The dynamic stiffness method for detection of concentrated damages in elastic frames

Annalisa Greco[1], Annamaria Pau[2]

[1]*Dipartimento di Ingegneria Civile e Ambientale, Università di Catania*
E-mail: *agreco@dica.unict.it*

[2]*Dipartimento di Ingegneria Strutturale e Geotecnica, Sapienza Università di Roma, Italy*
E-mail: *annamaria.pau@uniroma1.it*

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SUMMARY. The present paper examines the effect of a concentrated damage in elastic frames with reference to the variation induced in natural frequencies of vibration, evaluated by the dynamic stiffness method. First the direct problem for both undamaged and damaged frames is studied and some important considerations on the number of frequencies required in order to evaluate two damage parameters, i.e. location and intensity of the damage, are outlined. Second, an identification procedure based on the response comparison is presented. An optimal estimate of the parameters is obtained by minimizing an appropriate objective function which is defined as the sum of the squares of the differences between analytical and measured variations of frequencies in the undamaged and damaged state. Different damage configurations are considered to assess its reliability.

1 INTRODUCTION

Structural damage consists in a loss of stiffness inducing variations of both static and dynamic responses with respect to the undamaged structure and has been investigated in the literature using different techniques. These are based on the variation of dynamic characteristics, such as natural frequencies, mode shapes, dynamic flexibilities, or static quantities, such as displacements or strains induced by applied loads [1-4]. Particular attention has been devoted to the model of damage. In many studies, damage is represented by one or more fully open cracks along the axis of the beam and is modeled by a reduction in the rigidity at the correspondent abscissae [3, 5], others present one dimensional continuum theories [6, 7]. The present paper examines the problem of identification of one concentrated damage in elastic frames. The damage consists in a notch that reduces the height of the cross section at a given abscissa and is modeled by a reduction in the rigidity of the beam, described by means of an appropriate rotational spring [3].

The effect of damage is studied with reference to the variation induced in the natural frequencies of vibration. These are evaluated, for both undamaged and damaged structures, by means of an analytical method based on some properties of the dynamic stiffness matrix, i.e. the Wittrick and Williams algorithm [8-10]. The direct problem of the undamaged frame has been studied obtaining an explicit expression of the adimensional frequency parameters $a_i^4 = \omega_i^4\, mL^4/EI$ with respect to the geometrical and mechanical properties of the model such as distributed mass m, Young's modulus of the material E, moment of inertia I and length L. Assuming that I and L are known, the ratio m/E can be identified by means of the minimization of an objective function which measures the differences between analytical and measured frequencies. Once the mechanical parameters of the model have been reliably updated, the variation of the frequencies of the damaged frame are studied as a function of the damage parameters, i.e. adimensional location $l = x_d / L$ and stiffness $r = k_\varphi L/EI$ of the rotational spring. In the direct problem, for assumed values of damage parameters, the Wittrick and Williams algorithm provides the value of the i-th natural frequency ω_i. On the contrary, when ω_i is known for the damaged frame, for each possible damage position l, one value of stiffness r exists which corresponds to a value of the i-th natural frequency equal to ω_i. Therefore, for each frequency, a curve $r(l)$ can be obtained that describes r for all the possible positions of the damage. The analysis can be used to examine the uniqueness of the solution of the inverse problem, in fact, the curves $r(l)$ obtained for different ω_i cross at the abscissa where the damage is localized, providing the solution to the inverse problem. This fact allows to determine the minimum number of frequencies required to obtain a unique solution to the inverse problem. Then, the inverse problem is solved by the minimization of an objective function measuring the differences between analytical and measured variations of natural frequencies in the undamaged and damaged states [11]. Different damage configurations are considered to assess its reliability. The identification technique presented in the paper allows to reliably evaluate parameters also in the case in which the measured frequencies are affected by small instrumental errors [12].

