An analytical contour dynamics

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SUMMARY. The motion of a class of plane uniform vortices in an inviscid fluid is analytically investigated through a (nonlinear) integral equation, written in the Laplace transform (in time) of the Lagrangian Schwarz function of the vortex boundary. An approximate analytical technique based on successive substitutions for finding the solution of that equation is proposed. The first order approximation is preliminarily compared with numerical simulation and the progresses in the analytical calculations are illustrated.

1 INTRODUCTION

The study of the inviscid motion of uniform vortices in two dimensions is a quite old issue in Fluid Mechanics (see [1], cap. 9 and in particular the paragraph 9.2 about the use of the Schwarz function). The theoretical analyses has been almost entirely dedicated to relative equilibrium configurations of one ([2] for the Kirchhoff vortex) or more vortices [3, 4, 5, 6, 7, 8, 9] which are stationary solutions of Euler equations, while the study of the motion of nonequilibrium vortices have been faced essentially through numerical simulations. The approximate numerical integration is based on the rewriting of the Biot-Savart law as a line integral on the vortex boundary and on the Lagrangian integration of the motion of a discrete set of points on that curve. The resulting algorithm is called *contour dynamics* in Literature [10, 11].

In the last years, the present authors proposed a different analytical approach to the study of the motion of nonequilibrium uniform vortices [12, 13, 14]. It is still based on the use of the Schwarz function [15] of the vortex boundary, but it does not require the splitting of such a function in the sum of a function analytical inside and another one analytical outside the vortex, as described in [1]. Indeed, this splitting is almost impossible in real cases, unless for the class of quasi-circular vortices the motion of which is not so interesting. A new integral relation between the Schwarz function $\boldsymbol{\Phi}$ (a complex quantity is indicated with a bold symbol) and the conjugate velocity $\overline{\boldsymbol{u}}$ (overbar means conjugate) has been proposed:

$$\overline{\boldsymbol{u}}(\boldsymbol{x},\overline{\boldsymbol{x}}) = \frac{\omega}{2\boldsymbol{i}} \left[\chi_P(\boldsymbol{x})\, \overline{\boldsymbol{x}} + \frac{1}{2\pi\boldsymbol{i}} \int_{\partial P} d\boldsymbol{y}\, \frac{\boldsymbol{\Phi}(\boldsymbol{y})}{\boldsymbol{x}-\boldsymbol{y}} \right] \ , \tag{1}$$

ω being the vorticity level inside the vortex and the integral being understood as a Cauchy one (it will be indicated with a bar on the integral) when $\boldsymbol{x}$ lies on the vortex boundary ∂P. Moreover, χ_P is the characteristic function of the domain P: it holds 1 inside P, 0 outside P and just $1/2$ on the boundary. The link (1) enables us to analytically evaluate the velocity induced by any uniform vortex with standard tools of complex analysis. As briefly discussed in [14], the same relation opens the way to the analytical study of the vortex dynamics for nonequilibrium vortices.

The Lagrangian Schwarz function $\boldsymbol{\Phi}[\boldsymbol{x}(\boldsymbol{\xi},\tau),\tau] =: \boldsymbol{S}(\boldsymbol{\xi};\tau)$ ($\boldsymbol{\xi}$ is the position on the initial vortex boundary $\partial P(0)$ and τ is the time, nondimensionalized by $\omega/4$) is introduced, as well as its Laplace transform in time $\tilde{\boldsymbol{S}}(\boldsymbol{\xi};\boldsymbol{\sigma})$. This latter satisfies the integral equation [14]:

$$\underbrace{(\boldsymbol{i\sigma}-1)\tilde{\boldsymbol{S}}(\boldsymbol{\xi};\boldsymbol{\sigma}) + \frac{1}{\pi \boldsymbol{i}}\fint_{\partial P(0)} d\boldsymbol{\eta}\, \frac{\tilde{\boldsymbol{S}}(\boldsymbol{\eta};\boldsymbol{\sigma})}{\boldsymbol{\eta}-\boldsymbol{\xi}}}_{\mathbf{SD}[\tilde{\boldsymbol{S}}]:\ \textit{singular dominant part (linear)}} + \underbrace{\frac{1}{\pi \boldsymbol{i}}\int_{\partial P(0)} d\boldsymbol{\eta}\, \widetilde{\boldsymbol{g}\boldsymbol{S}}(\boldsymbol{\eta},\boldsymbol{\xi};\boldsymbol{\sigma})}_{\mathbf{R}[\tilde{\boldsymbol{S}}]:\ \textit{regular part (non linear)}} = \underbrace{\boldsymbol{i}\boldsymbol{S}_0(\boldsymbol{\xi})}_{\mathbf{ID}:\ \textit{data}} \tag{2}$$

which is easily deduced from the relation (1), $\boldsymbol{S}_0$ being the initial Schwarz function. Equation (2) is a singular nonlinear integral one [16], the nonlinearity being confined into the function $\boldsymbol{g}$:

$$\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{\xi};\tau) = \frac{\partial}{\partial \boldsymbol{\eta}} \log \frac{\boldsymbol{x}(\boldsymbol{\eta};\tau)-\boldsymbol{x}(\boldsymbol{\xi};\tau)}{\boldsymbol{\eta}-\boldsymbol{\xi}} \ ,$$

which accounts for the vortex geometry at the present time.

In [14] an iterative procedure to analytically solve equation (2) has been proposed and tested (through numerical integration). It is based on the solution of equation (2) for vanishing $\boldsymbol{g}$:

$$\mathbf{SD}[\tilde{\boldsymbol{S}}^{(0)}] = \mathbf{ID} \ , \tag{3}$$

which defines the so called 0-th order solution. Equation (3) has solution for $\boldsymbol{\xi} \in \partial P(0)$:

$$\boldsymbol{S}^{(0)}(\boldsymbol{\xi};\tau) = \boldsymbol{S}_0(\boldsymbol{\xi}) + \overline{\boldsymbol{\Delta}}(\tau)\overline{\boldsymbol{u}}_0(\boldsymbol{\xi}) \ , \tag{4}$$

