Regularity of solutions to quasilinear elliptic systems

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Elliptic Systems

\[ \sum_{i=1}^{n} D_{x_i} A_i^\alpha(x, u, Du) = B_i(x, u, Du) \quad \alpha = 1, \ldots, m \]

- \( \Omega \subset \mathbb{R}^n \) open bounded, \( n \geq 2 \);
- \( A_i : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \rightarrow \mathbb{R}^n, B_i : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \rightarrow \mathbb{R} \)
- \( u \in W^{1,1}(\Omega; \mathbb{R}^m) \) weak solution

\[ \sum_{\alpha=1}^{m} \sum_{i=1}^{n} \int_{\Omega} A_i^\alpha(x, u, Du) \varphi_{x_i}^\alpha + \sum_{\alpha=1}^{m} B^\alpha(x, u, Du) \varphi^\alpha dx = 0 \]

for all test function \( \varphi \).
The definition of **weak solution** leads to assign growth assumptions on $A_i^\alpha$ and $B_i$.

**Regularity of weak solution**

The situation is very different with respect to the single equation case.

There is a gap in the regularity scale for the solutions of systems and for the minimizers of integral vectorial functionals.
Historical Notes

We confine our presentation to the fundamental steps

Hadamard 1890, Bernstein 1904, \( n = m = 2 \)

Contributions of Caccioppoli 1933, Schauder 1934, Morrey 1938, Douglas-Nirenberg 1954

No real progress was made (except in two dimensional case) until

De Giorgi 1957

Nash 1958, Parabolic and Elliptic equations: "P. R. Garabedian writes from London of a manuscript by Ennio de Giorgi containing such a result"
Historical Notes

Very powerful theory of regularity

Linear elliptic equation and quadratic functionals

\[ a_{ij}(x) \in L^\infty \]
\[ a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \]

every weak solution \( u \in W^{1,2} \)
is locally H"older continuous
Historical Notes: Single Equation

De Giorgi methods are based on different steps

1. Caccioppoli type inequalities on level sets
2. Local boundedness
3. Local Hölder continuity

Moser 1960 generalizes Harnack inequality to general linear equations

Generalizations by:

- Stampacchìa 1958-1960
- Ladyzhenskaya and Ural’tseva 1968 papers and book
- Serrin 1964-1965 complete analysis in nonlinear case and a counterexample to the regularity when $u \notin W^{1,2}$
None of the new proofs given of the De Giorgi’s result could be extended to cover the case of systems

De Giorgi 1968 proved that his result cannot extended to systems

De Giorgi’s counterexample

$$\sum_{ij} \frac{\partial}{\partial x_i} (a^\alpha_{ij}(x) \frac{\partial u^\beta}{\partial x_j}) = 0, \quad n = m > 2$$

$$a^\alpha_{ij}(x) = \delta_{ij} \delta_{\alpha\beta} + [(n-2)\delta_{i\alpha} + \frac{x_{\alpha}x_i}{|x|^2}][(n-2)\delta_{j\beta} + \frac{x_{\beta}x_j}{|x|^2}]$$

$$u(x) = \frac{x}{|x|^\gamma} \text{ with } \gamma = \frac{n}{2} \left(1 - \frac{1}{\sqrt{4(n-1)^2 + 1}}\right)$$

is a solution in $\mathbb{R}^n - \{0\}$ and a weak solution:

$$u \in W^{1,2} \text{ but is not bounded}$$
Historical Notes: Counterexamples to regularity

- Giusti-Miranda 1968 and Maz'ja 1968 in the quasilinear case
- For extremals of integral functional (systems in variation) Nečas 1975 ($n, m = n^2$)
- Nonlinear case with different growth assumption: Freshe 1973, Hildebrandt-Widman 1975
- More recent contribution by Šverák-Yan 2000 ($n=3, m=5$)

Phenomenon purely vectorial

Weak solutions to nonlinear elliptic systems or extremals to vector valued regular integrals in general are not smooth
These counterexamples suggested two directions in the mathematical literature:

1. **Indirect approach to regularity**: partial regularity i.e. smoothness of solutions up to a set $\Omega_0$ of zero measure with the study of the properties of the singular set.

2. **Everywhere regularity in the interior of $\Omega$**, when it is possible, starting as usual from the local boundedness.

