Ceradini's approach in fracture mechanics

A. Salvadori, A. Carini
CeSiA - Centro di studio e ricerca di sismologia applicata e dinamica strutturale,
DICATA - Dipartimento di Ingegneria Civile, Architettura, Territorio e Ambiente
Università di Brescia, via Branze 43, 25123 Brescia, Italy
E-mail: angelo.carini@ing.unibs.it, alberto.salvadori@ing.unibs.it

Keywords: Fracture mechanics, plasticity, variational formulations, crack growth.

SUMMARY. The crack propagation problem for linear elastic fracture mechanics has been studied by some authors exploiting its analogy with plasticity theory (see e.g. [1, 2, 3]). This approach is here further pursued, by noting that Stress Intensity Factors (SIFs) asymptotic expansion allows a Colonnetti's approach. As a consequence, a minimum variational formulation is obtained in terms of crack tip velocity. It is reminiscent of Ceradini's theorem for plasticity.

1 INTRODUCTION

Fracturing process reveals three distinct phases [4]: loading without crack growth, stable crack growth an unstable crack growth. During crack advancing, energy dissipation takes place in the process-region, in the plastic region outside the process region, and eventually in the wake of plastic region. When the fracture process is idealized to infinitesimally small scale yielding, energy dissipation during crack growth is concentrated at the crack tip. This assumption together with linear elasticity is assumed in the present note, making use of Hooke's law without limitation of stress and strain magnitudes: the stress-strain fields in the crack tip vicinity is uniquely determined by the stress intensity factors (SIFs).

Similarly to the determination of the "elastic limit", the concept of incipient crack growth is difficult to identify: in both cases, the difficulty is solved by a convention. Onset of crack growth is governed theoretically by a local condition, describing when the process region reaches a critical state which, in most cases of engineering interest, is independent on body and loading geometry: this property is termed autonomy (see [5] but also the excellent description in [4]).

Even if the total amount of stable crack growth does not obey the property of autonomy, being dependent on the plastic region about the crack tip, stable crack growth is ruled by local conditions at the process region. The onset of unstable crack growth is, on the contrary, a result of a global instability. These issues are discussed in section 6.

The global quasi-static fracture propagation problem consists in seeking an expression of the crack propagation rate for all three phases of the fracturing process. The question can be posed in the following way: given the state of stress and the history of crack propagation (if any), express the crack propagation rate (if any) as a function of the stress and of the history. Indeed this path of reasoning seems quite natural: though, most of algorithm for crack propagation are designed in the opposite way: they express the external load history as a function of the crack propagation rate [6]. Whereas the latter approach is quite easy, it is not optimal in evaluating the critical point of the equilibrium path; further it seems to be unsuitable in the presence of many propagating cracks in multi-connected bodies.

For linear elastic fracture mechanics, the global quasi-static fracture propagation problem has been studied in [3] exploiting its analogy with plasticity theory. A maximum principle was stated, that expressed the maximum dissipation at the crack tip during propagation; from it, associated flow rule and loading/unloading conditions in Kuhn-Tucker complementarity form descend. Consistency conditions led to the formulation of an algorithm for crack advancing, which was driven by the increment of external actions (under the simplifying assumption of proportional loading) and allowed the evaluation of crack length increment and curvature at the crack tips of several cracks contemporarily advancing.

This idea is here further pursued, by noting that Amestoy-Leblond [7] Stress Intensity Factors (SIFs) asymptotic expansion has an effect superposition interpretation. Discussions at section 5 allow a Colonnetti's approach in fracture mechanics. As a consequence, a minimum variational formulation is obtained in section 7 in terms of crack tip velocity. It is reminiscent of Ceradini's theorem for plasticity.

2 NOTATION

Small strains hypothesis is assumed on a domain $\Omega = \bigcup_{n=1}^{N} \Omega_n \subset \mathbb{R}^2$, together with the isotropic linear elastic constitutive law in all the N homogeneous closed domains $\bar{\Omega}_n$. Interfaces between domains are assumed to be rigid, i.e. relative displacements along each interface are not allowed. Loci $\Upsilon_i, \ i = 1, 2, ...,$ of possible displacement discontinuities $\mathbf{w}_i(\mathbf{x})$ are defined as usual inside of each domain Ω: the issues of interface cracks and of intersections between moving cracks and interfaces fall beyond the purposes of the present note.

