Some sufficient conditions for lower semicontinuity in SBD

Giuliano Gargiulo[1], Elvira Zappale[2]
[1]*Dipartimento di Scienze Biologiche ed Ambientali, Universitá degli Studi del Sannio, Italy*
E-mail: giuliano.gargiulo@unisannio.it

[2]*Dipartimento di Ingegneria dell'Informazione e Matematica Applicata, Universitá degli Studi di Salerno, Italy*
E-mail: ezappale@unisa.it

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SUMMARY. Some lower semicontinuity results are given in the space of special fields of bounded deformation for fracture energetic models of the types

$$\int_{J_u} \Psi([u], \nu_u) d\mathcal{H}^{N-1}, \text{ and } \int_{J_u} \Theta(u^+, u^-, \nu_u) d\mathcal{H}^{N-1}$$

under the noninterpenetration constraint $[u] \cdot \nu_u \geq 0 \ \mathcal{H}^{N-1}-$ a. e. on J_u.

1 INTRODUCTION

Geometric Measure Theory, the direct methods of the Calculus of Variations, the mathematical approach of Free Discontinuity Problems, and, in particular, the structure of the Fields of Bounded Deformations have been used here in order to study the equilibrium configurations of some fracture models.

This study is motivated by the results contained in [1, 2] where it has been studied, both from the mechanical and computational view point, in the regime of linearized elasticity, the propagation of the fracture in a cracked body with a dissipative energy a la Barenblatt, i.e. of the type $\int_K \phi([u]\cdot\nu_u, [u]\cdot\tau_u) d\mathcal{H}^{N-1}$, where K denotes the unknown crack site, $[u]\cdot\nu_u$, $[u]\cdot\tau_u$ represent the detachment and the sliding components respectively, of the opening of the fracture $[u]$, and the energy density ϕ has the form $\phi([u]\cdot\nu_u, [u]\cdot\tau_u) = \begin{cases} 0 & \text{if } [u]\cdot\nu_u = [u]\cdot\tau_u = 0, \\ \text{constant} & \text{if } [u]\cdot\nu_u \geq 0, \\ +\infty & \text{if } [u]\cdot\nu_u < 0 \end{cases}$

It has to be emphasized that the form of the energy density ϕ also takes into account an infinitesimal noninterpenetration constraint, i.e. all the deformations u pertainining to the effective description of the energy must satisfy $[u]\cdot\nu_u \geq 0$ $\mathcal{H}^{N-1}$ a.e. on K.

In order to derive, from the mathematical view point, the properties of the energy ϕ above which guarantee lower semicontinuity with respect to the natural convergences (2.13) $\div$ (2.15) below, in order to generalize the models contained in [1, 2] and to extend the lower semicontinuity results for surface integrals contained in [3], the following results has been proved in [4]:

Theorem 1.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, Let*

$$\Phi := \{\varphi : [0, +\infty[\to [0, +\infty[, \varphi \text{ convex, subadditive and nondecreasing}\} \tag{1.1}$$

and let $\varphi \in \Phi$. Let $\{u_h\}$ be a sequence in $SBD(\Omega)$, such that $[u_h]\cdot\nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h, converging to u in $L^1(\Omega;\mathbb{R}^N)$ satisfying (2.12) below, with a function $\gamma : [0, +\infty[\to [0, \infty[$ nondecreasing and verifying the superlinearity condition (2.11) below. Then

$$[u]\cdot\nu_u \geq 0 \ \ \mathcal{H}^{N-1} - \text{ a.e. on } J_u, \tag{1.2}$$

and

$$\int_{J_u} \varphi([u]\cdot\nu_u) d\mathcal{H}^{N-1} \leq \liminf_{h\to+\infty} \int_{J_{u_h}} \varphi([u_h]\cdot\nu_{u_h}) d\mathcal{H}^{N-1}. \tag{1.3}$$

Clearly the class Φ in (1.1) includes functions of the type ϕ above, but it has also to be observed that, in general, the functions in Φ can be truly convex. In fact, typical examples of functions in Φ are given by $\varphi : s \in \mathbb{R}^+ \mapsto (1 + s^p)^{\frac{1}{p}}$, $p \geq 1$, but in practice this class of functions does not perfectly fit the mechanical framework, where actually a 'concave-type' behavior is expected.

With the aim of finding a wider class of functions containing the function ϕ in [1, 2], and also including energy densities with a more general dependence on the opening of the fracture $[u]$ and from the normal of the crack site ν_u, rather than just from their scalar product $[u]\cdot\nu_u$, we introduced in [5] the following type of functions. Let $\Psi : (a,b) \in \mathbb{R}^N \times S^{N-1} \to [0,+\infty[$ be defined as follows

$$\Psi : (a,b) \in \mathbb{R}^N \times S^{N-1} \mapsto \sup_{\xi \in S^{N-1}} |b\cdot\xi|\psi(|a\cdot\xi|), \tag{1.4}$$

where $\psi : [0,+\infty[\to [0,+\infty[$ is a lower semicontinuous, nondecreasing subadditive function (more generally a lower semicontinuos function such that $\psi(|\cdot|)$ is subadditive). Thus the following lower semicontinuity result with respect to convergences (2.13) $\div$ (2.15) below has been established:

