# Regularity results for optimal patterns in the branched transportation problem

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# Outline

# Branched transportation problems

- Many natural systems show a distinctive tree-shaped structure: plants, trees, drainage networks, root systems, bronchial and cardiovascular systems.
- These systems could be described in terms of mass transportation, but Monge-Kantorovich theory turns out to be the wrong mathematical model since the mass is carried from the initial to the final point on a straight line.

# Branched transportation problems

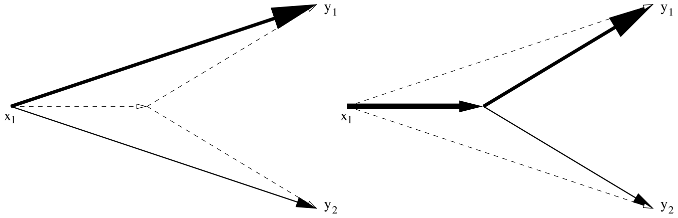
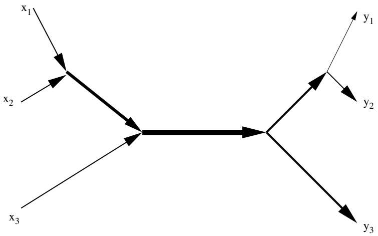



Figure: V-shaped versus Y-shaped transport.

# The functional (discrete case)

- $\Omega \subset \mathbb{R}^N$ compact, convex;
- $\mu^+ = \sum_{i=1}^m a_i \delta_{x_i}, \mu^- = \sum_{j=1}^n b_j \delta_{y_j}$ convex combinations of Dirac masses;
- $G$ weighted directed graph; $\operatorname{spt} \mu^+, \operatorname{spt} \mu^- \subseteq V(G)$;
- the mass flows from the initial measure $\mu^+$ to the final measure $\mu^-$ “inside” the edges of the graph $G$.



# The functional (discrete case)

- The point is now to provide to each transport path $G$ a suitable cost that makes keeping the mass together cheaper. The right cost function is

$$J_\alpha(G) := \sum_{e \in E(G)} [m(e)]^\alpha l(e),$$

$l(e)$ length of edge $e$, $0 \leq \alpha < 1$ fixed;

- this cost takes advantage of the subadditivity of the function $t \mapsto t^\alpha$ in order to make the tree-shaped graphs cheaper.

# The functional (continuous case)

- $e$ (oriented edge) $\mapsto \mu_e = (\mathcal{H}^1_{|e})\hat{e}$ (vector measure);
- $G \mapsto T_G = \sum_{e \in E(G)} m(e)\mu_e$;
- div $T_G = \mu^+ - \mu^-$ sums up all conditions;
- a general irrigation pattern is defined by density and the cost as a lower semicontinuous envelope:

$$J_\alpha(T) = \inf_{T_{G_i} \to T} \liminf_{i \to +\infty} J_\alpha(T_{G_i}).$$

# The Irrigation Problem

**Problem (Irrigation problem)**

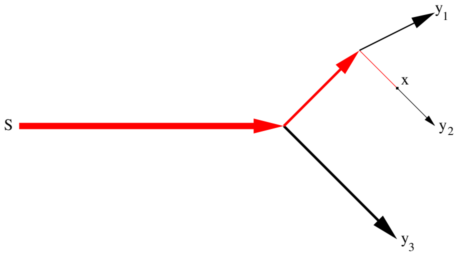
- *$\mu^+$, $\mu^-$ probability measures on $\mathbb{R}^N$;*
- *minimize $J_\alpha(G)$ among irrigation patterns $G$ such that* $\operatorname{div} G = \mu^+ - \mu^-$.

A pattern minimizing $J_\alpha$ is the best branched structure between the source $\mu^+$ and the irrigated measure $\mu^-$.

# The landscape function

$\mu^+ = \delta_S$. For optimal graphs the landscape function $Z$ is given by

$$Z(x) = \sum_{\text{path from } S \text{ to } x} [m(e)]^{\alpha-1} l(e).$$

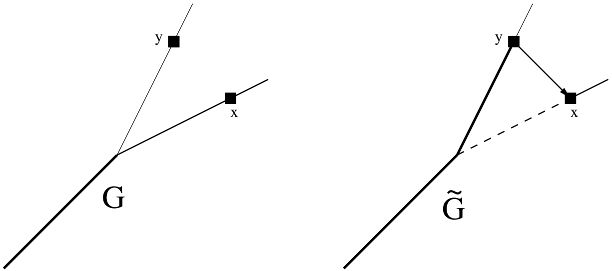



The landscape function can be defined also in the continuous setting.

# Why to consider the landscape function?

- In a discrete form, the landscape function was already introduced in geophysics and is related to the problem of erosion and landscape equilibrium;
- the landscape function is related to first order variations of the functional $J_\alpha$;
- the Hölder regularity of landscape function is related to the decay of the mass on the paths of the graph.

