The dynamic non-linear behaviour of beams with closing cracks

Salvatore Caddemi[1], Ivo Caliò[1], Massimo Marletta[1]

[1]*Department of Civil and Environmental Engineering, University of Catania, Italy*
E-mail: *scaddemi@dica.unict.it, icalio@dica.unict.it, marletta@dica.unict.it*

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SUMMARY. In this study the nonlinear dynamic response of beam in presence of multiple concentrated closing cracks is addressed. The overall behaviour of such a beam is nonlinear due to the opening and closing of the cracks during the dynamic response, however it can be regarded as a sequence of linear phases each of them characterised by different number and positions of the cracks in open state. The response of the beam is evaluated by determining the exact modal properties of the beam, in each linear phase, and evaluating the corresponding time history linear response through modal superposition analysis. Appropriate initial conditions at the instant of transition between two successive linear phases have been considered and an energy control criterion has been enforced with the aim of establishing the sufficient number of modes that must be taken into account in order to obtain suitable results. Some numerical applications are presented in order to investigate the dynamic non-linear behaviour of beams with closing cracks.

1 INTRODUCTION

In the last decades several authors devoted considerable interest in procedures aiming at the identification of the state of damage of a structure by processing its dynamic response. This increased interest has led to improvements of the existing methods as well as to developments of new procedures for the analysis of the dynamic response of damaged structures in terms of both forward and inverse problems. Most of the procedures proposed in the literature are based on the strong assumption that the damaged structure behaves linearly during the dynamic response, however various theoretical and experimental studies have demonstrated that in some cases a state of damage in a structure can cause a nonlinear behaviour in its dynamic response. A relevant problem within the context of the nonlinear response of damaged structure is that concerned with the so called '*closing crack*', i.e. a crack which opens and closes during the dynamic response causing nonlinear structural behaviour. This phenomenon was observed experimentally by Gudmunson [1] while performing dynamic tests in a cantilever beam aimed at correlating the position and the extension of the crack with the measure of the variation in natural frequencies.

There are different approaches for crack modelling in beam structures reported in the literature; a large part of the considered approaches can be attributed to one of the following categories: spring models or elastic hinges, local stiffness reduction, and finite element models. Friswell and Penny in [2] compare some different approaches for crack modelling and demonstrate that, for structural health monitoring using low frequency vibration, simple models of crack flexibility based on beam elements are adequate. The latter paper also addresses the effect of the excitation for the case of closing cracks, where the beam stiffness can be considered bilinear, depending on whether the crack is open or closed. In these cases vibration based identification procedures, based on the simplifying hypothesis that the structure under test behaves linearly, could lead to wrong results about the state of the damage of the structure. Both experimental and theoretical investigations show that the decrease of natural frequencies of beams in presence of closing cracks cannot be described through a model in which it is assumed that the cracks are always open. Furthermore several studies highlighted that in presence of closing cracks there is a significant change of the response spectrum that is characterised by the presence of sub-harmonics typical of non-linear systems. Although the study of the dynamic behaviour of a cracked beams has been investigated by several authors, the great part of the surveys are relative to open cracks, very few researches have been devoted to the study of beams with closing cracks and most of them consider only the presence of a single closing crack, a short comprehensive review of the adopted methods can be found in [3]. The bi-linear behaviour of a beam with a single closing crack was also recognized in [4,5]. In reference [6] Ostachowicz and Krawczuk used the harmonic balance method to determine the response of a cantilever beam with a single closing crack under harmonic excitation taking advantage of the great reduction of calculation time permitted by this technique with respect to numerical integration. Later, Pugno *et al*. [7] extended the latter method to the general case of several breathing cracks and, furthermore, by introducing a smooth crack closure.

In this study the problem of the evaluation of the nonlinear dynamic response of a beam under a generic excitation in presence of multiple concentrated closing cracks is addressed. The authors exploit a closed-form expression provided for the mode-shapes of a beam with an arbitrary number of open cracks [8]. In fact, the overall behaviour of a beam with several closing cracks can be regarded as a sequence of linear phases, each of them characterised by different number and positions of the cracks in the open state. Therefore, in this paper the response of the beam with closing cracks, is evaluated by determining the modal properties of the beam in each linear phase and calculating the time history responses through modal superposition analysis. Appropriate initial conditions at the instant of transition between two successive linear phases have been considered and an energy control has been enforced in order to establish the minimum number of modes that must be taken into account in order to obtain suitable results. Some numerical applications are presented to illustrate the proposed approach for beams with multiple closing cracks under different boundary conditions both for harmonic loading. In order to compare the proposed approach with other accurate results reported in the literature, the harmonic responses of two-cracked cantilever steel beams reported by Pugno *et al.* in [7] have been considered.