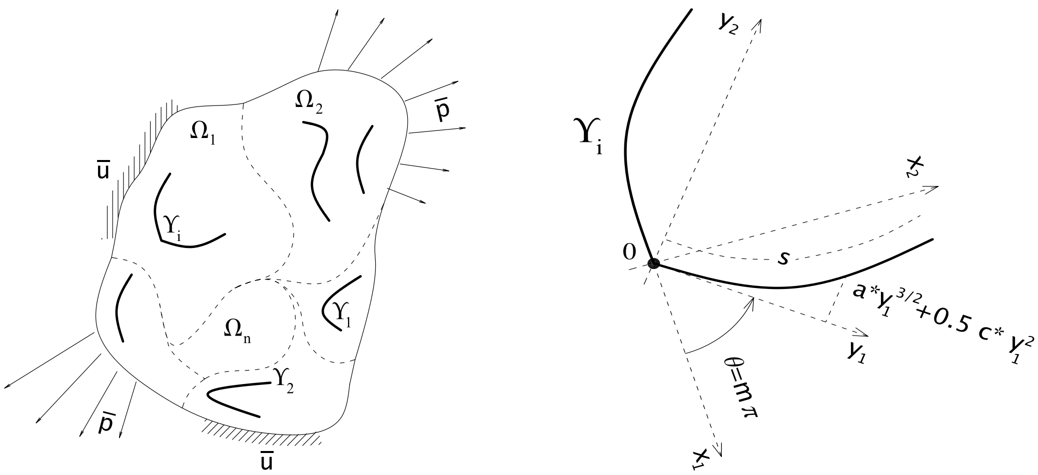

Figure 1: Notation. C denotes the curvature of the main branch at the crack tip, whereas a^* and C^* define the curvature of the elongated branch.

The structural response to the following quasi-static external actions is sought: tractions $\bar{\mathbf{p}}(\mathbf{x})$ on $\Gamma_p \subset \partial\Omega$, displacements $\bar{\mathbf{u}}(\mathbf{x})$ on $\Gamma_u \subset \partial\Omega$. Bulk forces are assumed to be zero. External actions are all assumed to be *proportional*, i.e. that they vary only through multiplication by a time-dependent scalar $\kappa(t)$, termed load factor, taken to be zero at initial time $t_0 = 0$ when the cracks attained their initial length. In the present note, "time" t represents any variable which monotonically increases in the physical time and merely orders events; the mechanical phenomena to study are time-independent.

The notation of [7], see also figure 1, will be used. In such a celebrated paper, Amestoy and Leblond established the general form of the expansion of the stress intensity factors (SIFs) in powers of the crack extension length s, for a crack propagating in a two-dimensional body along an arbitrary kinked (by an angle $\theta = m\pi$) and curved path. They evaluated the detailed form the functions of the geometric and mechanical parameters which appear in the expansion, too. Denoting with $\mathbf{K} = \{K_1, K_2\}$ the SIFs vector, the expansion of $\mathbf{K}$ at the extended crack tip in powers of s is of the general form:

$$\mathbf{K}(s) = \mathbf{K}^* + \mathbf{K}^{(1/2)}\sqrt{s} + \mathbf{K}^{(1)}\, s + O(s^{3/2}) \tag{1}$$

where $\mathbf{K}^*$, $\mathbf{K}^{(1/2)}$, $\mathbf{K}^{(1)}$ are given componentwise (using the Einstein summation convention) by

$$K_p^* = F_{pq}(m)\, K_q \tag{2}$$

$$K_p^{(1/2)} = G_p(m)\, T + a^* H_{pq}(m)\, K_q \tag{3}$$

$$\begin{aligned} K_p^{(1)} = \; & Z_p + I_{pq}(m)\, b_q + C\, J_{pq}(m)\, K_q + a^* Q_p(m) T + {a^*}^2 L_{pq}(m)\, K_q \\ & + C^* M_{pq}(m)\, K_q \end{aligned} \tag{4}$$

In these equations, T, and the b_qs are the non singular stress and coefficients of the $\sqrt{r}$ terms in the stress expansion at the original crack tip **0**. The F_{pq}s, G_ps, H_{pq}s, I_{pq}s, J_{pq}s, Q_ps, L_{pq}s, and M_{pq}s are functions of the kink angle θ, which are termed universal because they obey to the autonomy concept; finally, Z_p depends on the geometry of Ω.

