$

Here Tr means the contraction by g and $L_{\tilde{Y}}g$ is the Lie derivative of the tensor g in the direction of the horizontal lift $\tilde{Y}$ of Y. If α is a vertical 1-form, then for all $f \in C^\infty(M)$ we have $L_{f\tilde{Y}}(\alpha) = p^* f L_{\tilde{Y}}(\alpha)$, hence $\mathrm{Tr}(L_{\tilde{Y}}g)$ is tensorial in Y. Therefore, $\tilde{\nabla}$ can be thought of as a connection in the (infinite dimensional) vector bundle over M of fiber-wise sections of the spinor bundles, and the correction term ensures that this connection preserves the L^2 metric. It is $\tilde{\nabla}$ that occurs both in the Bismut-Freed connection form and in the lifted Dirac operator.

Let S^1 be a smooth loop embedded in M, and $N := p^{-1}(S^1)$. Fix the trivial (non-bounding) spin structure on S^1. Then N inherits a spin structure, and moreover the spinor bundle is isomorphic to $S^+ \oplus S^-$. The action of the unit horizontal vector (denoted Y) is simply $\mp i$ on $S^\pm$. This is one of the two possible choices, the other one being $\pm i$. We choose this sign in order to get Witten's original sign in the holonomy formula [9].

Let now δ_t be the Dirac operator for the base metric scaled by t^{-2}, i.e. for the metric

$$g_t^N = g + t^{-2} g^M. \tag{6}$$

Proposition 2.1. *In matrix form, using the splitting $S = S^+ \oplus S^-$, we have*

$$\delta_t = \begin{bmatrix} -ti\tilde{\nabla}_Y & D^* \\ D & ti\tilde{\nabla}_Y \end{bmatrix}.$$

Proof: The Levi-Civita connection ∇^t on (N, g_t^N) satisfies the identities

$$\begin{aligned} \nabla^t_U V &= \nabla_U V - \frac{t^2}{2}(L_{\tilde{Y}}g)(U,V)\tilde{Y} \\ (\nabla^t_{t\tilde{Y}}U, V) &= t(\nabla_{\tilde{Y}}U, V), \end{aligned}$$

where U, V are vertical vectors and $\tilde{Y}$ is the horizontal lift of the unit tangent vector to S^1 for the initial metric g^M. Let ϕ be a section of the spinor bundle $S = S^+ \oplus S^-$. Locally, ϕ is determined by a pair $[P, \sigma]$, where P is a local section of the Spin bundle which sits over a local orthonormal frame $(e_1, \ldots, e_{2n}, t\tilde{Y})$, and σ lives in the standard spinor representation Σ_{2n+1}. For any vector $V \in TN$, we have

$$\begin{aligned} \nabla^t_V(\phi) &= [P, V(\sigma)] + \frac{1}{2}\sum_{i<j\le 2n}(\nabla^t_V e_i, e_j)c(e_i)c(e_j)\phi \\ &\quad + \frac{1}{2}\sum_{i\le 2n}(\nabla^t_V e_i, t\tilde{Y})c(e_i)c(t\tilde{Y})\phi. \end{aligned}$$

Recall that $t\tilde{Y}$ has length 1 for g_t^N and that $c(t\tilde{Y})$ equals $\mp i$ on $S^{\pm}$. Therefore

$$\begin{aligned}
\delta_t\phi &= \sum_{k=1}^{2n} c(e_k)\Bigg([P, e_k(\sigma)] + \frac{1}{2}\sum_{i<j\leq 2n}(\nabla^t_{e_k}e_i, e_j)c(e_i)c(e_j)\phi \\
&\quad + \frac{1}{2}\sum_{i\leq 2n}(\nabla^t_{e_k}e_i, t\tilde{Y})c(e_i)c(t\tilde{Y})\phi\Bigg) + c(t\tilde{Y})\nabla^t_{t\tilde{Y}}(\phi) \\
&= \begin{bmatrix} -it\nabla_{\tilde{Y}} & D^* \\ D & it\nabla_{\tilde{Y}} \end{bmatrix}\phi + \frac{1}{2}\sum_{k=1}^{2n}\sum_{i\leq 2n}(\nabla^t_{e_k}e_i, t\tilde{Y})c(e_k)c(e_i)c(t\tilde{Y})\phi \\
&= \left(\begin{bmatrix} -it\nabla_{\tilde{Y}} & D^* \\ D & it\nabla_{\tilde{Y}} \end{bmatrix} - \frac{t}{4}c(t\tilde{Y})\sum_{k,i=1}^{2n}(L_{\tilde{Y}}g)(e_k, e_i)c(e_k)c(e_i)\right)\phi \\
&= \left(\begin{bmatrix} -it\nabla_{\tilde{Y}} & D^* \\ D & it\nabla_{\tilde{Y}} \end{bmatrix} + \frac{t}{4}c(t\tilde{Y})\mathrm{Tr}(L_{\tilde{Y}}g)\right)\phi.
\end{aligned}$$

□

3. Analytic extensions and regularized traces

This section contains a review of the basic results allowing one to define analytic extensions of zeta-type functions. Proofs are included for the benefit of the reader.

Recall that classical pseudo-differential operators on a closed manifold X have a characterization in terms of their Schwartz kernels on X^2. Namely, such a kernel is a distribution ψ on X^2 with *sing supp*(ψ) contained in the diagonal Δ_X; moreover, ψ must be *classical conormal* to Δ_X, in the sense that the Fourier transform in the normal directions of a cut-off of ψ near Δ_X is a classical symbol, i.e. a symbol admitting an asymptotic expansion in homogeneous components with step 1. Since we work with complex powers of operators, we allow classical symbols of any complex orders. By Peetre's Theorem, the differential operators are exactly those pseudo-differential operators with Schwartz kernel supported on the diagonal.

Proposition 3.1. *Let $P(s)$ be an entire family of classical pseudo-differential operators on X, such that $P(s) \in \Psi^s(X)$. Then $\mathrm{Tr}(P(s))$, which is* a priori *defined for $\mathrm{Re}(s) < -\dim X$, extends analytically to $\mathbb{C}$ with possible simple poles at integers $k \geq -\dim X$.*

Proof: For $\mathrm{Re}(s) < -\dim X$, $\mathrm{Tr}(P(s)$ equals the integral of the pointwise trace of the distributional kernel of $P(s)$ over the diagonal. By pulling back to the tangent bundle and then Fourier transforming in the fibers, this becomes the integral on T^*X of the pointwise trace of the full symbol of $P(s)$, which is an entire family $\{a(s)\}_{s\in\mathbb{C}}$ such that $a(s)$ is a classical symbol of order s. The claim follows from the next Lemma. □

Lemma 3.2. *Let $a(s)$ be an entire family of symbols of order s on a vector bundle V over X. Fix a density $d\mu$ on X and a Hermitian metric g in V. Then $\int_V a(s)d\mu dg$, which is well-defined for $\mathrm{Re}(s) < -\dim V$, extends analytically to $s \in \mathbb{C}$, with at most simple poles at real integers.*

Proof: If $a_k(s,v)$ is an entire family of symbols on V such that for $\|v\|\geq 1$, $a_k(s,v)=h\left(s,\frac{v}{\|v\|}\right)\|v\|^{k+s}$ where h is entire in s, then $\int_V a(s,v)dv$ splits in the integral on the unit ball bundle (which is entire) and the integral on the exterior of the unit ball, which can be computed explicitly in polar coordinates

$$\begin{aligned}\int_{r\geq 1} a(s,v)dv &= \int_1^\infty \int_{S(V)} h(s,\sigma) r^{k+s+\dim V-1} d\sigma dr\\ &= -\frac{1}{s+k+\dim V}\int_{S(V)} h(s,\sigma)d\sigma.\end{aligned}$$

