Regularity of solutions to quasilinear elliptic systems

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Elliptic Systems

$$\sum_{i=1}^{n} D_{x_i} A_i^{\alpha}(x, u, Du) = B_i(x, u, Du) \qquad \alpha = 1, ..., m$$

- $\Omega \subset \mathbb{R}^n$ open bounded, $n \geq 2$;
- $A_i : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \longrightarrow \mathbb{R}^n$, $B_i : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \longrightarrow \mathbb{R}$
- $u \in W^{1,1}(\Omega; \mathbb{R}^m)$ **weak solution**

$$\sum_{\alpha=1}^{m} \sum_{i=1}^{n} \int_{\Omega} A_i^{\alpha}(x, u, Du)\, \varphi_{x_i}^{\alpha} + \sum_{\alpha=1}^{m} B^{\alpha}(x, u, Du)\, \varphi^{\alpha}\, dx = 0$$

for all **test function** φ

Elliptic Systems

The definition of **weak solution** leads to assign growth assumptions on A_i^α and B_i

Regularity of weak solution

The situation is very different with respect to the single equation case

There is a gap in the regularity scale for the solutions of systems and for the minimizers of integral vectorial functionals

Historical Notes

We confine our presentation to the fundamental steps

Hadamard 1890, Bernstein 1904, $n = m = 2$

Contributions of Caccioppoli 1933, Schauder 1934, Morrey 1938, Douglas-Nirenberg 1954

No real progress was made (except in two dimensional case) until

De Giorgi 1957

Nash 1958, Parabolic and Elliptic equations: *"P. R. Garabedian writes from London of a manuscript by Ennio de Giorgi containing such a result"*

Historical Notes

Very powerful theory of regularity

Linear elliptic equation and quadratic functionals

- $a_{ij}(x) \in L^\infty$
- $a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$

every weak solution $u \in W^{1,2}$

is locally **Hölder continuous**

Historical Notes: Single Equation

De Giorgi methods are based on different steps

1. Caccioppoli type inequalities on level sets
2. Local boundedness
3. Local Hölder continuity

Moser 1960 generalizes Harnack inequality to general linear equations

Generalizations by:

- Stampacchia 1958-1960
- Ladyzhenskaya and Ural'tseva 1968 papers and book
- Serrin 1964-1965 complete analysis in nonlinear case and a counterexample to the regularity when $u \notin W^{1,2}$

Historical Notes: Conterexamples to regularity

None of the new proofs given of the De Giorgi's result could be extended to cover the case of systems

De Giorgi 1968 proved that his result cannot extended to systems

De Giorgi's counterexample

$$\sum_{ij} \frac{\partial}{\partial x_i}(a_{ij}^{\alpha\beta}(x)\frac{\partial u^\beta}{\partial x_j}) = 0, \quad n = m > 2$$

$$a_{ij}^{\alpha\beta}(x) = \delta_{ij}\delta_{\alpha\beta} + [(n-2)\delta_{i\alpha} + \frac{x_\alpha x_i}{|x|^2}][(n-2)\delta_{j\beta} + \frac{x_\beta x_j}{|x|^2}]$$

$$u(x) = \frac{x}{|x|^\gamma} \text{with } \gamma = \frac{n}{2}(1 - \frac{1}{\sqrt{4(n-1)^2+1}})$$

is a solution in $\mathbb{R}^n - \{0\}$ and a weak solution:

$u \in W^{1,2}$ but is not bounded

Historical Notes: Conterexamples to regularity

- Giusti-Miranda 1968 and Mazja 1968 in the quasilinear case
- For extremals of integral functional (sistems in variation) Nečas 1975 (n, $m = n^2$)
- Nonlinear case with different growth assumption: Freshe 1973, Hildebrandt-Widman 1975
- More recent contribution by Šverák-Yan 2000 (n=3, m=5)

Phenomenon purely vectorial

Weak solutions to nonlinear elliptic systems or extremals to vector valued regular integrals **in general are not smooth**

Regularity for systems

These counterexamples suggested **two directions** in the mathematical literature

1. **indirect approch to regularity**: partial regularity i.e. smootheness of solutions up to a set Ω_0 of zero measure with the study of the properties of the singular set
2. **everywhere regularity in the interior of Ω**, when it is possible, **starting as usual from the local boundedness**