3 A VARIATIONAL SETTING FOR CRACK PROPAGATION CRITERIA

Equation (2) is a milestone in the prediction of the kinking angle at a crack tip according to some crack propagation criteria, as the Local Symmetry [8] (shortened in LS) or the Maximum Energy Release Rate [9, 10] (shortened in MERR). As a distinctive peculiarity [11, 12], these two criteria are grounded on the stress and strain fields in the "propagated configuration" as the crack elongation approaches zero from above $s \to 0^+$. It is natural therefore to analyze these criteria into the $K_1^* - K_2^*$ plane, that from now on will be termed the "Amestoy-Leblond" plane. Several criteria (to cite but a few: Maximum Tensile Stress [13], Maximum Shear Stress [14], apparent Crack Extension Force [15], Strain Energy Density [16]) widely used in the computational fracture mechanics community, stem from the crack configuration "at the onset of propagation": they have been extensively represented in the plane $K_1 - K_2$.

The mathematical representation of the onset of fracture can be written in the following general form:

$$\varphi(K_1, K_2, \theta) = \vartheta(K_1, K_2, \theta) - \vartheta(K_1^C, 0, \theta^C) = 0 \tag{5}$$

whereas the safe equilibrium domain reads:

$$\varphi(K_1, K_2, \theta) < 0 \tag{6}$$

Criteria differs from the choice of function ϑ which is a measure of the safety of a pair $\{K_1, K_2\}$ with respect to a critical state, say $\{K_1^C, 0\}$. In (5), K_1^C is the fracture toughness and θ^C is the propagation angle attained when $K_2 = 0$ and $K_1 = K_1^C$, with function φ "usually[1]" defining the crack propagation criterion.

Inequality (6) has to be understood as follows: it exists a region around the origin in the $K_1 - K_2$ plane such that for all $\theta \in \mathbb{R}$ it does not exist any vector $\mathbf{K} = \{K_1, K_2\}$ for which φ vanishes,

[1]The use of term "usually" worths a better explanation. Whereas in plasticity the choice of a yield function is free, in fracture mechanics the release (better: dissipation) of energy at the crack tip during propagation poses a constraint to the

whatever the relationship between the angle of propagation θ and the SIFs might be. This idea can be given a mathematical picture. At a given time t, values of $\mathbf{K}$ can be evaluated as a linear function of κ, as the geometry is given; ratio $\alpha = \frac{K_2}{K_1}$, usually termed the "mode mixity" ratio, is therefore fixed; $\theta(t)$ is unknown as well as $\kappa(t)$ such that $\varphi = 0$. The pair $\{K_1, K_2\}$ is in fact equivalent to the pair $\{\kappa, \alpha\}$. The onset of propagation:

$$\varphi(\kappa, \alpha, \theta) = 0$$

implicitly defines a function $\kappa(\theta, \alpha)$ with α as a given parameter. The "actual" kinking angle θ is associated to the lowest value of κ. If the hypotheses of the implicit function theorem are fulfilled, then:

$$\left.\frac{\mathrm{d}\kappa}{\mathrm{d}\theta}\right|_{\alpha} = -\frac{\partial \varphi}{\partial \theta}\left(\frac{\partial \varphi}{\partial \kappa}\right)^{-1} = 0 \qquad \Rightarrow \frac{\partial \varphi}{\partial \theta} = 0 \tag{8}$$

The kinking angle θ is sought therefore as the one that, at any given α, minimizes κ. It turns out that this condition implies maximizing φ: in its complete form, the problem of finding $\{\kappa, \theta\}$ reads:

$$find \;\; \{\kappa, \theta\} \quad\;\; s.t. \quad \varphi = 0\,, \quad \frac{\partial \varphi}{\partial \theta} = 0 \tag{9}$$

As $\varphi \leq 0$, $\{\kappa, \theta\}$ in (9) are maximizers of φ.