Any family of classical symbols $a(s)$ of order s can be decomposed as $a(s)=a_0(s)+\tilde{a}(s)$, where $a_0(s,v)$ is a family as above for $k=0$, and $\tilde{a}(s)$ is an entire family of symbols of order $s-1$. Then $\int a_0(s,v)dv$ has just one pole at $s=-\dim V$, while $\int \tilde{a}(s,v)dv$ is well-defined for $\mathrm{Re}(s)<-\dim V+1$. The Lemma follows by iteration of this argument. □

Recall the definition of the Wodzicki residue trace: fix an elliptic first-order positive pseudo-differential operator Q on the m-dimensional manifold X. Then the complex powers of Q form an entire family of classical pseudo-differential operators. By Proposition 3.1, for any classical pseudo-differential operator $P\in\Psi^{\mathbb{Z}}(X)$ the function $s\mapsto \mathrm{Tr}(Q^{-s}P)$, *a priori* defined for $\mathrm{Re}(s)>m+\deg(P)$, extends analytically to $\mathbb{C}$ with possible simple poles at real integers $k\leq m+\deg P$. The map

$$P\mapsto \mathrm{Res}_{s=0}\,\mathrm{Tr}(Q^{-s}P):=\mathrm{Tr}_w(P)$$

defines a functional on the algebra $\Psi^{\mathbb{Z}}(X)$ which vanishes on $\Psi^{-\dim(X)-1}(X)$. Moreover, as the notation suggests, Tr_w is independent of the choice of Q. From the definition, it follows easily that Tr_w vanishes on commutators (i.e. Tr_w is a trace), and that it is given by the local expression

$$\mathrm{Tr}_w(P)=\int_{S^*X} p_{[-m]}\mathcal{R}\lrcorner\omega^m \tag{7}$$

where $p_{[-m]}$ is the homogeneous component of homogeneity $-m$ in the full symbol of P, S^*X is the sphere bundle of T^*X, $\mathcal{R}$ is the radial vector field on T^*X and ω is the standard symplectic form. As a consequence, the right-hand side of (7) is independent of the quantization defining the full symbol of P.

Corollary 3.3. *The functions $\overline{\zeta}(\delta,s)$, $\overline{\eta}(\delta,s)$, respectively the family of* 1*-forms defined in (9), admit analytic extensions to $\mathbb{C}$ with possible simple poles at integers $k\leq\dim N$, respectively $k\leq\dim N-1$.*

Proof: The claim follows directly from Proposition 3.1 with the exception of possible additional poles introduced by the Gamma factors. For $\Gamma(s/2)$ these poles occur at $s=2k$ for k a negative integer. At these values the zeta function is finite, since its residue equals the residue trace of a differential operator and hence vanishes. The function $\Gamma(s+1/2)$ introduces additional poles at $s=2k-1$ for k a negative integer. The eta function is finite at these points, since its residue equals $\mathrm{Tr}_w(\delta^{-2k+1})=0$. For the Bismut-Freed connection form, the poles of $\Gamma(s/2+1)$ occur at $s=2k-2$ for k a negative integer, and again one sees that the function $\mathrm{Tr}((D_U^* D_U)^{-\frac{s}{2}} D_U^{-1}\tilde{\nabla}(D_U))$ is regular at these points. $\square$

In fact some of these pole do not occur, as stated in the Introduction. We do not stress this point, since it is irrelevant for our purpose.

4. Review of the determinant line bundle over S^1

The determinant line bundle $\det(D)$ is a complex line bundle over S^1 and as such it is trivial. It has a canonical connection due to Bismut-Freed [3] that we recall below. Let $\mathcal{E}^{\pm}$ be the infinite-dimensional vector bundles over S^1 whose fiber over $x \in S^1$ is the space of smooth sections in $E^{\pm}$ over N_x. For every $x \in S^1$, let $\mathcal{E}^{\pm}_x = \mathcal{E}^{\pm}_{x,\leq\alpha} \oplus \mathcal{E}^{\pm}_{x,>\alpha}$ be the decomposition of $\mathcal{E}^{\pm}_x$ in subspaces spanned by eigensections of D^*D, respectively of DD^*, of eigenvalue smaller or strictly larger than α. These spaces do not form vector bundles in general. One can find a trivialization of $\det(D)$ in the following way [3]: there exist trivial finite-dimensional vector bundles $U^{\pm}$ over S^1 and maps of vector bundles $D_{12} : U^+ \to C^\infty(N/S^1, \mathcal{E}^-)$, $D_{21} : C^\infty(N/S^1, \mathcal{E}^+) \to U^-$, $D_{22} : U^+ \to U^-$, such that the operator

$$D_U := \begin{bmatrix} D & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{8}$$

is invertible. Moreover, one can ask that the following technical condition be fulfilled: there exists a finite cover $\mathcal{V} = \{V_{\alpha_1}, \ldots, V_{\alpha_k}\}$ of S^1 by sets $V_{\alpha_j} := \{x \in S^1; \alpha_j \notin Spec(D^*D)\}$, such that for $x \in V_\alpha \in \mathcal{V}$, D_U coincides with D on $\mathcal{E}^+_{x,>\alpha}$, and D_U maps $\mathcal{E}^+_{x,<\alpha} \oplus U^+$ onto $\mathcal{E}^-_{x,<\alpha} \oplus U^-$. We endow $U^{\pm}$ with the trivial connections.

Then the determinant line bundle with the Bismut-Freed connection is isomorphic to the trivial bundle $\mathbb{C} \times S^1 \to S^1$ with the connection $d + A(0)$, where $A(0)$ is the regularized value in $s = 0$ of the meromorphic family of 1-forms

$$A(D_U, s) := \Gamma\left(1 + \frac{s}{2}\right) \mathrm{Tr}((D_U^* D_U)^{-\frac{s}{2}} D_U^{-1} \tilde{\nabla}(D_U)). \tag{9}$$

In the case where D is invertible, one obtains a canonical family $A(D, s)$ by taking $U^{\pm} = 0$. In the general case, one can see nevertheless that for different choices of $U^{\pm}$ and D_{ij} the 1-form $A(0)$ changes by some exact term, thus leaving the holonomy unchanged. Indeed, if we enlarge D_U by the identity map of some trivial bundle, then (9) clearly does not change. Any two choices of D_U and D_{ij} are stably homotopic. Finally, if θ is a parameter, then

$$\partial_\theta A(D_U(\theta), 0) = d\mathrm{Tr}(D_U^{-1} \partial_\theta (D_U)).$$

The Bismut-Freed connection preserves the Quillen metric on $\det(D)$ [8]. In the complex setting, it is just the Hermitian connection associated to that metric. We will call its holonomy around S^1 the holonomy of $\det(D)$. A simple computation shows

$$\mathrm{hol}(\det(D)) = \exp\left(-\int_{S^1} A(0)\right).$$

5. The extended adiabatic algebra

We refer to [4] or [7] for a detailed discussion on the scalar adiabatic algebra; the extension to bundles is straightforward. The main feature of this algebra is that it realizes operators of the type we are interested in and their complex powers at the limiting value $t = 0$. Therefore, it will suffice to study these limiting operators, which belong to the so-called *suspended* algebra [6].