2 FORMULATION OF THE PROBLEM

The considered model is represented by an Euler-Bernoulli vibrating beam, of length L and uniform mass per unit length m, in presence of multiple concentrated closing cracks, with general boundary restraint conditions subjected to a generic load function $p(x,t)$. The basic concept adopted in this study is that the concentrated cracks may be open or closed; when the generic crack is open, it affects locally the flexural stiffness of the beam and its influence can be modelled by means of generalised functions. The adoption of generalised functions to treat singularities in the flexural stiffness both in the context of static and stability analyses has been previously considered by the authors in [9-12]. According to the latter model, if a finite number of open cracks N_c are considered along the span of the beam at abscissas, i=1, 2, ... , N_c, punctual reductions of the stiffness are introduced, so that the following expression of uniform flexural stiffness with Dirac's delta singularities is adopted to treat the concentrated open cracks:

$$EI(\xi) = E_o I_o \left[1 - \sum_{i=1}^{N_c} \gamma_i \cdot \delta\left(\xi - \xi_{o,i}\right)\right] \tag{1}$$

where $E_o I_o$ is the flexural stiffness of the uniform undamaged beam, $\xi = x/L$ is a normalised abscissa, γ_i is a dimensionless damage intensity parameter, and $\delta(\xi)$ is the Dirac's delta function. The exact explicit expressions of the vibration modes and the corresponding frequencies of a multi-cracked beam with open cracks has been presented in [8]. Here, the above mentioned solution is employed for analysing, through modal analysis, the nonlinear dynamic response of beams with closing cracks. In this context, a closing crack is intended as a crack that is open for a given sign of the curvature of the beam in the current position and is closed otherwise. In such a system the variability of the stiffness of each crack, associated to its state (closed or open), can be conveniently described by the following flexural stiffness model

$$EI(\xi, \mathbf{b}) = E_o I_o \left[1 - \sum_{i=1}^{N_c} b_i \cdot \gamma_i \cdot \delta\left(\xi - \xi_{o,i}\right)\right] \tag{2}$$

where the *i*-th component b_i of the state vector $\mathbf{b}$ is assumed equal to 1, if the *i*-th crack is open, or zero, if the integrity of the cross-section is assumed as the fracture surfaces close.

According to the flexural stiffness model represented by Eq. (2), the dynamic differential equation of the Euler-Bernoulli beam with an arbitrary number of closing cracks subjected to a general transversal load distribution $p(\xi,t)$ can be written as

$$\left\{\left[1 - \sum_{i=1}^{N_c} b_i\, \gamma_i\, \delta\left(\xi - \xi_{o,i}\right)\right] u''(\xi,t)\right\}'' + \frac{mL^4}{E_o I_o}\ddot{u}(\xi,t) = \frac{L^4}{E_o I_o}\, p(\xi,t) \tag{3}$$

where the apex indicates differentiation with respect to the normalised abscissa ξ, and the dot indicates differentiation with respect to time *t*.

In order to obtain the time-history response of equation (3) through the modal analysis, the eigen-properties of the beam in a generic state, identified by the Boolean vector $\mathbf{b}$, must be evaluated.

2.1 *Eigen-values of the beam in a generic crack configuration*

The classical mode shapes and the corresponding frequencies of the beam subjected to a generic cracked configuration can be evaluated by considering the dynamic differential equation that governs the free vibration of the beam that corresponds to Eq. (3) without the contribution loading term. The solution of the free vibration problem can be obtained with the use of separation of variables

$$u(\xi,t) = y(t)\;\; \phi(\xi) \tag{4}$$

Substitution of equation (4) into the free vibration equation of motion yields to the following differential equation for the modal displacements that, after some simple algebraic manipulation, can be written in the form:

$$\left\{\left[1 - \sum_{i=1}^{N_c} b_i \cdot \gamma_i \cdot \delta\left(\xi - \xi_{o,i}\right)\right] \phi''(\xi)\right\}'' - \alpha^4\, \phi(\xi) = 0 \tag{5}$$

where the frequency parameter $\alpha^4 = \omega^2 mL^4 / (E_o I_o)$ has been introduced.