Scheme (9) applies to all propagation criteria that authors are aware of, with only one notable exception, namely the local symmetry criterion. In such a case, $\varphi(K_1, K_2, \theta) = 0$ is independent on the load factor κ. At any given mode-mixity ratio corresponds a kinking angle trough formula (36) in appendix A.

4 A PLASTICITY FRAMEWORK FOR LEFM

4.1 Intuitive facts

The definition (6) of a "safe equilibrium domain" and of the "onset of crack propagation" (5) as its closure are reminiscent to the plasticity theory [17]: they appear as the counterpart of the elastic domain and of the yield surface. On the other hand, if cracks extension is considered irreversible, crack length s and crack tip velocity $\dot{s}$ must be taken as positive quantities. Furthermore, the following chain of linear complementary conditions:

$$\varphi \leq 0 \qquad \dot{s} \geq 0 \qquad \varphi\, \dot{s} = 0 \tag{10}$$

must hold. No propagation is allowed $\dot{s} = 0$ in the safe equilibrium domain and vice versa. Equations (10) clearly reproduce Kuhn-Tucker conditions of plasticity. All these similarities pushed towards setting a mathematical analogy between plasticity and fracture, which is shortly summarized in next section.

---

choice of φ: namely, the safe equilibrium domain must be defined in terms of

$$\varphi(\mathbf{K}^*) = \frac{1}{2}\,\frac{1-\nu^2}{E}\,\left(||\mathbf{K}^*||^2 - {K_1^C}^2\right) \tag{7}$$

which defines the maximum energy release rate criterion. Nevertheless, the question arises if from this choice of φ, crack propagation angles from any crack propagation criteria can still be recovered: this issue has been addressed in a companion paper.

4.2 Analogy between plasticity and fracture.

Provided that merely the crack tip is considered as a material point, one is tempted to state that a crack tip is not going to propagate if the SIFs vector $\mathbf{K}^*$ belongs to the set:

$$\mathbb{E} = \left\{ \{K_1^*, K_2^*\} \in \mathbb{R}_0^+ \times \mathbb{R} \mid \varphi(K_1^*, K_2^*) < 0 \right\} \tag{11}$$

which is termed the "safe equilibrium domain". When $\mathbf{K}^* \in \mathbb{E}$ the material surrounding the crack tip is experiencing a purely linear elastic behavior, eventually corresponding to an elastic unloading. The boundary of $\mathbb{E}$, $\partial\mathbb{E}$, is named the "onset of crack propagation surface":

$$\partial\mathbb{E} = \left\{ \{K_1^*, K_2^*\} \in \mathbb{R}_0^+ \times \mathbb{R} \mid \varphi(K_1^*, K_2^*) = 0 \right\} \tag{12}$$

and vectors $\mathbf{K}^* \notin \overline{\mathbb{E}}$ are ruled out. The definitions above implicity label the SIFs vector as an internal force for the LEFM problem, conjugated to a not yet specified internal variable.

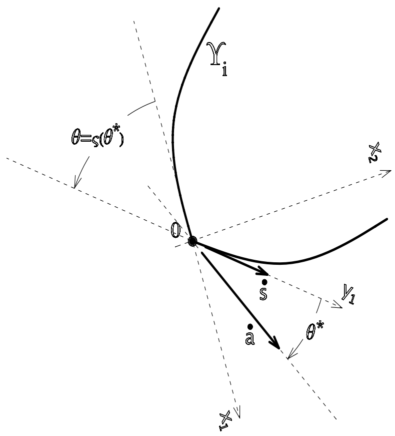

Figure 2: Definition of vector $\dot{\mathbf{a}}$ and of function ς.

At all material points experiencing plastic deformations, mechanical dissipation $\mathcal{D} > 0$ is induced; local dissipation inequality defines in plasticity (and more generally in standard dissipative systems) generalized strain rate as the conjugate to the generalized stress, as their product gives the rate of dissipation [1]. In LEFM, mechanical dissipation is due to the irreversible nature of crack extension [18]: it seems natural assuming as internal variable a quantity related to the quasi static crack tip velocity vector $\dot{\mathbf{s}}$, defined as the vector oriented with axis y_1 in figure 1 and with modulus equal to the quasi static velocity $\dot{s}|_{s\to 0^+}$ as the crack elongation s approaches zero from above. The internal variable is here termed "dissipation rate" vector $\dot{\mathbf{a}}$ and is defined as in figure 2: it is related to $\dot{\mathbf{s}}$ by its orientation defined through the kinking angle θ^* and by its length $\dot{a}$, defined as:

$$\dot{a} = \frac{G_c}{K_I^C} \dot{s} \tag{13}$$

where $G_c = 2\gamma_s$ stands for the surface energy density and γ_s the surface energy of each plane of the crack.