Let $p : X \to M$ a fibration of closed manifolds (in our case, this will be $N \to S^1$), and let $E^+, E^- \to X$ be two vector bundles. An *adiabatic vector field* is a family $v : [0, \infty) \to \mathcal{V}(X)$, such that $v(0)$ is tangent to the fibers of p. These vector fields form a locally free $C^\infty(X \times [0, \infty))$-module, so they are the sections of a vector bundle over $X \times [0, \infty)$ called ${}^{a}TX$. Let $\tilde{E}^\pm$ denote the lifts of $E^\pm$ to $X \times [0, \infty)$ and U the universal enveloping algebra functor. An adiabatic family of differential operator from E^+ to E^- is an element in $Hom(\tilde{E}^+, \tilde{E}^-) \otimes_{C^\infty(X \times [0,\infty))} U(C^\infty({}^{a}TX))$, i.e. a composition of adiabatic vector fields with matrix coefficients.

Let $X_a^2 = [X^2 \times [0, \infty); X \times_M X \times \{0\}]$ be the blow-up of the fiber diagonal of p at $t = 0$ inside $X^2 \times [0, \infty)$. This means replacing $X \times_M X \times \{0\}$ by the half-sphere bundle of its positive normal bundle, and then gluing along geodesic rays to get the smooth structure. The result is a smooth manifold with corners of codimension 2, and the smooth structure is independent of the metric on X used to define geodesics. The new face introduced by the blow-up (called the *front face*) is canonically diffeomorphic to $X \times_M X \times_M TM$ and therefore has a canonical vector bundle structure. Denote by β the canonical blow-down map from X_a^2 to $X^2 \times [0, \infty)$. The *lifted diagonal* Δ_a is by definition the closure of $\beta^{-1}(\Delta_X \times (0, \infty))$ inside X_a^2.

One can view adiabatic families of differential operators as distributions on $X^2 \times [0, \infty)$ having Schwartz kernels conormal to $\Delta_X \times (0, \infty)$ and supported on it, but, because of the degeneracy, not all such kernels define adiabatic operators. In fact, the Schwartz kernels of adiabatic operators lift to the manifold X_a^2 defined above, and these lifts span *all* the distributions conormal to and supported on the lifted diagonal. One must make precise the density bundle where these lifted kernels take values; this is $(p_2 \circ \beta)^*(\Omega({}^{a}TX))$, where p_2 is the projection on the second factor and Ω is the 1-density functor. In light of this fact, one makes the following

Definition 5.1. *An adiabatic pseudo-differential operator is a distributional section of $p_1^*(E^-) \otimes p_2^*(\Omega^1({}^{a}TX) \otimes E^+)^*$ over X_a^2, classical conormal to Δ_a, extendable across the front face and vanishing rapidly to all other boundary faces.*

We denote the space of these distributions by $\Psi_a(X, E^+, E^-)$. In Section 7 we need a slightly more general calculus. Let $U^\pm \to M$ be vector bundles. Let

$$\begin{aligned} S_{12} &= [X \times M \times [0, \infty); X \times_M M \times \{0\}] \\ S_{21} &= [M \times X \times [0, \infty); M \times_M X \times \{0\}] \end{aligned}$$

be manifolds with corners obtained through blow-up. Then $\Psi_{12}^{-\infty}(X, U^-, E^+)$ is defined as the space of smooth distributional kernels on S_{12} with values in the bundle $p_1^*(\otimes E^+) \otimes p_2^*(\Omega({}^a TM) \otimes U^-)^*$, which are rapidly vanishing to the boundary faces other than the front face. Similarly, $\Psi_{21}^{-\infty}(X, E^-, U^+)$ is the space of smooth sections of $p_1^*(U^+) \otimes p_2^*(\Omega({}^a TX) \otimes E^-)^*$ on S_{21} which are rapidly vanishing at all boundary faces other than the front face of S_{21}. Further, define $\Psi_{22}(M, U^+, U^-) := \Psi_a(M, U^+, U^-)$ to be the space of adiabatic operators corresponding to the identity fibration $M \to M$, and $\Psi_{11}(X, E^+, E^-) := \Psi_a(X, E^+, E^-)$.

Proposition 5.2. *There exists natural composition maps* $\Psi_{ij}^z \otimes \Psi_{jk}^w \to \Psi_{ik}^{z+w}$.

Proof: One constructs the so called *adiabatic triple spaces.* The proposition follows from the properties of pull-back and push-forward operations on conormal distributions, in the framework of Melrose's theory [5]. □

As a consequence, we get a pseudo-differential calculus $\Psi_{ae}(X, E^\pm, U^\pm)$ formed by matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{ij} belongs to Ψ_{ij}.

Melrose [6] introduced a notion of suspension for pseudo-differential operators. If W is any vector bundle over M, one can define $\Psi_{sus(W)}(X/M)$ as the space of families of pseudo-differential operators over the fibers of the fiber bundle $X \times_M W \to M$, which are translation invariant with respect to W, and whose Schwartz kernels decay rapidly away from the W-diagonal. These operators are identified (via Fourier transform in W) with families over M of parameter-dependent operators on the fibers of $X \to M$, with parameters in W^* having symbolic behavior.

For $E^+ = E^- = E$ and $U^+ = U^- = U$, the space $\mathcal{A} = \Psi_{ae}^{\mathbb{Z}}(X, E, U)$ is an algebra. We summarize below its properties.

- $\mathcal{A}$ has a natural double filtration $\{\mathcal{A}^{i,k}\}_{i\in\mathbb{Z},k\in\mathbb{N}}$, increasing in the first and decreasing in the second superscript, i.e. $\mathcal{A}^{i_1,k_1} \subset \mathcal{A}^{i_2,k_2} \Leftrightarrow i_1 \le i_2$ and $k_1 \ge k_2$. Namely, i denotes the operator order, while k indicates the order of vanishing at $t = 0$. This filtration is compatible with the product on $\mathcal{A}$.
- The quotient $\mathcal{A}_\sigma^k := \mathcal{A}^k / \mathcal{A}^{k-1}$ is canonically isomorphic to
$$S^{[k]}({}^a T^* X \setminus 0, E \otimes E^*) \oplus S^{[k]}({}^a T^* M \setminus 0, U \otimes U^*),$$
the space of symbols of pure homogeneity k.
- The quotient $\mathcal{A}_\partial := \mathcal{A}^{\mathbb{Z},0} / \mathcal{A}^{\mathbb{Z},1}$ is canonically isomorphic to the algebra of matrices $(a_{ij})_{1\le i,j\le 2}$ such that
  - $a_{11} \in \Psi_{sus({}^a T^* M_{|t=0})}(X/M, E)$. The suspending variables τ live naturally on ${}^a T^* M_{|t=0}$, which is canonically isomorphic to T^*M.
  - $a_{22} \in S^{\mathbb{Z}}({}^a T^* M_{|t=0}, U)$, the space of classical symbols on the vector bundle ${}^a T^* M_{|t=0}$.
  - a_{12}, respectively a_{21} belongs to the space of smooth kernels vanishing rapidly to infinity on $X \times_M T^*M$, respectively $T^*M \times_M X$, with values in the appropriate vector bundles.

The isomorphism is given by Fourier-transforming the restriction to the front face of the Schwartz kernel of an extended adiabatic operator, in the adiabatic tangent fibers of the front face.

