Prescribed Energy Solutions of some class of Semilinear Elliptic Equations

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joint work with Francesca Alessio

De Giorgi Conjecture ('78):

If $-\Delta u = u - u^3$ on $\mathbf{R}^n$, $-1 < u < 1$, $\partial_{x_n} u > 0$ and $n \leq 8$ then

$$u(x) = q(a \cdot x + b) \text{ for an } a \in \mathbf{R}^n \text{ and } b \in \mathbf{R}$$

where q solves $-\ddot{q} = q - q^3$ on $\mathbf{R}$

Gibbons conjecture: If $-\Delta u = u - u^3$ on $\mathbf{R}^n$, and

$$\lim_{x_n \to \pm\infty} u(x) = \pm 1 \text{ uniformly w.r.t } (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$$

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Proof of Gibbons Conjecture

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Infinitely many non monotone multidimensional solutions

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Existence of one multidimensional solution

Equations with potential depending on the x_n variable

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Systems of Allen Cahn type equations

Consider the system studied in [ABG97]

(S) $$-\Delta u(x,y) + \nabla W(u(x,y)) = 0, \quad (x,y) \in \mathbf{R}^2$$

where $W \in C^2(\mathbf{R}^2, \mathbf{R})$ is a double well potential:

(W1) $0 = W(\pm 1, 0) < W(\xi)$ for any $\xi \in \mathbf{R}^2 \setminus \{(\pm 1, 0)\}$, $D^2 W(\pm 1, 0) > 0$,

(W2) $\liminf_{|\xi| \to +\infty} W(\xi) > 0$,

(W3) $W(-x, y) = W(x, y)$.

Problem: find bidimensional solutions u satisfying

$$\lim_{x \to \pm\infty} u(x,y) = (\pm 1, 0) \quad \text{unif. w.r.t. } y \in \mathbf{R}.$$

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Associated ODE System: heteroclinic solutions

$$\begin{cases} -\ddot{q}(x) + \nabla W(q(x)) = 0, & x \in \mathbf{R} \\ q(\pm\infty) = (\pm 1, 0). \end{cases}$$

We look for the minima of the Action

$$V(q) = \int_{-\infty}^{+\infty} \frac{1}{2}|\dot{q}|^2 + W(q)\, dx$$

over the space

$$\Gamma = \{q - \psi \in H^1(\mathbf{R})^2 \,/\; q_1(x) = -q_1(-x)\}$$

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Discreteness Assumption (H_c): $K = K_- \cup K_+$ with $\mathrm{dist}_{L^2}(K_-, K_+) > 0$.

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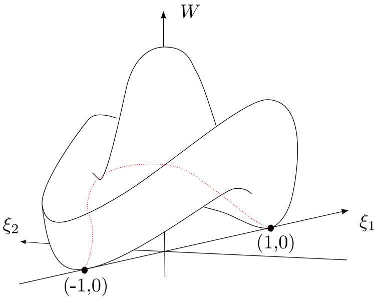

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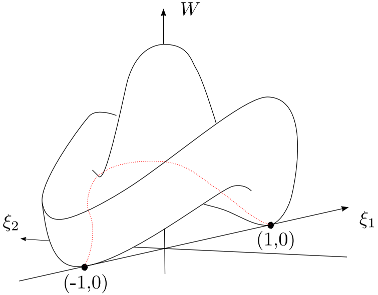

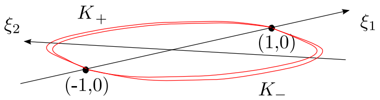

Looking for bidimensional solutions.

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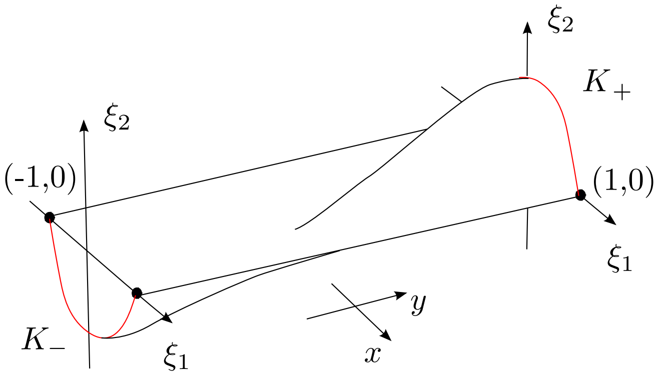

Theorem. If (W1)- (W3) and (H_c) are satisfied then there exists a bidimensional solution.