Equation (5), by performing double differentiation with respect to ξ of the first term containing the Dirac's delta distribution, and after simple algebra, may be given the following form:

$$\phi^{IV}\left(\xi\right)-\alpha^4\phi\left(\xi\right)=B\left(\xi\right) \tag{6}$$

where the function $B\left(\xi\right)$ collects all the terms with the Dirac's deltas and their derivatives. The general solution of equation (6) is reported in [8] and may be specialised, for the case under study, in explicit form as follows:

$$\begin{aligned}
\phi\left(\mathbf{b},\xi\right)=C_1&\left\{\frac{1}{2\alpha}\sum_{i=1}^{N_c}b_i\,\lambda_i\mu_i\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]U(\xi-\xi_{oi})+\sin\alpha\xi\right\}+\\
+C_2&\left\{\frac{1}{2\alpha}\sum_{i=1}^{N_c}b_i\,\lambda_i\upsilon_i\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]U(\xi-\xi_{oi})+\cos\alpha\xi\right\}+\\
+C_3&\left\{\frac{1}{2\alpha}\sum_{i=1}^{N_c}b_i\,\lambda_i\zeta_i\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]U(\xi-\xi_{oi})+\sinh\alpha\xi\right\}+\\
+C_4&\left\{\frac{1}{2\alpha}\sum_{i=1}^{N_c}b_i\,\lambda_i\eta_i\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]U(\xi-\xi_{oi})+\cosh\alpha\xi\right\}
\end{aligned} \tag{7}$$

where the terms $\mu_i,\nu_i,\zeta_i,\eta_i$ are given by the following recursive expressions.

$$\begin{aligned}
\mu_j&=\frac{\alpha}{2}\sum_{i=1}^{j-1}b_i\lambda_i\mu_i\left[-\sin\alpha\left(\xi_{oj}-\xi_{oi}\right)+\sinh\alpha\left(\xi_{oj}-\xi_{oi}\right)\right]-\alpha^2\sin\alpha\xi_{oj}\\
\upsilon_j&=\frac{\alpha}{2}\sum_{i=1}^{j-1}b_i\lambda_i\upsilon_i\left[-\sin\alpha\left(\xi_{oj}-\xi_{oi}\right)+\sinh\alpha\left(\xi_{oj}-\xi_{oi}\right)\right]-\alpha^2\cos\alpha\xi_{oj}\\
\zeta_j&=\frac{\alpha}{2}\sum_{i=1}^{j-1}b_i\lambda_i\zeta_i\left[-\sin\alpha\left(\xi_{oj}-\xi_{oi}\right)+\sinh\alpha\left(\xi_{oj}-\xi_{oi}\right)\right]+\alpha^2\sinh\alpha\xi_{oj}\\
\eta_j&=\frac{\alpha}{2}\sum_{i=1}^{j-1}b_i\lambda_i\eta_i\left[-\sin\alpha\left(\xi_{oj}-\xi_{oi}\right)+\sinh\alpha\left(\xi_{oj}-\xi_{oi}\right)\right]+\alpha^2\cosh\alpha\xi_{oj}
\end{aligned} \tag{8}$$

The dimensionless parameters λ_i appearing in equations (7) and (8) are related to damage extent as reported in [8,12] and will be adopted in the applications in order to represent the intensity of concentrated damages. The integration constants C_1,C_2,C_3,C_4, appearing in equation (7), can be easily evaluated in explicit form by imposing the boundary conditions involving the values of the eigen-mode and its derivatives.

2.2 *The evaluation of the non-linear time-history response*

The adopted model for the closing crack, in which the generic crack may be either open or closed, implies that the nonlinear response of the system can be considered as a sequence of linear states and each of them can be evaluated by means of a classical modal analysis. Therefore, by considering a time interval in which the system maintains the same crack configuration, i.e. the state vector **b** does not change, the corresponding displacement time-history response can be expressed exactly by an infinite series through modal superposition as follows:

$$u(t,\xi)=\sum_{n=1}^{\infty}\phi_n(\mathbf{b},\xi)\cdot y_n(t) \tag{9}$$

where $\phi_n(\mathbf{b},\xi)$ is the *n*-th mode shape corresponding to the state vector **b** and $y_n(t)$ is the *n*-th normal coordinate.