Theorem 1.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, let $\gamma : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function verifying the superlinearity condition (2.11), and let Ψ be as in (1.4) where $\psi : [0,+\infty[\to [0,+\infty[$ is a lower semicontinuous function such that $t \in]-\infty,+\infty[\to \psi(|t|)$ is subadditive. Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ satisfying the bound (2.12), such that $[u_h]\cdot\nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h and converging to u in $L^1(\Omega;\mathbb{R}^N)$. Then (1.2) holds and*

$$\int_{J_u} \Psi([u],\nu_u)d\mathcal{H}^{N-1} \leq \liminf_{h\to+\infty} \int_{J_{u_h}} \Psi([u_h],\nu_{u_h})d\mathcal{H}^{N-1} \tag{1.5}$$

We observe that the energy $\int_{J_u} \phi([u]\cdot\nu_u, [u]\cdot\tau_u)d\mathcal{H}^{N-1}$ in [1, 2] with ϕ as above can be recasted in the terms of a suitable Ψ as in (1.4) requiring that the noninterpenetration constraint (1.2) is verified, in fact it suffices to consider $\psi_{\rm const} : t \in [0,+\infty[\to K, K > 0$, from which we deduce that $\Psi = \Psi_{\rm const} : (a,b) \in \mathbb{R}^N \times S^{N-1} \to K$.

Moreover, as observed in [5] (see Remark 4.8 therein, that we summarize here for the reader's convenience), the intersection between the classes Ψ in (1.4) and Φ in (1.1) is reduced to ϕ. In fact we recall that the function φ of Theorem 1.1, admits the representation

$$\varphi(t) = \sup_{\alpha\in\mathcal{A}} \{c_\alpha t + d_\alpha\} \tag{1.6}$$

with $c_\alpha, d_\alpha \geq 0$. On the other hand by (1.4) one can deduce (see Theorem 4.5 in [5]) that

$$\Psi(a,b) = \sup_{\xi\in S^{N-1}} |b\cdot\xi|\Psi(|a\cdot\xi|\mathbf{e},\mathbf{e})$$

for any $\mathbf{e} \in S^{N-1}$, thus in order to find a ψ such that

$$\sup_{\xi\in S^{N-1}} |b\cdot\xi|\psi(|a\cdot\xi|) = \varphi(a\cdot b)$$

for every $a \in \mathbb{R}^N$ and $b \in S^{N-1}$, with $a\cdot b \geq 0$ we can assume

$$\psi(t) = \varphi(t), \text{ for every } t \in [0,+\infty[,$$

so that

$$\varphi(a\cdot b) = \sup_{\xi\in S^{N-1}} |b\cdot\xi|\varphi(|a\cdot\xi|) \text{ for every } (a,b) \in \mathbb{R}^N \times S^{N-1}$$

which implies

$$\varphi(a\cdot b) \geq \varphi(|a\cdot\xi|)|b\cdot\xi| \text{ for every } a \in \mathbb{R}^N, b,\xi \in \mathbb{S}^{N-1}, a\cdot b \geq 0. \tag{1.7}$$

Thus, taking $0 \leq x,y \leq r, (t,s) = \frac{1}{r}(x,y), a = r\mathbf{e}_1, b = t\mathbf{e}_1 + \sqrt{1-t^2}\mathbf{e}_2, \xi = s\mathbf{e}_1 + \sqrt{1-s^2}\mathbf{e}_2, \{\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_N\}$, the canonical basis of $\mathbb{R}^N$, and letting $r \to +\infty$, we obtain by (1.7)

$$\varphi(x) \geq \varphi(y), \text{ for every } x,y \geq 0,$$

which ensures that φ has to be constant. The difference among the two classes is not very surprising, and in fact, also the techniques adopted to prove the two lower semicontinuity results (Theorem 1.1 and Theorem 1.2) are very different, the first relying on Geometric Measure Theory and the second on the structure of the Special fields of Bounded Deformation, enlightened in [6, 3].

In order to further generalize the previous results, expecially with the aim of considering surface energies whose densities have explicit dependence on the two different *one side Lebesgue' s limits* (see Section 2 below) and on the normal to the jump site, we introduce the class Θ.

$$\Theta : (i, j, p) \in \mathbb{R}^N \times \mathbb{R}^N \times S^{N-1} \mapsto \sup_{\xi \in S^{N-1}} |p \cdot \xi||g(i \cdot \xi) - g(j \cdot \xi)| \tag{1.8}$$

where $g : \mathbb{R} \to]0, +\infty[$ is a continuous function. Still relying on the structure of the Special Fields of Bounded Deformation and on the fact that in dimension 1, SBD functions coincide with SBV ones, we prove the following result

Theorem 1.3. *Let Ω be a bounded open subset of $\mathbb{R}^N$, let $\gamma : [0, +\infty[\to [0, +\infty[$ be a non-decreasing function verifying the superlinearity condition (2.11), and let Θ be as in (1.8) where $g : \mathbb{R} \to]0, +\infty[$ is a continuous function. Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ satisfying the bound (2.12), such that $[u_h] \cdot \nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h and converging to u in $L^1(\Omega; \mathbb{R}^N)$. Then (1.2) holds and*

$$\int_{J_u} \Theta(u^+, u^-, \nu_u) d\mathcal{H}^{N-1} \leq \liminf_{h \to +\infty} \int_{J_{u_h}} \Theta(u_h^+, u_h^-, \nu_{u_h}) d\mathcal{H}^{N-1} \tag{1.9}$$

The structure of the paper is the following. In Section 2 the principal results from Geometric Measure Theory, concerning spaces of functions with bounded deformation and special functions of bounded variation, are recalled. Section 3 is devoted to the proof of Theorem 1.3.