$\boldsymbol{\Delta}(\tau)$ being $4\, e^{\boldsymbol{i}\tau} \sin\tau$. The above solution can be analytically continued in any point $\boldsymbol{\xi}$ external to $\partial P(0)$ through the new 0-th order function: $\boldsymbol{S}^{(0)}(\boldsymbol{\xi};\tau) = \boldsymbol{S}_0(\boldsymbol{\xi}) + \overline{\boldsymbol{\Delta}}(\tau)\, \overline{\boldsymbol{U}}_0(\boldsymbol{\xi})$, $\overline{\boldsymbol{U}}_0$ being the continued initial velocity (nondimensionalized with ω and an arbitrary length scale):

$$\overline{\boldsymbol{U}}_0(\boldsymbol{\xi}) = \frac{1}{2\boldsymbol{i}}\ \Big[\ \frac{1}{2}\ \boldsymbol{S}_0(\boldsymbol{\xi}) + \frac{1}{2\pi \boldsymbol{i}}\int_{\partial P(0)} d\boldsymbol{\eta}\ \frac{\boldsymbol{S}_0(\boldsymbol{\eta})}{\boldsymbol{\xi}-\boldsymbol{\eta}}\ \Big] \ ,$$

deduced from equation (1), via analytic continuation from the vortex boundary. In an equivalent way, the 0-th order vortex shape following from equation (5) through complex conjugation is analytically continued as: $\boldsymbol{x}^{(0)}(\boldsymbol{\xi};\tau) = \boldsymbol{\xi} + \boldsymbol{\Delta}(\tau)\ \boldsymbol{U}_0(\boldsymbol{\xi})$, where the (analytically continued) velocity field $\boldsymbol{U}_0$ is:

$$\boldsymbol{U}_0(\boldsymbol{\xi}) = \frac{1}{2\boldsymbol{i}}\ \Big[-\frac{1}{2}\ \boldsymbol{\xi} + \frac{1}{2\pi \boldsymbol{i}}\int_{\partial P(0)} d\boldsymbol{\eta}\ \frac{\boldsymbol{\eta}\ \boldsymbol{S}_0'(\boldsymbol{\eta})}{\boldsymbol{S}_0(\boldsymbol{\xi})-\boldsymbol{S}_0(\boldsymbol{\eta})}\ \Big] \ .$$

Higher order approximations of the solution of equation (2) are obtained through successive substitution ($k = 1, 2, \ldots$) inside the same equation (2) in which the nonlinear term is treated as a source one:

$$\mathbf{SD}[\tilde{\boldsymbol{S}}^{(k)}] = \mathbf{ID} - \mathbf{R}[\tilde{\boldsymbol{S}}^{(k-1)}] \ . \tag{5}$$

$\tilde{\boldsymbol{S}}^{(k)}$ will be named as the k-th order solution. The present paper deals with the analytical calculation of the first order ($k = 1$) solution of equation (5), which will be evaluated for a certain class of vortices, the kinematics of which has been previously analyzed [13].

2 FIRST ORDER SOLUTION (GENERAL CASE)

At the first order, the regular term $\mathbf{R}[\tilde{\boldsymbol{S}}^{(0)}]$ (5) must be evaluated in correspondence to the solution $\boldsymbol{S}^{(0)}$ (4) found just above. First of all, the function $\boldsymbol{h}$ is introduced:

$$\boldsymbol{h}(\boldsymbol{\eta},\boldsymbol{\xi}) := \frac{\boldsymbol{u}_0(\boldsymbol{\eta}) - \boldsymbol{u}_0(\boldsymbol{\xi})}{\boldsymbol{\eta}-\boldsymbol{\xi}} , \tag{6}$$

which is continuous together with all its derivatives on the curve $\partial P(0)$. The solution at the first order is evaluated by specifying the function $\boldsymbol{g}$ evaluated in correspondence to the 0-th order solution (4):

$$\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{\xi};\tau) = \partial_{\boldsymbol{\eta}} \log \left[\, 1 + \boldsymbol{\Delta}(\tau)\boldsymbol{h}(\boldsymbol{\eta},\boldsymbol{\xi}) \,\right] = \frac{\boldsymbol{\Delta}(\tau)\boldsymbol{h}'(\boldsymbol{\eta},\boldsymbol{\xi})}{1+\boldsymbol{\Delta}(\tau)\boldsymbol{h}(\boldsymbol{\eta},\boldsymbol{\xi})} , \tag{7}$$

the apex indicating a derivative with respect to the first argument (*i.e.*, $\boldsymbol{\eta}$). In this way the Laplace transform in time is given by the following formula:

$$\begin{aligned} \widetilde{\boldsymbol{g}\boldsymbol{S}^{(0)}}(\boldsymbol{\eta},\boldsymbol{\xi};\boldsymbol{\sigma}) &= \mathcal{L}\left\{ \left[\, \boldsymbol{S}_0(\boldsymbol{\eta}) + \overline{\boldsymbol{\Delta}}(\tau)\overline{\boldsymbol{U}}(\boldsymbol{\eta}) \,\right] \partial_{\boldsymbol{\eta}} \log \left[\, 1 + \boldsymbol{\Delta}(\tau)\boldsymbol{h}(\boldsymbol{\eta},\boldsymbol{\xi}) \,\right] \right\}(\boldsymbol{\sigma}) \\ &= \boldsymbol{h}'(\boldsymbol{\eta},\boldsymbol{\xi})\, \mathcal{L}\left\{ \frac{\boldsymbol{\Delta}(\tau)\, [\boldsymbol{S}_0(\boldsymbol{\eta}) + \overline{\boldsymbol{\Delta}}(\tau)\overline{\boldsymbol{U}}_0(\boldsymbol{\eta})]}{1+\boldsymbol{\Delta}(\tau)\boldsymbol{h}(\boldsymbol{\eta},\boldsymbol{\xi})} \right\}(\boldsymbol{\sigma}) , \end{aligned}$$

which leads to the following form of the regular term:

$$\mathbf{R}[\tilde{\boldsymbol{S}}^{(0)}](\boldsymbol{\xi},\boldsymbol{\sigma}) = \mathcal{L}\{\boldsymbol{N}(\boldsymbol{\xi};\tau)\}(\boldsymbol{\sigma}) , \tag{8}$$

having introduced the function:

$$\boldsymbol{N}(\boldsymbol{\xi};\tau) = \frac{\boldsymbol{\Delta}(\tau)}{\pi \boldsymbol{i}} \int_{\partial P(0)} d\boldsymbol{\chi}\, \boldsymbol{h}'(\boldsymbol{\chi},\boldsymbol{\xi})\, \frac{\boldsymbol{S}_0(\boldsymbol{\chi}) + \overline{\boldsymbol{\Delta}}(\tau)\overline{\boldsymbol{U}}_0(\boldsymbol{\chi})}{1+\boldsymbol{\Delta}(\tau)\boldsymbol{h}(\boldsymbol{\chi},\boldsymbol{\xi})} \tag{9}$$

In the following, the initial vortex boundary is defined through a map onto the unit circle $\boldsymbol{z} \mapsto \boldsymbol{\xi}$, the integral inside the function $\boldsymbol{N}$ (9) is rewritten as the following integral on $\mathcal{C}$:

$$\boldsymbol{N}[\boldsymbol{\xi}(\boldsymbol{z});\tau'] = \frac{1}{\pi \boldsymbol{i}} \int_{\mathcal{C}} d\boldsymbol{\zeta}\ \left\{ \boldsymbol{S}_0[\boldsymbol{\chi}(\boldsymbol{\zeta})] + \overline{\boldsymbol{\Delta}}(\tau')\overline{\boldsymbol{U}}_0[\boldsymbol{\chi}(\boldsymbol{\zeta})] \right\}\ \partial_{\boldsymbol{\zeta}} \log[1+\boldsymbol{\Delta}(\tau')\boldsymbol{H}(\boldsymbol{\zeta},\boldsymbol{z})] , \tag{10}$$

$\boldsymbol{H}(\boldsymbol{\zeta},\boldsymbol{z})$ indicating the composite function $\boldsymbol{h}[\boldsymbol{\chi}(\boldsymbol{\zeta}),\boldsymbol{\xi}(\boldsymbol{z})]$.

