On some variational problems in Riemannian and Fractal Geometry

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Abstract

We present some resent results, obtained in collaboration with G. Bonanno and V. Rădulescu on some variational problems arising from Geometry.
More precisely, in the first part of the talk, we deal with elliptic problems defined on compact Riemannian manifolds. This study is motivated by the Emden-Fowler equation that appears in mathematical physics, after a suitable change of coordinates, one obtains a new problem defined on the unit sphere $\mathbb{S}^d$ endowed of the standard metric.
In the second part, under an appropriate oscillating behavior either at zero or at infinity of the nonlinear term, the existence of a sequence of weak solutions for an eigenvalue Dirichlet problem, on a fractal domain, is proved.

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More precisely, in the first part of the talk, we deal with elliptic problems defined on compact Riemannian manifolds. This study is motivated by the Emden-Fowler equation that appears in mathematical physics, after a suitable change of coordinates, one obtains a new problem defined on the unit sphere $\mathbb{S}^d$ endowed of the standard metric.
In the second part, under an appropriate oscillating behavior either at zero or at infinity of the nonlinear term, the existence of a sequence of weak solutions for an eigenvalue Dirichlet problem, on a fractal domain, is proved.

We cite the following very recent monograph as general reference on this subject

A. Kristály, V. Rădulescu and Cs. Varga

Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems, Cambridge University press, 2010.

The problem

Let $(\mathcal{M}, g)$ be a compact d-dimensional Riemannian manifold without boundary, where $d \geq 3$. Let Δ_g denote the Laplace-Beltrami operator on $(\mathcal{M}, g)$ and assume that the functions $\alpha, K \in C^\infty(\mathcal{M})$ are positive. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a locally Hölder continuous function with *sublinear* growth and λ is a positive real parameter. We are interested in the existence of solutions to the following eigenvalue problem:

$$-\Delta_g w + \alpha(\sigma) w = \lambda K(\sigma) f(w), \qquad \sigma \in \mathcal{M},\ w \in H_1^2(\mathcal{M}) \qquad (P_\lambda)$$

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A problem on Riemannian manifolds

By using variational methods we find a well determined open interval of values of the parameter λ for which problem (P_λ) admits at least three solutions.

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A remarkable case of problem (P_λ) is

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma)f(w), \qquad \sigma \in \mathbb{S}^d,\ w \in H_1^2(\mathbb{S}^d), \quad (S_\lambda)$$

where $\mathbb{S}^d$ is the unit sphere in $\mathbf{R}^{d+1}$, h is the standard metric induced by the embedding $\mathbb{S}^d \hookrightarrow \mathbf{R}^{d+1}$, s is a constant such that $1-d < s < 0$, and Δ_h denotes the Laplace-Beltrami operator on $(\mathbb{S}^d, h)$.

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Existence results for problem (S_λ) yield, by using an appropriate change of coordinates, the existence of solutions to the following parameterized Emden-Fowler equation

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\}. \qquad (\mathfrak{F}_\lambda)$$

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Moreover, we observe that the existence of a smooth positive solution for problem (S_λ), when $s = -d/2$ or $s = -d/2 + 1$, and $f(t) = |t|^{\frac{4}{d-2}} t$, can be viewed as an affirmative answer to the famous Yamabe problem on $\mathbb{S}^d$.

For these topics we refer to Aubin, Cotsiolis and Iliopoulos, Hebey, Kazdan and Warner, Vázquez and Véron, and to the excellent survey by Lee and Parker.
In these cases the right hand-side of problem (S_λ) involves the critical Sobolev exponent.

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Further, Kristály, Rădulescu and Varga proved the existence of an open interval of positive parameters for which problem (P_λ) admits two distinct nontrivial solutions by using an abstract three critical points theorem due to Bonanno.

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Some remarks on a three critical points theorem, Nonlinear Anal. TMA **54** (2003), 651–665.