2 NOTATIONS AND PRELIMINARIES

Here and in the sequel, let Ω be a bounded open subset of $\mathbb{R}^N$. We shall usually suppose, when non explicitly mentioned, (essentially to avoid trivial cases) that $N > 1$. Let $u \in L^1(\Omega; \mathbb{R}^m)$, the set of Lebesgue points of u is denoted by Ω_u. In other words $x \in \Omega_u$ if and only if there exists $\tilde{u}(x) \in \mathbb{R}^m$ such that

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho^N} \int_{B_\varrho(x)} |u(y) - \tilde{u}(x)| dy = 0.$$

The space $BD(\Omega)$ of *vector fields with bounded deformation* is defined as the set of vector fields $u = (u^1, \ldots, u^N) \in L^1(\Omega; \mathbb{R}^N)$ whose distributional gradient $Du = \{D_i u^j\}$ has the symmetric part

$$Eu = \{E_{ij} u\}, E_{ij} u = (D_i u^j + D_j u^i)/2$$

which belongs to $\mathcal{M}_b(\Omega; M^{N \times N}_{sym})$, the space of bounded Radon measures in Ω with values in $M^{N \times N}_{sym}$, the space of symmetric $N \times N$ matrices. For $u \in BD(\Omega)$, the *jump set* J_u is defined as the set of points $x \in \Omega$ where u has two different *one sided Lebesgue limits* $u^+(x)$ and $u^-(x)$, with respect to a suitable direction $\nu_u(x) \in S^{N-1} = \{\xi \in \mathbb{R}^N : |\xi| = 1\}$, i.e.

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho^N} \int_{B^\pm_\varrho(x, \nu_u(x))} |u(y) - u^\pm(x)| dy = 0, \tag{2.1}$$

where $B^\pm_\varrho(x, \nu_u(x)) = \{y \in \mathbb{R}^N : |y - x| < \varrho, (y - x) \cdot (\pm \nu_u(x)) > 0\}$. In [6] it has been proved that for every $u \in BD(\Omega)$ the jump set J_u is Borel measurable and countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable and $\nu_u(x)$ is normal to the approximate tangent space to J_u at x for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure (see [7] and [19]).
Let $u \in BD(\Omega)$, the Lebesgue decomposition of Eu is written as

$$Eu = E^a u + E^s u$$

with $E^a u$ the absolutely continuous part and $E^s u$ the singular part with respect to the Lebesgue measure $\mathcal{L}^N$.
The density of $E^a u$ with respect to $\mathcal{L}^N$ is denoted by $\mathcal{E}u$, i.e. $E^a u = \mathcal{E}u \mathcal{L}^N$. We recall that $E^s u$ can be further decomposed as

$$E^s u = E^j u + E^c u$$

with $E^j u$, the *jump part* of Eu, i.e. the restriction of $E^s u$ to J_u and $E^c u$ the *Cantor part* of Eu, i.e. the restriction of $E^s u$ to $\Omega \setminus J_u$. Furthermore, in [6] it has been proved that

$$E^j u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{N-1} \lfloor J_u \tag{2.2}$$

where $\odot$ denotes the symmetric tensor product, defined by $a \odot b := (a \otimes b + b \otimes a)/2$ for every $a, b \in \mathbb{R}^N$, and $\mathcal{H}^{N-1} \lfloor J_u$ denotes the restriction of $\mathcal{H}^{N-1}$ to J_u, i.e. $(\mathcal{H}^{N-1} \lfloor J_u)(B) = \mathcal{H}^{N-1}(B \cap J_u)$ for every Borel set $B \subseteq \Omega$. Moreover in [6] it has been also proved that $|E^c u|(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $\mathcal{H}^{N-1}(B) < +\infty$, where $|\cdot|$ stands for the total variation. In the sequel, for every $u \in L^1_{loc}(\Omega; \mathbb{R}^N)$ we denote by $[u]$ the vector $u^+ - u^-$. For any $y, \xi \in \mathbb{R}^N$, $\xi \neq 0$, and any $B \in \mathcal{B}(\Omega)$ we define

$$\begin{array}{l} \pi_\xi := \{y \in \mathbb{R}^N : y \cdot \xi = 0\}, \\ B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\}, \\ B^\xi := \{y \in \pi_\xi : B^\xi_y \neq \emptyset\}, \end{array} \tag{2.3}$$

i.e. π_ξ is the hyperplane orthogonal to ξ , passing through the origin and $B^\xi = p_\xi(B)$, where p_ξ, denotes the orthogonal projection onto π_ξ. B^ξ_y is the one-dimensional section of B on the straight line passing through y in the direction of ξ.

Given a function $u : B \to \mathbb{R}^N$, defined on a subset B of $\mathbb{R}^N$, for every $y, \xi \in \mathbb{R}^N$, $\xi \neq 0$, the function $u^\xi_y : B^\xi_y \to \mathbb{R}$ is defined by

$$u^\xi_y(t) := u^\xi(y + t\xi) = u(y + t\xi) \cdot \xi \text{ for all } t \in B^\xi_y. \tag{2.4}$$

In [6] it has been proved that a vector field u belongs to $BD(\Omega)$ if and only if its '*projected sections*' u^ξ_y belong to $BV(\Omega^\xi_y)$. More precisely the following Structure Theorem (cf. Structure Theorem 4.5 in [6]) has been proved.