A maximum principle - termed $\mathcal{D}$-principle - for LEFM is postulated. *For given dissipation rate vector* $\dot{\mathbf{a}}$*, among all possible SIFs* $\mathbf{k}^* \in \overline{\mathbb{E}}$*, the function*

$$\mathcal{D}(\mathbf{k}^*; \dot{\mathbf{a}}) = \mathbf{k}^* \cdot \dot{\mathbf{a}} \tag{14}$$

attains its maximum for the actual SIF vector $\mathbf{K}^*$*:*

$$\mathcal{D}(\mathbf{K}^*; \dot{\mathbf{a}}) = \max_{\mathbf{k}^* \in \overline{\mathbb{E}}} \mathcal{D}(\mathbf{k}^*; \dot{\mathbf{a}}) \tag{15}$$

Analogously to maximum dissipation in plasticity, $\mathcal{D}$-principle implies: i) associative flow rule in the Amestoy-Leblond plane (normality law):

$$\dot{\mathbf{a}} = \frac{\partial \varphi}{\partial \mathbf{K}^*} \dot{\lambda} \tag{16}$$

ii) loading/unloading conditions in Kuhn-Tucker complementarity form:

$$\dot{\lambda} \geq 0, \qquad \varphi \leq 0, \qquad \dot{\lambda}\, \varphi = 0 \tag{17}$$

iii) convexity of safe equilibrium domain $\mathbb{E}$.

$\mathcal{D}$-principle has a neat physical interpretation. Inserting the Maximum Energy Release Rate onset of propagation (7) into (16), it comes out:

$$\dot{\mathbf{a}} = \frac{1-\nu^2}{E}\, \mathbf{K}^* \dot{\lambda} \tag{18}$$

and from (14)

$$\mathcal{D}(\mathbf{K}^*; \dot{\mathbf{a}}) = \frac{1-\nu^2}{E}\, ||\mathbf{K}^*||^2 \dot{\lambda} = G_c\, \dot{\lambda} \geq 0 \tag{19}$$

Owing to equation (13), it can be therefore concluded that: *i.* $\dot{\lambda} = \dot{s}$ is the actual "quasi-static crack propagation velocity" and $\lambda = s$ will coincide with the total crack propagation, provided that $\lambda = s = 0$ at the beginning of the crack propagation history; *ii.* function $\mathcal{D}$ equals the energy dissipation at the crack tip due to an infinitesimal crack propagation $\dot{\lambda} = \dot{s}$; consequently, $\mathcal{D}$-principle is the counterpart of the postulate of the maximum plastic work.

The last of conditions (17) is the rigorous counterpart of the intuitive description (10). *Consistency condition* can be deducted from (17) by time derivative at $\varphi = 0$; they read:

$$\text{When } \varphi = 0, \qquad \dot{\lambda} \geq 0, \qquad \dot{\varphi} \leq 0, \qquad \dot{\lambda}\, \dot{\varphi} = 0 \tag{20}$$

Vectors $\dot{\mathbf{a}}$ and $\dot{\mathbf{s}}$ materialize the kinking angle θ^*, that comes out from the normality law:

$$\frac{\partial \varphi}{\partial {K_1}^*} \tan\theta^* = \frac{\partial \varphi}{\partial {K_2}^*} \qquad \Rightarrow \qquad \tan\theta^* = \frac{K_2^*}{K_1^*} = \alpha^* \tag{21}$$

Angle θ^* is a measure of the mode mixity α^* in the Amestoy-Leblond plane. It's defined in the kinked reference $\{y_1, y_2\}$ as shown in figure 2.

Function ς, which is defined in figure 2, relates angle θ^* and the actual kinking angle θ according to (34) in appendix A. Other possibilities for map ς have been considered in a companion paper. It has been argued if, interpreting ς as a "constitutive" property, crack propagation angles from any criteria could have been eventually recovered.