- The "boundary" algebra $\mathcal{A}^{\mathbb{Z},\mathbb{N}}/\mathcal{A}^{\mathbb{Z},\infty}$ is isomorphic as a vector space to the formal series algebra $\mathcal{A}_\partial[[t]]$. An isomorphism is fixed by the choice of a Riemannian metric on M, a connection in $X \to M$ and the restriction of a connection in $E \to X$ to horizontal vectors. We illustrate this for the left upper corner. We use an analog of the so-called right quantization, i.e.
$$\begin{aligned} X \times_M X \times_M {}^{a}TM &\to X^2 \times [0,\infty) \\ (x_m, y_m, v_m(t)) &\mapsto (x_m, \tilde{c}_{v_m}(y_m)(1), t), \end{aligned}$$
where x_m, y_m are points in X over $m \in M$, v is an adiabatic vector field, c_v is the geodesic in the direction of v and $\tilde{c}(y_m)$ is the horizontal lift of c starting at y_m. This map lifts to X^2_a, and the desired isomorphism is given by pulling back adiabatic Schwartz kernels to $X \times_M X \times_M {}^{a}TM$ and then Fourier transforming in the fibers of ${}^{a}TM$. We need the connection in horizontal directions in order to trivialize the pull-back of the coefficient bundle over each fiber of ${}^{a}TM \to M$, so that we can compute Fourier transform.
- This isomorphism maps $it\nabla_{\partial/\partial x_i}$ to the constant series τ_i if ∇ is the connection used in the construction of the isomorphism.
- The product structure on $\mathcal{A}^{\mathbb{Z},0}/\mathcal{A}^{\mathbb{Z},\infty}$ induced by the above isomorphism is a deformation with parameter t of the suspended (fiber-wise) product:

$$A(x,\tau) * B(x,\tau) = AB + it \sum_{i=1}^{\dim M} \frac{\partial A}{\partial \tau_i} \nabla_{\frac{\partial}{\partial x_i}} B + O(t^2). \tag{10}$$

- For every $Q \in \mathcal{A}^{1,0}$ an adiabatic family of invertible positive operators, there exists the entire family $\{Q^s\}_{s\in\mathbb{C}}$ of complex powers of Q such that $Q^s \in \mathcal{A}^{s,0}$.

The trace function $\mathrm{Tr}(A(t))$ of an extended adiabatic operator $A \in \Psi^{\alpha,0}_{ae}(X)$ is smooth for $t > 0$, provided $\mathrm{Re}(\alpha) < -\dim X$. The following Proposition shows that the behavior at $t = 0$ of this function is controllable, and thus provides us with our main tool to study adiabatic limits.

Proposition 5.3. (1) *Let A be an extended adiabatic operator in $\Psi^{\alpha,0}_{ae}(X)$ with $\mathrm{Re}(\alpha) < -\dim X$. Then the function $t^{\dim M}\mathrm{Tr}(A(t))$ is smooth on $[0,\infty)$. Moreover, if $A \sim \sum_{i=0}^{\infty} t^i A_i$ is the image of A in $\mathcal{A}_\partial[[t]]$, then*

$$\mathrm{Tr}(A(t)) \sim (2\pi t)^{-\dim M} \sum_{i=0}^{\infty} \int_{T^*M} t^i \mathrm{Tr}(A_i)\omega^{\dim M}, \tag{11}$$

where ω is the standard symplectic form on T^*M, and $\mathrm{Tr}(A_i)$ is the fiber-wise trace of the family of suspended operators A_i.

(2) *Let $A(s,t)$ be an entire family of operators in $\mathcal{A}$, such that $A(s) \in \mathcal{A}^{s,0}$. For each integer $k \geq -1$ and $t > 0$, let $c_{k,s}(A(t))$ be the k-th Laurent*

coefficient of $\mathrm{Tr}(A(s,t))$ *around* s. *Then for each fixed* k *and* s, *the function*

$$t^{\dim M} c_{k,s}(A(t))$$

is smooth for $t \in [0,\infty)$, *and its Taylor expansion at* $t = 0$ *is given by*

$$(2\pi)^{-\dim M} \sum_{i=0}^{\infty} \int_{T^*M} t^i c_{k,s}(A_i) \omega^{\dim M},$$

where $c_{k,s}(A_i) : T^*M \to \mathbb{C}$ *is the* k*-th Laurent coefficient of* $\mathrm{Tr}(A_i)$ *around* s.

Proof: (1). Use the isomorphism between $\mathcal{A}^{\mathbb{Z},\mathbb{N}}/\mathcal{A}^{\mathbb{Z},\infty}$ and $\mathcal{A}_\partial[[t]]$. For fixed t, the trace of $A(t)$ is the integral of its distributional kernel over the diagonal. We pull back this kernel through the map $X \times_M X \times_M {}^a TM \to X_a^2$ constructed above, and then we Fourier transform in the fibers of ${}^a TM$. Recall that for the identity fibration $M \to M$ there exist two canonical bundle maps $I, J : {}^a TM \to TM \times [0,\infty)$. Namely, I is the canonical inclusion of adiabatic vector fields inside all vector fields, and J is the isomorphism given by $v(t) \mapsto v(t)/t$. We denote by the same letters the duals maps. Then $J^*(I_*\omega) = \omega/t$. The integral of the restriction of the kernel to the diagonal becomes (after Fourier transform) the integral on the total space, with a density factor $(I_*\omega/2\pi)^{\dim M}$. When we go back to T^*M via J^*, this becomes $(\omega/(2\pi t))^{\dim M}$ as claimed.

(2). It follows from the proof of Lemma 3.2 that if $a(s,t)$ is a smooth family of entire families of symbols, then the Laurent development of $\int_V a(s,t) d\mu dg$ around any complex point s depends smoothly on t. In particular, if $a(s,t) \sim \sum_{i=0}^{\infty} t^i a_i(s)$ is the Taylor expansion at $t = 0$ of $a(s,t)$ then $c_{k,s}(a(t)) \sim \sum_{i=0}^{\infty} t^i c_{k,s}(a_i)$. We apply this to the full adiabatic symbols $a(s,t)$ of the family $A(s,t)$, which consist of a symbol on ${}^a T^*X$ and a symbol on ${}^a T^*M$. The claim follows from (11) by noting that $a_i(s)$ is the full symbol of $A_i(s)$. □

6. The adiabatic limit: the invertible case

Throughout this section assume that the family D is invertible. With this assumption we only need to use the adiabatic algebra $\Psi_a(N)$ and not the extended calculus, since the operator δ_t defined in (2) is an elliptic operator in $\Psi_a(N)$ (although it is clearly not elliptic at $t = 0$ in the usual sense). As a first application of the formalism from Section 5, we reprove a result from [2].

Proposition 6.1. *There exists* $\epsilon > 0$ *such that the operator* δ_t *is invertible for all* $0 < t < \epsilon$.

Proof: In the quotient algebra $\mathcal{A}_\partial = \Psi_a^{\mathbb{Z},0}(N)/\Psi_a^{\mathbb{Z},1}(N)$, the operator

$$\delta_t^2 = \begin{bmatrix} \tau^2 + D^*D & 0 \\ 0 & \tau^2 + DD^* \end{bmatrix}$$

is invertible [6] since D is assumed to be invertible. We deduce from this the existence of an inverse B_0 of δ_t modulo $\Psi_a^{0,1}(X)$, i.e. $\delta_t B_0 = I - R_0$, with $B_0 \in \Psi_a^{-1,0}(N)$ and $R_0 \in \Psi_a^{0,1}(N)$. Let Q be an adiabatic operator which realizes the asymptotic sum $B := B_0(I + R_0 + R_0^2 + \ldots)$. Then $R(t) := \delta_t Q - I \in \Psi_a^{0,\infty}$. This ideal is canonically isomorphic to the algebra of families of classical pseudodifferential operators on N which are rapidly vanishing at $t=0$. Thus, the norm of $R(t)$ as a bounded operator on $L^2(N)$ tends to 0, which implies that $\delta_t Q = I + R(t)$ is invertible for sufficiently small t. □

From our choice of adiabatic right quantization, it follows that in the algebra $\Psi_a^{\mathbb{Z},0}(N)/\Psi_a^{\mathbb{Z},\infty}(N) \cong \mathcal{A}_\delta[[t]]$, we have

$$\delta_t \sim_{t\to 0} \begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}, \qquad \delta_t^2 \sim_{t\to 0} \begin{bmatrix} D^*D+\tau^2 & -it\tilde{\nabla}_{\partial_\sigma} D^* \\ it\tilde{\nabla}_{\partial_\sigma} D & DD^*+\tau^2 \end{bmatrix}.$$

Let $h : [0,\infty)\times\mathbb{C}\to\mathbb{C}$ be a family of meromorphic functions on $\mathbb{C}$ indexed by t. We say that $h(t)$ is a *smooth family of meromorphic functions* if for every $s\in\mathbb{C}$ and every $k\in\mathbb{Z}$, the Laurent coefficient $h_{k,s}(t)$ is a smooth function of t.