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Some basic facts

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Some basic facts

Let $(\mathcal{M}, g)$ be a smooth compact d-dimensional $(d \geq 3)$ Riemannian manifold without boundary and let g_{ij} be the components of the metric g. As usual, we denote by $C^\infty(\mathcal{M})$ the space of smooth functions defined on $\mathcal{M}$. Let $\alpha \in C^\infty(\mathcal{M})$ be a positive function and put $\|\alpha\|_\infty := \max_{\sigma \in \mathcal{M}} \alpha(\sigma)$.

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Some basic facts

For every $w \in C^\infty(\mathcal{M})$, set

$$\|w\|^2_{H^2_\alpha} := \int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^2 d\sigma_g,$$

where ∇w is the covariant derivative of w, and $d\sigma_g$ is the Riemannian measure. In local coordinates $(x^1, \ldots, x^d)$, the components of ∇w are given by

$$(\nabla^2 w)_{ij} = \frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial w}{\partial x^k},$$

where

$$\Gamma^k_{ij} := \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) g^{lk},$$

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Here, and in the sequel, the Einstein's summation convention is adopted. Moreover, the measure element $d\sigma_g$ assume the form $d\sigma_g = \sqrt{\det g}\, dx$, where dx stands for the Lebesgue's volume element of $\mathbf{R}^d$. Hence, let

$$\mathrm{Vol}_g(\mathcal{M}) := \int_{\mathcal{M}} d\sigma_g.$$

In particular, if $(\mathcal{M}, g) = (\mathbb{S}^d, h)$, where $\mathbb{S}^d$ is the unit sphere in $\mathbf{R}^{d+1}$ and h is the standard metric induced by the embedding $\mathbb{S}^d \hookrightarrow \mathbf{R}^{d+1}$, we set

$$\omega_d := \mathrm{Vol}_h(\mathbb{S}^d) := \int_{\mathbb{S}^d} d\sigma_h.$$

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The Sobolev space $H^2_\alpha(\mathcal{M})$ is defined as the completion of $C^\infty(\mathcal{M})$ with respect to the norm $\|\cdot\|_{H^2_\alpha}$. Then $H^2_\alpha(\mathcal{M})$ is a Hilbert space endowed with the inner product

$$\langle w_1, w_2\rangle_{H^2_\alpha} = \int_{\mathcal{M}} \langle \nabla w_1, \nabla w_2\rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)\langle w_1, w_2\rangle_g d\sigma_g,$$

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Some basic facts

Since α is positive, the norm $\|\cdot\|_{H^2_\alpha}$ is equivalent with the standard norm

$$\|w\|_{H^2_1} := \left(\int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} |w(\sigma)|^2 d\sigma_g\right)^{1/2}.$$

Moreover, if $w \in H^2_\alpha(\mathcal{M})$, the following inequalities hold

$$\min\{1, \min_{\sigma\in\mathcal{M}} \alpha(\sigma)^{1/2}\}\|w\|_{H^2_1} \leq \|w\|_{H^2_\alpha} \leq \max\{1, \|\alpha\|_\infty^{1/2}\}\|w\|_{H^2_1}. \qquad (1)$$

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From the Rellich-Kondrachov theorem for compact manifolds without boundary one has

$$H^2_\alpha(\mathcal{M}) \hookrightarrow L^q(\mathcal{M}),$$

for every $q \in [1, 2d/(d-2)]$. In particular, the embedding is compact whenever $q \in [1, 2d/(d-2))$.
Hence, there exists a positive constant S_q such that

$$\|w\|_q \leq S_q \|w\|_{H^2_\alpha}, \qquad \text{for all } w \in H^2_\alpha(\mathcal{M}). \tag{2}$$

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From now on, we assume that the nonlinearity f satisfies the following structural condition:

$f : \mathbb{R} \to \mathbb{R}$ is a locally Hölder continuous function sublinear at infinity, that is,

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Weak solutions of problem (P_λ)

A function $w \in H_1^2(\mathcal{M})$ is said a weak solution of (P_λ) if

$$\int_{\mathcal{M}} \langle \nabla w, \nabla v \rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) \langle w, v \rangle_g d\sigma_g - \lambda \int_{\mathcal{M}} K(\sigma) f(w(\sigma)) v(\sigma) d\sigma_g = 0,$$

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Weak solutions of problem (P_λ)

