Vortex formation and growth behind an orifice

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SUMMARY. The understanding of the vortex formation process behind an orifice is currently driving novel attempt to evaluate the fluid dynamic performances of the human heart. The concept of formation time, formerly developed for axially symmetric orifices and tentatively applied to mark the flow across cardiac valves, is here studied in a two-dimensional system. The theoretical inviscid vortex dynamics analysis is completed by numerical simulation of the Navier-Stokes equations. As a result, the inviscid vortex dynamics resolves the initial rapid acceleration of the separated vortex. Afterward, the numerical simulations demonstrate that a finite critical formation time, previously reported for vortex rings, does not exist in two-dimensional vortex formation. This qualitative difference between the two limit cases of axially symmetric and two-dimensional in vortex formation, implies the development of intermediate peculiar phenomena after three-dimensional orifices, and suggest some care before application of asymptotic concepts to real cardiac flow.

1 INTRODUCTION

The blood flow in the human left ventricle is known to develop a vortical motion that should facilitate the ejection of blood into the primary circulation [1, 2]. Although strongly connected with the system efficiency, the vortex dynamics inside the heart is of little diagnostic use and it is not taken into account in the design of therapeutic procedures. The presence of the intraventricular vorticity is central in the overall synchronicity of the beating heart; in fact, the quantification of such a vortex structure is driving novel attempts to improve understanding of cardiac diseases [3, 4, 5].

The diagnostic approach introduced in [3] is based on the ventricular vortex growth following the concept of universal formation time introduced in [6]. This concept, based on experimental evidence, stresses that the vortex rings that develops after an orifice cannot growth indefinitely as a compact structure. Its growth continues until the so called formation number Vt/D, where V is the velocity across the orifice, D its diameter, and t is time, is below a critical value; afterwards, the leading vortex stops to grow and secondary vortices develops in its wake. The transfer of this concept into the left ventricle [3, 5] suggests that an optimized vortex development must be such that the formation number is some below the critical value to avoid generation of incoherent, weakly turbulent, cardiac flow. The effective existence of such a critical formation number as a universal law emerging from vortex dynamics has been analyzed in a series of papers after its introduction [7, 8, 9]. In extreme synthesis, the forming vortex ring is continuously fed by the rolling-up vortex layer (of strength $\sim V$ and velocity $\sim V/2$) separating from the orifice edge. Therefore the circulation of the vortex ring grows with a rate proportional to V^2, and its self-induced translation velocity rises as well until it exceeds the velocity of the separating shear layer (*pinch-off*), at that point the newly separated vorticity cannot reach the vortex and eventually rolls-up in its wake [8].

Many cardiovascular valves present a tapered shape that could be more similar to a two-dimensional opening than a circular orifice. Despite this, the vortex formation process has received less attention in two-dimensional systems, when a vortex ring is replaced by a symmetric vortex pair. Two-dimensional vortex pairs still have a self-induced velocity, therefore the basic concept at the base of the formation process is commonly assumed to be analogous [10]. On the other side, 2D vortex dynamics does not present the vortex stretching and enstrophy increase due to the radial displacement of circular vortex lines, and allows, roughly speaking, a less constrained vortex motion. The dynamics of a two-dimensional vortex pair generated by a piston pushing fluid out of an orifice was analyzed in [11] by an inviscid vortex method, often known as the Brown & Michael model [12]. It showed that the separation and strength of the vortices strongly depend upon the time history of the piston motion although this dependence was not addressed and attention was posed on the formation of secondary vortices that develop when the piston is stopped. An experimental study of the formation of a vortex dipole from an orifice is reported in [13] where, unexpected by that author, the pinch-off was not observed in any of the experiments.

The present study analyzes two-dimensional vortex formation from sharp-edge orifices, and verifies whether the critical formation time concept can be extended or is modified in two-dimensions. The analysis is first performed theoretically, using techniques of inviscid vortex dynamics, to uncover the fundamental phenomena involved. It is then progressed by numerical integration of the Navier-Stokes equations in order to account for the extended distribution of the separated vorticity.