If classical modal damping is assumed, the structural dissipation can be considered by simply assigning a viscous modal damping ratio ζ_n to each considered mode in each non linear phase. In each linear phase, the response of the beam in terms of displacement can therefore be estimated by choosing a finite number of modes *N*, by solving the *N* independent equations of the normalised coordinates and superposing the modal responses. It is important to consider a sufficient number of vibration modes in order to maintain the error due to the modal truncation within an acceptable tolerance. In the following paragraph this aspect will be further discussed.

Let us now consider a beam with N_c closing cracks. The initial configuration, open or closed, of each crack is known and therefore the initial state vector **b** must be assigned. For this initial condition the first *N* natural frequency parameters and the corresponding modes of vibration can be derived by means of equation (7), and its derivatives, by enforcing the boundary condition, once the zeros of the corresponding frequency equations have been evaluated [8]. The response of the system during each linear phase, i.e. a phase characterised by the same state vector **b**, is obtained by using modal superposition. When one or more cracks open or close the system is subjected to a state change, in this case, the definition of the phase transition conditions is necessary in order to characterise the initial conditions for the new linear phase to be solved with a new set of modal coordinates.

2.3 *Phase transition conditions*

Without loss of generality it is assumed that a generic closed crack opens when the curvature at the crack position $\xi_{o,i}$ reaches the positive sign (upward concavity), while an open crack closes when the curvature at the crack position $\xi_{o,j}$ attains the negative sign (downward concavity). The incipient opening condition for a closed crack is characterised by the transition of the curvature from a negative toward a positive value, therefore this condition can be written as follows:

$$u''(t,\xi) = 0 \quad \frac{du''}{dt}(t,\xi) > 0 \tag{10}$$

viceversa, the incipient closing conditions for an open crack may be expressed as follows:

$$u''(t,\xi) = 0\,; \quad \frac{du''}{dt}(t,\xi) < 0 \tag{11}$$

Let us suppose that the opening/closing conditions (10)/(11) are satisfied for the *i*-th crack at the time instant t_o. At t_o a change in the *i*-th component of the state vector **b** occurs. Therefore a new set of *N* natural frequency parameters $\alpha_1^+, \alpha_2^+, ..., \alpha_N^+$ and the corresponding mode shapes $\phi_1(\mathbf{b}^+,\xi), \phi_2(\mathbf{b}^+,\xi), \ldots \phi_N(\mathbf{b}^+,\xi)$ must be evaluated for this new linear phase that, for simplicity, can be identified by the vector $\mathbf{b}^+$. Beyond the time instant t_o, the response of the beam must be evaluated in terms of the normalised coordinates $y_n^+(t)$ relative to the new mode shapes. Obviously this new solution is valid until a new event, associated to opening or closing of one or more cracks, occurs.

In the simplifying hypothesis that no dissipation energy is associated to the opening or closing of a crack, the initial conditions that must be enforced at the beginning of each linear phase must be determined by imposing the continuity of the displacement and velocity flexural response at time t_o. It must be noticed that, in view of the modal truncation error, the transition from one finite set of mode shapes (at the $\mathbf{b}^-$ state) to another (at the $\mathbf{b}^+$ state) to represent the same displacement and velocity configuration of the beam, introduces an additional error associated to the change of the mode shapes basis. An energy balance at the transition instant has been adopted in order to provide an estimation of the error due to the change of mode shape basis.

3 NUMERICAL APPLICATIONS

The numerical applications presented in this section are relative to beams with multiple closing cracks subject to different boundary conditions under harmonic loading. The frequency equation for a multi-cracked beam, in a fixed configuration, can be derived by means of the closed form solution reported in equation (7) by simply enforcing the standard boundary conditions. In particular, in this section the closed form solution for a general state of open/closed cracks is adopted to treat the cases of cantilever and simply supported Euler Bernoulli beams. The damage parameter λ has been chosen as representative of the damage intensities, and the correspondent crack depths can be easily inferred through existing damage models as reported in [12]. The response of the considered beams to harmonic loadings is analysed and the results are reported in terms of frequency response functions and compared with other results provided in the literature.