The regular term (8) is inserted into the first order equation $\mathbf{SD}[\tilde{\boldsymbol{S}}^{(1)}] = \mathbf{ID} - \mathbf{R}[\tilde{\boldsymbol{S}}^{(0)}]$, so that the Laplace transform of the solution at the first order follows as:

$$\begin{aligned} \tilde{\boldsymbol{S}}^{(1)}(\boldsymbol{\xi};\boldsymbol{\sigma}) &= \tilde{\boldsymbol{S}}^{(0)}(\boldsymbol{\xi},\boldsymbol{\sigma}) - \frac{\boldsymbol{i\sigma}-1}{\boldsymbol{i\sigma}(\boldsymbol{i\sigma}-2)}\, \mathcal{L}\{\boldsymbol{N}(\boldsymbol{\xi};\tau)\}(\boldsymbol{\sigma}) + \\ &\quad + \frac{1}{\boldsymbol{i\sigma}(\boldsymbol{i\sigma}-2)}\, \mathcal{L}\left\{ \frac{1}{\pi \boldsymbol{i}} \int_{\partial P(0)} d\boldsymbol{\eta}\, \frac{\boldsymbol{N}(\boldsymbol{\eta};\tau)}{\boldsymbol{\eta}-\boldsymbol{\xi}} \right\}(\boldsymbol{\sigma}) . \end{aligned} \tag{11}$$

The Laplace antitransform of $\tilde{\boldsymbol{S}}^{(1)}$ is evaluated by considering the following fact. If $\boldsymbol{F}(\boldsymbol{\sigma})$ is the Laplace transform of the function $\boldsymbol{f}(\tau)$, due to the fact that:

$$\frac{\boldsymbol{i\sigma}-1}{\boldsymbol{i\sigma}(\boldsymbol{i\sigma}-2)}\, \boldsymbol{F} = \frac{1}{2\boldsymbol{i}}\left(+ \frac{\boldsymbol{F}}{\boldsymbol{\sigma}} + \frac{\boldsymbol{F}}{\boldsymbol{\sigma}+2\boldsymbol{i}} \right) , \quad \frac{1}{\boldsymbol{i\sigma}(\boldsymbol{i\sigma}-2)}\, \boldsymbol{F} = \frac{1}{2\boldsymbol{i}}\left(- \frac{\boldsymbol{F}}{\boldsymbol{\sigma}} + \frac{\boldsymbol{F}}{\boldsymbol{\sigma}+2\boldsymbol{i}} \right) ,$$

the Laplace antitransfoms of the above terms can be written as:

$$\int_0^\tau d\tau' \, \boldsymbol{\theta}_{1,2}(\tau-\tau')\boldsymbol{f}(\tau') \ \text{ with: } \ \boldsymbol{\theta}_{1,2}(\tau) = \frac{\pm 1 + e^{-2\boldsymbol{i}\tau}}{2\boldsymbol{i}} \ ,$$

once Appendix A has been accounted for. As a consequence, the Laplace antitransform of the first order solution (11) is rewritten as:

$$\begin{aligned} \boldsymbol{S}^{(1)}(\boldsymbol{\xi};\tau) \quad = \quad & \boldsymbol{S}^{(0)}(\boldsymbol{\xi};\tau) - \int_0^\tau d\tau' \, \boldsymbol{\theta}_1(\tau-\tau')\boldsymbol{N}(\boldsymbol{\xi};\tau') + \\ & + \int_0^\tau d\tau' \, \boldsymbol{\theta}_2(\tau-\tau') \, \frac{1}{\pi \boldsymbol{i}} \int_{\partial P(0)} d\boldsymbol{\eta} \, \frac{\boldsymbol{N}(\boldsymbol{\eta};\tau')}{\boldsymbol{\eta}-\boldsymbol{\xi}} \ . \end{aligned} \tag{12}$$

The above first order solution is calculated for non-trivial vortex shapes in the following, firstly in a numerical fashion and then in an analytical form. This latter calculation is the main aim of the present paper and is not compled at the present time.

3 CALCULATION OF THE FIRST ORDER SOLUTION FOR A $(1,1)$-VORTEX

The results about the first order solution found in Section 2 will be now specified for a certain class of vortices, having their Schwarz function with two poles in the transformed $\boldsymbol{z}$-plane. Indeed, it has been proved in a previous paper [13] that a complete kinematical analysis of such a kind of vortices can be carried out, thanks to the new integral formula (1). These vortices have been classified into six classes, on the basis of the behaviour of the functions $\boldsymbol{x}(\boldsymbol{z})$ and $\boldsymbol{\zeta}^\star(\boldsymbol{z})$, this latter being the pseudo-inverse one (satisfying the relation: $\boldsymbol{x}[\boldsymbol{\zeta}^\star(\boldsymbol{z})] = \boldsymbol{x}(\boldsymbol{z})$ for any $\boldsymbol{z}$. In the present calculations, only the first class (*i.e.* the $(1,1)$-vortices) will be considered[1], due to the fact that their possess the simplest analytical structure of the self-induced velocity.

Introduced the complex time $\boldsymbol{\tau}(\tau') := \exp(2\boldsymbol{i}\tau') - 1$, for a vortex of kind $(1,1)$ the function $\boldsymbol{H}(\boldsymbol{\zeta},\boldsymbol{z})$ assumes the following form:

$$\boldsymbol{\Delta}(\tau')\boldsymbol{H}(\boldsymbol{\zeta},\boldsymbol{z}) = \boldsymbol{\tau}(\tau') \, \frac{(\boldsymbol{z}-\boldsymbol{w}_2)\,(\boldsymbol{\zeta}-\boldsymbol{w}_2)\,[(\boldsymbol{d}_0\boldsymbol{z}-\boldsymbol{d}_1)\boldsymbol{\zeta}-(\boldsymbol{d}_1\boldsymbol{z}-\boldsymbol{d}_2)]}{(\boldsymbol{z}-\boldsymbol{w}_2^\star)\,(\boldsymbol{\zeta}-\boldsymbol{w}_2^\star)\,[(\boldsymbol{\alpha}\boldsymbol{z}-\boldsymbol{\beta})\boldsymbol{\zeta}-(\boldsymbol{\beta}\boldsymbol{z}-\boldsymbol{\gamma})]} \ , \tag{13}$$

where the coefficients $\boldsymbol{d}_0$, $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ appear in the algebraic structure of the velocity. They are given by the following formulae

$$\boldsymbol{d}_0 = \overline{\boldsymbol{a}}_1\boldsymbol{w}_1^2 - \overline{\boldsymbol{a}}_2\boldsymbol{w}_2^2\boldsymbol{\mu} \ , \ \ \boldsymbol{d}_1 = \overline{\boldsymbol{a}}_1\boldsymbol{w}_1^2\boldsymbol{w}_2^\star - \overline{\boldsymbol{a}}_2\boldsymbol{w}_1\boldsymbol{w}_2^2\boldsymbol{\mu} \ , \ \ \boldsymbol{d}_2 = \overline{\boldsymbol{a}}_1\boldsymbol{w}_1^2{\boldsymbol{w}_2^\star}^2 - \overline{\boldsymbol{a}}_2\boldsymbol{w}_1^2\boldsymbol{w}_2^2\boldsymbol{\mu} \ ,$$

$\boldsymbol{\mu}$ being $\boldsymbol{a}_1\boldsymbol{a}_2 \, \{(\boldsymbol{z}_1-\boldsymbol{z}_2)/[\boldsymbol{a}_1(\boldsymbol{z}_2-\boldsymbol{w}_2)+\boldsymbol{a}_2(\boldsymbol{z}_1-\boldsymbol{w}_2)]\}^2$. The algebraic structure of the above coefficients will be a key-point in evaluating the first order solution: it implies a certain symmetry property that will be discussed later, see equation (16).