2 THE WITTRICK AND WILLIAMS ALGORITHM

The frequencies of vibration of both undamaged and damaged frames can be evaluated by means of an efficient analytical method such as the Wittrick & Williams algorithm, which is based on the dynamic stiffness matrix of the structure [8-10]. In this paragraph, the main characteristics of the considered algorithm are briefly summarized. Considering the equation of motion of the Euler Bernoulli beam

$$EIv(x,t)^{IV} + m\ddot{v}(x,t) = 0 \tag{1}$$

where EI is the bending stiffness and m the distributed mass per unit length, the solution can be expressed in the form:

$$v(x,t) = \varphi(x)y(t) \tag{2}$$

where the equation of the mode of vibration is

$$\varphi(\xi) = Q_1 \sin a\xi + Q_2 \cos a\xi + Q_3 \sinh a\xi + Q_4 \cosh a\xi \tag{3}$$

In (3), the constants Q_i depend on the boundary conditions; nondimensional abscissa $\xi = x/L$ and frequency parameter $a^4 = \omega^2 mL^4/EI$ have been introduced. Expressing either the vector of the generalized nodal forces **f** and displacements **v** in terms of the constants Q_i, the following relation holds, in which $K(\omega)$ is the dynamic stiffness matrix of the single beam:

$$\mathbf{f} = K(\omega)\mathbf{v}, \tag{4}$$

where

$$\mathbf{f}^T = \begin{bmatrix} T_i L & M_i & T_j L & M_j \end{bmatrix} \;;\; \mathbf{v}^T = \begin{bmatrix} v_i & L\gamma_i & v_j & L\gamma_j \end{bmatrix} \;;\; K(\omega) = \frac{EI}{L^2}\begin{bmatrix} \delta & \theta & -\delta^* & \theta^* \\ \theta & \alpha & -\theta^* & \beta \\ -\delta^* & -\theta^* & \delta & -\theta \\ \theta^* & \beta & -\theta & \alpha \end{bmatrix}$$

$$\delta = \frac{(sC + cS)a^3}{D} \;;\; \delta^* = \frac{(s + S)a^3}{D} \;;\; \theta = \frac{sSa^2}{D} \;;\; \theta^* = \frac{(c - C)a^2}{D} \;;\; \alpha = \frac{(sC - cS)a}{D} \;;\; \beta = \frac{(S - s)a}{D}$$

$$D = 1 - cC \;;\; s = \sin a \;;\; c = \cos a \;;\; S = \sinh a \;;\; C = \cosh a \,.$$

The Wittrick & Williams algorithm allows to evaluate the number $J=J_k+J_0$ of frequencies of vibration which are smaller than a trial value ω^* and, therefore, by means of an iterative procedure, to converge to any required accuracy.

The two terms in the evaluation of J are respectively J_k, number of negative eigenvalues of the matrix $K(\omega^*)$, and J_0, total number of natural frequencies of clamped clamped beams which are smaller than ω^*, given by:

$$J_0 = \sum_{k=1}^{n^\circ beams} J_{bk} \text{, where } J_b = i - \frac{1}{2}\left[1 - (-1)^i \, sign(D)\right] \text{, } i = \mathrm{int}\left(\frac{a(\omega^*)}{\pi}\right) \text{, } sign(D) = \begin{cases} 1 & \text{if } D > 0 \\ -1 & \text{if } D < 0 \end{cases}.$$

3 DIRECT PROBLEM

The considered frames are composed by two columns and a beam whose lengths are respectively H and L. The geometric characteristics of the cross sections are represented by the area and the moment of inertia which are denoted in the following equations as A_c ,I_c, for the columns and A_b, I_b for the beam. Besides the undamaged state, the case in which a concentrated damage represented by a notch reduces the height of the cross section from h^U to h^D is considered. It has been assumed that the width of the notch is such that it is possible to neglect the reduction in the total mass of the structure.

The damage determines, at a given abscissa x_d, a reduction in the flexural rigidity which can be modelled by means of a rotational spring whose non-dimensional stiffness is calculated with the following relation [3]:

$$k_\varphi = \frac{2L}{h^U}\frac{1-\beta}{\beta}, \tag{5}$$

where $\beta = EI^U - EI^D / EI^U$, EI^U and EI^D are respectively the flexural rigidities of the undamaged and damaged cross sections.

