Integrability for solutions to some quasilinear elliptic systems

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$$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$$

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) \right) = 0, \qquad x \in \Omega, \qquad \alpha = 1, ..., N$$

$x \to a_{ij}^{\alpha\beta}(x, u)$ measurable

$u \to a_{ij}^{\alpha\beta}(x, u)$ continuous

$a_{ij}^{\alpha\beta}(x, u)$ bounded and elliptic

De Giorgi's counterexample (1968) $\Longrightarrow$ $u = (u^1, ..., u^N)$ not regular

additional conditions

$$a_{ij}^{\gamma\beta}(x,u) = 0 \quad \text{for} \quad \beta \neq \gamma$$

$$\Downarrow$$

$$\sup_{\Omega} u^{\gamma} \leq \sup_{\partial\Omega} u^{\gamma}$$

$$\theta^\gamma \leq u^\gamma \Longrightarrow a_{ij}^{\gamma\beta}(x,u) = 0 \text{ for } \beta \neq \gamma$$

$$\Downarrow$$

$$\sup_\Omega u^\gamma \leq \max\left[\theta^\gamma; \sup_{\partial\Omega} u^\gamma\right]$$

Necas - Stara (1972)
Mandras (1976)
Leonetti - Petricca (2009)

$$|a_{ij}^{\gamma\beta}(x,u)| \leq \frac{c}{(1+|u|)^q} \quad \text{for} \quad \beta \neq \gamma$$

$$\theta \leq |u| \qquad \Longrightarrow \qquad \nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)\xi_j\xi_i$$

$$|a_{ij}^{\gamma\gamma}(x,u)| \leq \tilde{c}$$

$$u \in W^{1,2}(\Omega, \mathbb{R}^N) \text{ weak solution}$$

$$\Downarrow$$

$$u \in L_{loc}^{2^*(t+1)}(\Omega, \mathbb{R}^N)$$

for every t such that

$$0 < t < \frac{\nu}{n\tilde{c}2N}, \quad t \leq \frac{q}{2}$$

Leonetti - Petricca (2011)

additional assumption

$$a_{ij}^{\gamma\gamma}(x,u) = b_{ij}(x,u)$$

$$\Downarrow$$

$$u \in L_{loc}^{2^*(\frac{q}{2}+1)}(\Omega, \mathbb{R}^N)$$

Leonetti - Petricca (2011)

PROOF

$$0 = \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x,u) D_j u^\beta D_i v^\alpha$$

fix $\gamma \in \{1, ..., N\}$

$$v^\alpha = \begin{cases} 0 & \text{if} \quad \alpha \neq \gamma \\ \phi(|u|) u^\alpha \eta^2 & \text{if} \quad \alpha = \gamma \end{cases}$$

$\eta : \mathbb{R}^n \to \mathbb{R}$ cut-off function

$\phi(|u|) = (p+1)^2 |u|^{2p}, \qquad p > 0$

$$0=\int\limits_{\Omega}\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}a_{ij}^{\alpha\beta}(x,u)D_ju^{\beta}D_iv^{\alpha}=$$

$$=\int\limits_{\Omega}\sum_{i,j=1}^{n}\sum_{\beta=1}^{N}a_{ij}^{\gamma\beta}(x,u)D_ju^{\beta}D_i(\phi(|u|)u^{\gamma}\eta^2)=$$

$$=\int\limits_{\Omega}\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^{\gamma}D_i(\phi(|u|)u^{\gamma}\eta^2)+$$

$$+\int\limits_{\Omega}\sum_{i,j=1}^{n}\sum_{\beta\neq\gamma}^{N}a_{ij}^{\gamma\beta}(x,u)D_ju^{\beta}D_i(\phi(|u|)u^{\gamma}\eta^2)$$

$$D_i(\phi(|u|)u^{\gamma}\eta^2)=$$

$$=\phi'(|u|)1_{\{|u|>0\}}\sum_{\delta=1}^{N}\frac{u^{\delta}}{|u|}(D_iu^{\delta})u^{\gamma}\eta^2+\phi(|u|)(D_iu^{\gamma})\eta^2+\phi(|u|)u^{\gamma}D_i(\eta^2)$$

$$\int\limits_{\Omega} \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)(D_j u^{\gamma})\phi(|u|)(D_i u^{\gamma})\eta^2 =$$

$$-\int\limits_{\Omega} \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)(D_j u^{\gamma})\phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^{\delta}}{|u|}(D_i u^{\delta})u^{\gamma}\eta^2 -$$

$$-\int\limits_{\Omega} \sum_{i,j=1}^{n} \sum_{\beta\neq\gamma}^{N} a_{ij}^{\gamma\beta}(x,u)(D_j u^{\beta})\phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^{\delta}}{|u|}(D_i u^{\delta})u^{\gamma}\eta^2 +$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\nu \int_\Omega |Du^\gamma|^2 \phi(|u|)\eta^2 \leq \int_\Omega \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)(D_j u^\gamma)\phi(|u|)(D_i u^\gamma)\eta^2$$

$$-\int_\Omega \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma\eta^2 \leq$$