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[1] The notations used below are the ones in the paper [13]. The Schwarz function is assumed in the $\boldsymbol{z}$-plane as:

$$\boldsymbol{\Phi}(\boldsymbol{z}) = \frac{\boldsymbol{a}_1}{\boldsymbol{z}-\boldsymbol{z}_1} + \frac{\boldsymbol{a}_2}{\boldsymbol{z}-\boldsymbol{z}_2} \ ,$$

for suitable choice of the poles ($\boldsymbol{z}_{1,2}$) and of the corresponding residues ($\boldsymbol{a}_{1,2}$). The image points $\boldsymbol{w}_k = 1/\overline{\boldsymbol{z}}_k$ ($k = 1, 2$) are used, as well as the point $\boldsymbol{z}_2^\star = \boldsymbol{\zeta}^\star(\boldsymbol{z}_2)$ and its image $\boldsymbol{w}_2^\star = 1/\overline{\boldsymbol{z}}_2^\star$. Poles and residues are combined into the quantities $\boldsymbol{\alpha} = \overline{\boldsymbol{a}}_1\boldsymbol{w}_1^2 + \overline{\boldsymbol{a}}_2\boldsymbol{w}_2^2$, $\boldsymbol{\beta} = \boldsymbol{w}_1\boldsymbol{w}_2(\overline{\boldsymbol{a}}_1\boldsymbol{w}_1 + \overline{\boldsymbol{a}}_2\boldsymbol{w}_2)$, $\boldsymbol{\gamma} = \boldsymbol{w}_1^2\boldsymbol{w}_2^2(\overline{\boldsymbol{a}}_1 + \overline{\boldsymbol{a}}_2)$ and $\boldsymbol{\delta}^2 = \boldsymbol{\alpha}\boldsymbol{\gamma} - \boldsymbol{\beta}^2$.

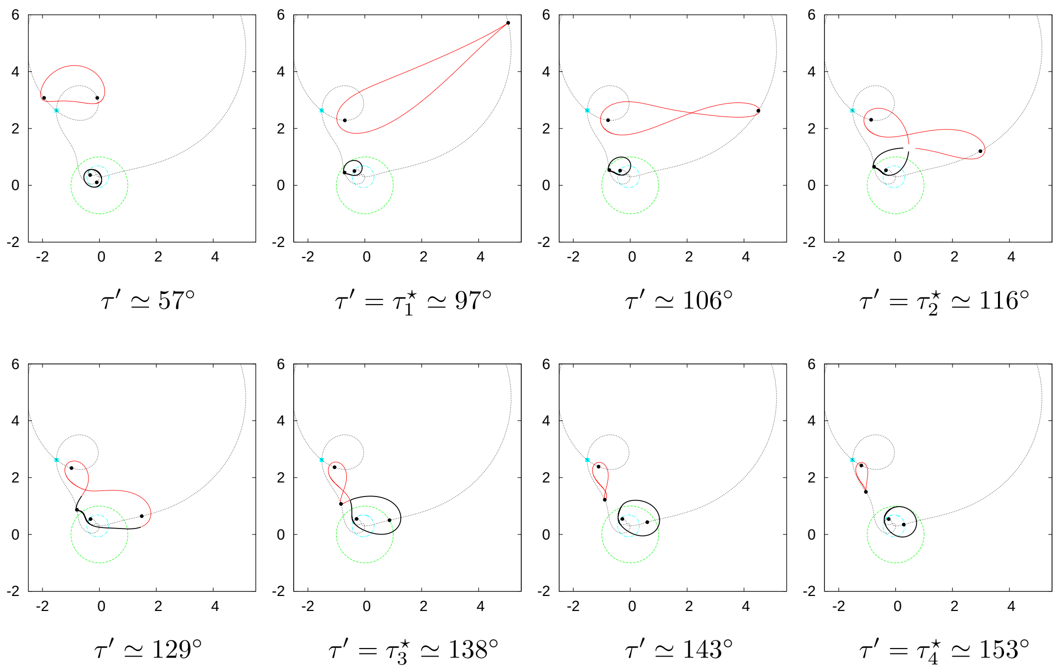

Figure 1: For a vortex of kind $(1,1)$ ($\boldsymbol{a}_1 = 1$, $\boldsymbol{a}_2 = 0.2 + \boldsymbol{i}$, $\boldsymbol{z}_1 = -0.7 + \boldsymbol{i}\,0.05$ and $\boldsymbol{z}_2 = -1 + \boldsymbol{i}\,1.5$) the curves $\mathcal{L}_1$ (red solid line) and $\mathcal{L}_2$ (black) are drawn at different times (expressed in degrees) in the first period. The positions of the branch points of the functions $\boldsymbol{\zeta}_{1,2}(\boldsymbol{z}, \tau')$ are indicated with black symbols, while with a dashed black line is drawn the curve (not simple) described by that points for $\boldsymbol{z}$ running on $\mathcal{C}$. $\mathcal{C}$ is drawn with green dashed line. The point $\boldsymbol{w}_\star$ as well as the circle $\mathcal{C}^\star$ are also drawn with turquoise symbol and dashed line, respectively.

The condition $1 + \boldsymbol{\Delta H} = 0$ is written by using equation (13) as:

$$
\begin{aligned}
&(\boldsymbol{z} - \boldsymbol{w}_2^\star)\ (\boldsymbol{\alpha z} - \boldsymbol{\beta})\ \ [\boldsymbol{\zeta}(\boldsymbol{z}, \tau') - \boldsymbol{w}_2^\star]\ \left[\boldsymbol{\zeta}(\boldsymbol{z}, \tau') - \frac{\boldsymbol{\beta z} - \boldsymbol{\gamma}}{\boldsymbol{\alpha z} - \boldsymbol{\beta}}\right] = \\
= \ & -\boldsymbol{\tau}(\tau')\ (\boldsymbol{z} - \boldsymbol{w}_2)\ (\boldsymbol{d}_0 \boldsymbol{z} - \boldsymbol{d}_1)\ \ [\boldsymbol{\zeta}(\boldsymbol{z}, \tau') - \boldsymbol{w}_2]\ \left[\boldsymbol{\zeta}(\boldsymbol{z}, \tau') - \frac{\boldsymbol{d}_1 \boldsymbol{z} - \boldsymbol{d}_2}{\boldsymbol{d}_0 \boldsymbol{z} - \boldsymbol{d}_1}\right] .
\end{aligned}
$$

