Partial Continuity for Vectorial Problems

Mikil Foss

Department of Mathematics
University of Nebraska-Lincoln

Optimization Days
Ancona 2011

Main Question

Local Minimizers

Given an open bounded $\Omega \subset \mathbb{R}^n$ and $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, we call $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ a local minimizer for

$$J[\mathbf{u}] := \int_\Omega g(\mathbf{x},\mathbf{u}(\mathbf{x}),D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

if

$$J[\mathbf{u}] \leq J[\mathbf{u}+\varphi] \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^N).$$

Main Question

Local Minimizers

Given an open bounded $\Omega \subset \mathbb{R}^n$ and $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, we call $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ a local minimizer for

$$J[\mathbf{u}] := \int_\Omega g(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

if

$$J[\mathbf{u}] \le J[\mathbf{u}+\varphi] \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^N).$$

Question

How does the smoothness of $(\mathbf{x},\mathbf{u}) \mapsto g(\mathbf{x},\mathbf{u},\mathbf{F})$ affect the regularity of a local minimizer $\mathbf{u}$?

Main Question

Local Minimizers

Given an open bounded $\Omega \subset \mathbb{R}^n$ and $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, we call $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ a local minimizer for

$$J[\mathbf{u}] := \int_\Omega g(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

if

$$J[\mathbf{u}] \leq J[\mathbf{u}+\varphi] \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^N).$$

Question

How does the smoothness of $(\mathbf{x}, \mathbf{u}) \mapsto g(\mathbf{x}, \mathbf{u}, \mathbf{F})$ affect the regularity of a local minimizer $\mathbf{u}$?

Main Question

Local Minimizers

Given an open bounded $\Omega \subset \mathbb{R}^n$ and $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, we call $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ a local minimizer for

$$J[\mathbf{u}] := \int_\Omega g(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

if

$$J[\mathbf{u}] \leq J[\mathbf{u}+\varphi] \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^N).$$

Question

How does the smoothness of $\mathbf{x} \mapsto g(\mathbf{x}, \mathbf{F})$ affect the regularity of a local minimizer $\mathbf{u}$?

General Assumptions

We assume that $g : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ satisfies the following

Hypotheses

For some $2 \leq p < \infty$ and $0 < \lambda \leq L < \infty$

1. $\mathbf{F} \mapsto g(\mathbf{x}, \mathbf{F})$ is $\mathscr{C}^3$

General Assumptions

We assume that $g : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the following

Hypotheses

For some $2 \le p < \infty$ and $0 < \lambda \le L < \infty$

1. $\mathbf{F} \mapsto g(\mathbf{x},\mathbf{F})$ is $\mathscr{C}^3$
2. $\lambda(1+|\mathbf{F}|)^{p-2}|\boldsymbol{\xi}|^2 \le \partial^2_{\mathbf{F}} g(\mathbf{x},\mathbf{F}) \cdot \boldsymbol{\xi} \otimes \boldsymbol{\xi} \le L(1+|\mathbf{F}|)^{p-2}|\boldsymbol{\xi}|^2$

$$\partial^2_{\mathbf{F}} g(\mathbf{x},\mathbf{F}) \cdot \boldsymbol{\xi} \otimes \boldsymbol{\xi} := \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 g(\mathbf{x},\mathbf{F})}{\partial F_i^\alpha F_j^\beta} \xi_i^\alpha \xi_j^\beta$$

General Assumptions

We assume that $g : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the following

Hypotheses

For some $2 \le p < \infty$ and $0 < \lambda \le L < \infty$

1. $\mathbf{F} \mapsto g(\mathbf{x},\mathbf{F})$ is $\mathscr{C}^3$
2. $\lambda(1+|\mathbf{F}|)^{p-2}|\xi|^2 \le \partial^2_{\mathbf{F}} g(\mathbf{x},\mathbf{F}) \cdot \xi \otimes \xi \le L(1+|\mathbf{F}|)^{p-2}|\xi|^2$
3. $|g(\mathbf{x},\mathbf{F}) - g(\mathbf{y},\mathbf{F})| \le L\omega(|\mathbf{x}-\mathbf{y}|)(1+|\mathbf{F}|)^p$

ω captures the continuity properties of $\mathbf{x} \mapsto \dfrac{g(\mathbf{x},\mathbf{F})}{(1+|\mathbf{F}|)^p}$.

- ω is continuous and $\omega(0) = 0$
- ω is nondecreasing
- ω is concave.

e.g. If $\omega(t) \le t^\gamma$, then $\mathbf{x} \mapsto g(\mathbf{x},\mathbf{F})$ is Hölder continuous with exponent γ.

A Scalar Result

In the scalar setting (with $N = 1$), we have the following

Theorem (Giaquinta & Giusti (1983))

If $u \in W^{1,1}(\Omega)$ *is a local minimizer for*

$$J[u] := \int_{\Omega} g(\mathbf{x}, Du(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

and $\omega(t) \leq t^{\gamma}$ *for some* $0 < \gamma < 1$, *then* $u \in C^{1,\gamma/2}_{\mathrm{loc}}(\Omega)$.

Counterexamples

Everywhere continuity cannot be expected in the vectorial setting

De Giorgi (1968) produced the first counterexample to everywhere continuity. (Problem was measurable with respect to $\mathbf{x}$.)

Sverak & Yan (2002) produced a $g \in \mathscr{C}^\infty(\mathbb{R}^{N\times n})$ such that for some $\lambda, L > 0$ the function g satisfies

$$\lambda|\boldsymbol{\xi}|^2 \leq \partial^2_{\mathbf{F}} g(\mathbf{F}) \cdot \boldsymbol{\xi} \otimes \boldsymbol{\xi} \leq L|\boldsymbol{\xi}|^2 \text{ for all } \mathbf{F}, \boldsymbol{\xi} \in \mathbb{R}^{N\times n},$$

and the unique minimizer (satisfying boundary conditions) for

$$J[\mathbf{u}] := \int_B g(D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

is unbounded ($n = 5$ and $N = 14$).

A Vectorial Result

Everywhere regularity cannot be expected in the vectorial setting (when $N > 1$), but it is possible to establish partial regularity

Theorem (Giaquinta & Modica (1979))

If $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ *is a local minimizer for*

$$J[\mathbf{u}] := \int_\Omega g(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

and $\omega(t) \leq t^\gamma$ *for some* $0 < \gamma < 1$, *then* $\mathbf{u} \in C^{1,\gamma/2}_{\mathrm{loc}}(\Omega_0)$ *for some open* $\Omega_0 \subseteq \Omega$ *satisfying* $|\Omega \backslash \Omega_0| = 0$.

Argument involves making comparisons to solutions of appropriate linear constant coefficient elliptic systems.

Another Scalar Result

In the scalar setting (when $N = 1$), Hölder continuity of $\mathbf{x} \mapsto g(\mathbf{x}, \mathbf{F})$ leads to Hölder continuity for the gradient of the minimizer.

What happens if $\mathbf{x} \mapsto g(\mathbf{x}, \mathbf{F})$ is only assumed to be continuous?

Theorem (Manfredi (1988))

If $u \in W^{1,1}(\Omega)$ is a local minimizer for

$$J[u] := \int_{\Omega} g(\mathbf{x}, Du(\mathbf{x}))\, \mathrm{d}\mathbf{x},$$

and $\omega(t)$ is

1. *continuous with $\omega(0) = 0$*
2. *non-decreasing*
3. *concave,*

then for each $0 < \gamma < 1$ we have $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$.

Vectorial Analogues

Question

If $N > 1$ and $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a local minimizer for

$$J[\mathbf{u}] := \int_{\Omega} g(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

and just continuity of $\mathbf{x} \mapsto g(\mathbf{x},\mathbf{F})$ is assumed,

must there be an open $\Omega_0 \subseteq \Omega$ such that $|\Omega \backslash \Omega_0| = 0$ and $\mathbf{u} \in \mathscr{C}^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$ for some $\gamma \in (0,1)$?

- Campanato: if dimension is small: $n < p+2$
- Acerbi & Fusco; Marcellini; Uhlenbeck: g has additional structure (e.g. $g(\mathbf{x},\mathbf{F}) = \widetilde{g}(|\mathbf{F}|)$)

Partial Continuity

Theorem ($p \geq 2$: Foss & Mingione (2008))

If $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$ *is a local minimizer for*

$$J[\mathbf{u}] := \int_\Omega g(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

and $\omega(t)$ *is*

1. *continuous with* $\omega(0) = 0$
2. *non-decreasing*
3. *concave,*

then for each $\gamma \in (0,1)$ *there is an open* $\Omega_0 \subseteq \Omega$ *such that* $|\Omega \backslash \Omega_0| = 0$ *and* $\mathbf{u} \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$.