Sketch of the Proof:

Variational settings: We choose the variational space prescribing the right limits at infinity:

$$\mathcal{M} = \{u \in H^1_{loc}(\mathbf{R}^2)^2 \mid u(\cdot, y) \in \Gamma \text{ for a.e. } y \in \mathbf{R}\}$$

$$\mathcal{X} = \{u \in \mathcal{M} \mid \liminf_{y \to \pm\infty} \text{dist}_{L^2}(u(\cdot, y), K_\pm) = 0\}$$

Remark: The Euler Lagrange functional if always infinite on $\mathcal{X}$:

$$\int_{\mathbf{R}} \int_{\mathbf{R}} \tfrac{1}{2}|\partial_y u|^2 + \tfrac{1}{2}|\partial_x u|^2 + W(u)\,dx\,dy$$

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The renormalized Action functional: bidimensional solutions are searched as minima on the space $\mathcal{X}$ of the functional

$$\phi(u) = \int_{\mathbf{R}} \frac{1}{2}\|\partial_y u(\cdot,y)\|^2_{L^2(\mathbf{R})^2} + (V(u(\cdot,y)) - c)\ dy$$

Main estimates:

1) $$\|u(\cdot,y_1) - u(\cdot,y_2)\|^2_{L^2(\mathbf{R})^2} \leq (y_2 - y_1)\int_{y_1}^{y_2} \|\partial_y u(\cdot,y)\|^2_{L^2(\mathbf{R})^2}\,dy$$

In particular, if $\phi(u) < +\infty$ then the map $y \in \mathbf{R} \to u(\cdot,y) \in \bar{\Gamma}$ is continuous with respect to the L^2 metric.

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In particular, if $\phi(u) < +\infty$ then the map $y \in \mathbf{R} \to u(\cdot, y) \in \bar{\Gamma}$ is continuous with respect to the L^2 metric.

2) if $y_1 < y_2$ and $u \in \mathcal{M}$ then

$$\phi(u) \ge \left(2 \tfrac{1}{y_2 - y_1} \int_{y_1}^{y_2} (V(u(\cdot, y)) - c) \; dy \right)^{1/2} \|u(\cdot, y_1) - u(\cdot, y_2)\|_{L^2(\mathbf{R})^2}.$$

By 1) and 2) we have control of the transition time from K_- to K_+ and so concentration in the y variable. Together with the symmetry in the x variable this allows to get existence.

The renormalized Action functional: bidimensional solutions are searched as minima on the space $\mathcal{X}$ of the functional

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Energy prescribed solutions

Heuristics. If $u \in \mathcal{M}$ solves (S) then

$$\partial_y^2 u(x,y) = \underbrace{-\partial_x^2 u(x,y) + \nabla W(u(x,y))}_{V'(u(\cdot,y))}$$

Then u defines a trajectory $y \in \mathbf{R} \to u(\cdot,y) \in \Gamma$ solution to the infinite dimensional Lagrangian system

$$\frac{d^2}{dy^2} u(\cdot,y) = V'(u(\cdot,y)).$$

Roles: y time variable. $-V$ potential Energy.One dim. sol. equilibria. Two dim. sol. heteroclinic solutions.

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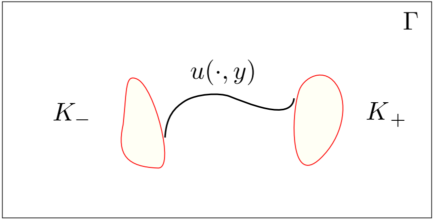

The Energy is conserved. If $u \in \mathcal{M}$ solves (S) on $\mathbf{R} \times (y_1, y_2)$ then

$$E_u(y) = \frac{1}{2}\|\partial_y u(\cdot, y)\|^2_{L^2(\mathbf{R})^2} - V(u(\cdot, y))$$

is constant on (y_1, y_2). ([AM, COCV, '05]; [Gui, JFA '08])

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$$(H_{\bar{c}}) \qquad \{V < \bar{c}\} = \mathcal{V}_- \cup \mathcal{V}_+ \text{ with } \mathrm{dist}_{L^2}(\mathcal{V}_-, \mathcal{V}_+) > 0.$$

Does there exist a solution $u \in \mathcal{M}$ with Energy $E_u = -\bar{c}$ which connects the sets $\mathcal{V}_-$ and $\mathcal{V}_+$?

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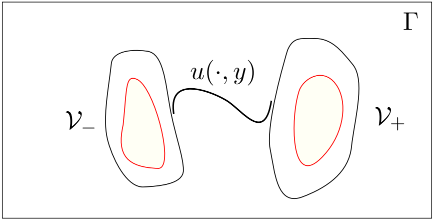

Some qualitative analysis.

Given a solution u at Energy $E_u = -\bar{c}$ we say that $u(\cdot, y_0)$ is a contact point if $V(u(\cdot, y_0)) = \bar{c}$.