The two roots of such an equation will be named as $\boldsymbol{\zeta}_1(\boldsymbol{z}, \tau')$ (which reduces to the point $\boldsymbol{w}_2^\star$ for any $\boldsymbol{z} \in \mathcal{C}$ at times $\tau' = k\pi$ with k non-negative integer) and as $\boldsymbol{\zeta}_2(\boldsymbol{z}, \tau')$ (satisfying the condition $\boldsymbol{\zeta}_1(\boldsymbol{z}, k\pi) = \boldsymbol{z}^\star(\boldsymbol{z})$ at the same times). The two functions $\boldsymbol{\zeta}_{1,2}(\boldsymbol{z}; \tau')$ are roots of the quadratic equation in $\boldsymbol{\zeta}$:

$$
\begin{aligned}
\boldsymbol{P}(\boldsymbol{z}, \tau' \mid \boldsymbol{\zeta}) \ = \ & \underbrace{[(\boldsymbol{z} - \boldsymbol{w}_2^\star)\ \boldsymbol{l}' + \boldsymbol{\tau}\ (\boldsymbol{z} - \boldsymbol{w}_2)\ \boldsymbol{l}'']}_{\boldsymbol{v}_0} \boldsymbol{\zeta}^2 - 2 \underbrace{[(\boldsymbol{z} - \boldsymbol{w}_2^\star)\ \boldsymbol{m}' + \boldsymbol{\tau}\ (\boldsymbol{z} - \boldsymbol{w}_2)\ \boldsymbol{m}'']}_{\boldsymbol{v}_1} \boldsymbol{\zeta} + \\
& + \underbrace{[(\boldsymbol{z} - \boldsymbol{w}_2^\star)\ \boldsymbol{n}' + \boldsymbol{\tau}\ (\boldsymbol{z} - \boldsymbol{w}_2)\ \boldsymbol{n}'']}_{\boldsymbol{v}_2} = 0 \, ,
\end{aligned}
\tag{14}
$$

the coefficients of which are given by the following formulae:

$$\begin{array}{lcllcl} \boldsymbol{l}' &=& \boldsymbol{\alpha}\boldsymbol{z}-\boldsymbol{\beta} & \boldsymbol{l}'' &=& \boldsymbol{d}_0\boldsymbol{z}-\boldsymbol{d}_1 \\ \boldsymbol{n}' &=& \boldsymbol{w}_2^\star(\boldsymbol{\beta}\boldsymbol{z}-\boldsymbol{\gamma}) & \boldsymbol{n}'' &=& \boldsymbol{w}_2(\boldsymbol{d}_1\boldsymbol{z}-\boldsymbol{d}_2) \\ \boldsymbol{m}' &=& (\boldsymbol{w}_2^\star\boldsymbol{l}'+\boldsymbol{n}'/\boldsymbol{w}_2^\star)/2 & \boldsymbol{m}'' &=& (\boldsymbol{w}_2\boldsymbol{l}''+\boldsymbol{n}''/\boldsymbol{w}_2)/2\ . \end{array}$$

It is worth noticing that in a fixed time τ' (being fixed, time dependences are omitted) the polynomial $\boldsymbol{P}$ satisfies the symmetry property:

$$\boldsymbol{P}(\boldsymbol{z}\mid\boldsymbol{\zeta})=\boldsymbol{P}(\boldsymbol{\zeta}\mid\boldsymbol{z})\ , \tag{15}$$

which comes from the corresponding property of $\boldsymbol{H}$. Moreover, their roots verify the identity:

$$\boldsymbol{P}[\boldsymbol{\zeta}_1(\boldsymbol{z})\mid\boldsymbol{\zeta}_2(\boldsymbol{z})]\equiv\boldsymbol{P}[\boldsymbol{\zeta}_2(\boldsymbol{z})\mid\boldsymbol{\zeta}_1(\boldsymbol{z})]\equiv 0\ , \tag{16}$$

for any $\boldsymbol{z}$. It will be called *reciprocity*.

3.1 *Behaviour of the roots $\boldsymbol{\zeta}_{1,2}$ as functions of $\boldsymbol{z}$. Critical times $\tau^\star_{1,4}$.*

At any fixed time τ' the roots $\boldsymbol{\zeta}_{1,2}(\boldsymbol{z},\tau')$ lie on the two curves $\mathcal{L}_{1,2}(\tau')$ for $\boldsymbol{z}$ running on $\mathcal{C}$. Due to the periodicity of $\boldsymbol{\tau}$, these curves move in a periodic way, with a period π in time. A sample sequence of configurations assumed by these curves at different times inside the period $[0,\pi)$ is shown in Fig. 1. $\mathcal{L}_1$ reduces to the point $\boldsymbol{w}_2^\star$ at the time 0, when $\mathcal{L}_2$ lies on the circle $\mathcal{C}^\star=\boldsymbol{\zeta}^\star(\mathcal{C})$ inside $\mathcal{C}$. $\mathcal{L}_2$ lies inside $\mathcal{C}$ up to the time $\tau_1^\star\simeq 0.557\pi$, when it touches $\mathcal{C}$, while $\mathcal{L}_1$ becomes a non-simple one. At later times the curve intersecates $\mathcal{C}$ in the two points: $\exp(\boldsymbol{i}\ \theta^\star_{1,2})=:\boldsymbol{\delta}_{1,2}$, see Fig. 2. Both curves $\mathcal{L}_{1,2}$ remain separated up to the time $\tau_2^\star\simeq 0.652\pi$, when they merge in a non-simple closed one called $\mathcal{L}$ below. In turn, $\mathcal{L}$ breaks at time $\tau_3^\star\simeq 0.778\pi$, where two closed curves $\mathcal{L}_1$ (non-simple and external to $\mathcal{C}$) and $\mathcal{L}_2$ (crossing $\mathcal{C}$) reappear. Finally, $\mathcal{L}_2$ returns inside the unit circle at the time $\tau_4'\simeq 0.862\pi$, when also $\mathcal{L}_1$ becomes simple again.

