Continuity of Solutions for a problem in the Calculus of Variations

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A basic problem in the Calculus of Variations

$$\text{To minimize} \qquad u \mapsto \int_\Omega L(\nabla u(x))\,dx$$

$$u_{|\partial\Omega} = \varphi$$

Standing Assumptions

- $\Omega \subset \mathbb{R}^n$ bounded open set
- $\varphi : \partial\Omega \to \mathbb{R}$ continuous
- $L : \mathbb{R}^n \to \mathbb{R}$ strictly convex and superlinear

The regularity problem

- Is the solution smooth in Ω ?
- Is the solution continuous on $\overline{\Omega}$?

A basic example

$$\text{To minimize} \qquad u \mapsto \int_\Omega |\nabla u(x)|^2\, dx$$

$$u_{|\partial\Omega} = \varphi$$

Regularity properties

- u is analytic on Ω
- u is continuous at any *regular* point $\gamma \in \partial\Omega$

De Giorgi's Theorem

Theorem

Assume

- $L \in C^2$, $\nabla^2 L > 0$
- *the solution u is locally Lipschitz in Ω*

Then u is locally $C^{1,\alpha}$ in Ω

The partial derivatives of u satisfy an elliptic equation of the form

$$\text{div}\,(A(x)\nabla v) = 0 \qquad A(x) = \nabla L(\nabla u(x))$$

By Schauder's Theory

- L smooth $\implies$ u smooth

By Bernstein's Theorem

- L analytic $\implies$ u analytic

A counterexample (Giaquinta, Marcellini)

A nice Lagrangian...

$$L(\xi) = \xi_1^2 + \cdots + \xi_{n-1}^2 + \frac{1}{2}\xi_n^4$$

...a singular minimum

$$u(x_1, \ldots, x_n) = c_n \frac{x_n^2}{\sqrt{\sum_{i=1}^{n-1} x_i^2}}$$

Two open problems

- u locally bounded $\implies$ u continuous ?
- φ continuous $\implies$ u continuous ?

Lipschitz regularity on uniformly convex sets

Theorem (Miranda)

Assume

- Ω *uniformly convex (= enclosing sphere condition)*
- φ *is* C^2

Then $u \in W^{1,\infty}(\Omega)$

Theorem (Clarke)

Assume

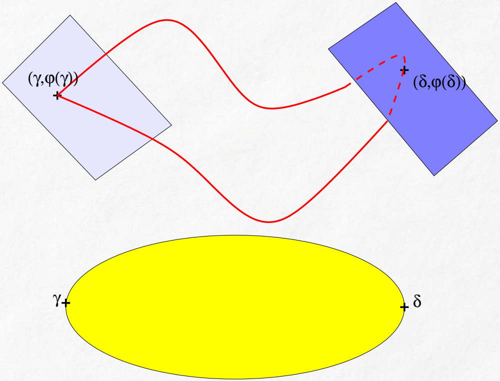
- Ω *uniformly convex*
- φ *is semiconvex*

Then $u \in W^{1,\infty}_{loc}(\Omega) \cap C^0(\overline{\Omega})$

Counterexample to global Lipschitzness

$$\Omega = B(0,1) \subset \mathbb{R}^2 \quad , \quad L(\xi) = |\xi|^2 \quad , \quad \varphi(x,y) = |y|$$

The bounded slope condition



Lipschitz regularity on convex sets

Theorem (Miranda)

Assume

- Ω *convex*
- φ *bounded slope condition*

Then $u \in W^{1,\infty}(\Omega)$

Theorem (Clarke)

Assume

- Ω *convex*
- φ *lower bounded slope condition*

Then $u \in W^{1,\infty}_{loc}(\Omega)$

A Lipschitz continuity result on a non convex domain

Theorem (Cellina)

Assume

- Ω *exterior sphere condition*
- $L(\xi) = l(|\xi|)$
- φ *constant on each connected components of* $\partial\Omega$

Then $u \in W^{1,\infty}(\Omega)$

Continuity up to the boundary

Theorem (B.)