If $u(\cdot, y_0)$ is a contact point then $E_u = -V(u(\cdot, y_0))$ and so

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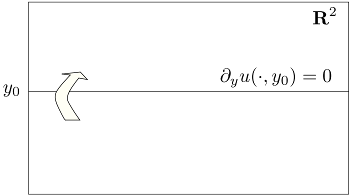

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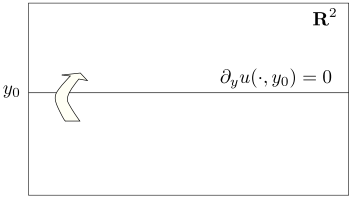

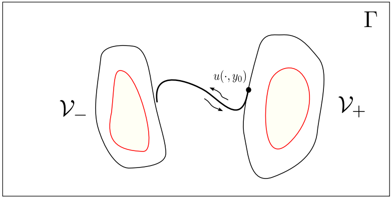

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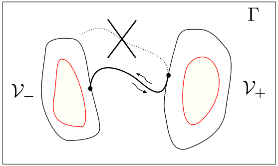

Brake orbit solution

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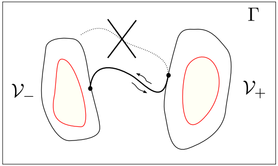

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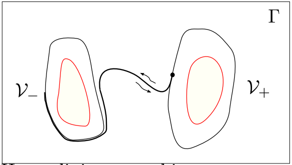

Homoclinic type orbit

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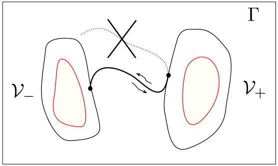

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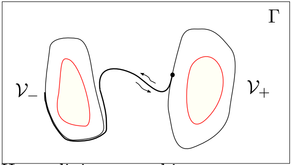

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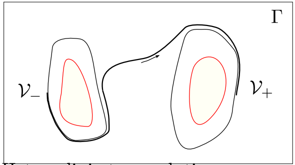

Heteroclinic type solution

The Variational setting.

We look for $-\bar{c}$-Energy connecting solutions looking for minima of the Functional

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on the space

$$\begin{aligned}\bar{\mathcal{X}} &= \{u \in \mathcal{M} \mid \liminf_{y\to\pm\infty} \operatorname{dist}_{L^2}(u(\cdot, y), \mathcal{V}_\pm) = 0 \\ &\qquad \text{and } V(u(\cdot, y)) \geq \bar{c} \text{ for a.e. } y \in \mathbf{R}\}\end{aligned}$$

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The above concentration arguments work even in this setting and the (suitable y-translated) minimizing sequences weakly converges to functions $\bar{u} \in \mathcal{M}$ (not a priori in $\bar{\mathcal{X}}$).

Set

$$\begin{aligned} \bar{s} &= \sup\{y \in \mathbf{R} / \mathrm{dist}_{L^2}(\bar{u}(\cdot, y), \mathcal{V}_-) \leq d_0 \text{ and } V(\bar{u}(\cdot, y)) \leq \bar{c}\} \\ \bar{t} &= \inf\{y > \bar{s} \,/\, V(\bar{u}(\cdot, y)) \leq \bar{c}\} \end{aligned}$$

then

- $-\infty \leq \bar{s} < \bar{t} \leq +\infty$
- if $[y_1, y_2] \subset (\bar{s}, \bar{t})$ then $\inf_{y \in [y_1, y_2]} V(\bar{u}(\cdot, y)) > \bar{c}$
- $\lim_{y \to \bar{s}^+} \mathrm{dist}_{L^2}(\bar{u}(\cdot, y), \mathcal{V}_-) = \lim_{y \to \bar{t}^-} \mathrm{dist}_{L^2}(\bar{u}(\cdot, y), \mathcal{V}_+) = 0$
- If $h \in C_0^\infty(\mathbf{R}^2)^2$ and $\mathrm{supp}(h) \subset \mathbf{R} \times (\bar{s}, \bar{t})$ then

$$\phi(\bar{u} + th) - \phi(\bar{u}) \geq 0 \text{ for } t \text{ small.}$$

This shows that $\bar{u}$ is a solution of (S) on $\mathbf{R} \times (\bar{s}, \bar{t})$ and so, by regularity, $V(u(\cdot, y)) \to \bar{c}$ whenever $y \to \bar{s}^+$ or $y \to \bar{t}^-$.

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$$\int_\sigma^\tau \frac{1}{2} \|\partial_y \bar{u}(\cdot, y)\|_{L^2}^2 \, dy = \int_\sigma^\tau V(\bar{u}(\cdot, y)) - \bar{c} \, dy,$$

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Last Remarks. If $\bar{s} > -\infty$ (resp. $\bar{t} < +\infty$) then it is a contact time.

If $\bar{s} = -\infty$ (resp. $\bar{t} = +\infty$) then the α-limit (resp. ω-limit) of $\bar{u}$ is constituted by critical points of V at level $\bar{c}$.

If $\bar{c}$ is a regular level of V then $-\infty < \bar{s} < \bar{t} < +\infty$ and $\bar{u}$ is of the brake orbit type.

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