2 INVISCID SINGLE VORTEX MODEL

Consider a two-dimensional incompressible flow across an orifice, take the orifice half-width $\mathcal{D}/2$ and the peak velocity $\mathcal{V}$ as unit scales of length and velocity, respectively. The dimensionless problem thus presents an orifice width $d = 2$ units and the velocity across the orifice $V(t)$ has unit peak velocity. The orifice has two sharp edges at the coordinate points $z_e = \pm i$ of the complex coordinate plane $z = x + iy$. Using the conformal transformation $\zeta(z) = F(z)$, that for the case of a symmetric orifice in unbounded space is (see, for example, [14], Art.66)

$$\zeta = F(z) = \frac{2}{\pi} \log\left(iz + i\sqrt{z^2 + i}\right) - i, \qquad z = F^{-1}(\zeta) = -i \cosh\left(\frac{2}{\pi}(\zeta + i)\right); \quad (1)$$

the physical plane is mapped onto a transformed plane, ζ, having the the geometry of a rectilinear duct where the sharp edges are mapped on the opposite facing walls at $\zeta_e = \pm i$.

In the limit of non-zero, vanishing viscosity the flow can be assumed as everywhere irrotational with the exception of 2 symmetric singular vortices, at $z_v(t)$ and $z_v^*(t)$, where * stands for complex-conjugate, of circulation $\Gamma(t)$ and $-\Gamma(t)$, respectively. The singular vortices represent a minimal model where the rolling-up vortex sheets separated from the sharp edges are concentrated at the vorticity centers [12]. The flow that satisfies the impermeability boundary conditions at the walls can be immediately evaluated in the transformed space where the complex potential is given by

$$W(\zeta, t) = V\zeta + i\frac{\Gamma}{2\pi} \log\left\{\frac{\sin\frac{i\pi}{4}(\zeta - \zeta_v)\sin\frac{i\pi}{4}(\zeta - \zeta_v + 2i)}{\sin\frac{i\pi}{4}(\zeta - \zeta_v^*)\sin\frac{i\pi}{4}(\zeta - \zeta_v^* + 2i)}\right\}; \quad (2)$$

and $\zeta_v = F(z_v)$ is the position of the vortex in the transformed space. The potential (2) accounts for a periodic vortex pair along the imaginary axis that satisfies the boundary condition on either walls. The flow potential in the physical space is $W(z) = W(\zeta(z))$, and the complex-conjugate velocity at a generic point z is $v^*(z) = \frac{dW}{dz} = \frac{dW}{d\zeta}F'$ is, from (2),

$$v^*(z) = \left\{V + \frac{\Gamma}{4}\left[\cot\frac{i\pi}{2}\left(\zeta(z) - \zeta_v^*\right) - \cot\frac{i\pi}{2}\left(\zeta(z) - \zeta_v\right)\right]\right\}F'(z). \quad (3)$$

The dynamic system specification seeks for the evolution of vortex position $z_v(t)$ and intensity $\Gamma(t)$ for a given flow $V(t)$ across the orifice. The dynamical equation of vortex evolution is the zero-force equation [12] (see also [15], pp.117-118)

$$\Gamma \frac{dz_v}{dt} + (z_v - z_e)\frac{d\Gamma}{dt} = \Gamma v(z_v). \tag{4}$$

The computation of the local velocity, $v(z)$, appearing in (4), at the vortex position is calculated by the Routh rule [16] to properly eliminate the singular vortex self-induction in physical space

$$\begin{aligned}
v^*(z_v) &= \lim_{z\to z_v} \frac{d}{dz}\left\{W - \frac{i\Gamma}{2\pi}\log(z - z_v)\right\} \\
&= \lim_{z\to z_v} \frac{d}{d\zeta}\left\{W - \frac{i\Gamma}{2\pi}\log(\zeta - \zeta_v)\right\} F' + \frac{i\Gamma}{2\pi}\lim_{z\to z_v}\frac{d}{dz}\log\left(\frac{\zeta-\zeta_v}{z-z_v}\right) \\
&= \left\{V + \frac{\Gamma}{4}\cot\frac{i\pi}{2}(\zeta_v - \zeta_v^*)\right\} + \frac{i\Gamma}{4\pi}\frac{F''}{F'}\bigg|_{z_v}.
\end{aligned} \tag{5}$$