Beck: subquadratic analouge

Partial Continuity for Elliptic Systems

We assume the field $\mathbf{a} : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfies

Hypotheses

For some $2 \leq p < \infty$ and $0 < \lambda \leq L < \infty$

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x}, \mathbf{F})$ is $\mathscr{C}^2$

Partial Continuity for Elliptic Systems

We assume the field $\mathbf{a} : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfies

Hypotheses

For some $2 \le p < \infty$ and $0 < \lambda \le L < \infty$

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x},\mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda\,(1+|\mathbf{F}|)^{p-2}\,|\boldsymbol{\xi}|^2 \le \partial_{\mathbf{F}}\mathbf{a}(\mathbf{x},\mathbf{F})\cdot\boldsymbol{\xi}\otimes\boldsymbol{\xi} \le L\,(1+|\mathbf{F}|)^{p-2}\,|\boldsymbol{\xi}|^2$

$$\partial_{\mathbf{F}}\mathbf{a}(\mathbf{x},\mathbf{F})\cdot\boldsymbol{\xi}\otimes\boldsymbol{\xi} := \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}\frac{\partial a_i^\alpha(\mathbf{x},\mathbf{F})}{\partial F_j^\beta}\xi_i^\alpha\xi_j^\beta$$

Partial Continuity for Elliptic Systems

We assume the field $\mathbf{a} : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfies

Hypotheses

For some $2 \le p < \infty$ and $0 < \lambda \le L < \infty$

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x},\mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda\,(1+|\mathbf{F}|)^{p-2}\,|\boldsymbol{\xi}|^2 \le \partial_{\mathbf{F}}\mathbf{a}(\mathbf{x},\mathbf{F})\cdot\boldsymbol{\xi}\otimes\boldsymbol{\xi} \le L\,(1+|\mathbf{F}|)^{p-2}\,|\boldsymbol{\xi}|^2$
3. $|\mathbf{a}(\mathbf{x},\mathbf{F}) - \mathbf{a}(\mathbf{y},\mathbf{F})| \le L\omega(|\mathbf{x}-\mathbf{y}|)\,(1+|\mathbf{F}|)^{p-1}$

ω satisfies

- ω is continuous and $\omega(0) = 0$
- ω is nondecreasing
- ω is concave.

Partial Continuity for Elliptic Systems

We assume the field $\mathbf{a} : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfies

Hypotheses

For some $2 \le p < \infty$ and $0 < \lambda \le L < \infty$

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x}, \mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda \left(1 + |\mathbf{F}|\right)^{p-2} |\xi|^2 \le \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}, \mathbf{F}) \cdot \xi \otimes \xi \le L \left(1 + |\mathbf{F}|\right)^{p-2} |\xi|^2$
3. $|\mathbf{a}(\mathbf{x}, \mathbf{F}) - \mathbf{a}(\mathbf{y}, \mathbf{F})| \le L\omega(|\mathbf{x} - \mathbf{y}|) \left(1 + |\mathbf{F}|\right)^{p-1}$

Theorem (Foss & Mingione (2008))

If $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$ satisfies

$$\int_\Omega \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega; \mathbb{R}^N)$$

then for each $\gamma \in (0,1)$ there is an open $\Omega_0 \subseteq \Omega$ such that $|\Omega \setminus \Omega_0| = 0$ and $\mathbf{u} \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$.

Partial $\mathscr{C}^{1,\alpha}$-Continuity for an Elliptic System

Hypotheses

For some $0 < \lambda \leq L < \infty$, the field $\mathbf{a} : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}^N$ satisfies

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x}, \mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda |\boldsymbol{\xi}|^2 \leq \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}, \mathbf{F}) \cdot \boldsymbol{\xi} \otimes \boldsymbol{\xi} \leq L |\boldsymbol{\xi}|^2$
3. $|\mathbf{a}(\mathbf{x}, \mathbf{F}) - \mathbf{a}(\mathbf{y}, \mathbf{F})| \leq L\omega(|\mathbf{x} - \mathbf{y}|)(1 + |\mathbf{F}|)$

Partial $\mathscr{C}^{1,\alpha}$-Continuity for an Elliptic System

Hypotheses

For some $0 < \lambda \leq L < \infty$, the field $\mathbf{a} : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ satisfies

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x},\mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda|\boldsymbol{\xi}|^2 \leq \partial_{\mathbf{F}}\mathbf{a}(\mathbf{x},\mathbf{F})\cdot\boldsymbol{\xi}\otimes\boldsymbol{\xi} \leq L|\boldsymbol{\xi}|^2$
3. $|\mathbf{a}(\mathbf{x},\mathbf{F}) - \mathbf{a}(\mathbf{y},\mathbf{F})| \leq L\omega(|\mathbf{x}-\mathbf{y}|)(1+|\mathbf{F}|)$

Theorem

If $\mathbf{u} \in W^{1,2}(\Omega;\mathbb{R}^N)$ *satisfies*

$$\int_\Omega \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\cdot D\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^N)$$

and $\omega(t) \leq t^\gamma$ *for some* $0 < \gamma < 1$, *then* $\mathbf{u} \in C^{1,\gamma/2}_{\mathrm{loc}}(\Omega_0)$ *for some open* $\Omega_0 \subseteq \Omega$ *satisfying* $|\Omega\setminus\Omega_0| = 0$.

Partial $\mathscr{C}^{1,\alpha}$-Continuity for an Elliptic System

Hypotheses

For some $0 < \lambda \leq L < \infty$, the field $\mathbf{a} : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ satisfies

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x},\mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda|\xi|^2 \leq \partial_{\mathbf{F}}\mathbf{a}(\mathbf{x},\mathbf{F})\cdot\xi\otimes\xi \leq L|\xi|^2$
3. $|\mathbf{a}(\mathbf{x},\mathbf{F}) - \mathbf{a}(\mathbf{y},\mathbf{F})| \leq L\omega(|\mathbf{x}-\mathbf{y}|)(1+|\mathbf{F}|)$

Theorem

If $\mathbf{u} \in W^{1,2}(\Omega;\mathbb{R}^N)$ *satisfies*

$$\int_\Omega \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\cdot D\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in \mathscr{C}^\infty_c(\Omega;\mathbb{R}^N)$$

and $\omega(t) \leq t^\gamma$ *for some* $0 < \gamma < 1$, *then* $\mathbf{u} \in C^{1,\gamma/2}_{\mathrm{loc}}(\Omega_0)$ *for some open* $\Omega_0 \subseteq \Omega$ *satisfying* $|\Omega\backslash\Omega_0| = 0$.

Partial $\mathscr{C}^{1,\alpha}$-Continuity for an Elliptic System

Hypotheses

For some $0 < \lambda \leq L < \infty$, the field $\mathbf{a} : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ satisfies

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x},\mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda|\boldsymbol{\xi}|^2 \leq \partial_{\mathbf{F}}\mathbf{a}(\mathbf{x},\mathbf{F}) \cdot \boldsymbol{\xi} \otimes \boldsymbol{\xi} \leq L|\boldsymbol{\xi}|^2$
3. $|\mathbf{a}(\mathbf{x},\mathbf{F}) - \mathbf{a}(\mathbf{y},\mathbf{F})| \leq L\omega(|\mathbf{x}-\mathbf{y}|)(1+|\mathbf{F}|)$

Theorem

If $\mathbf{u} \in W^{1,2}(\Omega;\mathbb{R}^N)$ *satisfies*

$$\int_\Omega \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^N)$$

and $\omega(t) \leq t^\gamma$ *for some* $0 < \gamma < 1$, *then* $\mathbf{u} \in C^{1,\gamma/2}_{\mathrm{loc}}(\Omega_0)$ *for some open* $\Omega_0 \subseteq \Omega$ *satisfying* $|\Omega \backslash \Omega_0| = 0$.

Will outline an argument used by Duzaar & Grotowski (2000): The $\mathbf{A}$-harmonic Approximation Method.

Campanato's Embedding Theorem

Theorem

If $f \in L^2(B_R)$ *and there is a constant* C *and an* $0 < \alpha < 1$ *such that*

$$\fint_{B_{\mathbf{z},r}} |f(\mathbf{x}) - (f)_{\mathbf{z},r}|^2 \, \mathrm{d}\mathbf{x} \le C r^{2\alpha} \qquad \text{for each } B_{\mathbf{z},r} \subset B_R$$

then $f \in \mathscr{C}^{0,\alpha}_{\mathrm{loc}}(B_R)$*.*

Notation: $B_{\mathbf{z},r}$ is the ball of radius r centered at $\mathbf{z}$.

$$(h)_{\mathbf{z},r} := \fint_{B_{\mathbf{z},r}} h(\mathbf{x}) \, \mathrm{d}\mathbf{x} \text{ is the mean-value of } h \text{ over } B_{\mathbf{z},r}.$$

Campanato's Embedding Theorem

Theorem

If $f \in L^2(B_R)$ *and there is a constant* C *and an* $0 < \alpha < 1$ *such that*

$$\fint_{B_{\mathbf{z},r}} |f(\mathbf{x}) - (f)_{\mathbf{z},r}|^2 \,\mathrm{d}\mathbf{x} \leq C r^{2\alpha} \qquad \text{for each } B_{\mathbf{z},r} \subset B_R$$

then $f \in \mathscr{C}^{0,\alpha}_{\mathrm{loc}}(B_R)$*.*

Suppose that $\mathbf{u} \in W^{1,2}(\Omega;\mathbb{R}^N)$ satisfies

$$\int_{\Omega} \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x}) \,\mathrm{d}\mathbf{x} \qquad \text{for all } \varphi \in \mathscr{C}^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^N)$$

Campanato's Embedding Theorem

Theorem

If $f \in L^2(B_R)$ *and there is a constant* C *and an* $0 < \alpha < 1$ *such that*

$$\fint_{B_{\mathbf{z},r}} |f(\mathbf{x}) - (f)_{\mathbf{z},r}|^2 \,\mathrm{d}\mathbf{x} \le Cr^{2\alpha} \qquad \text{for each } B_{\mathbf{z},r} \subset B_R$$

then $f \in \mathscr{C}^{0,\alpha}_{\mathrm{loc}}(B_R)$.