Bombieri 1976 *....it is an interesting open question to find "good conditions" which imply regularity everywhere*

Local boundedness: few contributions

Everywhere regularity needs additional assumptions

Local boudedness of solutions of *Linear* Elliptic Systems:

Ladyzhenskaya and Ural'tseva, 1968

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}(x)\, u^{\alpha}_{x_j} + \sum_{\beta=1}^{m} b_i^{\alpha\beta}(x)\, u^{\beta} + f_i^{\alpha}(x) \right) +$$

$$+ \sum_{i=1}^{n} \sum_{\beta=1}^{m} c_i^{\alpha\beta}(x)\, u^{\beta}_{x_i} + \sum_{\beta=1}^{m} d^{\alpha\beta}(x)\, u^{\beta} = f^{\alpha}(x)$$

$\forall\, \alpha = 1, 2, \ldots, m$, a_{ij}, $b_i^{\alpha\beta}$, $c_i^{\alpha\beta}$, $d^{\alpha\beta}$ bounded measurable and given functions f_i^{α}, f^{α},

Generalizations: Nečas-Stara 1972, Tomi 1973, Weigner 1977

Bonn School: Meier results

Meier, 1982 in his PhD thesis (supervisor Hildebrandt) and in a subsequent paper studied the boundedness (and integrability properties) of solutions to quasilinear elliptic systems:

$$\operatorname{div}(A^\alpha(x,u,Du)) = B^\alpha(x,u,Du) \quad \alpha = 1, ...m$$

under the natural conditions: p, p-growth

- $\sum_\alpha A^\alpha \xi^\alpha \geq |\xi|^p - b|u|^{p-1} - c_1$
- $|A^\alpha| \leq C(|\xi|^{p-1} + |u|^{p-1} + c_1)$
- $|B^\alpha| \leq C(|\xi|^{p-1} + |u|^{p-1} + c_1)$

Meier Theorem

Meier's result is obtained through the pointwise assumption for the indicator function

Positivity of indicator function

$$\mathbf{I}_A(x,u,Du) = \sum_{\alpha,\beta} \frac{u^\alpha u^\beta}{|u|^2} Du^\beta A^\alpha(x,u,Du) \geq 0$$

The arguments of the proof consist in a nontrivial generalization of the Serrin arguments for the single equation

The linear case considered by Ladyzhenskaya and Ural'tseva is included

Further contributions

Local boundedness for systems

- Under the same assuptions of Meier additional results by Landes 1989, 2000, 2005
- Following the ideas of Landes Krömer 2009 obtained similar results to Meier's ones (which however is not cited) for zero boundary data

The Meier's condition on $\mathbf{I}_A$ imposes structure conditions

Structure conditions

$$\mathrm{div}(|Du|^{p-2}Du) = 0 \quad \text{and} \quad I(u) = \int |Du|^p \, dx, \;\; p \geq 2$$

- Uhlenbeck 1975 gave a complete regularity result: $u \in C^{1,\tau}$
- Giaquinta-Modica 1986, Acerbi-Fusco 1989

Hölder continuity

Also for the systems the local boundedness is the first step to get more regularity

Hölder continuity for **BOUNDED** solution

Under additional structure assumptions:

- Wiegner 1976, 1981
- Hildebrant-Widman 1977: Green's function
- Caffarelli 1982 with different methods: weak Harnack inequality for supersolutions of a linear elliptic equation.

Test functions

The generalization to systems of the arguments used for a single equation are by no means obvious

Technical problems depend very often on

- the availability of **appropriate test functions**
- using the solution as a test function
- the way of truncating the vector valued solution: in the area of truncation the gradient is not vanishing as it does in the scalar case and can interfere in a bad way with the leading part

Quasilinear elliptic system

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}(x, u, Du)\, u^{\alpha}_{x_j} + b^{\alpha}_{i}(x, u, Du) \right) = f^{\alpha}(x, u, Du)$$

$\alpha = 1, 2, \ldots, m$

Generalization of Ladyzhenskaya and Ural'tseva system to the *quasilinear* case

Arises in many problems in differential geometry such as that harmonic mappings between manifolds or surfaces of prescribed mean curvature.