In the last passage of (5) the fact that the periodic extension of the self-induction term is zero by symmetry has also been used. The Routh rule derives from the proper evaluation of the limit

$$\lim_{z\to z_v}\frac{d}{dz}\log\left(\frac{\zeta-\zeta_v}{z-z_v}\right) = \frac{F''}{2F'}\bigg|_{z_v};$$

it can be demonstrated by substitution of the numerator of the logarithm argument with its expansion in Taylor series $(\zeta - \zeta_v) \simeq F'(z_v)(z - z_v) + \frac{1}{2}F''(z_v)(z-z_v)^2$, or by a centered estimation of the logarithm argument $\frac{\zeta-\zeta_v}{z-z_v} \simeq F'\left(\frac{z+z_v}{2}\right)$; both second order accurate when $z \to z_v$.

In the limit of a viscous flow with vanishing viscosity the boundary layer is everywhere of zero thickness and detaches from the sharp edge to ensure the physical requirement, known as Kutta condition, that the velocity $v^*(z) = \frac{dW}{d\zeta}F'$ must be finite at the sharp edge. The transformation F' is singular at the sharp edges and the Kutta condition reads $\frac{dW}{d\zeta}\Big|_{z_e} = 0$ that becomes, using (3),

$$\Gamma = 4V\frac{\cos\frac{i\pi}{2}\zeta_v \cos\frac{i\pi}{2}\zeta_v^*}{\sin\frac{i\pi}{2}(\zeta_v - \zeta_v^*)}. \tag{6}$$

The evolution of the separated vortex pair can thus be performed by (4, 5, 6).

The mathematical formulation can be progressed further by time differentiation of (6)

$$\frac{d\Gamma}{dt} = \left[\frac{d\Gamma}{d\zeta_v}F'(z_v)\right]\frac{dz_v}{dt} + \left[\frac{d\Gamma}{d\zeta_v}F'(z_v)\right]^*\frac{dz_v^*}{dt} + \frac{\Gamma}{V}\frac{dV}{dt},$$

that shows how it depends on evolution of the vortex position and of its complex-conjugate. Placing equation (4) in a system with its complex-conjugate version, inserting the expression for $\frac{d\Gamma}{dt}$, and solving for $\frac{dz_v}{dt}$ eventually leads to a single evolution equation for the vortex position

$$\frac{dz_v}{dt} = \frac{\left[\frac{d\Gamma}{d\zeta_v}F'\right]^*(z_v - z_e)^* v - \left[\frac{d\Gamma}{d\zeta_v}F'\right]^*(z_v - z_e)v^* + \Gamma v}{\left[\frac{d\Gamma}{d\zeta_v}F'\right]^*(z_v - z_e)^* + \left[\frac{d\Gamma}{d\zeta_v}F'(z_v)\right](z_v - z_e) + \Gamma}; \tag{7}$$

with F' and v the values at the vortex $F'(z_v)$ and $v(z_v)$ are intended and Γ is the function of z_v given by (6).

Inspection of (7), and of the functions of z_v contained therein (5, 6), shows that it can be normalized with respect to the external forcing $V(t)$. Define the normalized circulation and the normalized velocity

$$\gamma = \frac{\Gamma}{V}, \qquad u(z_v) = \frac{v(z_v)}{V} \tag{8}$$

both functions of z_v only (see 6, 5) that do not contain a dependence on $V(t)$ anymore, and define the *formation time*

$$\tau = \frac{1}{2}\int_0^t V(t)dt; \tag{9}$$

(the factor $\frac{1}{2}$ is included to agree with the definition in the literature), equation (7) can be rewritten in terms of the normalized variables

$$\frac{dz_v}{d\tau} = 2\frac{\left[\frac{d\gamma}{d\zeta_v}F'\right]^*(z_v - z_e)^* u - \left[\frac{d\gamma}{d\zeta_v}F'\right]^*(z_v - z_e)u^* + \gamma u}{\left[\frac{d\gamma}{d\zeta_v}F'\right]^*(z_v - z_e)^* + \left[\frac{d\gamma}{d\zeta_v}F'(z_v)\right](z_v - z_e) + \gamma}. \tag{10}$$