Suppose that $\mathbf{u} \in W^{1,2}(\Omega;\mathbb{R}^N)$ satisfies

$$\int_{\Omega} \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x}) \,\mathrm{d}\mathbf{x} \qquad \text{for all } \varphi \in \mathscr{C}^{\infty}_{\mathrm{c}}(\Omega;\mathbb{R}^N)$$

Objective

At a given Lebesgue point $\mathbf{x}_0$ of $D\mathbf{u}$, show there is $B_{\mathbf{x}_0,R} \subset \Omega$ and a constant C such that

$$\fint_{B_{\mathbf{z},r}} |D\mathbf{u}(\mathbf{x}) - (D\mathbf{u})_{\mathbf{z},r}|^2 \,\mathrm{d}\mathbf{x} \le Cr^{\gamma} \qquad \text{for each } B_{\mathbf{z},r} \subset B_{\mathbf{x}_0,R}$$

$\mathbf{A}$-Harmonic Mappings

We ultimately make comparisons to $\mathbf{A}$-harmonic mappings.

A-Harmonic Mappings

We ultimately make comparisons to $\mathbf{A}$-harmonic mappings.

Suppose that $\mathbf{A} \in \mathbb{R}^{(N\times n)\times(N\times n)}$ is positive definite.
i.e. For some $0 < \lambda \leq L < \infty$

$$\lambda|\boldsymbol{\xi}|^2 \leq \mathbf{A}\cdot\boldsymbol{\xi}\otimes\boldsymbol{\xi} \leq L|\boldsymbol{\xi}|^2 \qquad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^{N\times n}.$$

A-Harmonic Mappings

We ultimately make comparisons to $\mathbf{A}$-harmonic mappings.

Suppose that $\mathbf{A} \in \mathbb{R}^{(N\times n)\times(N\times n)}$ is positive definite.

Definition

A map $\mathbf{h} \in W^{1,2}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$ is called $\mathbf{A}$-*harmonic* if

$$\int_{B_{\mathbf{x}_0,R}} \mathbf{A}\cdot D\mathbf{h}(\mathbf{x})\otimes D\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in W_0^{1,2}(B_{\mathbf{x}_0,R};\mathbb{R}^N).$$

A-Harmonic Mappings

We ultimately make comparisons to $\mathbf{A}$-harmonic mappings.

Suppose that $\mathbf{A} \in \mathbb{R}^{(N\times n)\times(N\times n)}$ is positive definite.

Definition

A map $\mathbf{h} \in W^{1,2}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$ is called $\mathbf{A}$-*harmonic* if

$$\int_{B_{\mathbf{x}_0,R}} \mathbf{A}\cdot D\mathbf{h}(\mathbf{x})\otimes D\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in W_0^{1,2}(B_{\mathbf{x}_0,R};\mathbb{R}^N).$$

If $\mathbf{h}$ is $\mathbf{A}$-harmonic on $B_{\mathbf{x}_0,R}$, then for each $0 < \theta < 1/2$

$$\sup_{\mathbf{x}\in B_{\mathbf{x}_0,\theta R}} |\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}_0) - D\mathbf{h}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)|^2 \le C\theta^4 R^2 \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2\,\mathrm{d}\mathbf{x}.$$

Freezing the Coefficients

For each $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^N)$, we have

$$\int_{\Omega} \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

Freezing the Coefficients

For each $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^N)$, we have

$$\int_\Omega \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

Let $\mathbf{x}_0 \in \Omega$ is a point where

$$\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u}(\mathbf{x}) - (D\mathbf{u})_{\mathbf{x}_0,R}|^2\,\mathrm{d}\mathbf{x} \approx 0.$$

Freezing the Coefficients

For each $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^N)$, we have

$$\int_\Omega \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

Let $\mathbf{x}_0 \in \Omega$ is a point where

$$\fint_{B_{\mathbf{x}_0,R}} \left|D\mathbf{u}(\mathbf{x}) - (D\mathbf{u})_{\mathbf{x}_0,R}\right|^2 \mathrm{d}\mathbf{x} \approx 0.$$

We use the continuity of $\mathbf{x} \mapsto \dfrac{\mathbf{a}(\mathbf{x},\mathbf{F})}{1+|\mathbf{F}|}$, the divergence theorem and the fundamental theorem of calculus.

Put

$$\mathbf{F}_{\mathbf{x}_0,R} := (D\mathbf{u})_{\mathbf{x}_0,R}$$

Freezing the Coefficients

$$\int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}, D\mathbf{u}) \cdot D\boldsymbol{\varphi} \,\mathrm{d}\mathbf{x} = 0$$

Freezing the Coefficients

$$\int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}, D\mathbf{u}) \cdot D\varphi \,\mathrm{d}\mathbf{x} = 0 \Rightarrow \int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

Freezing the Coefficients

$$\int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}, D\mathbf{u}) \cdot D\varphi \,\mathrm{d}\mathbf{x} = 0 \Rightarrow \int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \Big\{ \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) - \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R}) \Big\} \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

Freezing the Coefficients

$$\int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}, D\mathbf{u}) \cdot D\varphi \, \mathrm{d}\mathbf{x} = 0 \Rightarrow \int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) \cdot D\varphi \, \mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \left\{ \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) - \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R}) \right\} \cdot D\varphi \, \mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \int_0^1 \left\{ \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R} + s(D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}))(D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}) \right\} \cdot D\varphi \, \mathrm{d}s \mathrm{d}\mathbf{x} \approx 0$$

Freezing the Coefficients

$$\int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}, D\mathbf{u}) \cdot D\varphi \,\mathrm{d}\mathbf{x} = 0 \Rightarrow \int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \left\{ \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) - \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R}) \right\} \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \int_0^1 \left\{ \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R} + s(D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}))(D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}) \right\} \cdot D\varphi \,\mathrm{d}s\mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \left\{ \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R})(D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}) \right\} \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

Freezing the Coefficients

$$\int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}, D\mathbf{u}) \cdot D\varphi \,\mathrm{d}\mathbf{x} = 0 \Rightarrow \int_{B_{\mathbf{x}_0,R}} \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \left\{ \mathbf{a}(\mathbf{x}_0, D\mathbf{u}) - \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R}) \right\} \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \int_0^1 \left\{ \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R} + s(D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}))(D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}) \right\} \cdot D\varphi \,\mathrm{d}s \mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \left\{ \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R})(D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}) \right\} \cdot D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

$$\Rightarrow \int_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot [D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}] \otimes D\varphi \,\mathrm{d}\mathbf{x} \approx 0$$

Linearization

We have the following estimate: for each $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot [D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}] \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{1 + |\mathbf{F}_{\mathbf{x}_0,R}|} \, \mathrm{d}\mathbf{x} + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|) \right\} \|D\varphi\|_{L^\infty}.$$

Linearization

We have the following estimate: for each $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot [D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}] \otimes D\varphi \, \mathrm{d}\mathbf{x} \right|$$

$$\leq C(L) \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{1 + |\mathbf{F}_{\mathbf{x}_0,R}|} \, \mathrm{d}\mathbf{x} + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|) \right\} \|D\varphi\|_{L^\infty}.$$

Here

$$\mathbf{A} := \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R})$$

and ω is the modulus of continuity for $\mathbf{x} \mapsto \dfrac{\mathbf{a}(\mathbf{x},\mathbf{F})}{1+|\mathbf{F}|}$.

Linearization

We have the following estimate: for each $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot [D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}] \otimes D\varphi \, \mathrm{d}\mathbf{x} \right|$$

$$\leq C(L) \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{1 + |\mathbf{F}_{\mathbf{x}_0,R}|} \, \mathrm{d}\mathbf{x} + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|) \right\} \|D\varphi\|_{L^\infty}.$$

Here

$$\mathbf{A} := \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R})$$

and ω is the modulus of continuity for $\mathbf{x} \mapsto \dfrac{\mathbf{a}(\mathbf{x},\mathbf{F})}{1+|\mathbf{F}|}$.

Linearization

We have the following estimate: for each $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot [D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}] \otimes D\varphi \, \mathrm{d}\mathbf{x} \right|$$

$$\leq C(L) \left\{ \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \, \mathrm{d}\mathbf{x} + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|) \right\} \|D\varphi\|_{L^\infty}.$$

Here

$$\mathbf{A} := \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}_0, \mathbf{F}_{\mathbf{x}_0,R})$$

and ω is the modulus of continuity for $\mathbf{x} \mapsto \dfrac{\mathbf{a}(\mathbf{x},\mathbf{F})}{1+|\mathbf{F}|}$.