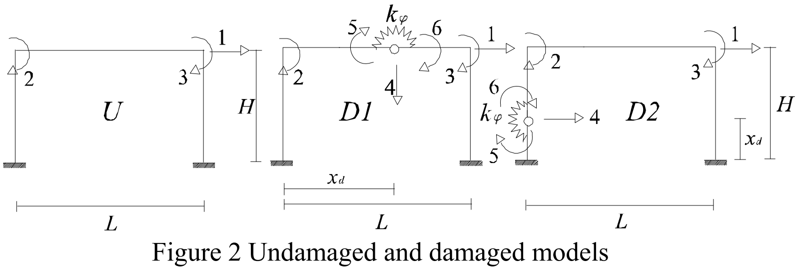

Figure 2 Undamaged and damaged models

In order to evaluate the natural frequencies of vibration of either the undamaged or damaged frame, three different dynamic stiffness matrices must be evaluated. In fact, for the damaged frame it is necessary to distinguish whether the damage is located on a column or the transversal beam. Figure 2 illustrates the considered cases and reports the nodal degrees of freedom with respect to which the dynamic stiffness matrices are assembled.

The three dynamic stiffness matrices turn out to be:

$$K^U(a_b)=\frac{EI_b}{L^2}\begin{bmatrix} \frac{q2\delta_c}{h^3}-a_b{}^4 & \frac{-\theta_c}{h^2} & \frac{-\theta_c}{h^2} \\ \frac{-\theta_c}{h^2} & \frac{\alpha_c}{h}+\alpha_b & \beta_b \\ \frac{-\theta_c}{h^2} & \beta_b & \frac{\alpha_c}{h}+\alpha_b \end{bmatrix}$$

$$K^{D1}(a_b,r,l)=\frac{EI_b}{L^2}\begin{bmatrix} \frac{2q\delta_c}{h^3}-a_b{}^4 & -\frac{q\theta_c}{h^2} & -\frac{q\theta_c}{h^2} & 0 & 0 & 0 \\ -\frac{q\theta_c}{h^2} & \frac{q\alpha_c}{h}+\frac{\alpha_{b1}}{l} & 0 & -\frac{\theta_{b1}{}^*}{l^2} & \frac{\beta_{b1}}{l} & 0 \\ -\frac{q\theta_c}{h^2} & 0 & \frac{q\alpha_c}{h}+\frac{\alpha_{b2}}{1-l} & \frac{\theta_{b2}{}^*}{(1-l)^2} & 0 & \frac{\beta_{b2}}{1-l} \\ 0 & -\frac{\theta_{b1}{}^*}{l^2} & \frac{\theta_{b2}{}^*}{(1-l)^2} & \frac{\delta_{b1}}{l^3}+\frac{\delta_{b2}}{(1-l)^3} & -\frac{\theta_{b1}}{l^2} & \frac{\theta_{b2}}{(1-l)^2} \\ 0 & \frac{\beta_{b1}}{l} & 0 & -\frac{\theta_{b1}}{l^2} & \frac{\alpha_{b1}}{l}+r & -r \\ 0 & 0 & \frac{\beta_{b2}}{1-l} & \frac{\theta_{b2}}{(1-l)^2} & -r & \frac{\alpha_{b2}}{(1-l)}+r \end{bmatrix}$$

$$K^{D2}(a_b,r,l)=\frac{EI_b}{L^2}\begin{bmatrix} \frac{q\delta_c}{h^3}+\frac{q\delta_{c2}}{(h-l)^3}-a_b{}^4 & -\frac{q\theta_{c2}}{(h-l)^2} & -\frac{q\theta_c}{h^2} & -\frac{q\delta^*_{c2}}{(h-l)^3} & 0 & -\frac{q\theta^*_{c2}}{(h-l)^2} \\ -\frac{q\theta_{c2}}{(h-l)^2} & \alpha_b+\frac{q\alpha_{c2}}{(h-l)} & \beta_b & \frac{q\theta^*_{c2}}{(h-l)^2} & 0 & \frac{q\beta_{c2}}{(h-l)} \\ -\frac{q\theta_c}{h^2} & \beta_b & \frac{q\alpha_c}{h}+\alpha_b & 0 & 0 & 0 \\ -\frac{q\delta^*_{c2}}{(h-l)^3} & \frac{q\theta^*_{c2}}{(h-l)^2} & 0 & \frac{q\delta_{c1}}{l^3}+\frac{q\delta_{c2}}{(h-l)^3} & -\frac{q\theta_{c1}}{l^2} & \frac{q\theta_{c2}}{(h-l)^2} \\ 0 & 0 & 0 & -\frac{q\theta_{c1}}{l^2} & \frac{q\alpha_{c1}}{l}+r & -r \\ -\frac{q\theta^*_{c2}}{(h-l)^2} & \frac{q\beta_{c2}}{(h-l)} & 0 & \frac{q\theta_{c2}}{(h-l)^2} & -r & \frac{q\alpha_{c2}}{(h-l)}+r \end{bmatrix}$$