Bombieri 1976: *...it is an interesting open question to find "good conditions" which imply regularity everywhere.*
Local boundedness: few contributions

Everywhere regularity needs additional assumptions

Local boundedness of solutions of *Linear* Elliptic Systems:

Ladyzhenskaya and Ural’tseva, 1968

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{n} a_{ij}(x) u_\alpha^x \right) + \sum_{\beta=1}^{\alpha} b_{i}^{\alpha\beta}(x) u^\beta + f_\alpha^i(x) \right) + \\
\sum_{i=1}^{n} \sum_{\beta=1}^{\alpha} c_i^{\alpha\beta}(x) u_\alpha^x + \sum_{\beta=1}^{\alpha} d^{\alpha\beta}(x) u^\beta = f_\alpha(x)
\]

\[\forall \alpha = 1, 2, \ldots, m, a_{ij}, b_{i}^{\alpha\beta}, c_i^{\alpha\beta}, d^{\alpha\beta} \text{ bounded measurable and given functions } f_\alpha^i, f_\alpha\]

Meier, 1982 in his PhD thesis (supervisor Hildebrandt) and in a subsequent paper studied the boundedness (and integrability properties) of solutions to quasilinear elliptic systems:

$$\text{div} (A^\alpha(x, u, Du)) = B^\alpha(x, u, Du) \quad \alpha = 1, \ldots, m$$

under the natural conditions: $p, p$-growth

- $\sum_\alpha A^\alpha \xi^\alpha \geq |\xi|^p - b|u|^{p-1} - c_1$
- $|A^\alpha| \leq C(|\xi|^{p-1} + |u|^{p-1} + c_1)$
- $|B^\alpha| \leq C(|\xi|^{p-1} + |u|^{p-1} + c_1)$
Meier’s result is obtained through the pointwise assumption for the indicator function

\[ I_A(x, u, Du) = \sum_{\alpha, \beta} \frac{u^\alpha u^\beta}{|u|^2} D u^\beta A^\alpha(x, u, Du) \geq 0 \]

The arguments of the proof consist in a nontrivial generalization of the Serrin arguments for the single equation

The linear case considered by Ladyzhenskaya and Ural’tseva is included
Further contributions

Local boundedness for systems

- Under the same assumptions of Meier additional results by Landes 1989, 2000, 2005
- Following the ideas of Landes Krömer 2009 obtained similar results to Meier’s ones (which however is not cited) for zero boundary data

The Meier’s condition on $I_A$ imposes structure conditions

Structure conditions

$$\text{div}(|Du|^{p-2}Du) = 0 \quad \text{and} \quad I(u) = \int |Du|^p \, dx, \quad p \geq 2$$

- Uhlenbeck 1975 gave a complete regularity result: $u \in C^{1,\tau}$
- Giaquinta-Modica 1986, Acerbi-Fusco 1989
Also for the systems the local boundedness is the first step to get more regularity.

Hölder continuity for **BOUNDDED** solution

Under additional structure assumptions:

- **Wiegner 1976, 1981**
- **Hildebrant-Widman 1977**: Green’s function
- **Caffarelli 1982** with different methods: weak Harnack inequality for supersolutions of a linear elliptic equation.
The generalization to systems of the arguments used for a single equation are by no means obvious.

Technical problems depend very often on

- the availability of **appropriate test functions**
- using the solution as a test function
- the way of truncating the vector valued solution: in the area of truncation the gradient is not vanishing as it does in the scalar case and can interfere in a bad way with the leading part
Quasilinear elliptic system

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{n} a_{ij}(x, u, Du) u_{x_j}^\alpha + b_i^\alpha(x, u, Du) \right) = f^\alpha(x, u, Du)
\]

\[\alpha = 1, 2, \ldots, m\]

Generalization of Ladyzhenskaya and Ural’tseva system to the quasilinear case

Arises in many problems in differential geometry such as that harmonic mappings between manifolds or surfaces of prescribed mean curvature.
Anisotropic growth conditions (simplified version)

$p_i$-ellipticity: $p_1, p_2, \ldots, p_n \in (1, +\infty)$

$$\sum_{i,j=1}^{n} a_{ij}(x, u, \xi) \lambda_i \lambda_j \geq M \sum_{i=1}^{n} \lambda_i^2 \left( \sum_{\alpha=1}^{m} (\xi_{i\alpha}^\alpha)^2 \right)^{\frac{p_i - 2}{2}}$$