The angles $\theta^\star_{1,2}$ are evaluated in the following way. Assume that in a fixed time (being fixed, time dependences are omitted) $\mathcal{L}_2$ intersecates $\mathcal{C}$ in a point $\boldsymbol{\delta}_1$. It follows that a point $\boldsymbol{\delta}_2\in\mathcal{C}$ exists, such that $\boldsymbol{\zeta}_2(\boldsymbol{\delta}_2)=\boldsymbol{\delta}_1$. The other root $\boldsymbol{\zeta}_1(\boldsymbol{\delta}_2)=:\boldsymbol{\zeta}_0$ lies outside the unit circle. Due to equation (15), if $\boldsymbol{P}(\boldsymbol{\delta}_2\mid\boldsymbol{\delta}_1)=0$, then also $\boldsymbol{P}(\boldsymbol{\delta}_1\mid\boldsymbol{\delta}_2)=0$ holds. As a consequence, $\boldsymbol{\delta}_2=\boldsymbol{\zeta}_2(\boldsymbol{\delta}_1)$, while from equation (16), $\boldsymbol{\zeta}_1(\boldsymbol{\delta}_1)=\boldsymbol{\zeta}_0$. In this way, the intersection points $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ verify the properties: $\boldsymbol{\zeta}_1(\boldsymbol{\delta}_1)=\boldsymbol{\zeta}_1(\boldsymbol{\delta}_2)=\boldsymbol{\zeta}_0$, $\boldsymbol{\zeta}_2(\boldsymbol{\delta}_1)=\boldsymbol{\delta}_2$ and the viceversa $\boldsymbol{\zeta}_2(\boldsymbol{\delta}_2)=\boldsymbol{\delta}_1$. As a consequence of the first property, $\mathcal{L}_1$ intersecates itself in correspondence to the point $\boldsymbol{\zeta}_0$, that will play a key role in the following. It follows also that $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ are the roots of the polynomial evaluated in $\boldsymbol{\zeta}_0$. This fact opens the way to calculate the point $\boldsymbol{\zeta}_0$ as the one for which both roots of the polynomial (14) lie on $\mathcal{C}$.

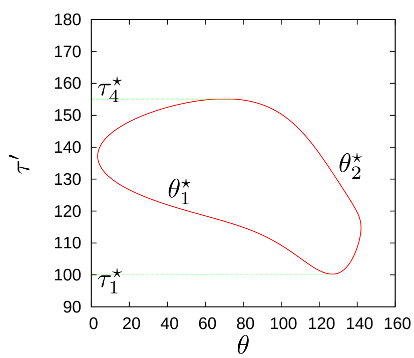

Figure 2: Angles $\theta^\star_{1,2}$ (degrees) vs. τ' (measured in degrees) for the vortex of Fig. 1.

In order to determine the position of the point $\boldsymbol{\zeta}_0$, which is a double point for the curve $\mathcal{L}_2$, the following ideas will be employed. The algebraic equation (14) has the form: $\boldsymbol{v}_0\boldsymbol{\zeta}^2-2\boldsymbol{v}_1\boldsymbol{\zeta}+\boldsymbol{v}_2=0$ (in order to avoid more complicate algebraic structures of the coefficients, this equation is not put in normal form), the coefficients $\boldsymbol{v}_{0,1,2}$ being complex functions of another point ($\boldsymbol{z}$) and of the time

(τ'). If one root $\boldsymbol{\zeta}$ of this equation lies on $\mathcal{C}$, a real number η exists such that $(1+\boldsymbol{i}\eta)/(1-\boldsymbol{i}\eta) = \boldsymbol{\zeta} \in \mathcal{C}$ and the quadratic equation:

$$\underbrace{(\boldsymbol{v}_0 + 2\boldsymbol{v}_1 + \boldsymbol{v}_2)}_{\boldsymbol{v}'_0}\ \eta^2 + 2\ \underbrace{\boldsymbol{i}(\boldsymbol{v}_2 - \boldsymbol{v}_0)}_{\boldsymbol{v}'_1}\ \eta + \underbrace{(-\boldsymbol{v}_0 + 2\boldsymbol{v}_1 - \boldsymbol{v}_2)}_{\boldsymbol{v}'_2} = 0 \tag{17}$$

possesses at least one root. As shown in Fig. 3, in order to satisfy equation (17) the vector $2\boldsymbol{v}'_1\eta =: \boldsymbol{v}''_1\eta$, once it starts from the endpoint of the vector $\boldsymbol{v}'_0\eta^2$ on the bold dashed line must have its endpoint on the origin O. As it is shown by Fig. 3, if the three vectors $\boldsymbol{v}'_0$, $\boldsymbol{v}''_1$ and $\boldsymbol{v}'_2$ are not parallel, only one solution of the equation can lie on the unit circle under suitable constraints on the complex numbers $\boldsymbol{v}'_0$, $\boldsymbol{v}''_1$ and $\boldsymbol{v}'_2$. Indeed, in Fig. 3 the regions which are forbidden to the vector $\boldsymbol{v}'_1$ are indicated: if $\boldsymbol{v}'_1$ (applied to the origin O) belongs to that regions, real solutions are not possible. On the contrary, if such vector lies in the region indicated with $\eta < 0$ solutions are possible only for negative η, as well as in the region $\eta > 0$ only for positive η. The point $\boldsymbol{\zeta}_0$, on the contrary, is such that both roots must lie on $\mathcal{C}$. As a consequence, $\boldsymbol{v}'_0(\boldsymbol{\zeta}_0)$, $\boldsymbol{v}''_1(\boldsymbol{\zeta}_0)$ and $\boldsymbol{v}'_2(\boldsymbol{\zeta}_0)$ must be parallel. By starting from this consideration, a way to collocate $\boldsymbol{\zeta}_0$ (and, as a consequence, the two roots on $\mathcal{C}$) which is based on that property can be developed. It leads to the solution of a quartic equation with real coefficients, which in turn can be interpreted in a geometrical fashion through the intersection of two conical curves.

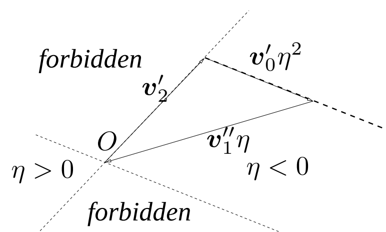

Figure 3: Discussion for the real solutions of equation (17), for non-parallel coefficients $\boldsymbol{v}'_0$, $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$.

3.2 *Opening and closing of the curves $\mathcal{L}_{1,2}$. Critical times $\tau^\star_{2,3}$*

The calculation of the time $\tau^\star_2$ in which $\mathcal{L}_{1,2}$ open and of the time $\tau^\star_3$ in which they close are performed by starting from the definition of the branch points of the functions $\boldsymbol{\zeta}_{1,2}$.
Indeed, when a branch point crosses $\mathcal{C}$, jumps in the square root of the discriminant of equation ({refe15) appear and, as a consequence, the curves $\mathcal{L}_{1,2}$ open.