where all the terms of the matrices have been expressed as functions of the non dimensional frequency parameter of the undamaged beam $a_b{}^4=\omega^2 m_b L^4/EI_b$ using the following relations:

$$a_c=\sqrt[4]{\frac{\mu}{q}}ha_b \quad a_{c1}=\sqrt[4]{\frac{\mu}{q}}la_b \quad a_{c2}=\sqrt[4]{\frac{\mu}{q}}(h-l)a_b \quad a_{b1}=la_b \quad a_{b2}=(1-l)a_b$$

and $h=H/L$; $l=x_d/L$; $r=k_\varphi L/EI_b$; $q=I_c/I_b$; $\mu=m_c/m_b=A_c/A_b$.

The Wittrick and Williams algorithm allows to evaluate infinite values of the frequency parameter a_b and can provide some deeper insight, with respect to finite element models, in the solution of the inverse problem, related to the identification of some model parameters, for its capability to provide exact solutions and, in general, for the synthetic form of the dynamic stiffness, which is used to solve several times the direct problem.

3.1 *Natural frequencies of the undamaged frame*

Once the desired number of frequency parameters a_b for the undamaged frame have been calculated, applying the considered algorithm to the stiffness matrix $K^u(a_b)$, the natural frequencies of the frame can be evaluated as functions of the mechanical properties of the material. Equation (6) shows how the natural frequencies depend on the ratio between the distributed mass and the Young's modulus of the material:

$$\omega_i(E/m_b) = \sqrt{\frac{a_{bi}^4 E I_b}{m_b L^4}} \,. \tag{6}$$

This equation allows to have a deep insight in the inverse problem related to the identification of material properties. In fact, if the values of some of the fundamental frequencies are plotted as functions of the ratio E/m_b it is easy to determine the abscissa, and therefore the ratio $E/m_{b,}$ corresponding to measured values of the natural frequencies.

Figure 3 refers to a frame in which L=800mm, H=1000mm, whose rectangular cross section is constant for columns and beam with sides 40x8 mm. The first three frequencies are plotted through equation (6). Three horizontal lines have been marked corresponding to pseudo-experimental frequencies evaluated by means of a FEM discretization of the frame with the following nominal values of mechanical properties: Young modulus E= $2*10^5$ N/mm^2, mass density per unit volume μ=7.849*10^{-9} Nsec^2/mm^4. The pseudo-experimental frequencies turn out to be ω_1=52.95 *Hz,* ω_2=167.92 *Hz,* ω_3=348.26 *Hz*. It can be noticed that the three pseudo-experimental frequencies are related to the same abscissa equal to 7.96 10^{10} s^{-2}.