5 ON THE AMESTOY-LEBLOND EXPANSION

Expansion (1) details the form of SIFs at the extended crack tip in powers of s. In other words, it shows the behavior of SIFs at a given crack tip due to an irreversible change in the geometry of the same crack tip. As the global quasi-static fracture propagation problem depends on geometry as well as on external loads, a complete expansion should read:

$$\mathbf{K}(\kappa, s) = \mathbf{K}^*(\kappa) + \mathbf{K}^{(1/2)}(\kappa)\,\sqrt{s} + \mathbf{K}^{(1)}(\kappa)\,s + O(s^{3/2}) \tag{22}$$

If the solution of the global problem at given load κ and geometry s is such that $\varphi < 0$, an increase of loads is not prone to elongate the crack:

$$at\ \kappa\ s.t.\ \varphi(\mathbf{K}, \theta) < 0 \quad \Rightarrow \quad \Delta\kappa \rightarrow \Delta\mathbf{K} = \frac{\mathbf{K}}{\kappa}\Delta\kappa\ , \quad \Delta\mathbf{K}^* = \frac{\mathbf{K}^*}{\kappa}\Delta\kappa \tag{23}$$

This describes the first phase of fracturing process, namely loading without crack growth. When the onset of crack propagation is reached, the second phase - when present - is triggered off: stable crack growth. A further increase of load causes therefore crack elongation. In the elongated configuration one writes:

$$at\ \kappa\ s.t.\ \varphi(\mathbf{K}, \theta) = 0 \quad \Rightarrow \quad \Delta\kappa \rightarrow \Delta\mathbf{K} = \Delta\mathbf{K}^* + \mathbf{K}^{(1/2)}\,\sqrt{s} + o(\Delta\kappa) \tag{24}$$

Equation (24) is a reminiscence of Ceradini's decomposition of stresses in plasticity. It decomposes the variation of SIFs as due to an elastic contribution ($\Delta\mathbf{K}^*$) and to a distortion (in fracture: crack elongation s; in plasticity: plastic strain) which reverses itself into SIFs (stresses in plasticity) by means of a stiffness factor (in fracture: $\mathbf{K}^{(1/2)}(\kappa)$, in plasticity the action of the Z matrix over the plastic part of the volume).

6 A GENERAL STABILITY CONDITION OF CRACK GROWTH

During stable crack growth, propagation is a sequence of equilibrium states. At each load corresponds a geometry configuration which propagates quasi-statically, keeping the system at the onset of fracture $\varphi(\mathbf{K}, \theta) = 0$. When the unstable propagation regime takes place, dynamic effects cannot be neglected. The transition between these two phases is a crucial information. Assuming in fact that unstable propagation leads to structural collapse, the safety of a structural components is measured against the stable/unstable crack growth transition.

From the consistency condition (20) one writes with a small abuse of notation:

$$0 = \dot{\varphi} = \frac{\partial\varphi}{\partial\mathbf{K}^*} \cdot \left(\dot{\mathbf{K}}^* + \mathbf{K}^{(1/2)}(\kappa)\sqrt{\dot{s}}\right) \tag{25}$$

The first term in brackets reflects the variation of $\mathbf{K}$ while keeping the initial geometry, because $\mathbf{K}^*$ is defined at $s \rightarrow 0^+$ and is not a function of the crack elongation. In Ceradini's decomposition spirit, it is an elastic contribution. As a consequence, (25) becomes:

$$0 = \frac{\partial\varphi}{\partial\mathbf{K}^*} \cdot \frac{\mathbf{K}^*}{\kappa}\dot{\kappa} + \frac{\partial\varphi}{\partial\mathbf{K}^*} \cdot \mathbf{K}^{(1/2)}(\kappa)\sqrt{\dot{s}} \tag{26}$$