Equation (10) is an evolutionary function for z_v, where the right hand side does not depend on the external forcing $V(t)$. It is therefore a universal equation for the formation of a vortex pair. The solution $z_v(\tau)$ includes implicitly all the solutions corresponding to different profiles $V(t)$ that are recovered simply by re-scaling (8,9). It shows that vortex formation can be described in terms of τ only, it therefore provides a mathematical support for the fundamental significance of the formation time and for the universality of vortex formation events independently from the details of $V(t)$.

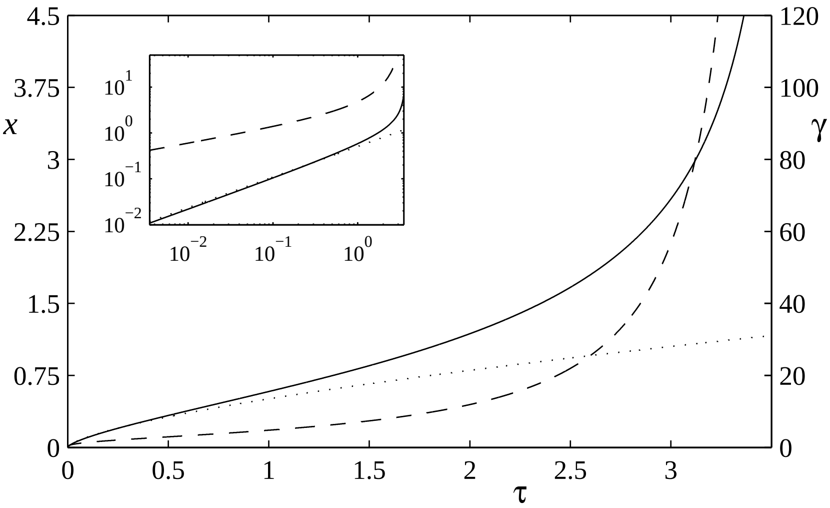

Figure 1: Vortex formation after a symmetric orifice. Time evolution of the normalized vortex x-position (continuous line) and of the normalized circulation $\gamma(\tau)$ (dashed line), the dotted line is the short-time asymptotic soluyion. The inset shows the same graph with logarithmic scales.

3 INVISCID MODEL RESULTS

Equation (10) can be integrated numerically by standard techniques. It presents a difficulty just at the initial condition, $z_v(0) = z_e$, where the equation is singular but can be integrated analytically to give the asymptotic small-time solution $z_v(t) = \pm i + 2^{\frac{2}{3}}\tau^{\frac{2}{3}}/\pi$ that is used to start the integration.

The time profile of the general solution is reported in figure 1, and the corresponding flow field at two instants is shown in figure 2. After the initial self-similar evolution, the external length scale due to the orifice size comes into play and the self-induced motion of the vortex pair becomes significant. While the vortex pair moves away from the orifice its circulation must grow to enforce the Kutta condition, the growth of circulation is progressively faster leading to a continuously accelerating vortex motion.

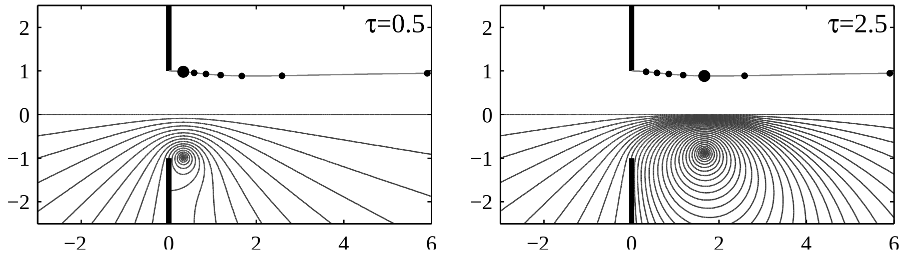

Figure 2: Vortex formation after a symmetric orifice. Streamlines (lower-half) and vortex trajectory (upper half) corresponding to $\tau = 0.5$ and $\tau = 2.5$. Stream-function levels are equispaced from 0 to 1 and logarithmic-spaced for higher values to cope with the singular behavior. The ticks on the vortex trajectories are spaced of $\Delta\tau = 0.5$, the larger tick indicates the current position.