Theorem 6.2. *If the family D is invertible then the family of eta functions $\overline{\eta}(\delta_t, s)$ – a priori defined for $t>0$ – extends to a smooth family for $t\in[0,\epsilon)$. Moreover,*

$$\lim_{t\to 0}\overline{\eta}(\delta_t,s) = \frac{1}{i\pi}\int_{S^1} A(D,s), \tag{12}$$

where $A(D,s)$ is the Bismut-Freed family of 1-forms defined by (9).

Proof: First, δ_t is invertible and so (2) becomes

$$\overline{\eta}(\delta_t,s) = \pi^{-1/2}\Gamma\left(\frac{1+s}{2}\right)\mathrm{Tr}((\delta_t^2)^{-\frac{s+1}{2}}\delta_t).$$

From Proposition 5.3 it follows that $t\overline{\eta}(\delta_t,s)$ is a smooth family of meromorphic functions. Further, we have

$$\begin{aligned}\overline{\eta}(\delta_t,s) \quad &\sim_{t\to 0} \quad \frac{\Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}}\frac{1}{2\pi t}\int_{T^*S^1}\mathrm{Tr}\left(\begin{bmatrix} D^*D+\tau^2 & -it\tilde{\nabla}_{\partial_\sigma} D^* \\ it\tilde{\nabla}_{\partial_\sigma} D & DD^*+\tau^2 \end{bmatrix}^{-\frac{s+1}{2}}\right. \\ &\left.\begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}\right) d\tau\wedge d\sigma.\end{aligned} \tag{13}$$

where all products and exponentials are in the sense of the product (10). For the singular coefficient of t^{-1} this implies

$$\begin{aligned}\lim_{t\to 0} t\overline{\eta}(\delta_t,s) \quad &= \quad \frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\int_{T^*S^1}\mathrm{Tr}\left(-\tau(D^*D+\tau^2)^{-\frac{s+1}{2}}\right) \\ &+\mathrm{Tr}\left(\tau(DD^*+\tau^2)^{-\frac{s+1}{2}}\right) d\tau\wedge d\sigma\end{aligned}$$

and this vanishes because it is the integral of an odd function of τ. We compute now the limit of $\overline{\eta}(\delta_t,s)$ as t tends to 0. Since $\begin{bmatrix} D^*D+\tau^2 & 0 \\ 0 & DD^*+\tau^2 \end{bmatrix}$ and

$\begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}$ commute modulo t, the constant term in t from (13) becomes

$$\begin{aligned}
\lim_{t\to 0}\overline{\eta}(\delta_t, s) &= \lim t^{-1}\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\int_{T^*S^1}\mathrm{Tr}\left(\begin{bmatrix} D^*D+\tau^2 & 0 \\ 0 & DD^*+\tau^2\end{bmatrix}^{-\frac{s+1}{2}}\right.\\
&\left.\begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}\right) d\tau\wedge d\sigma\\
&+\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\int_{T^*S^1} -\frac{s+1}{2}\mathrm{Tr}\left(\begin{bmatrix} D^*D+\tau^2 & 0 \\ 0 & DD^*+\tau^2\end{bmatrix}^{-\frac{s+3}{2}}\right.\\
&\left.\begin{bmatrix} 0 & -i\tilde{\nabla}_{\partial_\sigma}D^* \\ i\tilde{\nabla}_{\partial_\sigma}D & 0 \end{bmatrix}\begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}\right) d\tau\wedge d\sigma. \qquad (14)
\end{aligned}$$

The first term has two contributions from the diagonal, namely

$$\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\frac{s+1}{2}\frac{s+3}{2}\int_{T^*S^1} -\tau^2\mathrm{Tr}\left((D^*D+\tau^2)^{-\frac{s+5}{2}}i\tilde{\nabla}_{\partial_\sigma}(D^*D)\right)d\tau\wedge d\sigma, \qquad (15)$$

respectively

$$\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\frac{s+1}{2}\frac{s+3}{2}\int_{T^*S^1} \tau^2\mathrm{Tr}\left((DD^*+\tau^2)^{-\frac{s+5}{2}}i\tilde{\nabla}_{\partial_\sigma}(DD^*)\right)d\tau\wedge d\sigma. \qquad (16)$$

These factors cancel each other before integration; alternately, each of them equals 0 after integration; for instance, the quantity (15) equals

$$\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\frac{s+1}{2}\int_{T^*S^1} \tau^2 i\partial_\sigma\mathrm{Tr}\left((D^*D+\tau^2)^{-\frac{s+3}{2}}\right)d\tau\wedge d\sigma = 0.$$

We focus now on the second term. It is given by

$$\begin{aligned}
\lim_{t\to 0}\overline{\eta}(\delta_t, s) &= -\frac{s+1}{2}\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\int_{T^*S^1}\Big[\mathrm{Tr}\left((D^*D+\tau^2)^{-\frac{s+3}{2}}(-i\tilde{\nabla}_{\partial_\sigma}D^*)D\right)\\
&+\mathrm{Tr}\left((DD^*+\tau^2)^{-\frac{s+3}{2}}(i\tilde{\nabla}_{\partial_\sigma}D)D^*\right)\Big] d\tau\wedge d\sigma\\
&= i\frac{s+1}{2}\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}f\left(\frac{s+3}{2}\right)\int_{S^1}\Big[\mathrm{Tr}\left((D^*D)^{-\frac{s}{2}-1}\tilde{\nabla}_{\partial_\sigma}D^*)D\right)\\
&-\mathrm{Tr}\left((DD^*)^{-\frac{s}{2}-1}\tilde{\nabla}_{\partial_\sigma}D)D^*\right)\Big] d\sigma\\
&= \frac{\Gamma\left(\frac{s}{2}+1\right)}{2i\pi}\int_{S^1}\mathrm{Tr}\left((D^*D)^{-\frac{s}{2}-1}(D^*\tilde{\nabla}_{\partial_\sigma}(D)-\tilde{\nabla}_{\partial_\sigma}(D^*)D)\right)d\sigma \qquad (17)
\end{aligned}$$

where $f(s)$ is the function from (4). We have used the commutation formula $(D^*D)^sD^* = D^*(DD^*)^s$. Write the last expression as $I - II$. We claim that

$I+II=0$. Indeed,

$$\begin{aligned}
&\int_{S^1} \mathrm{Tr}\left((D^*D)^{-\frac{s}{2}-1}(D^*\tilde{\nabla}_{\partial_\sigma}(D)+\tilde{\nabla}_{\partial_\sigma}(D^*)D)\right)d\sigma\\
&= -\frac{s}{2}\int_{S^1}\mathrm{Tr}(\tilde{\nabla}_{\partial_\sigma}((D^*D)^{-\frac{s}{2}}))d\sigma\\
&= -\frac{s}{2}\int_{S^1}\partial_\sigma(\mathrm{Tr}(D^*D)^{-\frac{s}{2}})d\sigma\\
&= 0.
\end{aligned}$$

Thus, (17) is equivalent to (12). □

Corollary 6.3. [3] *The limit of* $\exp(-i\pi\overline{\eta}(\delta_t))$ *exists and equals the holonomy of* $\det(D)$.