Linearization

Here $\mathbf{w} := \dfrac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha}$ and $D\mathbf{w} = \dfrac{D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}}{\alpha}$

with

$$\alpha := \left\{ \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \, \mathrm{d}\mathbf{x} + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2 \right\}^{\frac{1}{2}}$$

This yields

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \left(\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty}.$$

Here $\mathbf{w} := \dfrac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha}$ and $D\mathbf{w} = \dfrac{D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}}{\alpha}$

with

$$\alpha := \left\{ \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \, \mathrm{d}\mathbf{x} + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2 \right\}^{\frac{1}{2}}$$

Linearization

This yields

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right|$$

$$\leq C(L) \left\{ \left(\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty}.$$

Here $\mathbf{w} := \dfrac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha}$ and $D\mathbf{w} = \dfrac{D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}}{\alpha}$

Observe that

$$\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{w}|^2 \, \mathrm{d}\mathbf{x} \leq 1.$$

This yields

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right|$$

$$\leq \underbrace{C(L) \left\{ \left(\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\}}_{=\delta} \|D\varphi\|_{L^\infty}.$$

Here $\mathbf{w} := \dfrac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha}$ and $D\mathbf{w} = \dfrac{D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}}{\alpha}$

Observe that

$$\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{w}|^2 \, \mathrm{d}\mathbf{x} \leq 1.$$

This yields

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \le \underbrace{C(L) \left\{ \left(\fint_{B_{\mathbf{x}_0,R}} \left| D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R} \right|^2 \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\}}_{=\delta} \|D\varphi\|_{L^\infty}.$$

Here $\mathbf{w} := \dfrac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha}$ and $D\mathbf{w} = \dfrac{D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}}{\alpha}$

We can make $\delta > 0$ as small as required throughout an open neighborhood of a given Lebesgue point.

$\mathbf{w}$ is not necessarily $\mathbf{A}$-harmonic.

It is, however, “almost” $\mathbf{A}$-harmonic provided

$$\delta = C(L)\left\{\left(\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}}\right\}$$

is small enough.

Lemma (A-Harmonic Approximation Lemma)

If $\mathbf{A} \in \mathbb{R}^{(N\times n)\times(N\times n)}$ *is a positive definite, then for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *with the following property:*

If $\mathbf{w} \in W^{1,2}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$ *satisfies*

$$\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{w}|^2 \,\mathrm{d}\mathbf{x} \le 1 \quad \text{and} \quad \left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A}\cdot D\mathbf{w}\otimes D\varphi \,\mathrm{d}\mathbf{x} \right| \le \delta \|D\varphi\|_{L^\infty}$$

for all $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$,

Lemma (A-Harmonic Approximation Lemma)

If $\mathbf{A} \in \mathbb{R}^{(N\times n)\times(N\times n)}$ *is a positive definite, then for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *with the following property:*

If $\mathbf{w} \in W^{1,2}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$ *satisfies*

$$\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{w}|^2\,\mathrm{d}\mathbf{x} \le 1 \quad \text{and} \quad \left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A}\cdot D\mathbf{w}\otimes D\varphi\,\mathrm{d}\mathbf{x} \right| \le \delta \|D\varphi\|_{L^\infty}$$

for all $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$, *then there is an* $\mathbf{A}$*-harmonic mapping* $\mathbf{h} \in W^{1,2}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$ *such that*

$$\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2\,\mathrm{d}\mathbf{x} \le 1 \quad \text{and} \quad \frac{1}{R^2}\fint_{B_{\mathbf{x}_0,R}} |\mathbf{h}-\mathbf{w}|^2\,\mathrm{d}\mathbf{x} \le \varepsilon.$$

A-Harmonic Comparison

Since $\mathbf{h}$ is $\mathbf{A}$-harmonic on $B_{\mathbf{x}_0,R}$ and $\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2 \,\mathrm{d}\mathbf{x} \leq 1$, for each

$0 < \theta < 1/2$

$$\sup_{\mathbf{x} \in B_{\mathbf{x}_0,\theta R}} |\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}_0) - D\mathbf{h}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)|^2 \leq C\theta^4 R^2 \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2 \,\mathrm{d}\mathbf{x} \leq C\theta^4 R^2.$$

A-Harmonic Comparison

Since $\mathbf{h}$ is $\mathbf{A}$-harmonic on $B_{\mathbf{x}_0,R}$ and $\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2 \, \mathrm{d}\mathbf{x} \leq 1$, for each $0 < \theta < 1/2$

$$\sup_{\mathbf{x} \in B_{\mathbf{x}_0,\theta R}} |\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}_0) - D\mathbf{h}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)|^2 \leq C\theta^4 R^2 \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2 \, \mathrm{d}\mathbf{x} \leq C\theta^4 R^2.$$

Since

$$\frac{1}{R^2} \fint_{B_{\mathbf{x}_0,R}} |\mathbf{h} - \mathbf{w}|^2 \, \mathrm{d}\mathbf{x} \leq \varepsilon,$$

for each $0 < \theta < 1/2$

$$\frac{1}{(\theta R)^2} \fint_{B_{\mathbf{x}_0,\theta R}} \left| \mathbf{w} - \mathbf{h}(\mathbf{x}_0) - D\mathbf{h}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \right|^2 \mathrm{d}\mathbf{x} \leq C \left\{ \frac{\varepsilon}{\theta^{n+2}} + \theta^2 \right\}.$$

with $\mathbf{w} = \frac{1}{\alpha} \left[\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0) \right]$

A-Harmonic Comparison

Since $\mathbf{h}$ is $\mathbf{A}$-harmonic on $B_{\mathbf{x}_0,R}$ and $\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2 \, \mathrm{d}\mathbf{x} \leq 1$, for each

$0 < \theta < 1/2$

$$\sup_{\mathbf{x} \in B_{\mathbf{x}_0,\theta R}} |\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}_0) - D\mathbf{h}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)|^2 \leq C\theta^4 R^2 \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2 \, \mathrm{d}\mathbf{x} \leq C\theta^4 R^2.$$

Since

$$\frac{1}{R^2} \fint_{B_{\mathbf{x}_0,R}} |\mathbf{h} - \mathbf{w}|^2 \, \mathrm{d}\mathbf{x} \leq \varepsilon,$$

for each $0 < \theta < 1/2$

$$\frac{1}{(\theta R)^2} \fint_{B_{\mathbf{x}_0,\theta R}} \left| \mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R} - \alpha [\mathbf{h}(\mathbf{x}_0) - D\mathbf{h}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)] \right|^2 \mathrm{d}\mathbf{x}$$

$$\leq C \left\{ \frac{\varepsilon}{\theta^{n+2}} + \theta^2 \right\} \alpha^2$$

A-Harmonic Comparison

Since $\mathbf{h}$ is $\mathbf{A}$-harmonic on $B_{\mathbf{x}_0,R}$ and $\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2 \, \mathrm{d}\mathbf{x} \leq 1$, for each

$0 < \theta < 1/2$

$$\sup_{\mathbf{x}\in B_{\mathbf{x}_0,\theta R}} |\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}_0) - D\mathbf{h}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)|^2 \leq C\theta^4 R^2 \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{h}|^2 \, \mathrm{d}\mathbf{x} \leq C\theta^4 R^2.$$

Since

$$\frac{1}{R^2} \fint_{B_{\mathbf{x}_0,R}} |\mathbf{h} - \mathbf{w}|^2 \, \mathrm{d}\mathbf{x} \leq \varepsilon,$$

for each $0 < \theta < 1/2$

$$\fint_{B_{\mathbf{x}_0,\theta R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\theta R}|^2 \, \mathrm{d}\mathbf{x} \leq C \left\{ \frac{\varepsilon}{\theta^{n+2}} + \theta^2 \right\} \alpha^2 + C\omega(R) \left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2.$$

Thus

$$\fint_{B_{\mathbf{x}_0,\theta R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\theta R}|^2 \,\mathrm{d}\mathbf{x}$$

$$\leq C\left\{\frac{\varepsilon}{\theta^{n+2}} + \theta^2\right\} \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \,\mathrm{d}\mathbf{x} + C\omega(R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2.$$

Thus

$$\fint_{B_{\mathbf{x}_0,\theta R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\theta R}|^2 \,\mathrm{d}\mathbf{x}$$

$$\leq C\left\{\frac{\varepsilon}{\theta^{n+2}} + \theta^2\right\}\fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \,\mathrm{d}\mathbf{x} + C\omega(R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2 .$$

We select $0 < \theta < \frac{1}{2}$ and $\varepsilon > 0$ so that

$$\fint_{B_{\mathbf{x}_0,\theta R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\theta R}|^2 \,\mathrm{d}\mathbf{x} \leq \theta^\gamma \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \,\mathrm{d}\mathbf{x} + C\omega(R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2$$

Decay Estimate

With

$$\Phi(\rho) := \fint_{B_{\mathbf{x}_0,\rho}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2 \,\mathrm{d}\mathbf{x} \qquad \text{and} \qquad \mathbf{F}_{\mathbf{x}_0,\rho} := (D\mathbf{u})_{\mathbf{x}_0,\rho},$$

the decay estimate

$$\fint_{B_{\mathbf{x}_0,\theta R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\theta R}|^2 \,\mathrm{d}\mathbf{x} \leq \theta^{\gamma} \fint_{B_{\mathbf{x}_0,R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2 \,\mathrm{d}\mathbf{x} + C\omega(R)\,(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2$$