Anisotropic growth conditions (simplified version)

p_i-ellipticity: $p_1, p_2, \dots, p_n \in (1, +\infty)$

$$\sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\lambda_i\lambda_j \geq M \sum_{i=1}^{n} \lambda_i^2 \left(\sum_{\alpha=1}^{m} (\xi_i^\alpha)^2 \right)^{\frac{p_i-2}{2}}$$

p_i- growth conditions

$$\left| \sum_{j=1}^{n} a_{ij}(x,u,\xi)\,\xi_j^\alpha \right| \leq M \left\{ \sum_{j=1}^{n} |\xi_j|^{p_j} + |u|^\gamma + 1 \right\}^{1-\frac{1}{p_i}} \qquad \forall i,\ \alpha$$

Anisotropic growth conditions

Growth condition on the perturbation term b_i^α

$$|b_i^\alpha(x,u,\xi)| \leq M \left\{ \sum_{j=1}^{n} |\xi|^{p_j(1-\epsilon)} + |u|^\gamma + 1 \right\}^{1-\frac{1}{p_i}} \quad \forall i,\ \alpha$$

Growth condition on data f^α

$$|f^\alpha(x,u,\xi)| \leq M \left\{ \sum_{j=1}^{n} |\xi|^{p_j(1-\delta)} + |u|^{\gamma-1} + 1 \right\} \quad \forall \alpha$$

for suitable γ, ϵ and δ

Anisotropic Sobolev spaces

Definition

$$W^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m) = \{u \in W^{1,1}(\Omega;\mathbb{R}^m),\ u_{x_i} \in L^{p_i}(\Omega;\mathbb{R}^m),\ \forall i\}$$

Norm

$$\|u\|_{W^{1,(p_1,\dots,p_n)}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}$$

$$W_0^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m) = W_0^{1,1}(\Omega;\mathbb{R}^m) \cap W^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$$

Embedding Theorem

Let $\overline{p}$ be the harmonic average of the $\{p_i\}$ i.e.

$$\frac{1}{\overline{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$$

Troisi's Theorem 1969

Let $u \in W_0^{1,(p_1,\ldots,p_n)}(\Omega;\mathbb{R}^m)$

$$\|u\|^n_{L^{\overline{p}^*}(\Omega)} \leq c\prod_{i=1}^{n}\|u_{x_i}\|_{L^{p_i}(\Omega)},$$

where $\overline{p}^*$ is the usual Sobolev exponent of $\overline{p}$

Definition of weak solution

$u \in W^{1,(p_1,\dots,p_n)}_{loc}(\Omega;\mathbb{R}^m)$ is a weak solution if for all $\alpha = 1, \dots, m$

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x,u,Du)u^\alpha_{x_j} + b^\alpha_i(x,u,Du) \right) \varphi^\alpha_{x_i} dx + \int_\Omega f^\alpha(x,u,Du)\varphi^\alpha dx = 0$$

for all $\varphi \in C^1_0(\Omega;\mathbb{R}^m)$ (for density also $\varphi \in W^{1,(p_1,\dots,p_n)}_0(\Omega;\mathbb{R}^m)$)

Assumptions allow the *good* definition of weak solution since

- $|\sum_{j=1}^n a_{ij}(\cdot,u,Du)u^\alpha_{x_j}|,\ \ b^\alpha_i(\cdot,u,Du) \in L^{(p_i)'}_{\mathrm{loc}}(\Omega)$
- $f^\alpha(\cdot,u,Du) \in L^{(\overline{p}^*)'}_{\mathrm{loc}}(\Omega)$

Local boundedness

Cupini-Marcellini-Mascolo 2011, to appear on Manuscripta Math.

Assume

$$\max\{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\} < \overline{\mathbf{p}}^*$$

and

$$1 < \gamma < \overline{p}^*, \quad 0 < \epsilon < 1, \quad \frac{1}{\overline{p}^*} < \delta < 1$$

then **every weak solution u is locally bounded** and there exist $c \geq 0$ and $\theta \geq 0$ such that

$$\sup_{B_{R/2}(x_0)} |u| \leq c \left\{ \int_{B_R(x_0)} (|u| + 1)^{\overline{p}^*} \, dx \right\}^{\frac{1}{\overline{p}^*}(1+\theta)}$$