$$n\tilde{c} \int_\Omega |Du^\gamma|\phi'(|u|)|Du||u^\gamma|\eta^2$$

$$-\int_\Omega \sum_{i,j=1}^{n} \sum_{\beta\neq\gamma}^{N} a_{ij}^{\gamma\beta}(x,u)(D_j u^\beta)\phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma\eta^2 \leq$$

$$\int_\Omega \frac{cn^2 N}{(1+|u|)^q}|Du|\phi'(|u|)|Du||u^\gamma|\eta^2$$

$$\nu \int_\Omega |Du^\gamma|^2 \phi(|u|)\eta^2 \leq n\tilde{c} \int_\Omega \phi'(|u|)|u||Du|^2\eta^2 +$$

$$cn^2 N \int_\Omega \frac{\phi'(|u|)|u|}{(1+|u|)^q} |Du|^2\eta^2 + \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\phi(|u|) = (p+1)^2|u|^{2p}, \qquad \phi'(|u|)|u| = (p+1)^2|u|^{2p-1}2p|u| = 2p\phi(|u|)$$

$$\nu \int_\Omega |Du^\gamma|^2 \phi(|u|)\eta^2 \leq n\tilde{c}2p \int_\Omega \phi(|u|)|Du|^2\eta^2 +$$

$$cn^2 N(p+1)^2 2p \int_\Omega \frac{|u|^{2p}}{(1+|u|)^q} |Du|^2\eta^2 + \quad . \quad . \quad . \quad . \quad . \quad .$$

sum upon γ

$$\nu \int_\Omega \sum_{\gamma=1}^N |Du^\gamma|^2 \phi(|u|)\eta^2 \le Nn\tilde{c}2p \int_\Omega \phi(|u|)|Du|^2\eta^2 +$$

$$cn^2N^2(p+1)^2 2p \int_\Omega \frac{|u|^{2p}}{(1+|u|)^q}|Du|^2\eta^2 + \quad . \quad . \quad . \quad . \quad . \quad .$$

$$n\tilde{c}N2p < \nu, \qquad 2p \le q$$

$$p < \frac{\nu}{n\tilde{c}N2}, \qquad p \le \frac{q}{2}$$

$$(\nu - Nn\tilde{c}2p)\int\limits_{\Omega}|Du|^2\phi(|u|)\eta^2 \leq cn^2N^2(p+1)^2 2p\int\limits_{\Omega}|Du|^2\eta^2 + \quad . \quad . \quad . \quad . \quad . \quad .$$

$$|Du|^2\phi(|u|)\eta^2 = |Du|^2(p+1)^2|u|^{2p}\eta^2$$

$$|D(|u|^{p+1}\eta)|^2 \leq 2(p+1)^2|u|^{2p}|Du|^2\eta^2 + 2n|u|^{2(p+1)}|D\eta|^2$$

$$\int_{B_\rho} |u|^{2^*(p+1)} \leq \left(c \int_\Omega |D(|u|^{p+1}\eta)|^2 \right)^{2^*/2} \leq$$

$$\left(c \int_\Omega (|u|^{2p}|Du|^2\eta^2 + |u|^{2(p+1)}|D\eta|^2) \right)^{2^*/2} \leq c \left(\int_\Omega |Du|^2 + \int_{B_R} (1+|u|)^{2(p+1)} \right)^{2^*/2}$$

$$|u|^{2(p+1)} \in L^1(B_R) \quad \Longrightarrow \quad |u|^{2^*(p+1)} \in L^1(B_\rho)$$

finite number of steps since $p < \frac{\nu}{n\tilde{c}N2}$ and $p \leq \frac{q}{2}$

with the additional assumption

$$a_{ij}^{\gamma\gamma}(x,u) = b_{ij}(x,u)$$

we get the new estimate

$$-\int_\Omega \sum_{\gamma=1}^{N} \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u)(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma \eta^2 =$$

$$-\int_\Omega \sum_{\gamma=1}^{N} \sum_{i,j=1}^{n} b_{ij}(x,u)(D_j u^\gamma)\phi'(|u|)1_{\{|u|>0\}} \sum_{\delta=1}^{N} \frac{u^\delta}{|u|}(D_i u^\delta)u^\gamma \eta^2 =$$

$$-\int_\Omega \sum_{i,j=1}^{n} b_{ij}(x,u) \sum_{\gamma=1}^{N}(D_j u^\gamma)u^\gamma \phi'(|u|)1_{\{|u|>0\}} \frac{1}{|u|} \sum_{\delta=1}^{N} u^\delta (D_i u^\delta)\eta^2 =$$

$$-\int_\Omega \sum_{i,j=1}^{n} b_{ij}(x,u)\langle u, D_j u\rangle \phi'(|u|)1_{\{|u|>0\}} \frac{1}{|u|}\langle u, D_i u\rangle \eta^2 \le 0$$

$$\nu \int_{\Omega} \sum_{\gamma=1}^{n} |Du^{\gamma}|^2 \phi(|u|)\eta^2 \le cn^2 N^2 (p+1)^2 2p \int_{\Omega} \frac{|u|^{2p}}{(1+|u|)^q} |Du|^2 \eta^2 + \quad . \quad . \quad . \quad .$$

$$2p \le q$$