In order to compare the proposed approach with other accurate results reported in the literature, the first application considered herein is relative to a prismatic cantilever steel beam, in presence of two closing cracks and subjected to a harmonic concentrated load at the free end, considered by Pugno et al. [7]. The beam has length L=0,70 m, square cross-section with height h=200 mm, and has been subjected to three different configurations of the crack depths and positions as reported in Table 1. In the analyses considered by Pugno et al. the structure has been discretised by using Euler-type finite elements with two nodes and two degrees of freedom per node, furthermore, they assumed that the transition from closed to open crack, and vice versa, is smooth rather than instantaneous. Assuming that the dynamic response is periodic, they employed the harmonic balance method to solve the equations of motion, furthermore, in order to demonstrate the efficiency of their procedure, in the same paper a comparative analysis with results, previously obtained by the same authors through direct numerical integration according to a different procedure presented in [13], has been also reported.

In Figure 1 the results obtained by Pugno et al. [7] in terms of maximum displacement u of the free end, normalised with respect to the maximum load value P, are compared with those obtained by the proposed approach. In order to perform the comparison, the values of damage intensity parameter λ, adopted in this work, corresponding to the relative crack depths is defined as follows [12]:

$$\lambda = \frac{h}{L} C(\beta) \tag{12}$$

where $C(\beta)$ is the local compliance due to the concentrated crack, which is here adopted according to the model proposed in [13].

Table 1. Cantilever beam configurations in presence of two cracks.

Configuration	Crack 1			Crack 2		
	Depth, d (mm)	Position (mm)	Intensity, λ	Depth, d (mm)	Position (mm)	Intensity, λ
1	4	50	0.0179	8	500	0.0564
2	6	50	0.033	8	500	0.0564
3	6	50	0.033	8	350	0.0564

The values of the damage intensity parameters corresponding to the cases considered by Pugno et al. [7] are reported in Table 1.

The cantilever beam has been subjected to a concentrated harmonic force at the free end section, the maximum displacement response of the beam, at the same section, has been evaluated by discarding the transient part of the response. Nine mode shapes have been considered for each linear phases of the analyses and a modal damping ratio equal to 2% has been set for all the needed vibration modes.

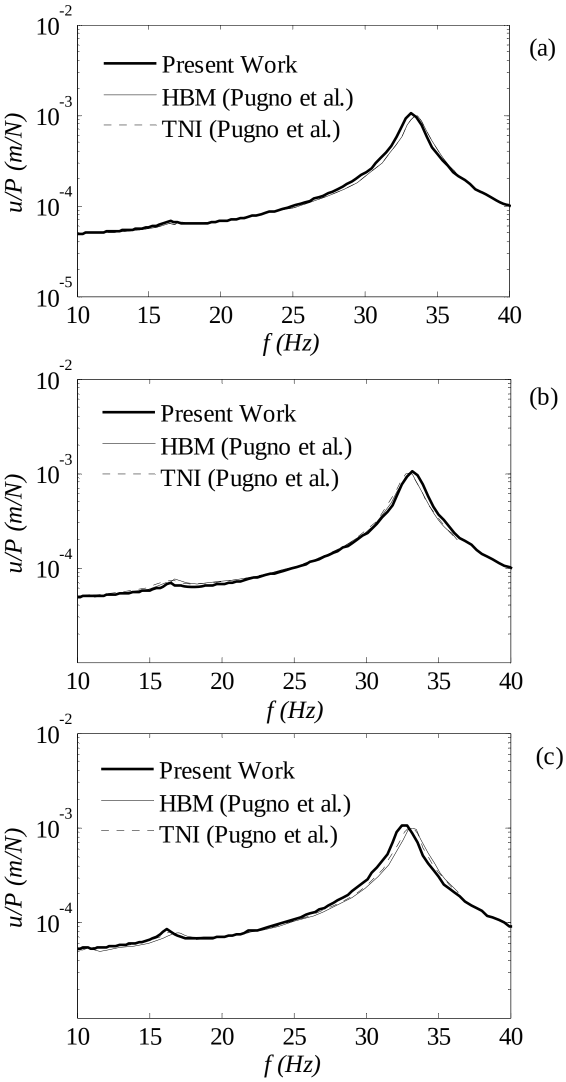

Figure 2. Frequency response functions u/P (u = maximum displacement of the free end, P = harmonic load amplitude) for the cantilever beam with two cracks of Table 1: a) configuration 1; b) configuration 2; c) configuration 3; Harmonic Balance Method (HBM) [12] (continuous line); Time Numerical Integration (TNI) [18] (broken line); proposed approach (bold line).