In view of definition (7) of the maximum energy release rate criterion, the amount

$$\frac{\partial\varphi}{\partial\mathbf{K}^*} \cdot \mathbf{K}^* = \frac{1-\nu^2}{E}\,||\mathbf{K}^*||^2 \tag{27}$$

is positive at $\varphi = 0$ where it equals G_c. Furthermore, $\sqrt{\dot{s}} > 0$ from the irreversible nature of crack growth. It turns out therefore that:

$$\dot{\varphi} = 0 \qquad \Rightarrow \qquad \frac{1-\nu^2}{E}\,||\mathbf{K}^*||^2 \dot{\kappa} = -\frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1/2)}(\kappa)\sqrt{\dot{s}} \tag{28}$$

which sets a condition for stable crack growth and, inherently, for the transition to the unstable phase:

$$\dot{\kappa} > 0 \;\rightarrow\; \frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1/2)}(\kappa) < 0 \tag{29}$$

Condition (29) can be restated in the easy form:

$$\dot{\kappa} > 0 \;\rightarrow\; \mathbf{K}^* \cdot \mathbf{K}^{(1/2)} < 0 \tag{30}$$

owing again to (7). The sign of $\mathbf{K}^* \cdot \mathbf{K}^{(1/2)}$ can easily be tested at any given crack tip. If the crack path is approximated to be piecewise linear, the evaluation of SIFs and T stresses are merely required in view of definitions (2)-(3).

7 A (LOCAL) VARIATIONAL STATEMENT FOR CRACK GROWTH

The following variational statement extends Ceradini's functional to fracture mechanics. *Denote with $\dot{\mu} = \sqrt{\dot{s}}$. The crack tip velocity that solves the global quasi-static fracture propagation problem minimizes the funcional*

$$\chi[\dot{\mu}] = -\frac{1}{2}\frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1/2)}\;\dot{\mu}^2 - \frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \dot{\mathbf{K}}^* \dot{\mu} \tag{31}$$

provided that:

$$\dot{\mu} \geq 0 \quad \textit{and} \quad \frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1/2)} < 0$$

The proof the theorem is here omitted for paucity of space. In the form above, Ceradini's functional is written for a single crack tip. Extension to several crack tips contemporarily propagating is quite straightforward; the only issue on this point appears to be the evaluation of expansion (1) at a crack tip (say i) when a different crack tip (say j) is advancing. Numerical studies have been put forward, showing that crack tips do interact but the asymptotical behavior, at authors' best knowledge, is still incomplete.

In the easy case of a single crack propagation, the minimum of Ceradini's functional reads:

$$\dot{\mu} = -\frac{\frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \dot{\mathbf{K}}^*}{\frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1/2)}} = -\frac{1}{\kappa}\;\frac{\frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \mathbf{K}^*}{\frac{\partial \varphi}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1/2)}}\;\dot{\kappa} = -\frac{1}{\kappa}\;\frac{||\mathbf{K}^*||^2}{\mathbf{K}^* \cdot \mathbf{K}^{(1/2)}}\;\dot{\kappa} \tag{32}$$

It is straightforward to extend functional (31) as well as stability condition (29) to the case of straight propagation (Mode I) at which $\mathbf{K}^{(1/2)}$ vanishes. A similar result was obtained in [3] by a different, way less rigorous, approach.

8 CONCLUSIONS

In the present note a variational formulation for the global quasi-static fracture propagation problem has been presented. It is rooted in a plasticity framework for linear elastic fracture mechanics, which stems itself from a maximum principle which is the counterpart of the maximum plastic work postulate. From such a cornerstone, Griffith's approach is recovered following a rigorous analogy setting.

This likely new way of looking at linear elastic fracture mechanics leads to some results that appear to worth attention. Several functionals have been extended, most of whom are not included in the present note. In it, the quasi-static crack tip velocity has been shown to be the minima of a quadratic functional which is analogous to Ceradini's one for plasticity. Conditions for stable crack growth and for the onset of instability have been investigated and major results gained.

Extension of the proposed framework to 3D fracture mechanics, cohesive, dynamics, fatigue is in progress.

References

[1] Q.S. Nguyen. *Stability and nonlinear solid mechanics*. John Wiley and Sons B.V., 2000.

[2] G. Maugin. *The Thermomechanics of Plasticity and Fracture*. Cambridge University Press, New York, 1992.

[3] A. Salvadori. A plasticity framework for (linear elastic) fracture mechanics. *J. Mech. Phys. Solids*, 56:2092–2116, 2008.

[4] K.B. Broberg. *Cracks and fracture*. Academic Press, 1999.