$p_i$-growth conditions

$$\left| \sum_{j=1}^{n} a_{ij}(x, u, \xi) \xi_{j}^\alpha \right| \leq M \left\{ \sum_{j=1}^{n} \left| \xi_{j} \right|^{p_j} + \left| u \right|^\gamma + 1 \right\}^{1 - \frac{1}{p_i}} \quad \forall i, \alpha$$
#### Anisotropic growth conditions

**Growth condition on the perturbation term** $b_i^\alpha$

$$|b_i^\alpha (x, u, \xi)| \leq M \left\{ \sum_{j=1}^{n} |\xi|^p_{j} (1-\epsilon) + |u|^\gamma + 1 \right\}^{1 - \frac{1}{p_i}} \quad \forall i, \alpha$$

**Growth condition on data** $f^\alpha$

$$|f^\alpha (x, u, \xi)| \leq M \left\{ \sum_{j=1}^{n} |\xi|^p_{j} (1-\delta) + |u|^\gamma^{-1} + 1 \right\} \quad \forall \alpha$$

for suitable $\gamma$, $\epsilon$ and $\delta$
Anisotropic Sobolev spaces

Definition

\[ W^{1,(p_1,\ldots,p_n)}(\Omega; \mathbb{R}^m) = \{ u \in W^{1,1}(\Omega; \mathbb{R}^m), \ u_{x_i} \in L^{p_i}(\Omega; \mathbb{R}^m), \ \forall i \} \]

Norm

\[ \| u \|_{W^{1,(p_1,\ldots,p_n)}(\Omega)} = \| u \|_{L^1(\Omega)} + \sum_{i=1}^{n} \| u_{x_i} \|_{L^{p_i}(\Omega)} \]

\[ W^{1,(p_1,\ldots,p_n)}_0(\Omega; \mathbb{R}^m) = W^{1,1}_0(\Omega; \mathbb{R}^m) \cap W^{1,(p_1,\ldots,p_n)}(\Omega; \mathbb{R}^m) \]
Embedding Theorem

Let $\bar{p}$ be the harmonic average of the $\{p_i\}$ i.e.

$$
\frac{1}{\bar{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}
$$

Troisi’s Theorem 1969

Let $u \in W_0^{1,(p_1,\ldots,p_n)}(\Omega; \mathbb{R}^m)$

$$
\|u\|_{L_{\bar{p}^*}}(\Omega) \leq c \prod_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}}(\Omega),
$$

where $\bar{p}^*$ is the usual Sobolev exponent of $\bar{p}$
Definition of weak solution

\[ u \in W^{1, (p_1, \ldots, p_n)}_{\text{loc}}(\Omega; \mathbb{R}^m) \text{ is a weak solution if for all } \alpha = 1, \ldots, m \]

\[ \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij}(x, u, Du) u_{x_j}^\alpha + b_i^\alpha(x, u, Du) \right) \varphi_{x_i}^\alpha dx + \int_{\Omega} f^\alpha(x, u, Du) \varphi^\alpha dx = 0 \]

for all \( \varphi \in C^1_0(\Omega; \mathbb{R}^m) \) (for density also \( \varphi \in W^{1, (p_1, \ldots, p_n)}_0(\Omega; \mathbb{R}^m) \))

Assumptions allow the \textit{good} definition of weak solution since

- \( | \sum_{j=1}^{n} a_{ij}(\cdot, u, Du) u_{x_j}^\alpha |, \ b_i^\alpha(\cdot, u, Du) \in L^{(p_i)'(\Omega)}_{\text{loc}} \)

- \( f^\alpha(\cdot, u, Du) \in L^{(\bar{p}^*)'(\Omega)}_{\text{loc}} \)
Local boundedness


Assume

$$\max \{p_1, p_2, \ldots, p_n\} < \bar{p}^*$$

and

$$1 < \gamma < \bar{p}^*, \quad 0 < \epsilon < 1, \quad \frac{1}{\bar{p}^*} < \delta < 1$$

then every weak solution \( u \) is locally bounded and there exist \( c \geq 0 \) and \( \theta \geq 0 \) such that

$$\sup_{B_{R/2}(x_0)} |u| \leq c \left\{ \int_{B_{R}(x_0)} (|u| + 1)\bar{p}^* \, dx \right\}^{\frac{1}{\bar{p}^*} (1+\theta)}$$
Sharp condition