From the comparisons reported in Figure 1, it can be observed a good agreement with the results obtained by Pugno et al. [7]. However, since the model adopted in this work considers instantaneous crack closure, the results obtained with the proposed procedure are closer to those obtained according to the procedure reported in [13] regarding the same hypothesis for the closing and the opening of the cracks. The small differences can also be associated to the different beam models adopted by each approach, to the different damage models adopted for the crack depth, and, finally, to the unavoidable errors associated to the modal truncation.

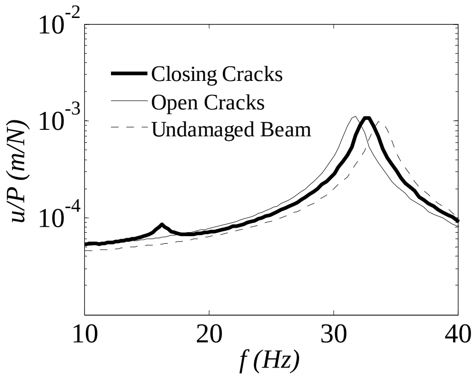

Figure 2. Frequency response function u/P (u = maximum displacement of the free end, P = harmonic load amplitude) with the proposed approach for the cantilever beam with two cracks in configuration 3 of Table 1: undamaged beam (continuous line); beam with open cracks (broken line); beam with closing cracks (bold line).

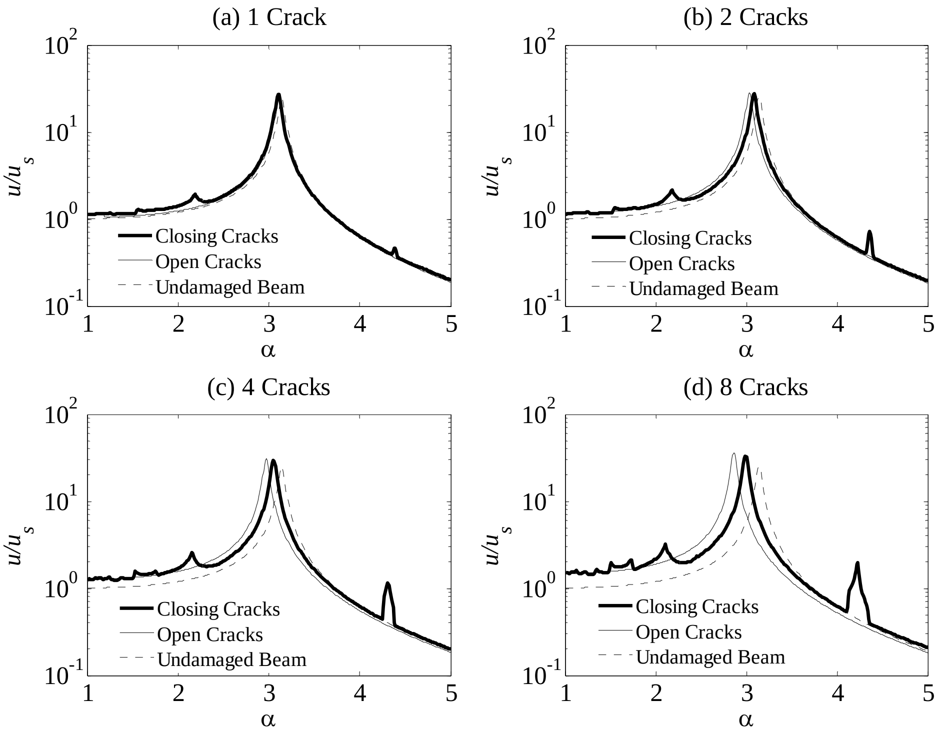

Figure 3**.** Frequency response functions u/u_S (u=maximum displacement of the middle span cross-section, u_S=static displacement of the undamaged beam due to the distributed load amplitude) for the simply-supported beam with equally spaced cracks with intensity $\lambda = 0.05$: a) 1 crack; b) 2 cracks; c) 4 cracks; d) 8 cracks.