Theorem 2.1. *Let $u \in BD(\Omega)$ and let $\xi \in \mathbb{R}^N$ with $\xi \neq 0$. Then*

(i) $E^a u\xi \cdot \xi = \int_{\Omega^\xi} D^a u^\xi_y d\mathcal{H}^{N-1}(y), |E^a u\xi \cdot \xi| = \int_{\Omega^\xi} |D^a u^\xi_y| d\mathcal{H}^{N-1}(y).$

(ii) For $\mathcal{H}^{N-1}$-almost every $y \in \Omega^\xi$, the functions u^ξ_y and $\tilde{u}^\xi_y$ (the Lebesgue representative of u, cf. formula (2.5) in [6]) belong to $BV(\Omega^\xi_y)$ and coincide $\mathcal{L}^1$-almost everywhere on Ω^ξ_y, the measures $|Du^\xi_y|$ and $V\tilde{u}^\xi_y$ (the pointwise variation of $\tilde{u}^\xi_t$, cf. formula (2.8) in [6]) coincide on Ω^ξ_y ,and $\mathcal{E}u(y+t\xi)\xi \cdot \xi = \nabla u^\xi_y(t) = (\tilde{u}^\xi_y)'(t)$ for $\mathcal{L}^1$-almost every $t \in \Omega^\xi_y$.

(iii) $E^j u\xi \cdot \xi = \int_{\Omega^\xi} D^j u^\xi_y d\mathcal{H}^{N-1}(y)$, $|E^j u\xi \cdot \xi| = \int_{\Omega^\xi} |D^j u^\xi_y| d\mathcal{H}^{N-1}(y)$.

(iv) $(J^\xi_u)^\xi_y = J_{u^\xi_y}$ *for $\mathcal{H}^{N-1}$-almost every $y \in \Omega^\xi$ and for every $t \in (J^\xi_u)^\xi_y$*

$$\begin{array}{l} u^+(y + t\xi) \cdot \xi = (u^\xi_y)^+(t) = \lim_{s \to t^+} \tilde{u}^\xi_y(s) \\ u^-(y + t\xi) \cdot \xi = (u^\xi_y)^-(t) = \lim_{s \to t^-} \tilde{u}^\xi_y(s), \end{array}$$

where the normals to J_u and $J_{u^\xi_y}$ are oriented so that $\nu_u \cdot \xi \geq 0$ and $\nu_{u^\xi_y} = 1$.

(v) $E^c u\xi \cdot \xi = \int_{\Omega^\xi} D^c u^\xi_y d\mathcal{H}^{N-1}(y), |E^c u\xi \cdot \xi| = \int_{\Omega^\xi} |D^c u^\xi_y| d\mathcal{H}^{N-1}(y).$

The space $SBD(\Omega)$ of *special vector fields with bounded deformation* is defined as the set of all $u \in BD(\Omega)$ such that $E^c u = 0$, or, in other words

$$Eu = \mathcal{E}u\mathcal{L}^N + (u^+ - u^-) \odot \nu_u \mathcal{H}^{N-1} \lfloor J_u$$

We also recall that if $\Omega \subset \mathbb{R}$, then the space $SBD(\Omega)$ coincides with the space of real valued special functions of bounded variations $SBV(\Omega)$, consisting of the functions whose distributional gradient is a Radon measure with no Cantor part (see [7] for a comprehensive treatment of the subject).

Here we recall Proposition 4.7 in [6] that will be exploited in the sequel.

Proposition 2.2. *Let $u \in BD(\Omega)$ and let $\xi_1, \ldots, \xi_N$ be a basis of $\mathbb{R}^N$. Then the following three conditions are equivalent:*

(i) $u \in SBD(\Omega)$.

(ii) For every $\xi = \xi_i + \xi_j$ with $1 \leq i, j \leq n$, we have $u^\xi_y \in SBV(\Omega^\xi_y)$ for $\mathcal{H}^{N-1}$-almost every $y \in \Omega^\xi$.

(iii) The measure $|E^s u|$ is concentrated on a Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{N-1}$.

Definition 2.3. *For any $u \in BD(\Omega)$ we define the non-negative Borel measure λ_u on Ω as*

$$\lambda_u(B) := \frac{1}{2\omega_{N-1}} \int_{S^{N-1}} \lambda_u^\xi(B) d\mathcal{H}^{N-1}(\xi) \ \ \forall B \in \mathcal{B}(\Omega), \tag{2.5}$$

where, for every $\xi \in S^{N-1}$

$$\lambda_u^\xi(B) := \int_{\Omega^\xi} \mathcal{H}^0(J_{u_y^\xi} \cap B_y^\xi) d\mathcal{H}^{N-1}(y) \ \ \forall B \in \mathcal{B}(\Omega). \tag{2.6}$$

Let

$$J_u^\xi := \left\{ x \in J_u : (u^+ - u^-) \cdot \xi \neq 0 \right\}, \tag{2.7}$$

we recall that

$$\mathcal{H}^{N-1}(J_u \setminus J_u^\xi) = 0 \text{ for } \mathcal{H}^{N-1} - \text{a.e. } \xi \in S^{N-1}. \tag{2.8}$$

The following result is a consequence of the Structure Theorem

Theorem 2.4. *For every $u \in BD(\Omega)$ and any $\xi \in S^{N-1}$,*

$$\lambda_u^\xi(B) = \int_{J_u^\xi \cap B} |\nu_u \cdot \xi| d\mathcal{H}^{N-1} \ \ \forall B \in \mathcal{B}(\Omega), \tag{2.9}$$

where ν_u is the approximate unit normal to J_u. Moreover $\lambda_u = \mathcal{H}^{N-1} \lfloor J_u$.