Decay Estimate

With

$$\Phi(\rho) := \fint_{B_{\mathbf{x}_0,\rho}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2 \, \mathrm{d}\mathbf{x} \qquad \text{and} \qquad \mathbf{F}_{\mathbf{x}_0,\rho} := (D\mathbf{u})_{\mathbf{x}_0,\rho},$$

the decay estimate can be rewritten as

$$\Phi(\theta R) \le \theta^{\gamma} \Phi(R) + C\omega(R) \left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2.$$

Decay Estimate

With

$$\Phi(\rho) := \fint_{B_{\mathbf{x}_0,\rho}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2 \, \mathrm{d}\mathbf{x} \qquad \text{and} \qquad \mathbf{F}_{\mathbf{x}_0,\rho} := (D\mathbf{u})_{\mathbf{x}_0,\rho},$$

the decay estimate can be rewritten as

$$\Phi(\theta R) \leq \theta^{\gamma}\Phi(R) + C\omega(R)\,(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2 .$$

We want to argue by induction that

$$\Phi(\theta^{k+1}R) \leq \theta^{\gamma}\Phi(\theta^k R) + C\omega(\theta^k R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^k R}|\right)^2 .$$

Decay Estimate

With

$$\Phi(\rho) := \fint_{B_{\mathbf{x}_0,\rho}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2 \, \mathrm{d}\mathbf{x} \qquad \text{and} \qquad \mathbf{F}_{\mathbf{x}_0,\rho} := (D\mathbf{u})_{\mathbf{x}_0,\rho},$$

the decay estimate can be rewritten as

$$\Phi(\theta R) \leq \theta^\gamma \Phi(R) + C\omega(R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2 .$$

We want to argue by induction that

$$\Phi(\theta^{k+1} R) \leq \theta^\gamma \Phi(\theta^k R) + C\omega(\theta^k R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^k R}|\right)^2 .$$

An iteration argument, the continuity of $\mathbf{x}_0 \mapsto \fint_{B_{\mathbf{x}_0,\rho}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2 \, \mathrm{d}\mathbf{x}$

and Campanato's embedding theorem then yields the partial Hölder continuity of $D\mathbf{u}$.

Decay Estimate

Recall

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \Phi(R)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty}.$$

with

$$\mathbf{w} := \frac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha}$$

and

$$\alpha := \left\{ \Phi(R) + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2 \right\}^{\frac{1}{2}}$$

Decay Estimate

The same linearization argument yields

$$\left| \fint_{B_{\mathbf{x}_0,\theta R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \Phi(\theta R)^{\frac{1}{2}} + \omega(\theta R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty}.$$

with

$$\mathbf{w} := \frac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,\theta R} - \mathbf{F}_{\mathbf{x}_0,\theta R}(\mathbf{x} - \mathbf{x}_0)}{\alpha}$$

and

$$\alpha := \left\{ \Phi(\theta R) + \omega(\theta R)(1 + |\mathbf{F}_{\mathbf{x}_0,\theta R}|)^2 \right\}^{\frac{1}{2}}$$

Decay Estimate

In general, for each $k \in \mathbb{N}$

$$\left| \fint_{B_{\mathbf{x}_0,\theta^k R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \Phi(\theta^k R)^{\frac{1}{2}} + \omega(\theta^k R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty}.$$

with

$$\mathbf{w} := \frac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,\theta^k R} - \mathbf{F}_{\mathbf{x}_0,\theta^k R}(\mathbf{x} - \mathbf{x}_0)}{\alpha}$$

and

$$\alpha := \left\{ \Phi(\theta^k R) + \omega(\theta^k R)(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^k R}|)^2 \right\}^{\frac{1}{2}}$$

Decay Estimate

Since

$$\left| \fint_{B_{\mathbf{x}_0,\theta^k R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \Phi(\theta^k R)^{\frac{1}{2}} + \omega(\theta^k R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty},$$

the $\mathbf{A}$-harmonic approximation lemma can be used to conclude that

$$\Phi(\theta^{k+1} R) \leq \theta^\gamma \Phi(\theta^k R) + C\omega(\theta^k R) \left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^k R}|\right)^2$$

Decay Estimate

Since

$$\left| \fint_{B_{\mathbf{x}_0,\theta^k R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq \underbrace{C(L) \left\{ \Phi(\theta^k R)^{\frac{1}{2}} + \omega(\theta^k R)^{\frac{1}{2}} \right\}}_{\text{needs to be less than } \delta} \|D\varphi\|_{L^\infty},$$

the $\mathbf{A}$-harmonic approximation lemma can be used to conclude that

$$\Phi(\theta^{k+1}R) \leq \theta^\gamma \Phi(\theta^k R) + C\omega(\theta^k R) \left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^k R}|\right)^2$$

provided that

$$\Phi(\theta^k R)^{\frac{1}{2}} + \omega(\theta^k R)^{\frac{1}{2}} \leq \frac{\delta}{C(L)}$$

Decay Estimate

Since

$$\left| \fint_{B_{\mathbf{x}_0,\theta^k R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \le \underbrace{C(L)\left\{ \Phi(\theta^k R)^{\frac{1}{2}} + \omega(\theta^k R)^{\frac{1}{2}} \right\}}_{\text{needs to be less than } \delta} \|D\varphi\|_{L^\infty},$$

the $\mathbf{A}$-harmonic approximation lemma can be used to conclude that

$$\Phi(\theta^{k+1}R) \le \theta^\gamma \Phi(\theta^k R) + C\omega(\theta^k R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^k R}|\right)^2$$

provided that

$$\Phi(\theta^k R)^{\frac{1}{2}} + \omega(\theta^k R)^{\frac{1}{2}} \le \frac{\delta}{C(L)}$$

- $\omega(\theta^k R) \le \omega(R)$, and we may assume $\omega(R) \le \frac{1}{4C(L)^2}\delta^2$.
- What about $\Phi(\theta^k R)$?

Decay Estimate

Assume, for inductive purposes, that

$$\Phi(\theta^k R) \le \theta^\gamma \Phi(\theta^{k-1} R) + C\omega(\theta^{k-1} R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^{k-1}R}|\right)^2.$$

Decay Estimate

Assume, for inductive purposes, that

$$\Phi(\theta^k R) \leq \theta^\gamma \Phi(\theta^{k-1}R) + C\omega(\theta^{k-1}R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^{k-1}R}|\right)^2.$$

If there were an $M < \infty$ such that

$$\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^{k-1}R}|\right)^2 \leq M\left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2,$$

then

$$\Phi(\theta^k R) \leq \frac{1}{4C(L)^2}\delta^2,$$

Decay Estimate

Assume, for inductive purposes, that

$$\Phi(\theta^k R) \leq \theta^\gamma \Phi(\theta^{k-1}R) + C\omega(\theta^{k-1}R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^{k-1}R}|\right)^2.$$

If there were an $M < \infty$ such that

$$\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^{k-1}R}|\right)^2 \leq M\left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2,$$

then

$$\Phi(\theta^k R) \leq \frac{1}{4C(L)^2}\delta^2,$$

under the additional inductive assumption

$$\Phi(\theta^{k-1}R) \leq \frac{1}{4C(L)^2}\delta^2$$

and the assumption that $\omega(\theta^{k-1}R) \leq \omega(R)$ is small enough.

The Crux

The argument relies on inductively establishing that there is an $M < \infty$ such that

$$\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^k R}|\right)^2 \leq M \left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2 ,$$

Estimate of $|(D\mathbf{u})_{\mathbf{x}_0,\theta^k\rho}|$

We have

$$\begin{aligned} |\mathbf{F}_{\mathbf{x}_0,\theta R}| &\leq |\mathbf{F}_{\mathbf{x}_0,R}| + \fint_{B_{\mathbf{x}_0,\theta R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|\,\mathrm{d}\mathbf{x} \\ &\leq |\mathbf{F}_{\mathbf{x}_0,R}| + \frac{C}{\theta^{\frac{n}{2}}}\Phi(R)^{\frac{1}{2}}. \end{aligned}$$

Estimate of $\left|(D\mathbf{u})_{\mathbf{x}_0,\theta^k\rho}\right|$

We have

$$\begin{aligned}\left|\mathbf{F}_{\mathbf{x}_0,\theta R}\right| &\leq \left|\mathbf{F}_{\mathbf{x}_0,R}\right| + \fint_{B_{\mathbf{x}_0,\theta R}} \left|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}\right| \mathrm{d}\mathbf{x} \\ &\leq \left|\mathbf{F}_{\mathbf{x}_0,R}\right| + \frac{C}{\theta^{\frac{n}{2}}}\Phi(R)^{\frac{1}{2}}.\end{aligned}$$

Assuming

$$\Phi(\theta R) \leq \theta^{\gamma}\Phi(R) + C\omega(R)\left(1 + \left|\mathbf{F}_{\mathbf{x}_0,R}\right|\right)^2$$