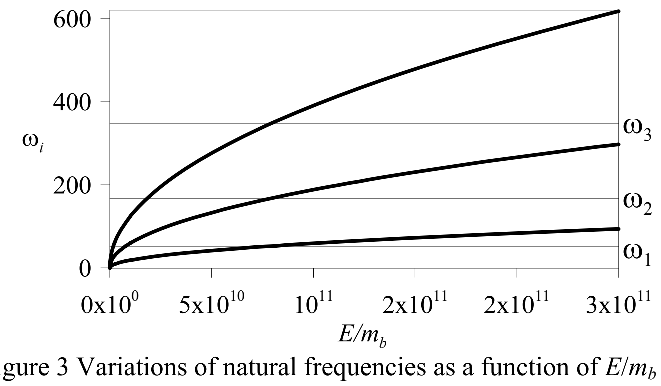

Figure 3 Variations of natural frequencies as a function of E/m_b

3.2 *Natural frequencies of the damaged frame*

When the Wittrick and Williams algorithm is applied to the stiffness matrices $K^{D1}(a_b,r,l)$ and $K^{D2}(a_b,r,l)$, the natural frequencies of the frame damaged respectively on the transverse beam or column can be evaluated. These frequencies depend on location and intensity of damage through the non dimensional parameters *l* and *r*. In this paragraph, the damage position is referred to by the nondimensional abscissa $l=x_d/(L+2H)$ that spans the frame from 0 (base) to 0.5, (middle of the transverse beam). For the case considered, H=0.8 m and L=1 m, the node between the transverse beam and the column is located at l= 0.31, marked with a vertical line in the following figures. The influence of damage parameters on natural frequencies can be studied observing Figure 4, that reports the variations of the first four natural frequencies with respect to the undamaged case as a function of *l* for three values of *r*. The maximum variation of frequency depends on the mode shape. The maximum variation of frequency coincides with the points where the modal curvature is maximum, as already observed for beams and arches [1,11].

Once again some important information on the solution to the inverse problem can be obtained through an advisement of the results of the parametric study. In fact, when ω_i is known, for each possible damage position l_j, a stiffness $r_i(l_j)$ exists which corresponds to a value of the i-th natural frequency equal to ω_i. Therefore, by considering all the possible positions of the damage either in the beam or column, discrete curves $r_i(l_j)$ can be plotted. Assuming, for example, that the first four natural frequencies of a damaged frame have been measured, these can be used to build four discrete curves $r_i(l_j)$ with *l* spanning the column and the transverse beam. These curves are evaluated with reference to a frame with the same geometry as described in paragraph 2.1 and different damage locations. For the sake of brevity, only two cases are reported in Figure 5 a) and b), which represent respectively one case of damage located along the column, at l=0.16, and transverse beam, l=0.35, for both cases r= 30. The curves show multiple intersections, however, in both cases and in absence of errors, by using three frequencies, the damage parameters can be determined univocally. It must be noted that in the diagrams there are areas where different couples of curves intersect, which indicates critical situations. In fact, in experimental cases, where errors occur, these points can provide solutions not corresponding to the actual values of damage parameters.

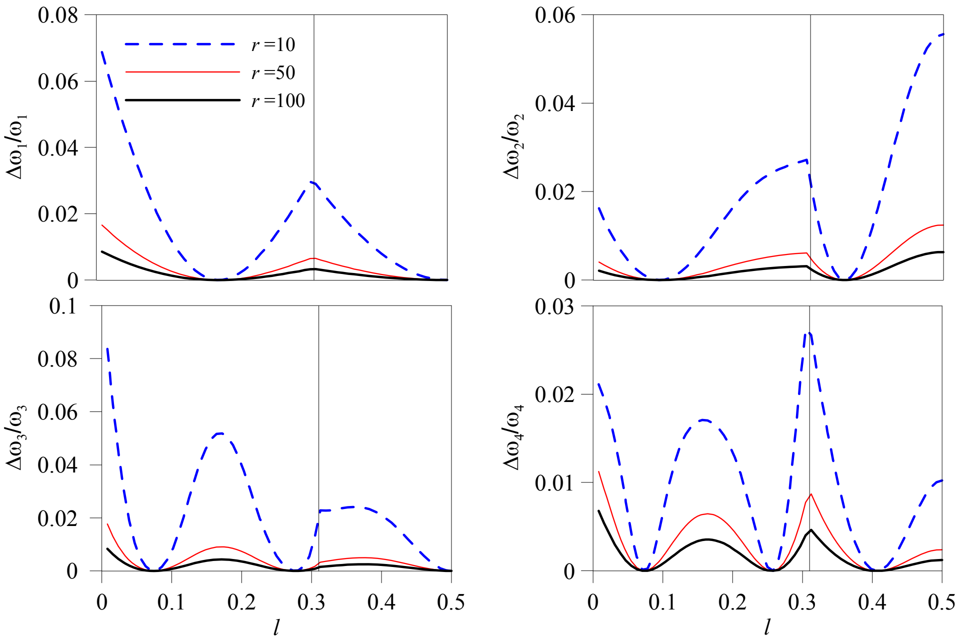

Figure 4 – Variations of natural frequencies as a function of *l* for different values of *r*