From a variational stand point the weak solutions of (P_λ) in $H_1^2(\mathcal{M})$, are the critical points of the C^1-functional given by

$$J_\lambda(u) := \frac{\|w\|^2_{H^2_\alpha}}{2} - \lambda \int_{\mathcal{M}} K(\sigma)F(w(\sigma))d\sigma_g,$$

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Theorem (G. Bonanno and S.A. Marano, Appl. Anal. 2010)

Let X be a reflexive real Banach space, $\Phi : X \to \mathbf{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbf{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:

(a_1) $$\frac{\sup_{\Phi(x)\leq r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})};$$

(a_2) *for each* $\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)} \right[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

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Main results on the existence of
at least three solutions

Notations

We set

$$\kappa_\alpha := \left(\frac{2}{\|\alpha\|_{L^1(\mathcal{M})}} \right)^{1/2},$$

and

$$K_1 := \frac{S_1}{\sqrt{2}} \|\alpha\|_{L^1(\mathcal{M})}, \qquad K_2 := \frac{S_q^q}{2^{\frac{2-q}{2}} q} \|\alpha\|_{L^1(\mathcal{M})}.$$

Further, let

$$F(\xi) := \int_0^\xi f(t)\, dt,$$

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Existence of three solutions

Theorem (G. Bonanno,—–, V. Rădulescu; Nonlinear Anal. (2011))

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that (h_∞) holds and assume that

(h_1) *There exist two nonnegative constants a_1, a_2 such that*

$$|f(t)| \leq a_1 + a_2|t|^{q-1}, \qquad \text{for all } t \in \mathbf{R},$$

where $q \in]1, 2d/(d-2)[$;

(h_2) *There exist two positive constants γ and δ, with $\delta > \gamma\kappa_\alpha$, such that*

$$\frac{F(\delta)}{\delta^2} > \frac{\|K\|_\infty}{\|K\|_{L^1(\mathcal{M})}}\left(a_1\frac{K_1}{\gamma} + a_2K_2\gamma^{q-2}\right).$$

Then, for each parameter $\lambda \in \Lambda_{(\gamma,\delta)}$ the problem (P_λ), possesses at least three solutions in $H_1^2(\mathcal{M})$.

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Existence of three solutions for the problem

$$-\Delta_h w + \alpha(\sigma)w = \lambda K(\sigma)f(\omega), \qquad \sigma \in \mathbb{S}^d,\ w \in H_1^2(\mathbb{S}^d)$$

Three solutions on the sphere

Let $\alpha, K \in C^\infty(\mathbb{S}^d)$ be positive and set

$$K_1^\star := \frac{\kappa_1 \|\alpha\|_{L^1(\mathbb{S}^d)}}{\sqrt{2}\min\left\{1, \min_{\sigma\in\mathbb{S}^d} \alpha(\sigma)^{1/2}\right\}}. \qquad (3)$$

Further, for $q \in]1, 2d/(d-2)[$, we will denote

$$K_2^\star := \frac{\kappa_q^q \|\alpha\|_{L^1(\mathbb{S}^d)}}{2^{\frac{2-q}{2}} q \min\left\{1, \min_{\sigma\in\mathbb{S}^d} \alpha(\sigma)^{q/2}\right\}}. \qquad (4)$$

Three solutions on the sphere

Let $\alpha, K \in C^\infty(\mathbb{S}^d)$ be positive and set

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Where

$$\kappa_q := \begin{cases} \omega_d^{\frac{2-q}{2q}} & \text{if} \quad q \in [1,2[, \\ \max\left\{ \left(\dfrac{q-2}{d\omega_d^{\frac{q-2}{q}}} \right)^{1/2}, \dfrac{1}{\omega_d^{\frac{q-2}{2q}}} \right\} & \text{if} \quad q \in \left[2, \dfrac{2d}{d-2}\right[. \end{cases}$$

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Corollary

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that (h_∞) and (h_1) hold. Further, assume that there exist two positive constants γ and δ, with $\delta > \gamma\kappa_\alpha$, and

$$(h_2^\star) \quad \frac{F(\delta)}{\delta^2} > \frac{\|K\|_\infty}{\|K\|_{L^1(\mathbb{S}^d)}} \left(a_1 \frac{K_1^\star}{\gamma} + a_2 K_2^\star \gamma^{q-2} \right),$$

where $K_1^\star$ and $K_2^\star$ are given respectively by (3) and (4). Then, for each parameter λ belonging to $\Lambda^\star_{(\gamma,\delta)}$ the problem (S_λ^α) possesses at least three distinct solutions.