The same argument of Theorem 2.4, i.e. (iv) of Theorem 2.1 and the fact that the $(N-1)$-dimensional area factor of p_ξ on J_u is $|\nu_u \cdot \xi|$ guarantees that for every Borel function $g : \Omega \to [0, +\infty]$, it results

$$\int_{J_u^\xi \cap B} g(y) |\nu_u \cdot \xi| d\mathcal{H}^{N-1}(y) = \int_{\Omega^\xi} \int_{p_\xi(J_u^\xi \cap B)} g(y + t\xi) d\mathcal{H}^0(t) d\mathcal{H}^{N-1}(y) \tag{2.10}$$

for any $\xi \in S^{N-1}$.

We recall the following compactness result for sequences in SBD proved in [3], (cf. Theorem 1.1 and Remark 2.3 therein).

Theorem 2.5. *Let $\gamma : [0, +\infty[\to [0, +\infty[$ be a non-decreasing function such that*

$$\lim_{t \to +\infty} \frac{\gamma(t)}{t} = +\infty. \tag{2.11}$$

Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ such that

$$\|u_h\|_{L^\infty(\Omega;\mathbb{R}^N)} + \int_\Omega \gamma(|\mathcal{E}u_h|) dx + \mathcal{H}^{N-1}(J_{u_h}) \leq K \tag{2.12}$$

for some constant K independent of h. Then there exists a subsequence, still denoted by $\{u_h\}$, and a function $u \in SBD(\Omega)$ such that

$$u_h \to u \text{ strongly in } L^1_{loc}(\Omega; \mathbb{R}^N), \tag{2.13}$$

$$\mathcal{E}u_h \rightharpoonup \mathcal{E}u \text{ weakly in } L^1(\Omega; M^{N\times N}_{sym}), \tag{2.14}$$

$$E^j u_h \rightharpoonup E^j u \text{ weakly* in } \mathcal{M}_b(\Omega; M^{N\times N}_{sym}), \tag{2.15}$$

$$\mathcal{H}^{N-1}(J_u) \leq \liminf_{h \to +\infty} \mathcal{H}^{N-1}(J_{u_h}) \tag{2.16}$$

The following result from Measure Theory will be exploited in the sequel, (cf. Lemma 2.35 in [7]).

Lemma 2.6. *Let λ be a positive σ-finite Borel measure in Ω and let $\varphi_i : \Omega \to [0, \infty]$, $i \in \mathbb{N}$, be Borel functions. Then*

$$\int_\Omega \sup_i \varphi_i d\lambda = \sup \left\{ \sum_{i\in I} \int_{A_i} \varphi_i d\lambda \right\}$$

where the supremum ranges over all finite sets $I \subset \mathbb{N}$ and all families $\{A_i\}_{i\in I}$ of pairwise disjoint open sets with compact closure in Ω.

Next we recall a sufficient condition to ensure lower semicontinuity in $SBV(\Omega)$ with respect to the convergence à la Ambrosio (cf. Definition 5.17 and Theorem 5.22 in [7]). Let $K \subset \mathbb{R}^d$ be compact and $f : K \times K \times \mathbb{R}^N \to [0, +\infty]$. The function f is said to be *jointly convex* if

$$f(i, j, p) = \sup_{h\in\mathbb{N}} \{(g_h(i) - g_h(j)) \cdot p\} \; \forall (i, j, p) \in K \times K \times \mathbb{R}^N \tag{2.17}$$

for some sequence $\{g_h\} \subset [C(K)]^p$. We emphasize that this notion plays for surface energy densities the same role as policonvexity for bulk energies.

A proof entirely analogous to the proof of Theorem 5.12 in [7] allows us to prove the following result.

Theorem 2.7. *Let $K \subset \mathbb{R}^d$ be a compact set and $f : K \times K \times \mathbb{R}^N \to [0, +\infty[$ be a jointly convex function Let $\{u_h\} \subset SBV(\Omega; \mathbb{R}^d)$ be a sequence converging in $L^1(\Omega; \mathbb{R}^d)$ to u such that $\{|\nabla u_h|\}$ is equiintegrable, $\mathcal{H}^{N-1}(J_{u_h}) \leq \mathrm{const}$ and, for any $h \in \mathbb{N}$, $u_h(x) \in K$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. Then (by virtue of Theorem 4.8 in [7]) $u \in SBV(\Omega; \mathbb{R}^d)$, $u(x) \in K$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and*

$$\int_{J_u} f(u^+, u^-, \nu_u) d\mathcal{H}^{N-1} \leq \liminf_h \int_{J_{u_h}} f(u_h^+, u_h^-, \nu_h) d\mathcal{H}^{N-1}.$$

3 LOWER SEMICONTINUITY OF $\int_{J_u} \Theta(u^+, u^-, \nu_u) d\mathcal{H}^{N-1}$

This section is devoted to the proof of Theorem 1.3. To this end we state and prove some preliminary lower semicontinuity results.