In the scalar case Boccardo-Marcellini-Sbordone 1990

Assumption \( q = \max \{ p_1, p_2, \ldots, p_n \} < p^* \) is sharp

Counterexamples

Counterexamples exist when \( q > p^* \)

- Marcellini 1987, Giaquinta 1987 \( m = 1, n > 3 \)

\[
\int_{\Omega} \left( \sum_{i=1}^{n-1} |u_{x_i}|^2 + c |u_{x_n}|^q \right) \, dx
\]

has an unbounded minimizer if \( \frac{q}{2} > \frac{n-1}{n-3} \)
Ellipticity assumption

\[ \sum_{i,j=1}^{n} a_{ij}(x, u, \xi) \lambda_i \lambda_j \geq M \sum_{i=1}^{n} \lambda_i^2 \left( \sum_{\alpha=1}^{m} (\xi_i^\alpha)^2 \right)^{\frac{p_i-2}{2}} \]

- is a **weaker** assumption with respect to the usual ellipticity and it reduces to the ordinary ellipticity only if \( p_1 = p_2 = \ldots = p_n = 2 \)

- implies that there exists \( M_1 > 0 \) such that

\[ \sum_{\alpha=1}^{m} \sum_{i,j=1}^{n} a_{ij}(x, u, \xi) \xi_i^\alpha \xi_j^\alpha \geq M_1 \sum_{i=1}^{n} \left( \sum_{\alpha=1}^{m} (\xi_i^\alpha)^2 \right)^{\frac{p_i}{2}} \]

- includes the scalar case \( m = 1 \) in full generality
Our analysis unifies the scalar case (one single GENERAL equation) and the vector valued one (system of equations)

**General quasilinear elliptic equation**

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (a_i (x, u, Du)) = f (x, u, Du), \quad a_i \in C^1
\]

\[
a_i (x, u, Du) - a_i (x, u, 0) = \int_0^1 \frac{d}{dt} a_i (x, u, t Du) \, dt
\]

\[
= \int_0^1 \sum_{j=1}^{n} \frac{\partial a_i}{\partial \xi_j} (x, u, t Du) \, u_{x_j} \, dt = \sum_{j=1}^{n} u_{x_j} \int_0^1 \frac{\partial a_i}{\partial \xi_j} (x, u, t Du) \, dt
\]
The original general equation becomes:

\[
\sum_{ij} \frac{\partial}{\partial x_i} \left( a_{ij}(x, u, Du)u_{x_j} + a_i(x, u, 0) \right) = f(x, u, Du)
\]

\[
a_{ij} = \int_0^1 \frac{\partial a_i}{\partial \xi_j}(x, u, t Du) \, dt, \quad b_i = a_i(x, u, 0)
\]

i.e. scalar case of the systems considered above

Ellipticity assumption on \( a_{ij} \) in term of the vector field \( a_i \) is

\[
\sum_{i,j=1}^n \frac{\partial a_i}{\partial \xi_j}(x, u, \xi)\lambda_i\lambda_j \geq M \min_i \left( \frac{1}{p_i - 1} \right) \sum_{i=1}^n |\xi_i|^{p_i - 2} \lambda_i^2
\]
Arguments of the proof: test function

The proof is linked with the possibility to exhibit test functions related with the solution

\[ \Phi_{\nu,k}^h(t) \in L^\infty \text{ a suitable approximation of } t^{\nu p_h} \]

Let \( u \in W^{1,(p_1,\ldots,p_n)} \) be a weak solution and \( \eta \) the usual cut-off function, define:

\[ \varphi_{\nu}^h(x) = \Phi_{\nu,k}^h(|u(x)|) u(x) \eta^\mu(x) \text{ depending on } h = 1,\ldots, n \]

\[ \varphi_{\nu}^h \in W^{1,(p_1,\ldots,p_n)}_0 \implies \varphi_{\nu}^h \text{ is a "good" test function} \]
Arguments of the proof