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$$\Lambda^{\star}_{(\gamma,\delta)} := \left] \frac{\delta^2 \|\alpha\|_{L^1(\mathbb{S}^d)}}{2F(\delta)\|K\|_{L^1(\mathbb{S}^d)}}, \frac{\|\alpha\|_{L^1(\mathbb{S}^d)}}{2\|K\|_\infty\left(a_1 \dfrac{K_1^\star}{\gamma} + a_2 K_2^\star \gamma^{q-2}\right)} \right[.$$

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Existence of three solutions for the Emden-Fowler problem

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\}$$

Emden-Fowler problems

Next, we consider the following parameterized Emden-Fowler problem that arises in astrophysics, conformal Riemannian geometry, and in the theories of thermionic emission, isothermal stationary gas sphere, and gas combustion:

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\}. \qquad (\mathfrak{F}_\lambda)$$

The equation $(\mathfrak{F}_\lambda)$ has been studied when f has the form $f(t) = |t|^{p-1} t$, $p > 1$, see Cotsiolis-Iliopoulos, Vázquez-Véron. In these papers, the authors obtained existence and multiplicity results for $(\mathfrak{F}_\lambda)$, applying either minimization or minimax methods.

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Emden-Fowler problems

The solutions of $(\mathfrak{F}_\lambda)$ are being sought in the particular form

$$u(x) = r^s w(\sigma), \tag{5}$$

where, $(r, \sigma) := (|x|, x/|x|) \in (0, \infty) \times \mathbb{S}^d$ are the spherical coordinates in $\mathbb{R}^{d+1} \setminus \{0\}$ and w be a smooth function defined on $\mathbb{S}^d$. This type of transformation is also used by Bidaut-Véron and Véron, where the asymptotic of a special form of $(\mathfrak{F}_\lambda)$ has been studied. Throughout (5), taking into account that

$$\Delta u = r^{-d} \frac{\partial}{\partial r}\left(r^d \frac{\partial u}{\partial r}\right) + r^{-2} \Delta_h u,$$

the equation $(\mathfrak{F}_\lambda)$ reduces to

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma) f(w), \qquad \sigma \in \mathbb{S}^d,\ w \in H_1^2(\mathbb{S}^d),$$

see also Kristály and Rădulescu.

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see also Kristály and Rădulescu.

Three solutions for Emden-Fowler problems

Corollary

Assume that d and s are two constants such that $1-d<s<0$. Further, let $K\in C^\infty(\mathbb{S}^d)$ be a positive function and $f:\mathbb{R}\to\mathbb{R}$ as in the previous Corollary. Then, for each parameter λ belonging to

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Example and Application

Example

Let $(\mathcal{M}, g)$ be a compact d-dimensional ($d \geq 3$) Riemannian manifold without boundary, fix $q \in]2, 2d/(d-2)[$ and let $K \in C^\infty(\mathcal{M})$ be a positive function. Moreover, let $h : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$h(t) := \begin{cases} 1 + |t|^{q-1} & \text{if } |t| \leq r \\ \dfrac{(1+r^2)(1+r^{q-1})}{1+t^2} & \text{if } |t| > r, \end{cases}$$

where r is a fixed constant such that

$$r > \max\left\{ \left(\frac{2}{\mathrm{Vol}_g(\mathcal{M})}\right)^{1/2}, q^{\frac{1}{q-2}} \left(\frac{\|K\|_\infty}{\|K\|_{L^1(\mathcal{M})}} (K_1 + K_2) \right)^{\frac{1}{q-2}} \right\}. \quad (6)$$