Estimate of $|(D\mathbf{u})_{\mathbf{x}_0,\theta^k\rho}|$

We have

$$|\mathbf{F}_{\mathbf{x}_0,\theta R}| \leq |\mathbf{F}_{\mathbf{x}_0,R}| + \fint_{B_{\mathbf{x}_0,\theta R}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|\,\mathrm{d}\mathbf{x}$$
$$\leq |\mathbf{F}_{\mathbf{x}_0,R}| + \frac{C}{\theta^{\frac{n}{2}}}\Phi(R)^{\frac{1}{2}}.$$

Assuming

$$\Phi(\theta R) \leq \theta^{\gamma}\Phi(R) + C\omega(R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,R}|\right)^2$$

Similarly

$$|\mathbf{F}_{\mathbf{x}_0,\theta^2 R}| \leq |\mathbf{F}_{\mathbf{x}_0,\theta R}| + \frac{C}{\theta^{\frac{n}{2}}}\Phi(\theta R)^{\frac{1}{2}}$$
$$\leq \left(1 + \frac{C}{\theta^{\frac{n}{2}}}\omega(R)^{\frac{1}{2}}\right)|\mathbf{F}_{\mathbf{x}_0,R}| + \boxed{\text{Other Positive Stuff}}$$

Estimate of $\left|(D\mathbf{u})_{\mathbf{x}_0,\theta^k\rho}\right|$

Assuming

$$\Phi(\theta^2 R) \leq \theta^\gamma \Phi(\theta R) + C\omega(\theta R)\left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta R}|\right)^2,$$

we continue with

$$\begin{aligned} |\mathbf{F}_{\mathbf{x}_0,\theta^3 R}| &\leq |\mathbf{F}_{\mathbf{x}_0,\theta^2 R}| + \frac{C}{\theta^{\frac{n}{2}}}\Phi(\theta^2 R)^{\frac{1}{2}} \\ &\leq \left(1 + \frac{C}{\theta^{\frac{n}{2}}}\omega(R)^{\frac{1}{2}} + \frac{C}{\theta^{\frac{n}{2}}}\omega(\theta R)^{\frac{1}{2}}\right)|\mathbf{F}_{\mathbf{x}_0,R}| + \boxed{\text{Other Positive Stuff}} \end{aligned}$$

Estimate of $|(D\mathbf{u})_{\mathbf{x}_0,\theta^k\rho}|$

Continuing

$$|\mathbf{F}_{\mathbf{x}_0,\theta^k R}| \leq |\mathbf{F}_{\mathbf{x}_0,R}| \left\{ \frac{C}{\theta^{\frac{n}{2}}} \sum_{j=0}^{k-2} \omega(\theta^j R)^{\frac{1}{2}} \right\} + \boxed{\text{Other Positive Stuff}}.$$

Estimate of $|(D\mathbf{u})_{\mathbf{x}_0,\theta^k\rho}|$

Continuing

$$|\mathbf{F}_{\mathbf{x}_0,\theta^k R}| \le |\mathbf{F}_{\mathbf{x}_0,R}| \left\{ \frac{C}{\theta^{\frac{n}{2}}} \sum_{j=0}^{k-2} \omega(\theta^j R)^{\frac{1}{2}} \right\} + \boxed{\text{Other Positive Stuff}}.$$

If $\omega(t) \searrow 0$ fast enough as $t \searrow 0$, then a uniform bound on $|\mathbf{F}_{\mathbf{x}_0,\theta^k R}|$ can be established.

Estimate of $|(D\mathbf{u})_{\mathbf{x}_0,\theta^k\rho}|$

Continuing

$$|\mathbf{F}_{\mathbf{x}_0,\theta^k R}| \leq |\mathbf{F}_{\mathbf{x}_0,R}| \left\{ \frac{C}{\theta^{\frac{n}{2}}} \sum_{j=0}^{k-2} \omega(\theta^j R)^{\frac{1}{2}} \right\} + \boxed{\text{Other Positive Stuff}}.$$

If $\omega(t) \searrow 0$ fast enough as $t \searrow 0$, then a uniform bound on $|\mathbf{F}_{\mathbf{x}_0,\theta^k R}|$ can be established.

In general, the decay estimate

$$\Phi(\theta^k R) \leq \theta^\gamma \Phi(\theta^{k-1} R) + C\omega(\theta^{k-1} R) \left(1 + |\mathbf{F}_{\mathbf{x}_0,\theta^{k-1} R}|\right)^2.$$

alone is insufficient to bound $|\mathbf{F}_{\mathbf{x}_0,\theta^k R}|$ uniformly.

Not enough to just assume ω is continuous.

A Modified Excess Function

Instead of defining the excess as

$$\Phi(\rho) := \fint_{B_{\mathbf{x}_0,\rho}} |D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2 \, \mathrm{d}\mathbf{x},$$

define

$$\Phi(\rho) := \fint_{B_{\mathbf{x}_0,\rho}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2}{\left(1 + |\mathbf{F}_{\mathbf{x}_0,\rho}|\right)^2} \, \mathrm{d}\mathbf{x}$$

Linearization Reviewed

Recall: for each $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot [D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}] \otimes D\varphi \, \mathrm{d}\mathbf{x} \right|$$

$$\leq C(L) \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{1 + |\mathbf{F}_{\mathbf{x}_0,R}|} \, \mathrm{d}\mathbf{x} + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|) \right\} \|D\varphi\|_{L^\infty}.$$

Linearization Reviewed

Recall: for each $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A}\cdot[D\mathbf{u}-\mathbf{F}_{\mathbf{x}_0,R}]\otimes D\varphi\,\mathrm{d}\mathbf{x} \right| \leq C(L)\left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u}-\mathbf{F}_{\mathbf{x}_0,R}|^2}{1+|\mathbf{F}_{\mathbf{x}_0,R}|}\,\mathrm{d}\mathbf{x} + \omega(R)(1+|\mathbf{F}_{\mathbf{x}_0,R}|) \right\} \|D\varphi\|_{L^\infty}.$$

Linearization Reviewed

Recall: for each $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A}\cdot[D\mathbf{u}-\mathbf{F}_{\mathbf{x}_0,R}]\otimes D\varphi \, \mathrm{d}\mathbf{x} \right|$$

$$\leq C(L)\left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u}-\mathbf{F}_{\mathbf{x}_0,R}|^2}{1+|\mathbf{F}_{\mathbf{x}_0,R}|}\,\mathrm{d}\mathbf{x} + \omega(R)(1+|\mathbf{F}_{\mathbf{x}_0,R}|) \right\} \|D\varphi\|_{L^\infty}.$$

Linearization Reviewed

Recall: for each $\varphi \in W_0^{1,\infty}(B_{\mathbf{x}_0,R};\mathbb{R}^N)$

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot [D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}] \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{1 + |\mathbf{F}_{\mathbf{x}_0,R}|} \mathrm{d}\mathbf{x} + \omega(R)(1 + |\mathbf{F}_{\mathbf{x}_0,R}|) \right\} \|D\varphi\|_{L^\infty}.$$

This implies

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot \frac{[D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}]}{1 + |\mathbf{F}_{\mathbf{x}_0,R}|} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2} \mathrm{d}\mathbf{x} + \omega(R) \right\} \|D\varphi\|_{L^\infty}.$$

Linearization Reviewed

This yields

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \left(\fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2} \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty}.$$

Here

$$\mathbf{w} := \frac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha\,(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)}$$

with

$$\alpha := \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2} \, \mathrm{d}\mathbf{x} + \omega(R) \right\}^{\frac{1}{2}}$$

Linearization Reviewed

This yields

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, d\mathbf{x} \right| \leq C(L) \left\{ \left(\fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2} \, d\mathbf{x} \right)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty}.$$

Here

$$\mathbf{w} := \frac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha \, (1 + |\mathbf{F}_{\mathbf{x}_0,R}|)}$$

with

$$\alpha := \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2} \, d\mathbf{x} + \omega(R) \right\}^{\frac{1}{2}}$$

Linearization Reviewed

This yields

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq C(L) \left\{ \left(\fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2} \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\} \|D\varphi\|_{L^\infty}.$$

Here

$$\mathbf{w} := \frac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha\,(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)}$$

with

$$\alpha := \left\{ \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2} \, \mathrm{d}\mathbf{x} + \omega(R) \right\}^{\frac{1}{2}}$$

Linearization Reviewed

This yields

$$\left| \fint_{B_{\mathbf{x}_0,R}} \mathbf{A} \cdot D\mathbf{w} \otimes D\varphi \, \mathrm{d}\mathbf{x} \right| \leq \underbrace{C(L) \left\{ \Phi(R)^{\frac{1}{2}} + \omega(R)^{\frac{1}{2}} \right\}}_{=\delta} \|D\varphi\|_{L^\infty}.$$

where

$$\Phi(R) := \fint_{B_{\mathbf{x}_0,R}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,R}|^2}{(1 + |\mathbf{F}_{\mathbf{x}_0,R}|)^2} \, \mathrm{d}\mathbf{x}$$

Here

$$\mathbf{w} := \frac{\mathbf{u} - (\mathbf{u})_{\mathbf{x}_0,R} - \mathbf{F}_{\mathbf{x}_0,R}(\mathbf{x} - \mathbf{x}_0)}{\alpha \, (1 + |\mathbf{F}_{\mathbf{x}_0,R}|)}$$

with

$$\alpha := \{\Phi(R) + \omega(R)\}^{\frac{1}{2}}$$

A Modified "Decay" Estimate

With

$$\Phi(\rho) := \fint_{B_{\mathbf{x}_0,\rho}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2}{\left(1 + |\mathbf{F}_{\mathbf{x}_0,\rho}|\right)^2} \, \mathrm{d}\mathbf{x},$$

the decay estimate takes the form

$$\Phi(\theta R) \leq \theta^{\beta} \Phi(R) + C\omega(R).$$

Here $0 < \beta < 1$ is fixed.