Proof: By specializing to the constant coefficient $\overline{\eta}(\delta_t)$ of $\overline{\eta}(\delta_t,s)$ at $s=0$, we get

$$\lim_{t\to 0} e^{-i\pi\overline{\eta}(\delta_t)} = e^{-\int_{S^1}A(D,0)}.$$

The right-hand side equals the holonomy along the circle of the connection $d+A(D,0)$ in the trivial bundle. □

We turn now to the zeta function.

Theorem 6.4. *Assume that the family of operators D is invertible. Then the family $t\overline{\zeta}(\delta_t,s)$ extends smoothly to $t=0$. Moreover,*

$$\lim_{t\to 0} t\overline{\zeta}(\delta_t,s) = \frac{1}{\sqrt{\pi}}\int_{S^1}\overline{\zeta}(D,s-1)d\sigma, \tag{18}$$

while

$$\frac{d(t\overline{\zeta}(\delta_t,s))}{dt}|_{t=0} = 0.$$

Proof: As for the eta function, we have

$$\begin{aligned}
t\overline{\zeta}(\delta_t,s) &= t\Gamma\left(\frac{s}{2}\right)\mathrm{Tr}\left((\delta_t^2)^{-\frac{s}{2}}\right)\\
&\sim \Gamma\left(\frac{s}{2}\right)\frac{1}{2\pi}\int_{T^*S^1}\mathrm{Tr}\begin{bmatrix} D^*D+\tau^2 & -it\tilde{\nabla}_{\tilde{Y}}D^*\\ it\tilde{\nabla}_{\tilde{Y}}D & DD^*+\tau^2\end{bmatrix}^{-\frac{s}{2}} d\tau\wedge d\sigma. \qquad (19)
\end{aligned}$$

The leading term in this expression is

$$\begin{aligned}
&\pi^{-1}\Gamma\left(\frac{s}{2}\right)\int_{T^*S^1}\mathrm{Tr}(D^*D+\tau^2)^{-s/2}d\tau\wedge d\sigma\\
&= \pi^{-1}\Gamma\left(\frac{s}{2}\right) f\left(\frac{s}{2}\right)\int_{S^1}\mathrm{Tr}(D^*D)^{-\frac{s-1}{2}}d\sigma.
\end{aligned}$$

Together with (4), this implies (18). It is clear that the off-diagonal terms in δ_t^2 contribute to (19) only modulo t^2. It follows that the coefficient of t in (19) is a sum of two terms coming from the diagonal. These terms are similar to (15) and (16), only that they are odd in τ and thus both equal 0 even before integrating over S^1. □

We can deduce from this result the adiabatic limit behavior of the usual quantities $\zeta(\delta_t,0)$ and of $\det(\delta_t) := e^{-\zeta'(\delta_t,0)}$. Let $\mathrm{Tr}_\zeta|D|$ denote the regularized value $\zeta(D,-1)$.

Theorem 6.5. *The functions* $t\zeta(\delta_t, 0)$ *and* $t\log\det(\delta_t)$ *extend smoothly to* $t = 0$. *Moreover,*

$$\lim_{t\to 0} t\zeta(\delta_t, 0) = -\int_{S^1} \mathrm{Tr}_w|D|d\sigma \tag{20}$$

$$\lim_{t\to 0} t\log\det(\delta_t) = \int_{S^1} \left(-\mathrm{Tr}_\zeta|D| + (2\log 2 - 2)\mathrm{Tr}_w|D|\right)d\sigma. \tag{21}$$

Proof: In terms of ζ, formula (18) reads

$$\lim_{t\to 0} t\Gamma\left(\frac{s}{2}\right)\zeta(\delta_t, s) = \pi^{-1/2}\Gamma\left(\frac{s-1}{2}\right)\int_{S^1}\zeta(D, s-1)d\sigma. \tag{22}$$

Recall that

$$\Gamma(0) = -\gamma, \quad \Gamma'\left(-\frac{1}{2}\right) = 2\pi^{1/2}(\gamma + 2\log 2 - 2),$$

where γ is the Euler constant, and $\Gamma(0)$ is the finite part of $\Gamma(s)$ at $s = 0$. Thus

$$\begin{aligned}
&\lim_{t\to 0} t\left(\frac{2}{s} - \gamma\right)\left(\zeta(\delta_t, 0) + s(\zeta)'(\delta_t, 0)\right)\\
&\equiv \frac{1}{\sqrt{\pi}}\left(-2\pi^{1/2} + \pi^{1/2}(\gamma + 2\log 2 - 2)s\right)\\
&\quad\left(\frac{1}{s}\int_{S^1}\mathrm{Tr}_w(D^*D)^{1/2}d\sigma + \int_{S^1}\zeta(D, -1)d\sigma\right) \mod (s).
\end{aligned}$$

Identifying the coefficients of s^{-1} and s^0 we obtain (20) and (21). □

Assume that N is odd-dimensional. In this case $\zeta(\delta_t, 0)$ must vanish and by (20) this implies that $\int_{S^1}\mathrm{Tr}_w(D^*D)^{\frac{1}{2}}d\sigma = 0$. From Theorem 6.5,

$$\lim_{t\to 0}\det(\delta_t)^t = e^{-\int_{S^1}\mathrm{Tr}_\zeta(|D|)d\sigma}.$$

7. Adiabatic limits in the non-invertible case

Consider now an elliptic family D as in Section 6, without the invertibility hypothesis. Then δ_t is also non-invertible in general. One can define the complex powers of δ_t^2 as being 0 on the null-space of δ_t. This works fine for a fixed value of t. However, the function $(0, \infty) \times \mathbb{C} \ni (\lambda, s) \mapsto \lambda^s$ does not converge to 0 as λ tends to 0, and this means that the family $(\delta_t^2)^s$ is discontinuous at values of t where an eigenvalue crosses 0. This turns out to be a major difficulty when trying to extend the analysis from the previous section to the family δ_t. We need to use the extended adiabatic algebra to overcome this problem.

Let U^+, U^- be trivial finite-dimensional Hermitian vector spaces over S^1, such that there exists an invertible family of operators D_U extending D by the formula (8). As in Section 4, we extend $\tilde{\nabla}_{\partial_\sigma}$ trivially to an operator

$$\tilde{\nabla}_{\partial_\sigma} : C^\infty(N/S^1, E^+) \oplus U^+ \longrightarrow C^\infty(N/S^1, E^-) \oplus U^-.$$

Let ϕ be a strictly positive Schwartz function on $\mathbb{R}$ such that $\phi(0) = 1$. Then

$$Q := \begin{bmatrix} P & A_{12}\phi(\tau) \\ A_{21}\phi(\tau) & A_{22}\phi(\tau) \end{bmatrix}$$

belongs to $\Psi^{1,0}_{ae}(N; E^+ \oplus U^+, E^- \oplus U^-)/\Psi^{1,1}_{ae}(N; E^+ \oplus U^+, E^- \oplus U^-)$. The operator Q can be realized as the full Taylor expansion at $t = 0$ of an extended adiabatic operator R_t, which differs from $\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$ by a perturbation in $\Psi^{-\infty,0}_{ae}(N)$. Let

$$d_t = \begin{bmatrix} -ti\tilde{\nabla}_Y & R_t^* \\ R_t & ti\tilde{\nabla}_Y \end{bmatrix} \in \Psi^{1,0}_{ae}(N, E, U).$$

The boundary symbol of d_t^2 is $\begin{bmatrix} Q^*Q + \tau^2 & 0 \\ 0 & QQ^* + \tau^2 \end{bmatrix}$, an operator invertible in $\mathcal{A}_\partial(N, E, U)$. Proposition 6.1 shows that d_t is invertible for small enough t. The results of Section 6 hold when replacing δ_t with d_t and D with Q, with the exception of the vanishing of the terms corresponding to (15), (16). Such results are, of course, uninteresting, since d_t and Q depend on the choice of $U^\pm$ and of the Schwartz function ϕ, and moreover the terms (15), (16) are generally not explicitly computable. However, there is one instance when these results can be linked to δ_t and D, namely when we restrict our attention to the constant coefficient in s at $s = 0$.