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From our Theorem, for each parameter

$$\lambda \in \left] \frac{qr^2 \operatorname{Vol}_g(\mathcal{M})}{2(qr + r^q)\|K\|_{L^1(\mathcal{M})}}, \frac{\operatorname{Vol}_g(\mathcal{M})}{2\|K\|_\infty (K_1 + K_2)} \right[,$$

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$$-\Delta_g w + w = \lambda K(\sigma) h(w), \qquad \sigma \in \mathcal{M},\ w \in H_1^2(\mathcal{M}),$$

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Three solutions for Elliptic problems

We just mention that similar results for elliptic problems on bounded domains of the Euclidean space are contained in

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Infinitely many weak solutions for Elliptic problems

Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbf{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

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Then, one has

Theorem (G. Bonanno,—-; Bound. Value Probl. 2009)

(a) *For every* $r > \inf_X \Phi$ *and every* $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, *the restriction of the functional* $I_\lambda := \Phi - \lambda\Psi$ *to* $\Phi^{-1}(]-\infty, r[)$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X.

(b) *If* $\gamma < +\infty$ *then, for each* $\lambda \in \left]0, \frac{1}{\gamma}\right[$, *the following alternative holds: either*
(b_1) I_λ *possesses a global minimum,*
or
(b_2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that* $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(c) *If* $\delta < +\infty$ *then, for each* $\lambda \in \left]0, \frac{1}{\delta}\right[$, *the following alternative holds: either*
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Let $f : \mathbf{R} \to \mathbf{R}$ *be a continuous non-negative function and* $p > N$. *Put*

$$\sigma(N,p) := \inf_{\mu \in]0,1[} \frac{1-\mu^N}{\mu^N(1-\mu)^p}, \qquad \tau := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega),$$

$$m := \frac{N^{-\frac{1}{p}}}{\sqrt{\pi}} \left[\Gamma\left(1+\frac{N}{2}\right)\right]^{\frac{1}{N}} \left(\frac{p-1}{p-N}\right)^{1-\frac{1}{p}} |\Omega|^{\frac{1}{N}-\frac{1}{p}},$$

and $\kappa := \dfrac{\tau^p}{m^p|\Omega|\sigma(N,p)}$. *Assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} < \kappa \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}. \qquad (g)$$

Then, for each $\lambda \in \left]\dfrac{\sigma(N,p)}{p\tau^p \limsup\limits_{\xi\to+\infty} \dfrac{F(\xi)}{\xi^p}}, \dfrac{1}{m^p p|\Omega| \liminf\limits_{\xi\to+\infty} \dfrac{F(\xi)}{\xi^p}}\right[$, *the problem* (D_λ^f) *admits a sequence of positive weak solutions which is unbounded in* $W_0^{1,p}(\Omega)$.

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Assumption (g) could be replaced by

(g') There exist two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$0 \leq a_n < \frac{1}{m\overline{\mu}^{N/p}\dfrac{\sigma^{1/p}(N,p)}{\tau}\omega_\tau^{1/p}} b_n$$

for every $n \in \mathbf{N}$ and $\lim_{n\to+\infty} b_n = +\infty$ such that

$$\lim_{n\to+\infty} \frac{|\Omega|F(b_n) - \overline{\mu}^N\omega_\tau F(a_n)}{b_n^p - m^p a_n^p \omega_\tau \dfrac{\sigma(N,p)}{\tau^p}\overline{\mu}^N} < \kappa|\Omega| \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p},$$

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Example

Assume that $p \in \mathbf{N}$ and $1 \leq N < p$. Put

$$a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!},$$

for every $n \in \mathbf{N}$.
Let $\{g_n\}$ be a sequence of non-negative functions such that:

g_1) $g_n \in C^0([a_n, b_n])$ such that $g_n(a_n) = g_n(b_n) = 0$ for every $n \in \mathbf{N}$;

g_2) $\int_{a_n}^{b_n} g_n(t)dt \neq 0$ for every $n \in \mathbf{N}$.