Sharp condition

In the scalar case Boccardo-Marcellini-Sbordone 1990

Assumption $\mathbf{q} = \max\{\mathbf{p_1}, \mathbf{p_2}, \dots, \mathbf{p_n}\} < \overline{\mathbf{p}}^*$ is sharp

Counterexamples

Counterexamples exist when $q > \overline{p}^*$

- Marcellini 1987, Giaquinta 1987 $m = 1$, $n > 3$

$$\int_\Omega \left(\sum_{i=1}^{n-1} |u_{x_i}|^2 + c|u_{x_n}|^q \right) dx$$

has an unbounded minimizer if $\frac{q}{2} > \frac{n-1}{n-3}$

Ellipticity assumption

$$\sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\lambda_i\lambda_j \geq M \sum_{i=1}^{n} \lambda_i^2 \left(\sum_{\alpha=1}^{m} (\xi_i^\alpha)^2 \right)^{\frac{p_i-2}{2}}$$

- is a **weaker** assumption with respect to the usual ellipticity and it reduces to the ordinary ellipticity only if $p_1 = p_2 = ... = p_n = 2$

- implies that there exists $M_1 > 0$ such that

$$\sum_{\alpha=1}^{m} \sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\xi_i^\alpha \xi_j^\alpha \geq M_1 \sum_{i=1}^{n} \left(\sum_{\alpha=1}^{m} (\xi_i^\alpha)^2 \right)^{\frac{p_i}{2}}$$

- includes the scalar case $m = 1$ in full generality

Equations and/or Systems 1

Our analysis unifies the scalar case (one single GENERAL equation) and the vector valued one (system of equations)

General quasilinear elliptic **equation**

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(a_i \left(x, u, Du \right) \right) = f \left(x, u, Du \right), \quad a_i \in C^1$$

$$a_i \left(x, u, Du \right) - a_i \left(x, u, 0 \right) = \int_0^1 \frac{d}{dt} a_i \left(x, u, t\, Du \right) \, dt$$

$$= \int_0^1 \sum_{j=1}^{n} \frac{\partial a_i}{\partial \xi_j} \left(x, u, t\, Du \right) \, u_{x_j} \, dt = \sum_{j=1}^{n} u_{x_j} \int_0^1 \frac{\partial a_i}{\partial \xi_j} \left(x, u, t\, Du \right) \, dt$$

Equation and/or Systems 2

The original general equation becomes:

$$\sum_{ij} \frac{\partial}{\partial x_i} \left(a_{ij}(x,u,Du) u_{x_j} + a_i(x,u,0) \right) = f\,(x,u,Du)$$

$$a_{ij} = \int_0^1 \tfrac{\partial a_i}{\partial \xi_j}\,(x,u,t\,Du)\;dt, \quad b_i = a_i\,(x,u,0)$$

i.e. scalar case of the systems considered above

Ellipticity assumption on a_{ij} in term of the vector field a_i is

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \xi_j}(x,u,\xi)\lambda_i\lambda_j \geq M\,\min_i\left(\frac{1}{p_i-1}\right)\sum_{i=1}^{n} |\xi_i|^{p_i-2}\,\lambda_i^2$$

Arguments of the proof: test function

The proof is linked with the possibility to exibit test functions related with the solution

$\Phi^h_{\nu\, k}(t) \in L^\infty$ a suitable approximation of $t^{\nu p_h}$

Let $u \in W^{1,(p_1,\dots,p_n)}$ be a weak solution and η the usual cut-off function, define:

$$\varphi^h_\nu(x) = \Phi^h_{\nu\, k}(|u(x)|)\, u(x)\, \eta^\mu(x) \quad \text{depending on } h = 1\dots., n$$

$\varphi^h_\nu \in W_0^{1,(p_1,\dots,p_n)} \implies \varphi^h_\nu$ is a "**good**" test function