3.3 *Source term $\boldsymbol{N}$*

The quantity $\boldsymbol{N}$ (10) is rewritten through the using of the equation (13) in the following way:

$$\begin{aligned}\boldsymbol{N}[\boldsymbol{\xi}(\boldsymbol{z});\tau'] \quad = \quad & \frac{1}{\pi\boldsymbol{i}}\int_{\mathcal{C}} d\boldsymbol{\zeta}\ \Big[\ \Big(\frac{\boldsymbol{a}_1}{\boldsymbol{\zeta}-\boldsymbol{z}_1}+\frac{\boldsymbol{a}_2}{\boldsymbol{\zeta}-\boldsymbol{z}_2}\ \Big)+\overline{\boldsymbol{\Delta}}(\tau')\ \Big(\frac{\boldsymbol{A}^{11}_1}{\boldsymbol{\zeta}-\boldsymbol{z}_1}+\frac{\boldsymbol{B}^{11}_2}{\boldsymbol{\zeta}-\boldsymbol{z}^\star_2}+\boldsymbol{C}^{11}\ \Big)\ \Big]\ \times \\ & \times\ \Big(\frac{1}{\boldsymbol{\zeta}-\boldsymbol{\zeta}_1}+\frac{1}{\boldsymbol{\zeta}-\boldsymbol{\zeta}_2}-\frac{1}{\boldsymbol{\zeta}-\boldsymbol{w}^\star_2}-\frac{1}{\boldsymbol{\zeta}-\boldsymbol{z}^\star}\ \Big)\ .\end{aligned} \tag{18}$$

Once the positions of the poles $\boldsymbol{\zeta}_{1,2}$ are evaluated as functions of the point $\boldsymbol{z} \in \mathcal{C}$ and of the time τ', straightforward residue calculation enables us to evaluate $\boldsymbol{N}$ through the use of equation (18). As an example, consider a point $\boldsymbol{z} \in \mathcal{C}$ (and eventually a time τ') such that $\boldsymbol{\zeta}_2(\boldsymbol{z},\tau')$ lies inside $\mathcal{C}$, the source term is:

$$\frac{1}{2}\boldsymbol{N}[\boldsymbol{\xi}(\boldsymbol{z}),\tau'] \quad = \quad -\frac{\boldsymbol{a}_1}{\boldsymbol{\tau}(\tau')+1}\,\frac{1}{\boldsymbol{\zeta}_1(\boldsymbol{z},\tau')-\boldsymbol{z}_1} - \frac{\boldsymbol{a}_2\boldsymbol{\delta}^2}{\boldsymbol{\alpha}^2(\boldsymbol{z}_2-\boldsymbol{\zeta}_\infty^\star)^2}\,\frac{\boldsymbol{\tau}(\tau')}{\boldsymbol{\tau}(\tau')+1}\,\frac{1}{\boldsymbol{\zeta}_1(\boldsymbol{z},\tau')-\boldsymbol{z}_2^\star}+$$
$$+\frac{\boldsymbol{a}_2}{\boldsymbol{\zeta}_2(\boldsymbol{z},\tau')-\boldsymbol{z}_2}+\frac{\boldsymbol{a}_2}{\boldsymbol{z}_2-\boldsymbol{\zeta}_\infty^\star}\,\frac{\boldsymbol{z}-\boldsymbol{\zeta}_\infty^\star}{\boldsymbol{z}-\boldsymbol{z}_2^\star}-\frac{\boldsymbol{a}_1}{\boldsymbol{z}_1-\boldsymbol{w}_2^\star}\,\frac{1}{\boldsymbol{\tau}(\tau')+1}+$$
$$-\frac{\boldsymbol{a}_2}{\boldsymbol{z}_2^\star-\boldsymbol{w}_2^\star}\,\frac{\boldsymbol{\delta}^2}{\boldsymbol{\alpha}^2(\boldsymbol{z}_2-\boldsymbol{\zeta}_\infty^\star)^2}\,\frac{\boldsymbol{\tau}(\tau')}{\boldsymbol{\tau}(\tau')+1}\,.$$

At the present stage of this reasearch activity, the above form of $\boldsymbol{N}$ is put in equation (12) and then *numerically* integrated in time and, eventually, in $\boldsymbol{z}$: its analytical calculation is under investigation, as discussed in the following section.

4 PRELIMINARY RESULTS AND FUTURE WORK

The numerical evealuation of the first order solution (12) exhibits a close agreement with the results of the numerical simulation through a contour dynamics code. A sample case is shown in Figs. 4-a and b. Analogous agreement is obtained for the other five classes of vortices having Schwarz function with two simple poles, as discussed in the paper [14] where the present first order solutions are calculated numerically.

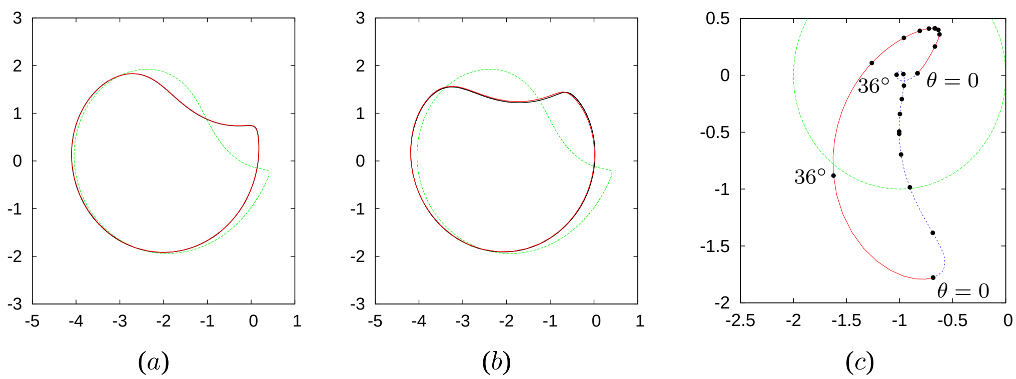

Figure 4: Comparisons between the results at times 0.3 (a) and 0.6 (b) of numerical simulation (black solid lines) and of the present integral (red solid) approach for the vortex of Fig. 1. The time $\tau = 0$ is also shown (green dashed line). The curves described by the branch points $\tilde{\boldsymbol{\tau}}_1$ (blue dashed line) and $\tilde{\boldsymbol{\tau}}_2$ (red solid) for $\boldsymbol{z}$ running on $\mathcal{C}$ are drawn in (c), together with the circle $\mathcal{C}-1$ described by $\boldsymbol{\tau}(\tau')$ in a period.

The present research activity regards the evaluation of the integrals in time in equation (12). They are based on a rewriting of the poles $\boldsymbol{\zeta}_{1,2}$ in the time-form:

$$\boldsymbol{\zeta}_{1,2}(\boldsymbol{z},\tau') = \frac{\boldsymbol{v}_1(\boldsymbol{z},\tau') \mp \boldsymbol{c}(\boldsymbol{z}-\boldsymbol{w}_2)(\boldsymbol{z}-\tilde{\boldsymbol{z}})\sqrt{[\boldsymbol{\tau}-\tilde{\boldsymbol{\tau}}_1(\boldsymbol{z})][\boldsymbol{\tau}-\tilde{\boldsymbol{\tau}}_2(\boldsymbol{z})]}}{\boldsymbol{v}_0(\boldsymbol{z},\tau')}\,. \tag{19}$$

In equation (19), $\boldsymbol{c}$ is a constant and $\tilde{\boldsymbol{z}} = (\boldsymbol{w}_2\boldsymbol{d}_1-\boldsymbol{d}_2)/(\boldsymbol{w}_2\boldsymbol{d}_0-\boldsymbol{d}_1)$. The two branch points $\tilde{\boldsymbol{\tau}}_{1,2}$ are functions of $\boldsymbol{z}$, they move on the curves shown in Fig. 4-c for $\boldsymbol{z}$ running on $\mathcal{C}$. Once the suitable branch of the square root in the formula (19) has been defined, the integrals in τ of the first order solution (12) can be analytically evaluated. This part of the calculation is under investigation at the present time.