Assume perturbation $b_i = 0$ and data $f = 0$

Insert $\varphi^h_{\nu}$ in the systems

$$I_1 + I_2 + I_3 = \int_{B_R} \sum_{\alpha=1}^m \sum_{ij=1}^n a_{ij}(x, u, Du) u^\alpha_{x_i} u^\alpha_{x_j} \Phi^h_{\nu} (|u|) \eta^\mu \, dx +$$

$$\int_{B_R} \sum_{\alpha, \beta=1}^m \sum_{ij=1}^n a_{ij}(x, u, Du) u^\alpha |u| u^\beta_{x_i} u^\beta_{x_j} (\Phi^h_{\nu} k)' (|u|) \eta^\mu \, dx +$$

$$\mu \int_{B_R} \sum_{\alpha=1}^m \sum_{ij=1}^n a_{ij}(x, u, Du) u^\alpha_{x_j} u^\alpha \Phi^h_{\nu} (|u|) \eta^{\mu-1} \eta_{x_i} \, dx = 0$$
\[ I_1 = \int_{B_R} \sum_{i,j=1}^{n} a_{ij}(x, u, Du) u^{\alpha}_{x_i} u^{\alpha}_{x_i} \Phi_{\nu}^{h} (|u|) \eta^{\mu} \, dx \]

\[ \geq M_1 \int_{B_R} \sum_{i=1}^{n} |u_{x_i}|^{p_i} \Phi_{\nu}^{h} (|u|) \eta^{\mu} \, dx \]

\[ \sum_{\alpha} \sum_{i,j=1}^{n} a_{ij}(x, u, Du) u^{\alpha}_{x_i} u^{\alpha}_{x_i} \geq M_1 \sum_{i=1}^{n} |u_{x_i}|^{p_i} \]
Elliptic Systems

Historical Notes

Local boundedness

Anisotropic behavior

\( p, q \)-growth

Systems with \( p, q \)-growth

General growth

The Last slide

Ellipticity

\[ l_2 = \int_{B_R} \sum_{i=1}^{n} \sum_{\alpha, \beta = 1}^{m} a_{ij}(x, u, Du) u^\alpha \frac{u^\beta}{|u|} u_{x_j}^\alpha u_{x_i}^\alpha (\Phi^h_{\nu k})'(|u|) \eta^\mu \, dx \geq 0 \]

\[ \sum_{i,j = 1}^{n} \sum_{\alpha, \beta = 1}^{m} a_{ij}(x, u, Du) u_{x_j}^\alpha u_{x_i}^\alpha u^\beta u^\beta = \]

\[ = \sum_{i,j = 1}^{n} a_{ij}(x, u, Du) \left\{ \sum_{\alpha = 1}^{m} u^\alpha u_{x_j}^\alpha \right\} \left\{ \sum_{\alpha = 1}^{m} u^\alpha u_{x_i}^\alpha \right\} \geq 0 \]

\[ \lambda_i = \left\{ \sum_{\alpha = 1}^{m} u^\alpha u_{x_i}^\alpha \right\} \]
Caccioppoli’s Estimates

**growth conditions:** $q = \max \{p_1, p_2, \ldots, p_n\}$

$$ |l_3| = \left| \int_{B_R} \sum_{i=1}^{n} \sum_{\alpha} a_{ij}(x, u, Du) u^\alpha u^{\alpha} \Phi^h_{\nu k}(|u|) \eta^{\mu-1} \eta x_i \, dx \right| \leq$$

$$\epsilon \int_{B_R} \sum_{i=1}^{n} |u_{x_i}|^{p_i} \Phi^h_{\nu k}(|u|) \eta^{\mu} \, dx + \frac{C_\epsilon}{(R-\rho)^q} \int_{B_R} \{|u|^q \Phi^h_{\nu k}(|u|)\} \, dx$$

as $k \to \infty \implies \Phi^h_{\nu k}(|u(x)|)$ goes to $|u(x)|^{\nu p_h}$

$$ \int_{B_R} |u_{x_h}|^{p_h} |u|^{p_h \nu} \eta^{\mu} \, dx \leq \frac{C}{(R-\rho)^q} \int_{B_R} |u|^q |u|^{p_h \nu} \, dx$$

$$(|u_{x_h}| |u|^\nu)^{p_h} \sim |D_{x_h} (|u|^{\nu+1})|^{p_h}$$
Iteration methods