Arguments of the proof

Assume pertubation $b_i = 0$ and data $f = 0$

Insert φ_ν^h in the systems

$$I_1 + I_2 + I_3 = \int_{B_R} \sum_{\alpha=1}^{m} \sum_{i\,j=1}^{n} a_{ij}(x,u,Du) u^\alpha_{x_i} u^\alpha_{x_j} \Phi^h_{\nu\,k}(|u|)\, \eta^\mu \, dx +$$

$$\int_{B_R} \sum_{\alpha,\beta=1}^{m} \sum_{i\,j=1}^{n} a_{ij}(x,u,Du)\, u^\alpha \frac{u^\beta}{|u|} u^\alpha_{x_j} u^\beta_{x_i} \, (\Phi^h_{\nu\,k})'(|u|)\, \eta^\mu \, dx +$$

$$\mu \int_{B_R} \sum_{\alpha=1}^{m} \sum_{i\,j=1}^{n} a_{ij}(x,u,Du) u^\alpha_{x_j} u^\alpha \Phi^h_{\nu\,k}(|u|)\, \eta^{\mu-1} \eta_{x_i} \, dx = 0$$

Ellipticity

$$I_1 = \int_{B_R} \sum_{i\,j=1}^{n} a_{ij}(x,u,Du) u^{\alpha}_{x_i} u^{\alpha}_{x_i} \Phi^{h}_{\nu\,k}(|u|)\, \eta^{\mu}\, dx$$

$$\geq M_1 \int_{B_R} \sum_{i=1}^{n} |u_{x_i}|^{p_i} \Phi^{h}_{\nu\,k}(|u|) \eta^{\mu}\, dx$$

$$\sum_{\alpha} \sum_{i,j=1}^{n} a_{ij}(x,u,Du) u^{\alpha}_{x_i} u^{\alpha}_{x_i} \geq M_1 \sum_{i=1}^{n} |u_{x_i}|^{p_i}$$

Ellipticity

$$I_2 = \int_{B_R} \sum_{i=1}^{n} \sum_{\alpha,\beta=1}^{m} a_{ij}(x,u,Du)\, u^\alpha \frac{u^\beta}{|u|}\, u^\alpha_{x_j} u^\alpha_{x_i} \, (\Phi^h_{\nu\,k})'(|u|)\, \eta^\mu \, dx \geq \mathbf{0}$$

$$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} a_{ij}(x,u,Du) u^\alpha_{x_j} u^\alpha u^\beta \, u^\beta_{x_i} =$$

$$= \sum_{i,j=1}^{n} a_{ij}(x,u,Du) \left\{ \sum_{\alpha=1}^{m} u^\alpha \, u^\alpha_{x_j} \right\} \left\{ \sum_{\alpha=1}^{m} u^\alpha \, u^\alpha_{x_i} \right\} \geq 0$$

$$\lambda_{\mathbf{i}} = \left\{ \sum_{\alpha=1}^{\mathbf{m}} \mathbf{u}^\alpha \, \mathbf{u}^\alpha_{\mathbf{x_i}} \right\}$$

Caccioppoli's Estimates

growth conditions: $q = \max\{p_1, p_2, \ldots, p_n\}$

$$|I_3| = \left| \int_{B_R} \sum_{i=1} \sum_{\alpha,} a_{ij}(x, u, Du) u^{\alpha}_{x_j} u^{\alpha} \Phi^h_{\nu\, k}(|u|)\, \eta^{\mu-1} \eta_{x_i}\, dx \right| \leq$$

$$\epsilon \int_{B_R} \sum_{i=1}^{n} |u_{x_i}|^{p_i} \Phi^h_{\nu\, k}(|u|)\, \eta^{\mu}\, dx + \frac{C_\epsilon}{(R-\rho)^q} \int_{B_R} \left\{ |u|^q \Phi^h_{\nu\, k}(|u|) \right\}\, dx$$

as $k \to \infty \Longrightarrow \Phi^h_{\nu\, k}(|u(x)|)$ goes to $|u(x)|^{\nu p_h}$

$$\int_{B_R} |u_{x_h}|^{p_h} |u|^{p_h \nu} \eta^{\mu}\, dx \leq \frac{C}{(R-\rho)^q} \int_{B_R} |u|^q |u|^{p_h \nu}\, dx$$

$$(|\mathbf{u}_{\mathbf{x}_h}||\mathbf{u}|^{\nu})^{\mathbf{p}_h} \sim |\mathbf{D}_{\mathbf{x}_h}(|\mathbf{u}|^{\nu+1})|^{\mathbf{p}_h}$$