Lemma 3.1. *Let $g : \mathbb{R} \to]0, +\infty[$ be a continuous function. Let $\gamma : [0, +\infty[\to [0, +\infty[$ be a nondecreasing function such that the superlinearity condition (2.11) holds. Let I be an open interval of $\mathbb{R}$. Let $\{u_j\} \subset SBV(I)$ such that*

$$\|u_j\|_{L^\infty(I)} + \int_I \gamma(|u_j'|) dx + \mathcal{H}^0(J_{u_j}) \leq C$$

(here u_j' denotes the absolutely continuous part of Du_j with respect to the Lebesgue measure). Assume also that $u_j \to u$ in $L^1(I)$. Then

$$\int_{J_u} |g(u^+) - g(u^-)| d\mathcal{H}^0 \leq \liminf_{j\to+\infty} \int_{J_{u_j}} |g(u_j^+) - g(u_j^-)| d\mathcal{H}^0.$$

Proof. First consider a subsequence $\{u_{j_k}\}$ such that $\liminf_{j\to+\infty} \int_{J_{u_j}} |g(u_j^+) - g(u_j^-)| d\mathcal{H}^0$ is a limit on k. By virtue of Theorem 2.5, it results that it admits a further subsequence, still denoted by $\{u_{j_k}\}$, such that all the convergence relations (2.13)÷(2.16) hold in I. Consequently, since $\{u_j\}$ is bounded in L^∞, the function $f : (i, j, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \sup_{\{g_\beta : g_\beta = g \text{ or} -g\}} \{(g_\beta(i) - g_\beta(j))p\}$, can be considered restricted to the product of a compact set for itself, as regards the first two components, and thus it turns out to be jointly convex. Thus Theorem 2.7, applied to f, ensures that

$$\int_{J_u} |g(u^+) - g(u^-)| d\mathcal{H}^0 \leq \lim_{k\to+\infty} \int_{J_{u_{j_k}}} |g(u_{j_k}^+) - g(u_{j_k}^-)| d\mathcal{H}^0 = \liminf_{j\to+\infty} \int_{J_{u_j}} |g(u_j^+) - g(u_j^-)| d\mathcal{H}^0.$$

□

Remark 1.

We observe that the lower semicontinuity result proved in Lemma 3.1 still holds when replacing the open interval I, by any open set of $\mathbb{R}$, thanks to the superadditivity of the liminf operator, at least on non-negative families.

The proof of the following result exploits the structure of SBD functions enlightened in Theorem 1.1 in [3].

Lemma 3.2. *Let $g : \mathbb{R} \to [0,+\infty[$ be a continuous function. Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $\gamma : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function verifying the superlinearity condition (2.11). Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ satisfying the bound (2.12), such that $[u_h]\cdot\nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h and converging to u in $L^1(\Omega;\mathbb{R}^N)$. Then*

$$\int_{J_u} |\xi\cdot\nu_u||g(u^+(y)\cdot\xi)-g(u^-(y)\cdot\xi)|d\mathcal{H}^{N-1}(y) \leq \liminf_{h\to+\infty}\int_{J_{u_h}} |\xi\cdot\nu_{u_h}||g(u_h^+(y)\cdot\xi)-g(u_h^-(y)\cdot\xi)|d\mathcal{H}^{N-1}(y) \quad (3.1)$$

for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$.

Proof. Let $\{u_h\} \subset SBD(\Omega)$ satisfying the bound (2.12) and converging to $u \in L^1(\Omega;\mathbb{R}^N)$. From Theorem 2.5 $u \in SBD(\Omega)$.

Let $\xi \in S^{N-1}$, and let $p_\xi : J_u \to \pi_\xi$ be the orthogonal projection onto π_ξ. First we observe that (iv) in Theorem 2.1 and Proposition 2.2 ensure that for $\mathcal{H}^{N-1}$- a.e. $y \in \Omega^\xi$ it results $(u_y^\xi)^+(t) = (u\cdot\xi)^+(y+t\xi)$ and $(u_y^\xi)^-(t) = (u\cdot\xi)^-(y+t\xi)$ for every $t \in J_{u_y^\xi}$ and $({u_h}_y^\xi)^+(t) = (u_h\cdot\xi)^+(y+t\xi)$ and $({u_h}_y^\xi)^-(t) = (u_h\cdot\xi)^-(y+t\xi)$ for every $t \in J_{{u_h}_y^\xi}$, with $u_y^\xi, {u_h}_y^\xi \in SBV(\Omega_y^\xi)$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$.

On the other hand, by (2.7) and (2.8), it results that

$$\begin{aligned}&\int_{J_u} |\xi\cdot\nu_u||g(u^+(y)\cdot\xi)-g(u^-(y\cdot\xi))|d\mathcal{H}^{N-1}(y) = \int_{J_u^\xi} |\xi\cdot\nu_u||g(u^+(y)\cdot\xi)-g(u^-(y)\cdot\xi)|d\mathcal{H}^{N-1}(y),\\ &\int_{J_{u_h}} |\xi\cdot\nu_{u_h}||g(u_h^+(y)\cdot\xi)-g(u_h^-(y)\cdot\xi)d\mathcal{H}^{N-1}(y) = \int_{J_{u_h}^\xi} |\xi\cdot\nu_{u_h}||g(u_h^+(y)\cdot\xi)-g(u_h^-(y)\cdot\xi)|d\mathcal{H}^{N-1}(y)\end{aligned} \quad (3.2)$$

for every $h \in \mathbb{N}$ and for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$. Formulas (3.2), (2.10) guarantee that there exists $N \subset S^{N-1}$ such that $\mathcal{H}^{N-1}(N) = 0$ and it results:

$$\int_{J_u} |\xi\cdot\nu_u||g(u^+(y)\cdot\xi)-g(u^-(y)\cdot\xi)|d\mathcal{H}^{N-1}(y) = \int_{\Omega^\xi}\Big[\int_{J_{u_y^\xi}} |g((u_y^\xi)^+(t))-g((u_t^\xi)^-(t))|d\mathcal{H}^0(t)\Big]d\mathcal{H}^{N-1}(y),$$

and

$$\int_{J_{u_h}} |\xi\cdot\nu_{u_h}||g(u_h^+(y)\cdot\xi)-g(u_h^-(y)\cdot\xi)|d\mathcal{H}^{N-1}(y) = \int_{\Omega^\xi}\Big[\int_{J_{{u_h}_y^\xi}} |g(({u_h}_y^\xi)^+(t))-g(({u_h}_y^\xi)^-(t))|d\mathcal{H}^0(t)\Big]d\mathcal{H}^{N-1}(y),$$

for every $h \in \mathbb{N}$ and for every $\xi \in S^{N-1}\setminus N$.