Suitable application of Troisi’s embedding theorem

\[
\left\{ \int_{B_{\rho}} (1 + |u|)^{\frac{p^*}{p}(\nu+1)} \, dx \right\}^{\frac{1}{p^*}} \leq \left\{ \frac{C(\nu + 1)}{[R - \rho]^q} \right\}^{\frac{1}{p}} \left\{ \int_{B_R} (1 + |u|)^{q(\nu+1)} \right\}^{\frac{1}{q}}
\]

which permits the application of Moser iteration methods since

\[ q = \max \{p_1, p_2, \ldots, p_n\} < p^* \]

The presence of the perturbation \(b_i\) and the data \(f\) make the proof much more complex.
Overview on the regularity results under non natural growth conditions

There are many integral functionals (and the related Euler-Lagrange systems) whose integrands do not satisfy *natural* growth condition

- small perturbation of polynomial growth
  \[
  f(z) = |\xi|^p \log^\alpha (1 + |\xi|), \quad p \geq 1, \quad \alpha > 0
  \]

- anisotropic growth
  \[
  f(\xi) = (1 + |\xi|^2)^{\frac{p}{2}} + \sum_{i=1}^{n} |\xi_i|^p, \quad p_i \geq p, \quad \forall i = 1, \ldots, n
  \]

\[
  f(x, \xi) = |(\xi_1, \ldots, \xi_j)|^q + a(x)|(\xi_{j+1}, \ldots, \xi_n)|^p, \quad 0 \leq a(x) \leq M
  \]
$p - q$, anisotropic and general growth

- **variable exponent**
  \[ f(\xi) = |\xi|^{p(x)}, \quad f(\xi) = [h(|\xi|)]^{p(x)}, \quad 1 < p \leq p(x) \leq q \]

  Model proposed by Rajagopal- Růžička 2001 for electrorheological fluids

- **large perturbation of polynomial growth (exponential)**
  \[ f(\xi) \sim e^{\alpha |\xi|}, \quad \alpha > 0 \]

- **general growth**: there exists $g_1$ and $g_2$ convex functions such that
  \[ g_1(|\xi|) - c_1 \leq f(x, s, \xi) \leq c_2(1 + g_2(|\xi|)), \]
$p - q$, anisotropic and general growth

- $p, q$-growth (eventually anisotropic)
  - $q$ and $p$ linked together by a condition depending on $n$
    \[
    \frac{q}{p} \leq c(n) \to_{n \to \infty} 1
    \]

- General growth
  - Theory of $N$-function and Orlicz Spaces
$p, q$-growth: scalar case

**Single Equation and/or integral functional**

- Talenti 1990 [$L^\infty$ regularity]
- Fusco-Sbordone 1990, 1993 [$p_i, p_i$, De Giorgi Methods]
- Lieberman 1992 [$p, q$]
- Moscariello-Nania 1991 [$p - q, L^\infty_{loc}, C^{0,\alpha}$]
- Choe 1992 [$p, q, L^\infty_{loc} \Rightarrow C^{1,\alpha}_{loc} (q < p + 1)$]
- Fan et al. 1996-2010 [$p(x)$-growth $C^{0,\alpha}$]
- Cupini-Marcellini-Mascolo 2009 [anisotropic functional $p_i - q$]
- ....many others authors and papers
Anisotropic functionals: $p_i, q$-growth

$$I(u) = \int_{\Omega} f(x, Du) \, dx$$

Cupini-Marcellini-Mascolo: Examples

- $f(\xi) = |\xi|^p \log(1 + |\xi|) + |\xi_n|^q$
- $f(\xi) = [g(|\xi|)]^p + [g(|\xi_n|)]^q$

For example $g(t) = t^{[a+b+(b-a) \sin \log \log (e+t)]}$ (Talenti 1990)

- $f(x, \xi) = (|\xi|^\alpha + |\xi_n|^\beta(x))^\gamma$
- $f(x, \xi) = \left( \sum_{i=1}^n |\xi_i|^{r_i(x)} \right)^\gamma$
- $f(x, \xi) = F \left( \sum_{i=1}^n [h(|\xi_i|)]^{r_i(x)} \right)$
- $f(x, \xi) = F \left( \sum_{i=1}^n f_i(x, |\xi_i|) \right)$
Local boundedness