Iteration methods

Suitable application of Troisi's embbeding theorem

$$\left\{ \int_{B_\rho} (1+|u|)^{\overline{p}^*(\nu+1)}\, dx \right\}^{\frac{1}{\overline{p}^*}} \leq \left\{ \frac{C(\nu+1)}{[R-\rho]^q} \right\}^{\frac{1}{p}} \left\{ \int_{B_R} (1+|u|)^{q(\nu+1)} \right\}^{\frac{1}{q}}$$

which permits the application of Moser iteration methods since

$$q = \max\{p_1, p_2, \ldots, p_n\} < \overline{p}^*$$

The presence of the perturbation b_i and the data f make the proof much more complex

$p - q$, anisotropic and general growth

Overview on the regularity results under non natural growth conditions

There are many integral functionals (and the related Euler-Lagrange systems) whose integrands do not satisfy *natural* growth condition

- small perturbation of polynomial growth

$$f(z) = |\xi|^p \log^\alpha(1 + |\xi|), \quad p \geq 1,\ \alpha > 0$$

- anisotropic growth

$$f(\xi) = (1 + |\xi|^2)^{\frac{p}{2}} + \sum_{i\,\alpha} |\xi_i^\alpha|^{p_i}, \quad p_i \geq p,\ \forall i = 1, \dots, n$$

$$f(x, \xi) = |(\xi_1, ..., \xi_j)|^q + a(x)|(\xi_{j+1}, ..., \xi_n)|^p, \quad 0 \leq a(x) \leq M$$

$p - q$, anisotropic and general growth

- variable exponent

$$f(\xi) = |\xi|^{p(x)}, \quad f(\xi) = [h(|\xi|)]^{p(x)}, \quad 1 < p \leq p(x) \leq q$$

Model proposed by Rajagopal- Růžička 2001 for electrorheological fluids

- large perturbation of polynomial growth (exponential)

$$f(\xi) \sim e^{|\xi|^{\alpha}}, \quad \alpha > 0$$

- general growth: there exists g_1 and g_2 convex functions such that

$$g_1(|\xi|) - c_1 \leq f(x, s, \xi) \leq c_2(1 + g_2(|\xi|)),$$

$p - q$, anisotropic and general growth

- p, q-growth (eventually anisotropic)

  q and p linked together by a condition depending on n

$$\frac{q}{p} \leq c(n) \to_{n\to\infty} 1$$

- General growth

  Theory of N-function and Orlicz Spaces

p, q-growth: scalar case

Single Equation and/or integral functional

- Marcellini 1989, 1991, 1993, 1996
- Talenti 1990 [L^∞ regularity]
- Boccardo-Marcellini-Sbordone 1990 [p_i, p_i] Stroffolini 1991
- Fusco-Sbordone 1990,1993 [p_i, p_i, De Giorgi Methods]
- Lieberman 1992 [p, q]
- Moscariello-Nania 1991 [$p - q$, L^∞_{loc}, $C^{0,\alpha}$]
- Choe 1992 [p, q, $L^\infty_{\mathrm{loc}} \Rightarrow C^{1,\alpha}_{\mathrm{loc}}$ $(q < p + 1)$]
- Bhattacharya-Leonetti 1991,1993,1996
- Fan et al.1996-2010 [$p(x)$-growth $C^{0,\alpha}$]
- **Cupini-Marcellini-Mascolo 2009** [anisotropic functional $p_i - q$]
- **....many others authors and papers**

Anisotropic functionals: p_i, q-growth

$$\mathcal{I}(u) = \int_\Omega f(x, Du)\, dx$$

Cupini-Marcellini-Mascolo: Examples

- $f(\xi) = |\xi|^p \log(1 + |\xi|) + |\xi_n|^q$
- $f(\xi) = [g(|\xi|)]^p + [g(|\xi_n|)]^q$

For example $g(t) = t^{[a+b+(b-a)\sin\log\log(e+t)]}$ (Talenti 1990)

- $f(x, \xi) = (|\xi|^\alpha + |\xi_n|^{\beta(x)})^\gamma$
- $f(x, \xi) = (\sum_{i=1}^n |\xi_i|^{r_i(x)})^\gamma$
- $f(x, \xi) = F\left(\sum_{i=1}^n [h(|\xi_i|)]^{r_i(x)}\right)$
- $f(x, \xi) = F\left(\sum_{i=1}^n f_i\,(x, |\xi_i|)\right)$