Consequently the proof will be completed once we show that

$$\begin{aligned}&\int_{\Omega^\xi}\Big[\int_{J_{u_y^\xi}} |g((u_y^\xi)^+(t))-g((u_t^\xi)^-)(t)|d\mathcal{H}^0(t)\Big]d\mathcal{H}^{N-1}(y) \leq\\ &\liminf_{h\to+\infty}\int_{\Omega^\xi}\Big[\int_{J_{{u_h}_y^\xi}} |g(({u_h}_y^\xi)^+(t))-g(({u_h}_y^\xi)^-(t))|d\mathcal{H}^0(t)\Big]d\mathcal{H}^{N-1}(y)\end{aligned} \quad (3.3)$$

for every $\xi \in S^{N-1}\setminus N$.

To this end, for each $\xi \in S^{N-1}\setminus N$ consider a subsequence $\{u_k\} \equiv \{u_{h_k}\}$ such that

$$\liminf_{h\to+\infty}\int_{J_{{u_h}_y^\xi}} |g(({u_h}_y^\xi)^+(t))-g(({u_h}_y^\xi)^-(t))|d\mathcal{H}^0(t) = \lim_{k\to+\infty}\int_{J_{{u_k}_y^\xi}} |g(({u_k}_y^\xi)^+(t))-g(({u_k}_y^\xi)^-(t))|d\mathcal{H}^0(t). \quad (3.4)$$

Next consider a further subsequence (denoted by $\{u_j\} \equiv \{u_{k_j}\}$) such that

$$\lim_{j\to+\infty}\mathcal{H}^{N-1}(J_{u_j}) = \liminf_{k\to+\infty}\mathcal{H}^{N-1}(J_{u_k}). \quad (3.5)$$

We want to show that the assumptions of Lemma 3.1 are satisfied. Let $I_{y,\xi}(u_j) = \int_{\Omega_y^\xi}\gamma(|{u'_j}_y^\xi(t)|)dt$, where ${u_j}_y^\xi(t) = u_j(y+t\xi)\cdot\xi$. From (ii) in Theorem 2.1 (i.e. $\mathcal{E}u_j(y+t\xi)\cdot\xi = ({u_j}^\xi)'_y(t)$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$ and for $\mathcal{L}^1$-a.e. $t \in \Omega_y^\xi$) and from Fubini-Tonelli's theorem, for any $\xi \in S^{N-1}\setminus N$ we have

$$\int_{\pi_\xi} I_{y,\xi}(u_j)d\mathcal{H}^{N-1}(y) = \int_\Omega \gamma(|\mathcal{E}u_j(x)\xi\cdot\xi|)dx.$$

Since $\{u_j\}$ satisfies the bound (2.12) and γ is non-decreasing, it follows that

$$\int_{\pi_\xi} I_{y,\xi}(u_j)d\mathcal{H}^{N-1}(y) \leq \int_\Omega \gamma(|\mathcal{E}u_j(x)|)dx \leq K, \tag{3.6}$$

for every $\xi \in S^{N-1} \setminus N$ and for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$. It is also easily seen that, from the bound on $\|u_j\|_{L^\infty}$, deriving from the global bound (2.12),

$$\|{u_j}^\xi_y\|_{L^\infty(\Omega^\xi_y)} \leq K. \tag{3.7}$$

From (3.6), (2.10) for every $\xi \in S^{N-1} \setminus N$ it results that there exists a constant $C \equiv C(K)$ such that

$$\liminf_{j\to+\infty} \int_{\pi_\xi} [I_{y,\xi}(u_j) + \mathcal{H}^0(J_{{u_j}^\xi_y})]d\mathcal{H}^{N-1}(y) \leq C < +\infty.$$

Let us fix $\xi \in S^{N-1} \setminus N$ (such that the previous inequality holds). Using Fubini-Tonelli's theorem we can extract a subsequence $\{u_m\} = \{u_{j_m}\}$ (depending on ξ) such that

$$\begin{aligned} &\lim_{m\to+\infty} \int_{\pi_\xi} [I_{y,\xi}(u_m) + \mathcal{H}^0(J_{{u_m}^\xi_y})]d\mathcal{H}^{N-1}(y) = \\ &\liminf_{j\to+\infty} \int_{\pi_\xi} [I_{y,\xi}(u_j) + \mathcal{H}^0(J_{{u_j}^\xi_y})]d\mathcal{H}^{N-1}(y) \leq C < +\infty, \end{aligned} \tag{3.8}$$

and for a.e. $y \in \Omega^\xi$, $u^\xi_{m,y} \in SBV(\Omega^\xi_y)$ and ${u_m}^\xi_y \to u^\xi_y$ in $L^1_{loc}(\Omega^\xi_y)$, with $u^\xi_y \in SBV(\Omega^\xi_y)$.