Special anisotropic growth conditions

\[ \sum_{i=1}^{n} [g(|\xi_i|)]^{p_i} \leq f(x, \xi) \leq L \left(1 + \sum_{i=1}^{n} [g(|\xi_i|)]^{q_i}\right), \quad 1 \leq p_i \leq q \]

- \( g : \mathbb{R}_+ \to \mathbb{R}_+ \), \( C^1 \), convex, increasing, \( g \in \Delta_2 \) i.e.
  \[ g(\lambda t) \leq \lambda^m g(t) \] for \( m, \lambda > 1 \) and \( t \geq t_0 \)

Cupini-Marcellini-Mascolo, 2009

If \( \max \{p_1, p_2, \ldots, p_n\} < \bar{p}^* \) the local minimizers of \( I \) are locally bounded and the following estimate holds:

\[ \|u - u_R\|_{L^\infty(B_{R/2})} \leq c \left\{ 1 + \int_{B_R} f(x, Du) \, dx \right\}^{\frac{1+\theta}{p}} \]

Condition on \( p_i \) is independent of \( g \)
$L^\infty$ first step to get regularity

Consider

$$\mathcal{F}(u) = \int_{\Omega} \sum_{i=1}^{n} |u_{x_i}(x)|^{p_i(x)} \, dx$$

- Lieberman 2005: $u \in L^\infty \Rightarrow u$ is Lipschitz continuous

**Application of Lieberman's results**

Let $p_1, p_2, .., p_n$ be Lipschitz continuous and for some $x_0$ we have

$$p_i(x_0) < (\overline{p}_i)(x_0), \quad \forall i = 1, 2, .., n$$

then the local minimizer $u$ of $\mathcal{F}$ is **Lipschitz continuous near $x_0$**
$p, q$: vector valued case

**Structure assumption:** $f(x, |Du|)$

- Acerbi-Fusco 1994 [partial regularity, $p_i - q$]
- Coscia-Mingione 1999 [$|Du|^{p(x)}$]
- Acerbi-Mingione 2000-2001 [partial regularity $|Du|^{p(x)}$]
- Leonetti-Mascolo-Siepe 2001, 2003 [Higher integrability, $Du \in L^\infty$, $1 < p < 2$]
- Bildhauer-Fuchs (et al.) 2002, 2003 [Higher integrability]
- Cupini-Guidorzi-Mascolo 2003 [Local Lipschitz continuity, new approximation methods]
- Foss-Passarelli-Verde, 2010 [Almost minimizers]
- De Maria -Passarelli 2010-2011 [partial regularity]
- Leonetti-Mascolo 2011
- ....many others authors and papers
Systems with $p, q$-growth (simplified version)

$p$-ellipticity condition

$$\sum_{i,j=1}^{n} \sum_{\alpha=1}^{m} a_{ij}(x, u, \xi) \lambda_i \lambda_j \geq M \sum_{i=1}^{n} \lambda_i^2 |\xi_i|^{p-2},$$

$p, q$-growth conditions

- $$\left| \sum_{j} a_{ij}(x, u, \xi) \xi_j^\alpha \right| \leq M |\xi|^{q-1} + |u|^{\gamma} + 1 \quad \forall i, \alpha$$
- $$|b_i^\alpha(x, u, \xi)| \leq M |\xi|^{p(1-\epsilon)} + |u|^{\gamma} + 1, \quad \forall i, \alpha$$
- $$|f^{\alpha}(x, u, \xi)| \leq M \left\{ |\xi|^{p(1-\delta)} + |u|^{\gamma-1} + 1 \right\}, \quad \forall \alpha$$

with suitable $\gamma$, $\epsilon$ and $\delta$. 
Systems with $p, q$-growth

**Assumptions**

- $\sum_{j=1}^{n} a_{ij}(x, u, Du) u_{x_j}^\alpha$ monotone and $\frac{q}{p} < \frac{n-1}{n-p}$

- $a_{ij} = A(x, u, |\xi|) \delta_{ij}$, $A(x, u, t) t$ increasing and $\frac{q}{p} < \frac{n}{n-p}$