Local boundedness

Special anisotropic growth conditions

$$\sum_{i=1}^{n} [g(|\xi_i|)]^{p_i} \leq f(x,\xi) \leq L\,(1 + \sum_{i=1}^{n} [g(|\xi_i|)]^{q}), \quad 1 \leq p_i \leq q$$

- $g : \mathbb{R}_+ \to \mathbb{R}_+$, C^1, convex, increasing, $g \in \Delta_2$ i.e.

$$g(\lambda t) \leq \lambda^m g(t) \text{ for } m, \lambda > 1 \text{ and } t \geq t_0$$

Cupini-Marcellini-Mascolo, 2009

If $\max\{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\} < \overline{\mathbf{p}}^*$ the local minimizers of $\mathcal{I}$ are locally bounded and the following estimate holds:

$$\|u - u_R\|_{L^\infty(B_{R/2})} \leq c \left\{1 + \int_{B_R} f(x, Du)\, dx\right\}^{\frac{1+\theta}{p}}$$

Condition on $\mathbf{p_i}$ is independent of g

L^∞ first step to get regularity

Consider

$$\mathcal{F}(u) = \int_\Omega \sum_{i=1}^{n} |u_{x_i}(x)|^{p_i(x)}\, dx$$

- Lieberman 2005 : $u \in L^\infty \Rightarrow u$ is Lipschitz continuous

Application of Lieberman's results

Let $p_1, p_2, ..p_n$ be Lipschitz continuous and for some x_0 we have

$$p_i(x_0) < (\overline{p_i})^*(x_0), \quad \forall i = 1, 2, .., n$$

then the local minimizer u of $\mathcal{F}$ is **Lipschitz continuous near $\mathbf{x_0}$**

p, q: vector valued case

Structure assumption: $f(x, |Du|)$

- Acerbi-Fusco 1994 [partial regularity, $p_i - q$]
- Coscia-Mingione 1999 [$|Du|^{p(x)}$]
- Esposito-Leonetti-Mingione 1999, 2002, 2004 [$Du \in L^\infty$]
- Acerbi-Mingione 2000-2001 [partial regularity $|Du|^{p(x)}$]
- Leonetti-Mascolo-Siepe 2001, 2003 [Higher integrability, $Du \in L^\infty$, $1 < p < 2$]
- Bildhauer-Fuchs (et al.) 2002, 2003 [Higher integrability]
- Cupini-Guidorzi-Mascolo 2003 [Local Lipschitz continuity, new approximation methods]
- Foss-Passarelli-Verde, 2010 [Almost minimizers]
- De Maria -Passarelli 2010-2011 [partial regularity]
- **Leonetti-Mascolo 2011**
- **....many others authors and papers**

Systems with p, q-growth (simplified version)

p-ellipticity condition

$$\sum_{i,j=1}^{n} \sum_{\alpha=1}^{m} a_{ij}(x, u, \xi)\lambda_i\lambda_j \geq M \sum_{i=1}^{n} \lambda_i^2 |\xi_i|^{\mathbf{p-2}},$$

p, q-growth conditions

- $\left| \sum_j a_{ij}\left(x, u, \xi\right) \xi_j^\alpha \right| \leq M \left|\xi\right|^{q-1} + \left|u\right|^\gamma + 1 \quad \forall\, i,\, \alpha$
- $|b_i^\alpha(x, u, \xi)| \leq M\, |\xi|^{p(1-\epsilon)} + |u|^\gamma + 1, \quad \forall\, i,\, \alpha$
- $|f^\alpha(x, u, \xi)| \leq M\, \left\{ |\xi|^{p(1-\delta)} + |u|^{\gamma-1} + 1 \right\}, \quad \forall\, \alpha$

with suitable γ, ϵ and δ.