Let $\xi \in S^{N-1} \setminus N$, by (3.8), Fatou's lemma, for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$, it results

$$\liminf_{m\to+\infty} [I_{y,\xi}(u_m) + \mathcal{H}^0(J_{{u_m}^\xi_y})] < +\infty. \tag{3.9}$$

Let us fix $N_{\Omega^\xi} \subset \Omega^\xi$ and a point $y \in \Omega^\xi \setminus N_{\Omega^\xi}$, such that $\mathcal{H}^{N-1}(N_{\Omega^\xi}) = 0$, (3.9) and (3.7) hold and such that ${u_m}^\xi_y \in SBV(\Omega^\xi_y)$ for any m. Passing to a further subsequence $\{u_l\} \equiv \{u_{m_l}\}$ we can assume that there exists a constant C' such that

$$\liminf_{m\to+\infty} [I_{y,\xi}(u_m) + \mathcal{H}^0(J_{{u_m}^\xi_y})] = \lim_{l\to+\infty} [I_{y,\xi}(u_l) + \mathcal{H}^0(J_{{u_l}^\xi_y})] \leq C'.$$

This means that $\{{u_l}^\xi_y\} \in SBV(\Omega^\xi_y)$ and satisfies all the assumptions of Lemma 3.1 for each interval (connected component) $I \subset \Omega^\xi_y$. Consequently (3.4) and Lemma 3.1 guarantee that

$$\begin{aligned} &\int_{J_{u^\xi_y}} |g((u^\xi_y)^+(t)) - g((u^\xi_y)^-(t))|d\mathcal{H}^0(t) \leq \lim_{l\to+\infty} \int_{J_{{u_l}^\xi_y}} |g(({u_l}^\xi_y)^+(t)) - g(({u_l}^\xi_y)^-(t))|d\mathcal{H}^0(t) = \\ &\liminf_{h\to+\infty} \int_{J_{{u_h}^\xi_y}} |g(({u_h}^\xi_y)^+(t)) - g(({u_h}^\xi_y)^-(t))|d\mathcal{H}^0(t) \end{aligned} \tag{3.10}$$

for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$.

The lower semicontinuity stated in (3.3) follows now from Fatou's lemma, which completes the proof.

□

Now we are in position to prove Theorem 1.3.

Proof of Theorem 1.3. (1.2) has been proved in [4] (cf. Lemma 3.1 therein). It remains to prove (1.5). The continuity of g allows us to assume ξ in (1.8) varying in any countable subset of S^{N-1}. It will be chosen in $S^{N-1} \setminus N$, N being the $\mathcal{H}^{N-1}$ exceptional set introduced in Lemma 3.2, and it will be denoted by $\mathcal{A}$, with elements ξ_α.

By superadditivity of liminf:

$$\liminf_{h\to+\infty} \int_{J_{u_h}} \Theta(u^+_h, u^-_h, \nu_{u_h})d\mathcal{H}^{N-1} \geq \sum_\alpha \liminf_{h\to+\infty} \int_{J_{u_h}\cap A_\alpha} |\xi_\alpha \cdot \nu_{u_h}|g(\xi_\alpha \cdot u^+_h) - g(\xi_\alpha \cdot u^-_h)|d\mathcal{H}^{N-1}$$

for any finite family of pairwise disjoint open sets $A_\alpha \subset \Omega$.

By Lemma 3.2 we have

$$\liminf_{h\to+\infty}\int_{J_{u_h}} |\xi_\alpha \cdot \nu_{u_h}||g(\xi_\alpha \cdot u_h^+) - g(\xi_\alpha \cdot u_h^-)|d\mathcal{H}^{N-1} \geq \int_{J_u} |\xi_\alpha \cdot \nu_u||g(\xi_\alpha \cdot u^+) - g(\xi_\alpha \cdot u_h^-)|d\mathcal{H}^{N-1}$$

for every $\xi_\alpha \in \mathcal{A}$. Therefore

$$\liminf_{h\to+\infty}\int_{J_{u_h}} \Theta(u_h^+, u_h^-, \nu_{u_h})d\mathcal{H}^{N-1} \geq \sum_\alpha \int_{J_u\cap A_\alpha} |\xi_\alpha \cdot \nu_u||g(\xi_\alpha \cdot u^+) - g(\xi_\alpha \cdot u^-)|d\mathcal{H}^{N-1}$$

for every $\xi_\alpha \in \mathcal{A}$ and for any finite family of pairwise disjoint open sets $A_\alpha \subset \Omega$.

By Theorem 2.6 we can interchange integration and supremum over all such families, thus getting

$$\liminf_{h\to+\infty}\int_{J_{u_h}} \Theta(u_h^+, u_h^-, \nu_{u_h})d\mathcal{H}^{N-1} \geq \int_{J_u} \Theta(u^+, u^-, \nu_u)d\mathcal{H}^{N-1},$$

whence (1.9) follows and that concludes the proof.

□

Remark 2. We emphasize that Theorem 1.3 still holds with obvious adaptations if one replaces the integrand Θ in (1.8) by

$$\Theta(i,j,p) := \sup_{\xi\in S^{N-1}} |b\cdot\xi||g_\xi(i\cdot\xi) - g_\xi(j\cdot\xi)|$$

with $g_\xi : \mathbb{R} \to [0, +\infty[$ continuously depending on $\xi \in S^{N-1}$, continuous functions.

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