[5] G.I. Barenblatt. On equilibrium cracks forming during brittle fracture (in russian). *Prikladnaya Matematika i Mekhanika*, 23:434–444, 1959. See also, *The mathematical theory of equilibrium cracks in brittle fracture*, Advances in appl. mech., **7** 55-129 (1962).

[6] T.J. Stone and I. Babuska. A numerical method with a posteriori error estimation for determining the path taken by a propagating crack. *Comp. Methods Appl. Mech. Eng.*, 160:245–271, 1998.

[7] M. Amestoy and J.B. Leblond. Crack paths in plane situations - ii. detailed form of the expansion of the stress intensity factord. *International Journal of Solids and Structures*, 29:465–501, 1992.

[8] R.V. Goldstein and R.L. Salganik. Brittle fracture of solids with arbitrary cracks. *Int. J. Fract.*, 10:507–523, 1974.

[9] A.A. Griffith. The phenomena of rupture and flow in solids. *Phil. Trans. R. Soc.*, 221:163–198, 1921.

[10] C.W. Wu. Maximum-energy-release-rate criterion applied to a tension-compression specimen with crack. *Journal of Elasticity*, 8:235–257, 1978.

[11] K.C. Le. Variational problems of crack equilibrium and crack propagation. In D.R. Owen G. Del Piero, editor, *Multiscale modeling in continuum mechanics and structured deformations*. Springer-Verlag, 2004.

[12] A. Chambolle, G.A. Francfort, and J.J. Marigo. When and how do cracks propagate? *Journal of the Mechanics and Physics of Solids*, 57(9):1614–1622, 2009.

[13] G. Erdogan and G.C. Sih. On the crack extension in plates under plane loading and trasverse shear. *ASME J. Basic Engng*, 85:519–527, 1963.

[14] A. Otsuka, K. Mori, and T. Miyata. The condition of fatigue crack growth in mixed mode condition. *Eng. Fracture Mech.*, 7:429–439, 1975.

[15] H.C. Strifors. A generalized force measure of conditions at crack tips. *Int. J. Solids Structures*, 10:1389–1404, 1973.

[16] G.C. Sih. Strain-energy-density factor applied to mixed-mode crack problems. *Int. J. Fracture*, 10:305–321, 1973.

[17] W. Han and B.D. Reddy. *Plasticity*. Springer-Verlag, New York, 1999.

[18] J. Rice and D.C. Drucker. Energy changes in stressed bodies due to void and crack growth. *Int. J. of Fracture*, 3:19–27, 1967.

A KINKING ANGLE

A.1 MERR

The kink angle predicted by the MERR is such that:

$$\frac{\partial}{\partial\theta}||\mathbf{F}\,\mathbf{K}||^2 = 0 \tag{33}$$

Matrix **F** has been defined in terms of the ratio $m = \frac{\theta}{\pi}$ as a series approximation of integral equation (39) pag. 476 in reference [7]. It is straightforward to show that equation (33) implies:

$$(F_{11} + \alpha F_{12})\left(\frac{\partial F_{11}}{\partial m} + \alpha\frac{\partial F_{12}}{\partial m}\right) + (F_{21} + \alpha F_{22})\left(\frac{\partial F_{21}}{\partial m} + \alpha\frac{\partial F_{22}}{\partial m}\right) = 0 \tag{34}$$

where $\alpha = \frac{K_2}{K_1}$. For any α equation (34) provides the kink angle θ_{MERR}. Within the present note, the solution of (34) was found numerically. The limit value for $K_1 = 0$ was attained at:

$$\theta^{II}_{MERR} = 1^{rad}.3222 \tag{35}$$

A.2 LS

The LS criterion is the only exception to the mathematical representation of the onset of fracture in the general form (5). It gives the kink angle θ_{LS} through the equation $K^*_2 = 0$:

$$F_{21} + \alpha F_{22} = 0 \tag{36}$$

where $\alpha = \frac{K_2}{K_1}$. For any α equation (36) provides the kink angle θ_{LS}. Within the present note, the solution of (36) was found numerically. The limit value for $K_1 = 0$ was attained at:

$$\theta^{II}_{LS} = 1^{rad}.34966 \tag{37}$$