For instance, we can choose the sequence $\{g_n\}$ as follows

$$g_n(\xi) := \sqrt{\frac{1}{16(n+1)!^2} - \left(\xi - \frac{n!(n+2)}{2}\right)^2}, \quad \forall n \in \mathbf{N}.$$

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From our result, for each $\lambda > \dfrac{\sigma(N,p)}{p 2^p \tau^p}$ the problem

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the authors, assuming that $f(0) \geq 0$ and that

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$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} = 0 \quad \text{and} \quad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} = +\infty,$$

proved problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

admits a sequence of non-negative solutions which is unbounded in $C^0(\overline{\Omega})$.

Infinitely many positive weak solutions for quasilinear elliptic systems involving the (p, q)–Laplacian

Theorem (G. Bonanno,——, D. O'Regan; Math. Comput. Modelling 2010)

Let $\Omega \subset \mathbf{R}^2$ be a non-empty bounded open set with boundary of class C^1. Let $f, g : \mathbf{R}^2 \to \mathbf{R}$ be two positive $C^0(\mathbf{R}^2)$-functions such that the differential 1-form $\omega := f(\xi,\eta)d\xi + g(\xi,\eta)d\eta$ is integrable and let F be a primitive of ω such that $F(0,0) = 0$. Fix $p, q > 2$, with $p \leq q$, and assume that

$$\liminf_{y\to+\infty} \frac{F(y,y)}{y^p} = 0; \quad \limsup_{y\to+\infty} \frac{F(y,y)}{y^q} = +\infty.$$

Then, the problem

$$\begin{cases} -\Delta_p u = f(u,v) & \text{in} \quad \Omega, \\ -\Delta_q v = g(u,v) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \\ v|_{\partial\Omega} = 0, \end{cases} \tag{$S^\star$}$$

admits a sequence $\{(u_n, v_n)\}$ of weak solutions which is unbounded in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and such that $u_n(x) > 0$, $v_n(x) > 0$ for all $x \in \Omega$ and for all $n \in \mathbf{N}$.

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Let $\Omega \subset \mathbf{R}^2$ be a non-empty bounded open set with boundary of class C^1. Let $f, g : \mathbf{R}^2 \to \mathbf{R}$ be two positive $C^0(\mathbf{R}^2)$-functions such that the differential 1-form $\omega := f(\xi,\eta)d\xi + g(\xi,\eta)d\eta$ is integrable and let F be a primitive of ω such that $F(0,0) = 0$. Fix $p, q > 2$, with $p \leq q$, and assume that

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Elliptic problems and Orlicz-Sobolev spaces

The problem

In this framework we have studied the non-homogeneous problem (under either Neumann or Dirichlet boundary conditions)

$$-\mathrm{div}(\alpha(|\nabla u|)\nabla u) = \lambda f(x,u) \quad \text{in} \quad \Omega,$$

where, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, while $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, λ is a positive parameter and $\alpha : (0,\infty) \to \mathbb{R}$ is such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & \text{for} \quad t \neq 0 \\ 0, & \text{for} \quad t = 0\,, \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

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Some results

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Infinitely many weak solutions for the Sierpiński fractal

The problem

We study the following Dirichlet problem

$$\begin{cases} \Delta u(x) + a(x)u(x) = \lambda g(x) f(u(x)) & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases} \qquad (S^{f,g}_{a,\lambda})$$

where V stands for the Sierpiński gasket, V_0 is its intrinsic boundary, Δ denotes the weak Laplacian on V and λ is a positive real parameter. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that the variable potentials a, $g : V \to \mathbb{R}$ satisfy the following conditions:

(h_1) $a \in L^1(V, \mu)$ and $a \leq 0$ almost everywhere in V;

(h_2) $g \in C(V)$ with $g \leq 0$ and such that the restriction of g to every open subset of V is not identically zero.

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The problem

The Sierpiński gasket has the origin in a paper by Sierpiński. In a very simple manner, this fractal domain can be described as a subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of 1/4 of the area, removing the corresponding open triangle from each of the three constituent triangles and continuing in this way. This fractal can also be obtained as the closure of the set of vertices arising in this construction.