A Modified "Decay" Estimate

With

$$\Phi(\rho) := \fint_{B_{\mathbf{x}_0,\rho}} \frac{|D\mathbf{u} - \mathbf{F}_{\mathbf{x}_0,\rho}|^2}{\left(1 + |\mathbf{F}_{\mathbf{x}_0,\rho}|\right)^2} \, \mathrm{d}\mathbf{x},$$

the decay estimate takes the form

$$\Phi(\theta R) \leq \theta^\beta \Phi(R) + C\omega(R).$$

Here $0 < \beta < 1$ is fixed.

New Objective

Instead of showing that $\Phi(\rho)$ decreases with ρ, we only show that if $\Phi(R)$ is small enough, then $\Phi(\rho)$ remains small for $0 < \rho < R$.

An Induction Argument

Suppose that $C\omega(R) < \left(1-\theta^{\beta}\right)\varepsilon_0$ and

$$\Phi(\theta^k R) \le \varepsilon_0.$$

An Induction Argument

Suppose that $C\omega(R) < \left(1 - \theta^{\beta}\right)\varepsilon_0$ and

$$\Phi(\theta^k R) \leq \varepsilon_0.$$

Then

$$\begin{aligned} \Phi(\theta^{k+1} R) &\leq \theta^{\beta}\Phi(\theta^k R) + C\omega(\theta^k R) \\ &\leq \theta^{\beta}\varepsilon_0 + \left(1 - \theta^{2\beta}\right)\varepsilon_0 \\ &= \varepsilon_0. \end{aligned}$$

An Induction Argument

Suppose that $C\omega(R) < \left(1 - \theta^{\beta}\right)\varepsilon_0$ and

$$\Phi(\theta^k R) \le \varepsilon_0.$$

Then

$$\begin{aligned}\Phi(\theta^{k+1}R) &\le \theta^{\beta}\Phi(\theta^k R) + C\omega(\theta^k R)\\ &\le \theta^{\beta}\varepsilon_0 + \left(1 - \theta^{2\beta}\right)\varepsilon_0\\ &= \varepsilon_0.\end{aligned}$$

As long as $\varepsilon_0 > 0$ is selected small enough, the $\mathbf{A}$-harmonic approximation lemma can be used, and the estimate

$$\Phi(\theta^{k+1}R) \le \theta^{\beta}\Phi(\theta^k R) + C\omega(\theta^k R)$$

is established by induction.

A Morrey Estimate

We now show that a Morrey-type estimate holds for $|D\mathbf{u}|$ at $\mathbf{x}_0$. We have

$$\Phi(\theta^k R) \leq \varepsilon_0 \qquad \text{for all } k \in \mathbb{N}.$$

Thus

$$\int_{B_{\theta^{k+1}R}} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} \leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2\int_{B_{\theta^{k+1}R}} |D\mathbf{u} - \mathbf{F}_{\theta^k R}|^2 \,\mathrm{d}\mathbf{x}$$

A Morrey Estimate

We now show that a Morrey-type estimate holds for $|D\mathbf{u}|$ at $\mathbf{x}_0$. We have

$$\Phi(\theta^k R) \leq \varepsilon_0 \qquad \text{for all } k \in \mathbb{N}.$$

Thus

$$\int_{B_{\theta^{k+1}R}} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2 \int_{B_{\theta^{k+1}R}} |D\mathbf{u} - \mathbf{F}_{\theta^k R}|^2 \, \mathrm{d}\mathbf{x}$$

$$\leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2\left(1 + |\mathbf{F}_{\theta^k R}|\right)^2 \int_{B_{\theta^k R}} \frac{|D\mathbf{u} - \mathbf{F}_{\theta^k R}|^2}{\left(1 + |\mathbf{F}_{\theta^k R}|\right)^2} \, \mathrm{d}\mathbf{x}$$

A Morrey Estimate

We now show that a Morrey-type estimate holds for $|D\mathbf{u}|$ at $\mathbf{x}_0$. We have

$$\Phi(\theta^k R) \leq \varepsilon_0 \qquad \text{for all } k \in \mathbb{N}.$$

Thus

$$\begin{aligned}
\int_{B_{\theta^{k+1}R}} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} &\leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2\int_{B_{\theta^{k+1}R}} |D\mathbf{u} - \mathbf{F}_{\theta^k R}|^2 \,\mathrm{d}\mathbf{x} \\
&\leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2\left(1 + |\mathbf{F}_{\theta^k R}|\right)^2 \int_{B_{\theta^k R}} \frac{|D\mathbf{u} - \mathbf{F}_{\theta^k R}|^2}{\left(1 + |\mathbf{F}_{\theta^k R}|\right)^2} \,\mathrm{d}\mathbf{x} \\
&\leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2|B_{\theta^k R}|\left(1 + |\mathbf{F}_{\theta^k R}|\right)^2 \Phi(\theta^k R)
\end{aligned}$$

A Morrey Estimate

We now show that a Morrey-type estimate holds for $|D\mathbf{u}|$ at $\mathbf{x}_0$. We have

$$\Phi(\theta^k R) \leq \varepsilon_0 \qquad \text{for all } k \in \mathbb{N}.$$

Thus

$$\begin{aligned}
\int_{B_{\theta^{k+1}R}} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} &\leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2 \int_{B_{\theta^{k+1}R}} |D\mathbf{u} - \mathbf{F}_{\theta^k R}|^2 \, \mathrm{d}\mathbf{x} \\
&\leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2\left(1 + |\mathbf{F}_{\theta^k R}|\right)^2 \int_{B_{\theta^k R}} \frac{|D\mathbf{u} - \mathbf{F}_{\theta^k R}|^2}{\left(1 + |\mathbf{F}_{\theta^k R}|\right)^2} \, \mathrm{d}\mathbf{x} \\
&\leq 2|B_{\theta^{k+1}R}||\mathbf{F}_{\theta^k R}|^2 + 2|B_{\theta^k R}| \left(1 + |\mathbf{F}_{\theta^k R}|\right)^2 \Phi(\theta^k R) \\
&\leq C\left(\theta^n + \varepsilon_0\right) \int_{B_{\theta^k R}} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + C\varepsilon_0 \left(\theta^k R\right)^n .
\end{aligned}$$

A Morrey Estimate

Assuming $\varepsilon_0 \leq \theta^n$, given $0 < \alpha < n$ we may select $0 < \theta < \frac{1}{2}$ so that

$$\int_{B_{\theta^{k+1}R}} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} \leq \theta^{\alpha} \int_{B_{\theta^k R}} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} + C\left(\theta^k R\right)^n .$$

A Morrey Estimate

Assuming $\varepsilon_0 \leq \theta^n$, given $0 < \alpha < n$ we may select $0 < \theta < \frac{1}{2}$ so that

$$\int_{B_{\theta^{k+1}R}} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \leq \theta^\alpha \int_{B_{\theta^k R}} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + C\left(\theta^k R\right)^n.$$

A standard iteration argument yields

$$\int_{B_r} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \leq \frac{Cr^\alpha}{R^\alpha} \int_{B_R} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + Cr^\alpha \qquad \text{for all } 0 < r \leq R$$

A Campanato Estimate

Using Poincaré's inequality

$$\int_{B_r} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \leq \frac{Cr^\alpha}{R^\alpha} \int_{B_R} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + Cr^\alpha \qquad \text{for all } 0 < r \leq \rho$$

implies

$$\int_{B_r} |\mathbf{u} - (\mathbf{u})_r|^2 \, \mathrm{d}\mathbf{x} \leq Cr^2 \int_{B_r} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \leq \frac{Cr^{\alpha+2}}{R^\alpha} \int_{B_R} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + Cr^{\alpha+2}$$

for all $0 < r \leq R$.

A Campanato Estimate

Using Poincaré's inequality

$$\int_{B_r} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} \leq \frac{Cr^\alpha}{R^\alpha} \int_{B_R} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} + Cr^\alpha \qquad \text{for all } 0 < r \leq \rho$$

implies

$$\int_{B_r} |\mathbf{u} - (\mathbf{u})_r|^2 \,\mathrm{d}\mathbf{x} \leq Cr^2 \int_{B_r} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} \leq \frac{Cr^{\alpha+2}}{R^\alpha} \int_{B_R} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} + Cr^{\alpha+2}$$

for all $0 < r \leq R$.