A LAPLACE ANTITRANSFORM OF $\boldsymbol{F}(\boldsymbol{y})/(\boldsymbol{y}+\boldsymbol{i}a)$ FOR a REAL

If $\boldsymbol{F}(\boldsymbol{y})$ is the Laplace transform of $\boldsymbol{f}(x)$ and a is an arbitrary real number, we have:

$$\begin{aligned} \frac{d}{dx}\left\{ e^{\boldsymbol{i}ax}\mathcal{L}^{-1}\left[\frac{\boldsymbol{F}(\boldsymbol{y})}{\boldsymbol{y}+\boldsymbol{i}a}\right](x)\right\} &= \frac{1}{2\pi \boldsymbol{i}}\frac{d}{dx}\int_{\mu-\boldsymbol{i}\infty}^{\mu+\boldsymbol{i}\infty} d\boldsymbol{y}\, \frac{\boldsymbol{F}(\boldsymbol{y})}{\boldsymbol{y}+\boldsymbol{i}a}\, e^{(\boldsymbol{y}+\boldsymbol{i}a)\,x} \\ &= e^{\boldsymbol{i}ax}\, \frac{1}{2\pi\boldsymbol{i}}\int_{\mu-\boldsymbol{i}\infty}^{\mu+\boldsymbol{i}\infty} d\boldsymbol{y}\, \boldsymbol{F}(\boldsymbol{y})\, e^{\boldsymbol{y}x} = e^{\boldsymbol{i}ax}\, \boldsymbol{f}(x)\ , \end{aligned}$$

μ being an arbitrary real and positive number. So that by integrating in x, the following relation is obtained:

$$\mathcal{L}^{-1}\left[\frac{\boldsymbol{F}(\boldsymbol{y})}{\boldsymbol{y}+\boldsymbol{i}a}\right](x) = e^{-\boldsymbol{i}ax}\left\{\int_0^x d\xi\, \boldsymbol{f}(\xi)\, e^{\boldsymbol{i}a\xi} + \mathcal{L}^{-1}\left[\frac{\boldsymbol{F}(\boldsymbol{y})}{\boldsymbol{y}+\boldsymbol{i}a}\right](0)\right\}\ . \tag{20}$$

But the second term in the right hand side is zero. Indeed, by exchanging the singular and the regular integrals that term becomes:

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{\boldsymbol{F}(\boldsymbol{y})}{\boldsymbol{y}+\boldsymbol{i}a}\right](0) &= \frac{1}{2\pi\boldsymbol{i}}\int_{\mu-\boldsymbol{i}\infty}^{\mu+\boldsymbol{i}\infty} d\boldsymbol{y}\, \frac{\boldsymbol{F}(\boldsymbol{y})}{\boldsymbol{y}+\boldsymbol{i}a} \\ &= \frac{1}{2\pi\boldsymbol{i}}\int_{\mu-\boldsymbol{i}\infty}^{\mu+\boldsymbol{i}\infty} \frac{d\boldsymbol{y}}{\boldsymbol{y}+\boldsymbol{i}a}\int_0^{+\infty} d\xi\, \boldsymbol{f}(\xi)\, e^{-\boldsymbol{y}\xi} \\ &= \int_0^{+\infty} d\xi\, \boldsymbol{f}(\xi)\, \frac{1}{2\pi\boldsymbol{i}}\int_{\mu-\boldsymbol{i}\infty}^{\mu+\boldsymbol{i}\infty} d\boldsymbol{y}\, \frac{e^{-\boldsymbol{y}\xi}}{\boldsymbol{y}+\boldsymbol{i}a}\ , \end{aligned} \tag{21}$$

where it is important that $\xi > 0$, in the last integral. It results to be zero, as easily proved by integrating on the rectangle in Fig. 5 the function $\boldsymbol{F}(\boldsymbol{y})/(\boldsymbol{y}+\boldsymbol{i}a)$. One obtains in the limit $M \to +\infty$:

$$\frac{1}{2\pi\boldsymbol{i}}\int_{\mu-\boldsymbol{i}\infty}^{\mu+\boldsymbol{i}\infty} d\boldsymbol{y}\, \frac{e^{-\boldsymbol{y}\xi}}{\boldsymbol{y}+\boldsymbol{i}a} + \frac{1}{2\pi\boldsymbol{i}}\int_{-\mu+\boldsymbol{i}\infty}^{-\mu-\boldsymbol{i}\infty} d\boldsymbol{y}\, \frac{e^{-\boldsymbol{y}\xi}}{\boldsymbol{y}+\boldsymbol{i}a} = e^{\boldsymbol{i}a\xi}\ .$$

From the above relation the first integral is evaluated through a change of variable in the second integral (from $\boldsymbol{y}$ to $\boldsymbol{y}' = -\boldsymbol{y}$), which leads to the following result:

$$\frac{1}{2\pi\boldsymbol{i}}\int_{\mu-\boldsymbol{i}\infty}^{\mu+\boldsymbol{i}\infty} d\boldsymbol{y}\, \frac{e^{-\boldsymbol{y}\xi}}{\boldsymbol{y}+\boldsymbol{i}a} = e^{\boldsymbol{i}a\xi} - \frac{1}{2\pi\boldsymbol{i}}\int_{\mu-\boldsymbol{i}\infty}^{\mu+\boldsymbol{i}\infty} d\boldsymbol{y}'\, \frac{e^{\boldsymbol{y}'\xi}}{\boldsymbol{y}'-\boldsymbol{i}a}\ ,$$

which vanishes. The relation (21) now gives:

$$\mathcal{L}^{-1}\left[\frac{\boldsymbol{F}(\boldsymbol{y})}{\boldsymbol{y}+\boldsymbol{i}a}\right](0) = 0\ . \tag{22}$$

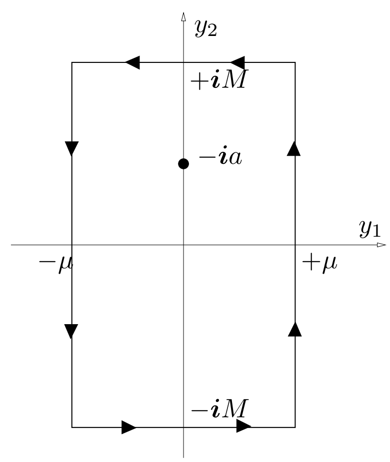

Figure 5: Integration path for the function $\boldsymbol{F}(\boldsymbol{y})/(\boldsymbol{y}+\boldsymbol{i}a)$.

Once the estimate (22) has been accounted for, the required Laplace antitransform follows by equation (20) as:

$$\mathcal{L}^{-1}\left[\frac{\boldsymbol{F}(\boldsymbol{y})}{\boldsymbol{y}+\boldsymbol{i}a}\right](x) = e^{-\boldsymbol{i}ax}\int_0^x d\xi\ \boldsymbol{f}(\xi)\ e^{\boldsymbol{i}a\xi}\ . \tag{23}$$

It is worth noting that for $a = 0$ the well known rule is still obtained from equation (23).

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