# First order variations

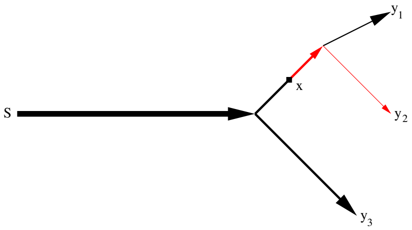



**Theorem (First order gain formula)**

*The pattern $\tilde{G}$ satisfies*

$$J_\alpha(\tilde{G}) - J_\alpha(G) \leq \alpha m(Z(y) - Z(x)) + m^\alpha |x - y|.$$

# Mass decay on the graph




**Theorem (Mass decay)**

*Suppose that G is optimal. If Z is Hölder continuous of exponent $\beta$, then the mass decay is exponent is $(1-\beta)/(1-\alpha)$:*

$$m(x) \gtrsim l(x)^{\frac{1-\beta}{1-\alpha}},$$

*and vice versa.*

# Hölder continuity when the irrigated measure is LAR

## Definition

A measure $\mu$ is lower Ahlfors regular in dimension $h$ if there exist $r_0 > 0$ and $c_A > 0$ such that:

$$\mu(B_r(x)) \geq c_A r^h \text{ for all } x \in \operatorname{spt} \mu \text{ and } 0 < r < r_0.$$

## Theorem

*Suppose that the irrigated measure is LAR in dimension h. Then, the landscape function Z is Hölder with exponent* $\beta = 1 + h(\alpha - 1)$.

Some example shows that the landscape regularity may be better and may depend on the source of irrigation.

# Best estimate on the Hölder exponent

## Definition

A measure $\mu$ is upper Ahlfors regular in dimension $h$ if there exists $C_A > 0$ such that:

$$\mu(B_r(x)) \leq C_A r^h \text{ for all } r > 0.$$

## Theorem

*Suppose that the irrigated measure is UAR above in dimension $h$ and the landscape function $Z$ is Hölder with exponent $\beta$. Then, $\beta \leq 1 + h(\alpha - 1)$.*

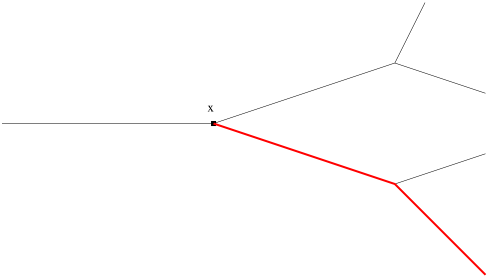
If the irrigated measure is Ahlfors regular in dimension $h$ (both LAR and UAR), the best Hölder exponent is $1 + h(\alpha - 1)$.

# Main branches from a point

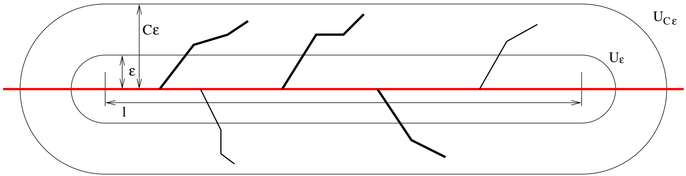
In the next the irrigated measure will always be *Ahlfors regular* in dimension $h$.

**Definition (Main branches from a point $x$)**

A *main branch* starting from a point $x$ is the branch maximizing the residual length.

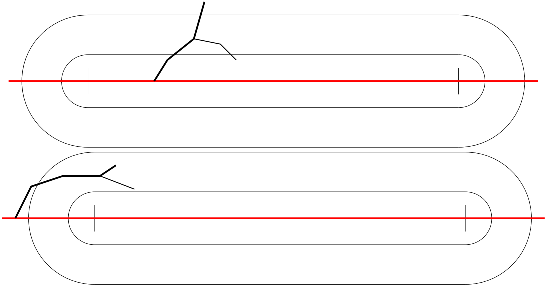


# Fractal regularity




- $N$ = number of branches bifurcating of residual length between $\varepsilon$ and $C\varepsilon$.
- mass carried by one of such branches $\gtrsim \varepsilon^h$,
- mass of the tubolar neighbourhood of radius $C\varepsilon \sim l\varepsilon^{h-1}$,
- mass balance: $\varepsilon^h N \lesssim l\varepsilon^{h-1}$,
- $N \lesssim \frac{l}{\varepsilon}$.

# Fractal regularity



It can be proved that for small $\varepsilon$ and a suitable choice of $C$ the measure irrigated by "long branches" and by "far away branches" is a fraction of the measure of $U_{C\varepsilon} \setminus U_\varepsilon$. Then, we also have

$$N \gtrsim \frac{l}{\varepsilon}.$$

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