This model does not include the possibility of vortex break-down and cannot give answers about the existence or the value of the critical formation time. The Brown & Michael approximation for the rolling-up vortex sheet is acceptable until the vortex sheet maintains the shape of an approximately circular spiral. Here, when the vortex pair translates downstream remaining attached to the sharp edges, the rolling-up vortex spiral becomes more deformed and extended longitudinally, and the singular model becomes a less and less a realistic representation of the separated wake. Looking at the right pictures in figure 2, it is evident that the vortex-induced circulating streamlines are hardly consistent with those of a longitudinally extended vortex structure.

Therefore the single vortex model contains the basic elements of the early vortex formation process, witnessing the universal significance of the formation time concept. On the other side it does not permit to draw any conclusion on the following phenomena that require to properly account for distributed nature of the vorticity field.

4 VISCOUS NUMERICAL SIMULATION

The vortex formation process after a sharp orifice, corresponding to the geometry (1), is here studied numerically by solution of the two-dimensional Navier-Stokes equation written in the vorticity-streamfunction, $\omega - \psi$, formulation in physical space

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{2}{Re}\nabla^2\omega; \tag{11}$$

with the Poisson relation $\nabla^2\psi = -\omega$, and the Reynolds number defined by $Re = \mathcal{VD}/\nu$, ν being the kinematic viscosity. The domain extends in the range $-\infty < x < +\infty$ and $-\infty < y \leq 0$, $y = 0$ being a symmetry axis, and a rigid wall of zero thickness is located on the plane $x = 0$ extending infinitely downward from the sharp edge at $y = -1$.

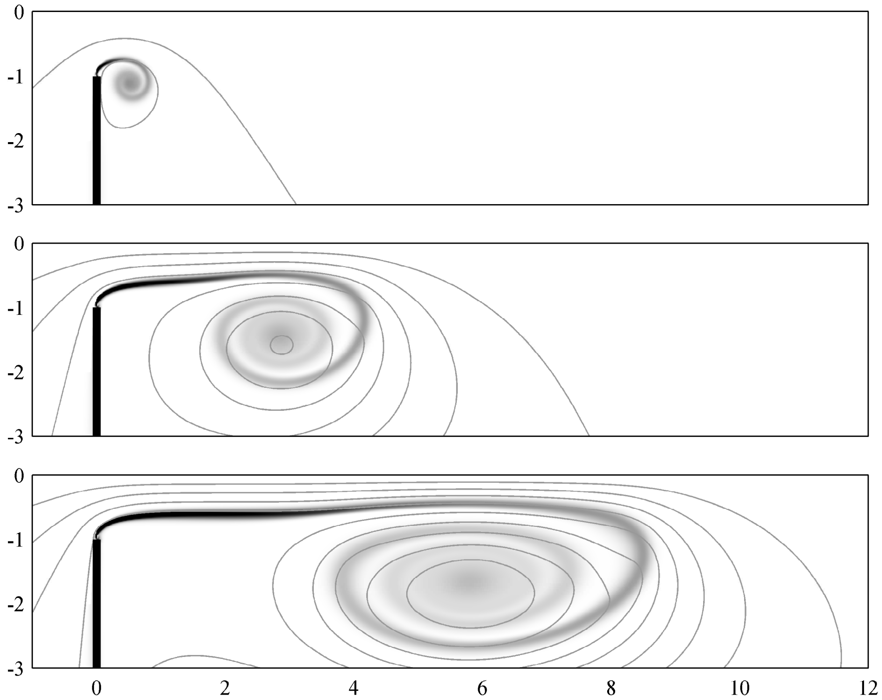

Figure 3: Vortex formation at the exit of a two-dimensional orifice at $Re = 2000$ for an impulsively started flow $V = 1$, $\tau = 1, 5, 10$. Distribution of the vorticity (white is $\omega = 0$, black is $\omega = 10$), and streamlines.