Proposition 7.1. *Up to $2\mathbb{Z}$, $\overline{\eta}(d_t)$ is independent of the perturbation involved in the definition of d_t. Moreover,*

$$\overline{\eta}(d_t) \equiv \overline{\eta}(\delta_t) + \mathrm{Ind}(D) \mod 2\mathbb{Z}.$$

Proof: Let $d_t(\theta)$ be a perturbation of δ_t by a 1-parameter family of symmetric operators, such that $\partial_\theta(d_t)$ belongs to $\Psi^{-\infty}_{ae}(N)$. We first show that $\overline{\eta}(d_t)$ is constant in θ as long as $d_t(\theta)$ remains invertible. Indeed,

$$\begin{aligned} \partial_\theta \mathrm{Tr}\left((d_t^2)^{-\frac{s+1}{2}} d_t\right) &= \mathrm{Tr}\left((d_t^2)^{-\frac{s+3}{2}}\left(-\frac{s+1}{2}\right)(\partial_\theta(d_t)d_t + d_t\partial_\theta(d_t))d_t \right. \\ &\quad \left. +(d_t^2)^{-\frac{s+1}{2}}\partial_\theta(d_t)\right) \\ &= -s\mathrm{Tr}\left((d_t^2)^{-\frac{s+1}{2}}\partial_\theta(d_t)\right). \end{aligned}$$

The last expression is the trace of an entire family of order $-\infty$ operators, and it vanishes at $s = 0$ because of the s factor. When θ crosses values where 0 is an eigenvalue of $d_t(\theta)$, one sees by separating large and small eigenvalues that the jumps are even integers.

Consider the operator

$$q_t = \begin{bmatrix} \delta_t + P_{\ker(\delta_t)} & 0 & 0 \\ 0 & -it\partial_\sigma - P_{\ker(\partial_\sigma)} & 0 \\ 0 & 0 & it\partial_\sigma + P_{\ker(\partial_\sigma)} \end{bmatrix}$$

acting on $C^\infty(N, E^+ \oplus E^-) \oplus C^\infty(S^1, U^+) \oplus C^\infty(S^1, U^-)$. Note that q_t is written in another basis than p_t. For fixed $t > 0$, q_t is an invertible perturbation of δ_t by operators of order $-\infty$, so $\overline{\eta}(q_t) \equiv \overline{\eta}(\delta_t) \mod 2\mathbb{Z}$. Now $\overline{\eta}(q_t, s) = \overline{\eta}(\delta_t, s) + (\dim U^- - \dim U^+)\overline{\eta}(it\partial_\sigma + P_{\ker(\partial_\sigma)}, s)$. At this stage we must check that $\overline{\eta}(it\partial_\sigma + P_{\ker(\partial_\sigma)}, s) = \pi^{-\frac{1}{2}}\Gamma\left(\frac{1+s}{2}\right)$ is regular at $s = 0$, and takes the value 1. The result follows then by noting that $\dim U^- - \dim U^+ = \mathrm{Ind}(D)$. $\square$

Similarly, we prove that $\zeta(d_t) = \zeta(\delta_t)$. First,

$$\begin{aligned}\partial_\theta \mathrm{Tr}\left((d_t^2)^{-\frac{s}{2}}\right) &= -\frac{s}{2}\mathrm{Tr}\left((d_t^2)^{-\frac{s}{2}-1}(\partial_\theta(d_t)d_t + d_t\partial_\theta(d_t))\right)\\ &= -s\mathrm{Tr}\left((d_t^2)^{-\frac{s}{2}-1}d_t\partial_\theta(d_t)\right)\end{aligned}$$

equals s times an entire function, so it vanishes at $s = 0$. Secondly, there are no jumps in $\zeta(d_t(\theta))$ when eigenvalues of d_t cross 0 (on finite parts of the spectrum, ζ at $s = 0$ is just the number of eigenvalues). Thirdly, the function $\zeta(it\partial_\sigma + P_{\ker(\partial_\sigma)}, s)$ vanishes at $s = 0$. Indeed, this is equivalent to the fact that for the Riemann zeta function,

$$\zeta(0) = -\frac{1}{2}.$$

We can now prove our main result for non-invertible families. The first formula was proved in [3] when D is a Dirac operator. In the second formula $|D|$ is defined only on the orthogonal complement of the null-space of D.

Theorem 7.2. *Let D be a family of elliptic differential operators of order 1 over S^1, and δ_t the operator defined by (2). Then $\exp(-i\pi\overline{\eta}(\delta_t))$ and $t\zeta(\delta_t)$ extend smoothly to $t = 0$. Moreover,*

$$\lim_{t\to 0} e^{-i\pi\overline{\eta}(\delta_t)} = (-1)^{\mathrm{Ind}(D)}\mathrm{hol}(\det(D)) \tag{23}$$

$$\lim_{t\to 0} t\zeta(\delta_t, 0) = -\int_{S^1} \mathrm{Tr}_w|D|d\sigma \tag{24}$$

and

$$\lim_{t\to 0}\left(\zeta(\delta_t, 0) + \frac{1}{t}\int_{S^1}\mathrm{Tr}_w|D|d\sigma\right) = 0.$$

Proof: Working in the extended adiabatic algebra, one sees as in Section 6 that $\overline{\eta}(d_t, s)$ and $t\zeta(d_t, s)$ are smooth down to $t = 0$. For the constant term at $s = 0$, this says exactly that $\exp(-i\pi\overline{\eta}(\delta_t))$ and $t\zeta(\delta_t)$ are smooth for $t \in [0, \infty)$. To find $\lim_{t\to 0}\exp(-i\pi\overline{\eta}(\delta_t))$, we decompose formula (14) where D has been replaced by Q, by separating large and small eigenvalues:

$$\lim_{t\to 0}\overline{\eta}(d_t, s) = \sum_{j=1}^{k}\int_{W_{\alpha_j}}(P_{>\alpha_j}(Q) + P_{<\alpha_j}(Q))d\sigma$$

where $\{W_{\alpha_j}\}$ is a partition of S^1 in closed intervals such that $W_{\alpha_j} \subset V_{\alpha_j}$, and the V_{α_j}'s satisfy the technical condition of Section 4. From the choice of the cover $\{V_{\alpha_j}\}$, it follows as in Theorem 6.2 that

$$P_{>\alpha_j}(Q) = P_{>\alpha_j}(D) = \frac{1}{i\pi}\int_{W_{\alpha_j}} A(D, s)_{>\alpha_j}.$$

The quantity $P_{<\alpha_j}(Q)$ is continuous with respect to variations of the Schwartz function ϕ. We define a deformation for $0 < y \leq 1$ by $\phi_y(\tau) := \phi(\tau y)$. Note that $\phi_1 = \phi$ and ϕ_y tends to 1 in L^∞ as y tends to 0. The integrand in $P_{<\alpha_j}(Q(y))$ is uniformly bounded by $(1+\tau^2)^{\frac{s}{2}-1}$ as $y \to 0$, hence for $s = 0$, $P_{<\alpha_j}(Q)$ converges to $P_{<\alpha_j}(D_U)$. Again by the same computations as in Theorem 6.2, this equals $\frac{1}{i\pi}\int_{W_{\alpha_j}} A(D,s)_{<\alpha_j}$. So

$$\lim_{y\to 0}\lim_{t\to 0}\overline{\eta}(d_t) = \frac{1}{i\pi}\int_{S^1} A(D,0).$$

Then (23) follows from Proposition 7.1 since the above limit is constant in y.