Assume that φ is continuous and one of the following

- Ω *convex*
- Ω *smooth and* $L(\xi) = l(|\xi|)$

Then u is continuous

Theorem (Mariconda-Treu)

Assume

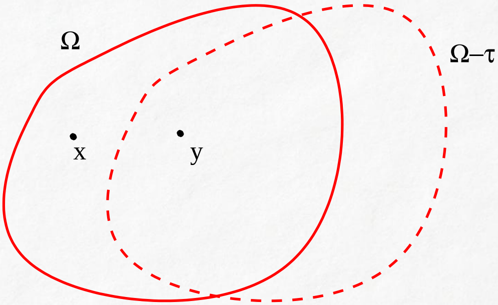
- Ω *convex*
- φ *Lipschitz continuous*
- L *coercive of order* $p > 1$

Then u is Hölder continuous (of order $\frac{p-1}{n+p-1}$)

A maximum principle : the Rado-Haar Lemma

Let x, $y \in \Omega$ and $\tau := x - y$. Compare the minimum u with

$$u_\tau(x) := u(x + \tau)$$



τ = x−y

An estimate on the modulus of continuity

$$|u(x) - u(y)| \leq \sup_{\substack{x' \in \Omega, y' \in \partial\Omega \\ |x'-y'| \leq |x-y|}} |u(x') - \varphi(y')|$$

Lower and upper barriers

Definition

$v : \Omega \to \mathbb{R}$ is an upper barrier at $\gamma \in \partial\Omega$ if

- $v \in W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$
- $v(\gamma) = \varphi(\gamma)$
- $v \geq u$ *a.e. on* Ω

Example: concave functions

Rado Haar Lemma $+$ barriers $\Longrightarrow$ continuity on $\overline{\Omega}$

Implicit barriers and continuity I

Lemma

Assume

- Ω *convex*
- φ *Lipschitz continuous*

Then u *continuous*

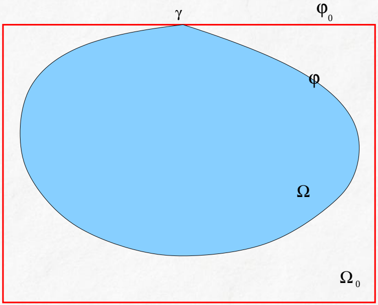
Proof: Let u be the solution of the original problem and $\gamma \in \partial\Omega$.

1^{st} **step** prove that u is continuous at γ when Ω is a cube

2^{nd} **step** an auxiliary variational problem

$$(P_0) \qquad \text{To minimize} \quad v \mapsto \int_{\Omega_0} L(\nabla v)\,, v_{|\partial\Omega_0} = \varphi_0$$

Implicit barriers and continuity II



$$\varphi_0(x) = \varphi(\gamma) + K_\varphi |x - \gamma| \geq \varphi(x)$$

- φ_0 convex $\implies$ φ_0 lower barrier for (P_0)
- the solution u_0 for $(P_0) \geq \varphi_0 \geq \varphi$
- u_0 is an implicit upper barrier at γ: $u_0 \geq u$ on Ω

More general Lagrangians

$$\text{To minimize} \qquad u \mapsto \int_\Omega \left(L(\nabla u(x)) + G(x,u(x))\right)\, dx$$

$$u_{|\partial\Omega} = \varphi$$

Standing Assumptions

- L uniformly convex: $\exists \alpha > 0$ s.t. $\forall\ \theta \in (0,1), \quad \xi,\ \xi' \in \mathbb{R}^n$

$$\theta L(\xi) + (1-\theta)L(\xi') - L(\theta\xi + (1-\theta)\xi') \geq \alpha|\xi - \xi'|^2$$

- G measurable in x and locally Lipschitz in u

Lipschitz continuity results

Theorem (Stampacchia, B.-Clarke)

Assume that Ω *is convex and* u *is bounded. Then*

- φ *satisfies the bounded slope condition* $\implies$ $u \in W^{1,\infty}(\Omega)$
- φ *satisfies the lower bounded slope condition* $\implies$ $u \in W^{1,\infty}_{loc}(\Omega) \cap C^0(\overline{\Omega})$

Continuity results

Theorem (B.)

Assume that Ω *is smooth,* $L(\xi) = l(|\xi|)$ *and* u *is bounded. Then*

- φ *continuous* $\quad\Longrightarrow\quad u \in C^0(\overline{\Omega})$
- φ *Lipschitz continuous* $\quad\Longrightarrow\quad u \in C^{0,\frac{1}{n+1}}(\overline{\Omega})$

A final counterexample

(Esposito-Leonetti-Mingione, Fonseca-Malý-Mingione)

$$\text{To minimize} \qquad u \mapsto \int_\Omega |\nabla u(x)|^p + a(x)|\nabla u(x)|^q)\, dx$$

$$1 < p < n < n+1 < q < +\infty$$

$$\Omega = \text{a cube}, \quad a \in C^1, \quad a \geq 0, \quad \varphi \text{ linear}$$

The set of non-Lebesgue points of the solution has (almost) dimension $N - p$!

A final open problem : What about autonomous Lagrangians ?