Thus

$$\fint_{B_r} |\mathbf{u} - (\mathbf{u})_r|^2 \,\mathrm{d}\mathbf{x} \leq \frac{Cr^{\alpha+2-n}}{R^\alpha} \int_{B_R} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} + Cr^{\alpha+2-n} \text{ for all } 0 < r \leq R.$$

Recall that $0 < \alpha < n$ was arbitrary.

A Campanato Estimate

Given $0 < \gamma < 1$, we select $\alpha = n - 2(1-\gamma)$, then

$$\fint_{B_r} |\mathbf{u} - (\mathbf{u})_r|^2 \, \mathrm{d}\mathbf{x} \le \frac{Cr^{\alpha+2-n}}{R^\alpha} \int_{B_R} |D\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + Cr^{\alpha+2-n} \qquad \text{for all } 0 < r \le R.$$

implies

$$\fint_{B_r} |\mathbf{u} - (\mathbf{u})_r|^2 \, \mathrm{d}\mathbf{x} \le Cr^{2\gamma} \qquad \text{for all } 0 < r \le R.$$

A Campanato Estimate

Given $0 < \gamma < 1$, we select $\alpha = n - 2(1-\gamma)$, then

$$\fint_{B_r} |\mathbf{u} - (\mathbf{u})_r|^2 \,\mathrm{d}\mathbf{x} \le \frac{Cr^{\alpha+2-n}}{R^\alpha} \int_{B_R} |D\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} + Cr^{\alpha+2-n} \qquad \text{for all } 0 < r \le R.$$

implies

$$\fint_{B_r} |\mathbf{u} - (\mathbf{u})_r|^2 \,\mathrm{d}\mathbf{x} \le Cr^{2\gamma} \qquad \text{for all } 0 < r \le R.$$

This estimate can be established in an open neighborhood of each Lebesgue point of $D\mathbf{u}$.

Partial Continuity for Elliptic Systems: $p = 2$

Assume the field $\mathbf{a} : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ satisfies

Hypotheses

For some $0 < \lambda \leq L < \infty$

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x}, \mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda |\xi|^2 \leq \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}, \mathbf{F}) \cdot \xi \otimes \xi \leq L |\xi|^2$
3. $|\mathbf{a}(\mathbf{x}, \mathbf{F}) - \mathbf{a}(\mathbf{y}, \mathbf{F})| \leq L\omega(|\mathbf{x} - \mathbf{y}|)(1 + |\mathbf{F}|)$

Theorem

If $\mathbf{u} \in W^{1,2}(\Omega; \mathbb{R}^N)$ *satisfies*

$$\int_\Omega \mathbf{a}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega; \mathbb{R}^N)$$

then for each $\gamma \in (0,1)$ *there is an open* $\Omega_0 \subseteq \Omega$ *such that* $|\Omega \backslash \Omega_0| = 0$ *and* $\mathbf{u} \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$.

Partial Continuity for Elliptic Systems: $p \geq 2$

Assume the field $\mathbf{a} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfies

Hypotheses

For some $0 < \lambda \leq L < \infty$ and $2 \leq p < \infty$

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x}, \mathbf{u}, \mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda (1 + |\mathbf{F}|)^{p-2} |\xi|^2 \leq \partial_{\mathbf{F}} \mathbf{a}(\mathbf{x}, \mathbf{u}, \mathbf{F}) \cdot \xi \otimes \xi \leq L (1 + |\mathbf{F}|)^{p-2} |\xi|^2$
3. $|\mathbf{a}(\mathbf{x}, \mathbf{u}, \mathbf{F}) - \mathbf{a}(\mathbf{y}, \mathbf{v}, \mathbf{F})| \leq L\omega(|\mathbf{x}-\mathbf{y}| + |\mathbf{u}-\mathbf{v}|)(1 + |\mathbf{F}|)^{p-1}$

Theorem

If $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$ *satisfies*

$$\int_\Omega \mathbf{a}(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x})\, \mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega; \mathbb{R}^N)$$

then for each $\gamma \in (0,1)$ *there is an open* $\Omega_0 \subseteq \Omega$ *such that* $|\Omega \backslash \Omega_0| = 0$ *and* $\mathbf{u} \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$.

Partial Continuity for Elliptic Systems: $p \geq 2$

Assume the field $\mathbf{a} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ satisfies

Hypotheses

For some $0 < \lambda \leq L < \infty$ and $2 \leq p < \infty$

1. $\mathbf{F} \mapsto \mathbf{a}(\mathbf{x}, \mathbf{u}, \mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda(1 + |\mathbf{F}|)^{p-2}|\boldsymbol{\xi}|^2 \leq \partial_{\mathbf{F}}\mathbf{a}(\mathbf{x}, \mathbf{u}, \mathbf{F}) \cdot \boldsymbol{\xi} \otimes \boldsymbol{\xi} \leq L(1 + |\mathbf{F}|)^{p-2}|\boldsymbol{\xi}|^2$
3. $|\mathbf{a}(\mathbf{x}, \mathbf{u}, \mathbf{F}) - \mathbf{a}(\mathbf{y}, \mathbf{v}, \mathbf{F})| \leq L\omega(|\mathbf{x} - \mathbf{y}| + |\mathbf{u} - \mathbf{v}|)(1 + |\mathbf{F}|)^{p-1}$

Theorem

If $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$ *satisfies*

$$\int_\Omega \mathbf{a}(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x})) \cdot D\varphi(\mathbf{x})\, \mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega; \mathbb{R}^N)$$

then for each $\gamma \in (0, 1)$ *there is an open* $\Omega_0 \subseteq \Omega$ *such that* $|\Omega \backslash \Omega_0| = 0$ *and* $\mathbf{u} \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$.

Partial Continuity for Minimizers

Assume that $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies

Hypotheses

For some $0 < \lambda \leq L < \infty$ and $2 \leq p < \infty$

1. $\mathbf{F} \mapsto g(\mathbf{x},\mathbf{u},\mathbf{F})$ is $\mathscr{C}^3$
2. $\lambda|\mathbf{F}|^p \leq g(\mathbf{x},\mathbf{u},\mathbf{F}) \leq L(1+|\mathbf{F}|)^p$
3. g is uniformly strictly quasiconvex
4. $|g(\mathbf{x},\mathbf{u},\mathbf{F}) - g(\mathbf{y},\mathbf{v},\mathbf{F})| \leq L\omega(|\mathbf{x}-\mathbf{y}| + |\mathbf{u}-\mathbf{v}||)(1+|\mathbf{F}|)^p$

Theorem

If $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$ *is a local minimizer for*

$$J[\mathbf{u}] := \int_\Omega g(\mathbf{x},\mathbf{u}(\mathbf{x}),D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

then for each $\gamma \in (0,1)$ *there is an open* $\Omega_0 \subseteq \Omega$ *such that* $|\Omega\backslash\Omega_0| = 0$ *and* $\mathbf{u} \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$.

Partial Continuity for Parabolic Systems

Assume the field $\mathbf{a} : (-T,0)\times\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}^{N\times n}$ satisfies

Hypotheses

For some $0<\lambda\leq L<\infty$ and $2\leq p<\infty$

1. $\mathbf{F}\mapsto\mathbf{a}(t,\mathbf{x},\mathbf{u},\mathbf{F})$ is $\mathscr{C}^2$
2. $\lambda(1+|\mathbf{F}|)^{p-2}|\boldsymbol{\xi}|^2\leq\partial_{\mathbf{F}}\mathbf{a}(t,\mathbf{x},\mathbf{u},\mathbf{F})\cdot\boldsymbol{\xi}\otimes\boldsymbol{\xi}\leq L(1+|\mathbf{F}|)^{p-2}|\boldsymbol{\xi}|^2$
3. $|\mathbf{a}(s,\mathbf{x},\mathbf{u},\mathbf{F})-\mathbf{a}(t,\mathbf{y},\mathbf{v},\mathbf{F})|\leq L\omega(\sqrt{|s-t|}+|\mathbf{x}-\mathbf{y}|+|\mathbf{u}-\mathbf{v}|)(1+|\mathbf{F}|)^{p-1}$

Theorem (Bögelein, Foss & Mingione (2011))

If $\mathbf{u}\in L^p((-T,0);W^{1,p}(\Omega;\mathbb{R}^N))\cap\mathscr{C}^0((-T,0);L^2(\Omega;\mathbb{R}^N))$ *satisfies*

$$\int_{(-T,0)\times\Omega}\left\{\mathbf{u}(t,\mathbf{x})\cdot\frac{\partial}{\partial t}\varphi(t,\mathbf{x})-\mathbf{a}(t,\mathbf{x},\mathbf{u}(t,\mathbf{x}),D\mathbf{u}(t,\mathbf{x}))\cdot D\varphi(t,\mathbf{x})\right\}\mathrm{d}(t,\mathbf{x})=0$$

for all $\varphi\in\mathscr{C}_{\mathrm{c}}^\infty((-T,0)\times\Omega;\mathbb{R}^N)$*, then for each* $\gamma\in(0,1)$ *there is an open* $Q_0\subseteq(-T,0)\times\Omega$ *with full measure such that* $\mathbf{u}\in C^{0;\gamma,\gamma/2}_{\mathrm{loc}}(Q_0)$*.*