The numerical method is a standard one. Cartesian coordinates are made discrete using second-order centered finite differences with constant grid-spacing. In order to facilitate the imposition of the boundary conditions on the finite computational domain, the streamfunction is subdivided into the sum of the known irrotational contribution ([14], Art.66) and a correction that satisfies the Poisson equation with homogeneous boundary conditions at all boundaries. The vorticity is evaluated at the two sides of the wall by imposition of the no-slip condition. Further details on the numerical method can be found in [17] from which the numerical code has been extracted after simplifications.

The numerical solution for the case of a constant flow, $V(t) = 1$, at $Re = 2000$ is shown in figure 3 at three different times. The vortex dynamics does not present the phenomenon of pinch-off, rather it tends to an apparently steadily growing solution. This solution is qualitatively different from that found in the case of axially symmetric vortex formation. In order to permit a careful comparison, the axisymmetric solution is computed with exactly the same numerical code just changing the equations to cylindrical axially symmetric, and the basic irrotational solution ([14], Art.107-108).

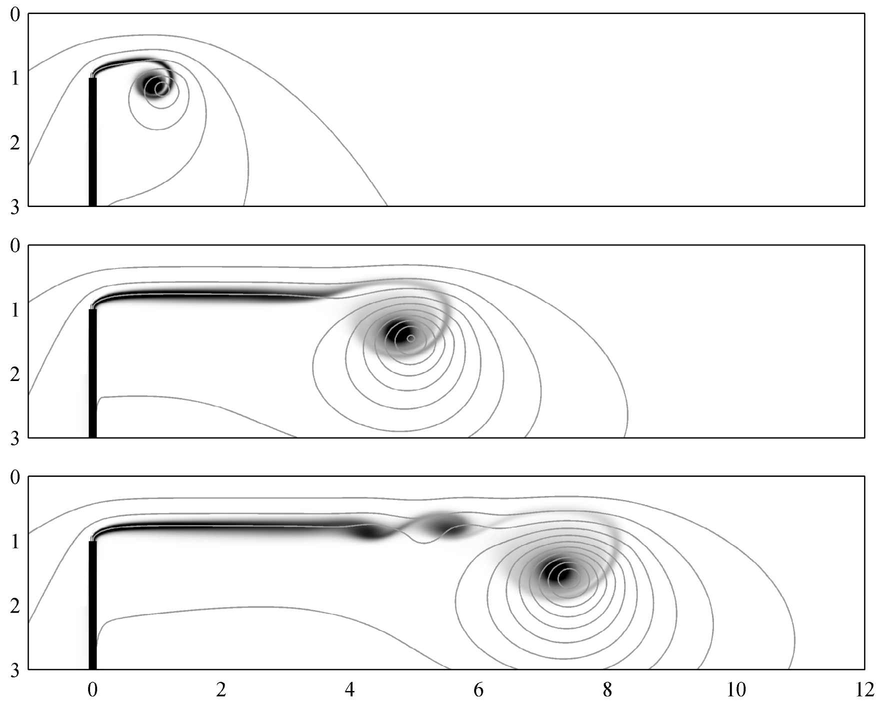

Figure 4: Vortex formation at the exit of a circular orifice at $Re = 2000$ for an impulsively started flow $V = 1$, $\tau = 1, 3.5, 5$. Distribution of the vorticity (white is $\omega = 0$, black is $\omega = 10$), and streamlines.

The corresponding axisymmetric solution for the case of a constant flow, $V(t) = 1$, at $Re = 2000$, is shown in figure 4. The behavior is in perfect agreement with experimental and numerical literature results [6, 7]: the vortex ring increases its velocity until, at a critical value of the formation time $\tau_{cr} \simeq 4$, it is larger than that of the separating vortex sheet that rolls-up in a novel trailing vortex.