The results plotted in Figure 2 are relative to the beam, in presence of two cracks, corresponding to the case 3 of Table 1 and are compared with the response of both the undamaged beam and the beam with open cracks. It can be observed as the beam with closing cracks shows the fundamental frequency collocated between those of the undamaged beam and the beam with open cracks.

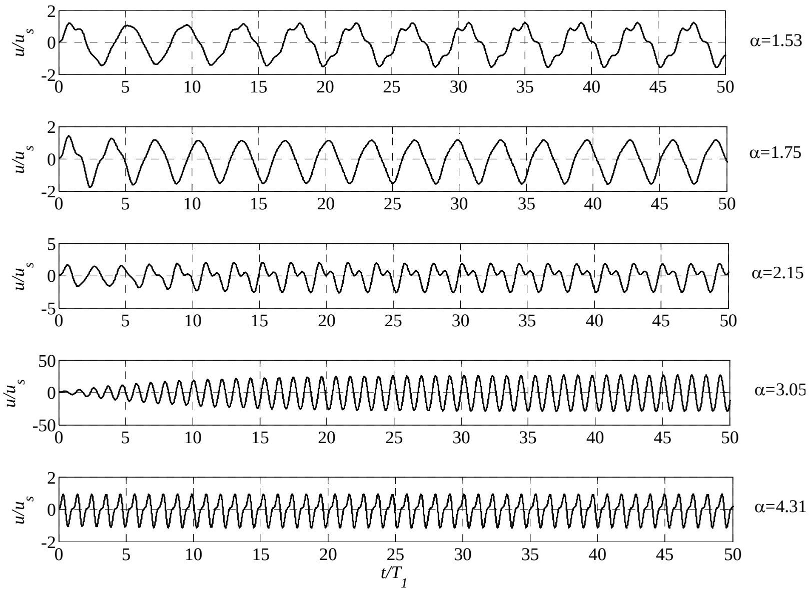

Figure 4. Time response function u/u_S (u=maximum displacement of the middle span cross-section, u_S=static displacement of the undamaged beam due to the distributed load amplitude) for the simply-supported beam with 4 cracks with intensity λ=0.05 subjected to an uniform harmonic load at the peak frequencies.

The results reported in Figure 3 are relative to a simply supported beam with different number of equally spaced cracks subjected to a harmonic uniform load. The latter results are reported in terms of maximum displacement u of the middle span cross-section normalised with respect to the value of the static displacement u_S due to the amplitude of the distributed load for the undamaged beam. The figure is relative to 1, 2, 4 and 8 cracks characterised by equal damage intensity parameters $\lambda = 0.05$, and reports a comparison of the proposed approach, for beams with closing cracks, with the response of the corresponding undamaged beam and the beam with open cracks. As highlighted by many authors in previous studies [2,6,15], the fundamental natural frequencies of the beam with closing cracks are collocated between those corresponding to the always-open and to the always-closed (undamaged) beams. Furthermore, the amplitudes of the fundamental resonant peaks of the beam with closing cracks show intermediate values with respect to the peaks of the undamaged and the always-open crack models for all the considered cases. For the four analysed beams with closing cracks, it can be also observed the reduction of the fundamental frequency with the increase of the number of equally spaced and equally damaged cracks. A further significant difference in the frequency response functions is associated to the presence of peaks, for the case of closing cracks, at lower and higher frequencies with respect to the fundamental one, indicating that the structures behave non-linearly.

In Figure 4 the time histories corresponding to each peak of the frequency response function of the simply supported beam with 4 cracks are reported. The analysis of Figure 4 reveals, except for the fundamental frequency, the presence of higher harmonic components in the frequency response spectrum indicating the non-linear behaviour of the structures.

4 CONCLUSIONS

In this work the non-linear dynamic behaviour of beams with multiple concentrated cracks has been analysed. The cracks have been modelled by means of Dirac's deltas which allowed the closed form evaluation of the beam mode shapes for a generic crack configuration. An integration procedure has been proposed to compute the time history through modal analysis by considering the sequence of crack opening/closing phenomena. Numerical analyses regarding beams with different boundary conditions have been presented for the case of harmonic loading in order to compare the results with others available in the literature.

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