Systems with p, q-growth

Assumptions

- $\sum_{j=1}^{n} a_{ij}(x, u, Du)\, u^{\alpha}_{x_j}$ monotone and $\frac{\mathbf{q}}{\mathbf{p}} < \frac{\mathbf{n-1}}{\mathbf{n-p}}$

or

- $a_{ij} = A(x, u, |\xi|)\, \delta_{ij}$, $A(x, u, t)t$ increasing and $\frac{\mathbf{q}}{\mathbf{p}} < \frac{\mathbf{n}}{\mathbf{n-p}}$

Cupini-Marcellini-Mascolo 2011: A priori estimate

Let u be a weak solution in $W^{1,q}(\Omega; \mathbb{R}^m)$

$$\sup_{B_{R/2}(x_0)} |u| \le c \left\{ \int_{B_R(x_0)} (|u|+1)^{p^*}\, dx \right\}^{\frac{1+\theta}{p^*}}$$

It remains an open problem whether the quasilinear system admits a weak solution in $W^{1,q}$

Vector valued integrals with p, q growth

$$\mathcal{I}(u) = \int_\Omega f(x, Du)\, dx$$

$$|z|^p \le f(x,z) \le |z|^q + C, \quad z \in R^{nm}$$

Leonetti-Mascolo: Examples

- $f_1(x, Du) = g(x, |Du|)$, $g(x,t)$ convex, Δ_2 functions in t.

Leonetti-Mascolo: Examples with no structure assumptions

- $f_2(x, Du) = \sum_{i=1}^n h_i(x, |D_{x_i} u|)$
- $f_3(x, Du) = a(x, |(u_{x_1}, ..., u_{x_{j-1}})|) + b(x, |(u_{x_j}, ..., u_{x_n})|)$

  $h_j(x,t)$, $a(x,t)$ and $b(x,t)$ are convex, Δ_2 functions in t

Vector valued integrals with p, q growth

Despite the difference in the shape of these functionals we identify common assumptions that allow to obtain *an unified proof* of regularity

Leonetti-Mascolo 2011

Under the sharp assumption:

$$\mathbf{q} < \mathbf{p}^*$$

the local minimizers of $\mathcal{I}$ are locally bounded and the following local estimate holds:

$$||u||_{L^\infty(B_{\frac{R}{2}})} \leq C \left(\int_{B_R} (1 + |u|^{p^*})\, dx \right)^{\frac{p^*-p}{p^*(p^*-q)}}$$

Bildhauer-Fuchs 2007, 2009 For the special splitting form $f_3 = f_3(Du)$: $u \in L^\infty \Rightarrow$ higher integrability $u \in L^\infty$.

General growth

- Mascolo-Papi 1994, 1996 [scalar $f = ...g(|Du|)$: boundedness, Harnack Inequality]
- Marcellini 1996 [vector: exponential growth $Du \in L^\infty$]
- Dall'Aglio-Mascolo-Papi 1998 [scalar: $f = f(x, u, Du)$]
- Mingione-Siepe 1999 [vector: $t\, logt$ growth]
- Cianchi 2000 [scalar: boudedness Orlicz spaces]
- **Dall'Aglio-Mascolo 2002** [vector: boudedness $f = g(x, |Du|)$]
- Mascolo-Migliorini 2003 [vector: $f = f(x, |Du|)$ exponential growth, $Du \in L^\infty$]
- Marcellini-Papi 2006 [vector: slow and fast behaviour $Du \in L^\infty$]
- Apushkinskya-Bildhauer-Fuchs 2009 [vector: $u \in L^\infty \Rightarrow C^{1,\tau}$]
- **....many others authors and papers**

Recommended survey on this field, Mingione 2006: **Regularity of minima: an invitation to the dark side of the calculus of variations**

General growth

Dall'Aglio-Mascolo, 2002

$$\mathcal{I}(u) = \int_\Omega g(x, |Du|)\, dx$$

- $g \in C^1$, convex, increasing, $g \in \Delta_2$ and growth assumptions on g_x.

Then all local minimizer of $\mathcal{I}$ are locally bounded

Here we *do not estimate the integrand with powers of the gradient* and the arguments of the proof are strictly related with the properties of g which permit to consider a "suitable approximation" of $\mathbf{g}^\nu(\mathbf{x}, |\mathbf{u}(\mathbf{x})|)$ as a test function

Camillo Benso, conte di Cavour (1810-1861)

He studied mathematics for many years at the military academy

...Dallo studio dei triangoli e delle formule algebriche sono passato a quelle degli uomini e delle cose; comprendo quanto quello studio mi sia stato utile per quello che ora vado facendo degli uomini e delle cose

...From studing triangles and algebraic formulas I switched to studing men and things, I realize how that study was useful for what I'm doing now about men and things