For the zeta function the argument is similar. We decompose as above formula (19) with D replaced by Q. The contributions of small eigenvalues can be ignored, since the residue trace does not "see" finite rank operators. This gives (24), i.e. the coefficient of t^{-1} in the adiabatic limit. The coefficient of t^0 at $s = 0$ does not depend on ϕ. One sees that this coefficient vanishes if one chooses ϕ to be an even function of τ. In that case, the integrand giving the coefficient of t^0 is odd in τ. □

Note that the sign in formula (23) disappears if we change the definition of δ_t to be consistent with the choice of the bounding spin structure on S^1.

Also, when the dimension of N is even, the eta invariant is a $\pmod{2\mathbb{Z}}$ homotopy invariant of elliptic operators. Thus for every t we get:

$$e^{-i\pi\overline{\eta}(\delta_t)} = (-1)^{\mathrm{Ind}(D)}\mathrm{hol}(\det(D)).$$

We have deliberately avoided mentioning the regularity of the eta function at $s = 0$ since it plays no role in the proof. In light of this regularity however, the previous theorem gives a generalization of the Bismut-Freed holonomy theorem to any family of elliptic first-order differential operators D.

8. Closing remarks

The adiabatic algebra is a powerful tool for studying such degenerate families of operators as δ_t. Compared to previous results in this direction, one needs to make only minimal assumptions about the family D, while obtaining significant regularity in the adiabatic limit "for free" from the adiabatic formalism. The smoothness of the eta and zeta functions in the adiabatic limit is by no means obvious; it can actually fail if D is non-invertible even if δ_t is invertible for all t. For example, consider the operator on S^1

$$\delta_t := \begin{bmatrix} -ti\partial_\sigma - t\alpha & 0 \\ 0 & ti\partial_\sigma + t\alpha \end{bmatrix}.$$

For general α and s, the expansion of $\eta(\delta_t, s)$ and $\zeta(\delta_t, s)$ at $t = 0$ will contain $\log t$ terms. The reason is that δ_t is not invertible as an adiabatic operator.

One can consider other elliptic adiabatic operators constructed from the family D, for instance

$$P_t := t\tilde{\nabla}_{\partial_\sigma} + D$$

in the case where D is self-adjoint. If the fibers of $N \to S^1$ are spin, N is even-dimensional and D is the family of Dirac operators on the fibers then P_t is the chiral Dirac operator for the metric (3) on N. We can apply the previous analysis to $\zeta(P_t, 0)$. Formula (24) and Theorems 6.4, 6.5 hold for P_t if we divide the right-hand side by 2.

In the particular case where $N = S^1$ is the identity fibration and $D = 1$, Theorem 6.4 says that the function $t\sum_{k=-\infty}^{\infty}(t^2k^2+1)^{-s/2}$, which is well-defined for $(s,t) \in \{\mathrm{Re}(s) > 1\} \times (0,\infty)$, extends as a meromorphic function in $s \in \mathbb{C}$ and smooth in $t \in [0,\infty)$. Moreover,

$$\lim_{t\to 0} t\sum_{k=-\infty}^{\infty}(t^2k^2+1)^{-s/2} = \pi^{1/2}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

This identity could also be checked directly.

It is evident that $\mathrm{Ind}(P_t) = \zeta(P_t, 0) - \zeta(P_t^*, 0)$. The left-hand side is constant in t, in particular

$$\mathrm{Ind}(P_t) = \lim_{t\to 0}(\zeta(P_t, 0) - \zeta(P_t^*, 0)).$$

Using the adiabatic algebra methods developed in this paper, we obtain

$$\begin{aligned}
\zeta(P_t, 0) \quad &\sim_{t\to 0} \quad -\frac{1}{2t}\int_{S^1}\mathrm{Tr}_w(D^2)^{\frac{1}{2}}d\sigma + \frac{1}{4}\int_{S^1}\mathrm{Tr}_w\left((D^2)^{-\frac{1}{2}}\tilde{\nabla}_{\partial_\sigma}(D)\right)d\sigma\\
\zeta(P_t^*, 0) \quad &\sim_{t\to 0} \quad -\frac{1}{2t}\int_{S^1}\mathrm{Tr}_w(D^2)^{\frac{1}{2}}d\sigma - \frac{1}{4}\int_{S^1}\mathrm{Tr}_w\left((D^2)^{-\frac{1}{2}}\tilde{\nabla}_{\partial_\sigma}(D)\right)d\sigma.
\end{aligned}$$

We are therefore led to the formula

$$\mathrm{Ind}(P_t) = \frac{1}{2}\int_{S^1}\mathrm{Tr}_w\left((D^2)^{-\frac{1}{2}}\tilde{\nabla}_{\partial_\sigma}(D)\right)d\sigma. \tag{25}$$

Formally, the right-hand side of (25) is $\frac{1}{2}\int_{S^1}\mathrm{Tr}_w(\tilde{\nabla}_{\partial_\sigma}(\log|D|))d\sigma$, and it is easily proved to equal the net flow of eigenvalues through 0 around the circle. We are therefore led to a purely analytical proof of the identity

$$\mathrm{Ind}(P_t) = \mathrm{sf}_{S^1}(D) \tag{26}$$

where sf denotes the spectral flow [1].

In the spirit of [6] we can interpret the right-hand side of (25) as the integral on S^1 of a closed 1-form defined on S^1 (or more generally on the basis M of the family D) by

$$\mathrm{var}(\eta) := \mathrm{Tr}_w\left((D^2)^{-\frac{1}{2}}\tilde{\nabla}(D)\right).$$

In [6] Melrose defined the analogous quantity in the more general setting of 1-suspended operators, and proved subsequently the extension of (26) to that setting. Our methods can be used to reprove Melrose's result for an elliptic adiabatic family in $\Psi_a(N)$.

References

[1] M.F. Atiyah, V.K. Patodi, and I.M. Singer. Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Philos. Soc.*, 1976.

[2] J.-M. Bismut and J. Cheeger. η-invariants and their adiabatic limits. *J. Am. Math. Soc.*, 2:33–70, 1989.

[3] J.-M. Bismut and D.S. Freed. The analysis of elliptic families II: Dirac operators, eta invariants and the holonomy theorem of Witten. *Commun. Math. Phys.*, 107:103–163, 1986.

[4] Rafe R. Mazzeo and Richard B. Melrose. The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration. *J. Differential Geom.*, 31(1):185–213, 1990.

[5] R.B. Melrose. Pseudodifferential operators, corners and singular limits. In *Proceedings of the International Congress of Mathematicians (Kyoto,1990)*, pages 217–234, Tokyo, 1991. Math. Soc. Japan.

[6] R.B. Melrose. The eta invariant and families of pseudodifferential operators. *Math. Res. Letters*, 2:541–561, 1995.

[7] S. Moroianu. *Residue functionals on the algebra of adiabatic pseudo-differential operators.* PhD thesis, MIT, 1999.

[8] D. Quillen. Determinants of Cauchy-Riemann operators over a Riemann surface. *Funk. Anal. iprilozen*, 19:37–41, 1985.

[9] E. Witten. Global gravitational anomalies. *Commun. Math. Phys.*, 100:197–229, 1985.

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