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The problem

Over the years, the Sierpiński gasket showed both to be extremely useful in representing roughness in nature and man's works. This geometrical object is one of the most familiar examples of fractal domains and it gives insight into the turbulence of fluids. According to Kigami this notion was introduced by Mandelbrot in 1977 to design a class of mathematical objects which are not collections of smooth components. We refer to Strichartz for an elementary introduction to this subject and to Strichartz for important applications to differential equations on fractals.

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The problem

The study of the Laplacian on fractals was originated in physics literature, where so-called *spectral decimation method* was developed in Alexander and Rammal *et al.*. The Laplacian on the Sierpiński gasket was first constructed as the generator of a diffusion process by Kusuoka and Goldstein.

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The problem

Finally, we recall that Breckner, Rădulescu and Varga in

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proved the existence of infinitely many solutions of problem $(S^{f,g}_{a,\lambda})$ under the key assumption, among others, that the non-linearity f is non-positive in a sequence of positive intervals. We point out that our results are mutually independent compared to those achieved in the above mentioned manuscript.

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Further References

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Abstract setting

Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V) := \{u \in C(V) \mid u|_{V_0} = 0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $||\cdot||_\infty$. For a function $u\colon V \to \mathbb{R}$ and for $m \in \mathbb{N}$ let

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m \\ |x-y|=2^{-m}}} (u(x)-u(y))^2. \tag{7}$$

We have $W_m(u) \leq W_{m+1}(u)$ for very natural m, so we can put

$$W(u) = \lim_{m\to\infty} W_m(u). \tag{8}$$

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Define

$$H_0^1(V) := \{u \in C_0(V) \mid W(u) < \infty\}.$$

It turns out that $H_0^1(V)$ is a dense linear subset of $L^2(V,\mu)$ equipped with the $||\cdot||_2$ norm. We now endow $H_0^1(V)$ with the norm

$$||u|| = \sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: for $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$ let

$$\mathcal{W}_m(u,v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y|=2^{-m}}} (u(x)-u(y))(v(x)-v(y)).$$

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$$\mathcal{W}(u,v) = \lim_{m\to\infty} \mathcal{W}_m(u,v).$$

Then $\mathcal{W}(u,v) \in \mathbb{R}$ and the space $H_0^1(V)$, equipped with the inner product $\mathcal{W}$, which induces the norm $||\cdot||$, becomes a real Hilbert space.
Moreover,

$$||u||_\infty \leq (2N+3)||u||, \text{ for every } u \in H_0^1(V), \tag{9}$$

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Main result

Theorem (G. Bonanno,—–, V. Rădulescu; preprint 2011)

Let $f : \mathbf{R} \to \mathbf{R}$ be a non-negative continuous function. Assume that

$$\liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} < +\infty \quad \textit{and} \quad \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = +\infty. \tag{h$_0$}$$

Then, for every

$$\lambda \in \left] 0, \frac{1}{2(2N+3)^2 \left(\int_V g(x) d\mu \right) \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2}} \right[,$$

there exists a sequence $\{v_n\}$ of pairwise distinct weak solutions of problem $(S^{f,g}_{a,\lambda})$ such that $\lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \|v_n\|_\infty = 0$.

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Remark

We explicitly observe that our result also holds for sign-changing functions $f : \mathbf{R} \to \mathbb{R}$ just requiring that

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Example

Set

$$a_1 := 2, \qquad a_{n+1} := (a_n)^{\frac{3}{2}},$$

for every $n \in \mathbf{N}$ and $S := \bigcup_{n \geq 0}]a_{n+1} - 1, a_{n+1} + 1[$. Define the continuous function $h : \mathbf{R} \to \mathbf{R}$ as follows

$$h(t) := \begin{cases} e^{\frac{1}{(t-(a_{n+1}-1))(t-(a_{n+1}+1))}+1} \frac{2(a_{n+1}-t)(a_{n+1})^3}{(t-(a_{n+1}-1))^2(t-(a_{n+1}+1))^2} & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

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and $H(a_{n+1})=(a_{n+1})^3$ for every $n\in\mathbf{N}$.
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Further perspectives

- To study the existence of multiple solutions for non-homogeneous Neumann problem on Riemannian manifolds with boundary;
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