**Cupini-Marcellini-Mascolo 2011: A priori estimate**

Let $u$ be a weak solution in $W^{1,q}(\Omega; \mathbb{R}^m)$

$$
\sup_{B_{R/2}(x_0)} |u| \leq c \left\{ \int_{B_R(x_0)} (|u| + 1)^{p^*} \, dx \right\}^{\frac{1+\theta}{p^*}}
$$

It remains an open problem whether the quasilinear system admits a weak solution in $W^{1,q}$.
Vector valued integrals with $p, q$ growth

\[ I(u) = \int_{\Omega} f(x, Du) \, dx \]

\[ |z|^p \leq f(x, z) \leq |z|^q + C, \quad z \in \mathbb{R}^{nm} \]

Leonetti-Mascolo: Examples

- $f_1(x, Du) = g(x, |Du|)$, $g(x, t)$ convex, $\Delta_2$ functions in $t$.

Leonetti-Mascolo: Examples with no structure assumptions

- $f_2(x, Du) = \sum_{i=1}^{n} h_i(x, |D_{x_i}u|)$
- $f_3(x, Du) = a(x, |(u_{x_1}, \ldots, u_{x_{j-1}})|) + b(x, |(u_{x_j}, \ldots, u_{x_n})|)$

$h_j(x, t)$, $a(x, t)$ and $b(x, t)$ are convex, $\Delta_2$ functions in $t$
Despite the difference in the shape of these functionals we identify common assumptions that allow to obtain *an unified proof* of regularity.

**Leonetti-Mascolo 2011**

Under the sharp assumption:

$$q < p^*$$

the local minimizers of $\mathcal{I}$ are locally bounded and the following local estimate holds:

$$\|u\|_{L^\infty(B_R \mathbb{Z})} \leq C \left( \int_{B_R} (1 + |u|^{p^*}) \, dx \right)^{\frac{p^* - p}{p^* (p^* - q)}}$$

**Bildhauer-Fuchs 2007, 2009** For the special splitting form

$$f_3 = f_3(Du): \ u \in L^\infty \Rightarrow \text{higher integrability } u \in L^\infty.$$
General growth

- **Mascolo-Papi 1994, 1996** [scalar \( f = \ldots g(|Du|) \): boundedness, Harnack Inequality]
- **Marcellini 1996** [vector: exponential growth \( Du \in L^\infty \)]
- **Dall’Aglio-Mascolo-Papi 1998** [scalar: \( f = f(x, u, Du) \)]
- **Mingione-Siepe 1999** [vector: \( t \log t \) growth]
- **Cianchi 2000** [scalar: boundedness Orlicz spaces]
- **Dall’Aglio-Mascolo 2002** [vector: boundedness \( f = g(x, |Du|) \)]
- **Mascolo-Migionini 2003** [vector: \( f = f(x, |Du|) \) exponential growth, \( Du \in L^\infty \)]
- **Marcellini-Papi 2006** [vector: slow and fast behaviour \( Du \in L^\infty \)]
- **Apushkinskya-Bildhauer-Fuchs 2009** [vector: \( u \in L^\infty \Rightarrow C^{1,\tau} \)]
- **....many others authors and papers**

Recommended survey on this field, Mingione 2006: *Regularity of minima: an invitation to the dark side of the calculus of variations*
Dall’Aglio-Mascolo, 2002

\[ \mathcal{I}(u) = \int_\Omega g(x, |Du|) \, dx \]

- \( g \in C^1 \), convex, increasing, \( g \in \Delta_2 \) and growth assumptions on \( g_x \).

Then all local minimizer of \( \mathcal{I} \) are locally bounded

Here we do not estimate the integrand with powers of the gradient and the arguments of the proof are strictly related with the properties of \( g \) which permit to consider a "suitable approximation" of \( g^\nu(x, |u(x)|) \) as a test function.
Camillo Benso, conte di Cavour (1810-1861)

He studied mathematics for many years at the military academy

...Dallo studio dei triangoli e delle formule algebriche sono passato a quelle degli uomini e delle cose; comprendo quanto quello studio mi sia stato utile per quello che ora vado facendo degli uomini e delle cose

...From studing triangles and algebraic formulas I switched to studing men and things, I realize how that study was useful for what I’m doing now about men and things