Analogous results of the two-dimensional vortex formation are found in correspondence of different time profiles of the external forcing $V(t)$, once properly scaled with the formation time. They are summarized in figure 5 that reports the time evolution of the vortex velocity, normalized with the free-stream velocity $V(t)$. In general, the two-dimensional vortex formation process presents a continuous growth without any critical bound. The vortex translates downstream, with a velocity that initially increases, as suggested by the inviscid model, but then settles to a value well below the vortex sheet velocity (that is about one half the free-stream velocity); therefore the leading vortex is always slower than the separating vortex sheet. The vortex sheet is able to continuously feed the vortex with new vorticity that does not concentrate into the vortex core but contributes to increase the longitudinal size of the extended vortex (at a rate proportional to V). The average vorticity value remains about constant (beside viscous diffusion effects), and circulation grows with the vortex size.

The two-dimensional vortex formation process does not reveal the pinch-off phenomenon that is present in axially symmetric vortex ring formation. This result is in agreement with the experiments [13] at lower Reynolds number. The behavior at larger Reynolds number ($Re = 5000$) is more prone

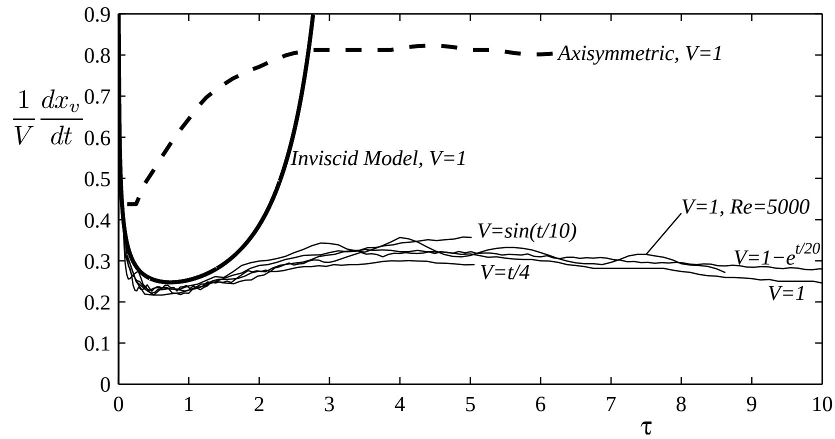

Figure 5: Time evolution of the vortex velocity, normalized by the free-stream velocity ($\frac{1}{V}\frac{dx_v}{dt}$) for the five numerical solutions, the thick line represent the single vortex solution, the thick dashed line is the small time asymptotic solution.

to appearance of Kelvin-Helmholtz instability of the thin vortex sheet, analogous to that observed for a single vortex [18], that eventually destroys the coherence of the vortex structure.

5 CONCLUSION

The vortex formation process is studied in two-dimensional flow after a sharp orifice. In the inviscid point vortex model the evolutionary equation can be written in self-similar terms, independently from the external velocity profile, where the *formation time* naturally arises as the quantity that describes vortex formation at a fundamental level.

The single vortex model is an acceptable approximation for the initial formation phase, while the vortex structure is a nearly circular spiraling vortex sheet. It shows that the vortex pair initially grows as two independent vortices, and then accelerates because of the self-induced velocity. Afterward, when the vortex is expected to become spatially extended, the single vortex model is unable to properly represents the effective vortex motion giving rise to a unrealistic unbounded accelerated dynamics.

The numerical solutions of the two-dimensional Navier-Stokes equations show that the vortex pair tends to a stably growing evolution: it translates downstream, extends longitudinally, and remains connected to the sharp edge by a vortex sheet that continuously feeds the vortex with new vorticity that does not accumulate in the vortex center but increases the vortex circulation by increasing its longitudinal extension only.

A critical formation time cannot be defined for two-dimensional vortex formation after orifices. The qualitatively different behavior between the formation of vortex rings (pinch-off) and two-dimensional vortex pair (stable growth) suggests multiple possibilities in the formation of three-dimensional vortex structures from orifices with variable curvature or with a tapered mouth, as it could be the case of cardiac valves.

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