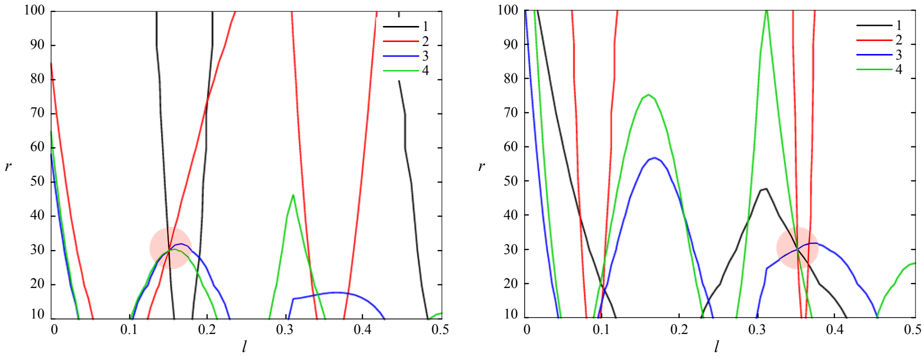

Figure 5 Curves $r_i(l_j)$ for damage located at l=0.15 a) and at l=0.35 b), r=30.

4 INVERSE PROBLEM

The study of the direct problem presented in the paragraph 3 allowed to evaluate the minimum number of frequencies required to solve the inverse problem. This is here solved by minimizing an objective function measuring the differences between analytical and measured natural frequencies. The identification technique presented in the paper allows to reliably evaluate the required parameters also when the measured frequencies are affected by small instrumental errors.

4.1 *Undamaged frame*

In order to have a good estimate of material's parameters and then a reliable structural model, the ratio between Young's modulus E and distributed mass m, can be identified by the minimizing the objective function (7), where ω_i^a and ω_i^e are analytical and experimental natural frequencies.

$$G^U(E/m) = \sum_{i=1}^{n} \sqrt{\left(\frac{\omega_i^a(E/m) - \omega_i^e}{\omega_i^e} \right)^2} \,. \tag{7}$$

For the same frame as described in paragraph 2.1, the following values of the first three natural frequencies are considered as pseudo-experimental data: ω_1=52.955 *Hz*, ω_2=167.924 *Hz,* ω_3=348.264 *Hz*. The Wittrick and Williams algorithm provides the first three frequency parameters a_1^4 =2.1313^4, a_2^4 = 3.7957^4 , a_3^4 = 5.4660^4. Figure 6 reports the objective function $G^U(E/m)$ and clearly shows that the absolute minimum, correspondent to zero value, is attained at the correct value of the ratio E/m .

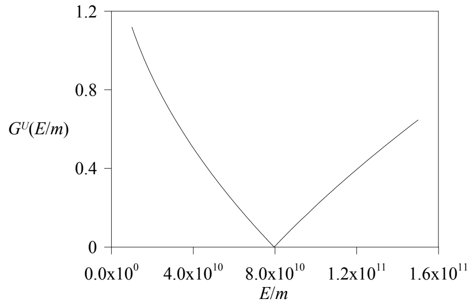

Figure 6 Objective function for the undamaged frame

4.2 *Sensitivity to experimental noise*

A real damage identification procedure is based on experimental data which are affected by unavoidable experimental noise. In order to assess the performance of the proposed identification procedure when frequency measurements are affected by instrumental errors, the experimental frequencies are here modelled as random variables:

$$\omega_i^e = \omega_i^*(1+\varepsilon R_i)\,, \tag{8}$$

where ω_i^* is the actual value of the *i*-th natural frequency; R_i are uniformly distributed random variables in $[-1,1]$ with zero mean and independent of each other; ε is a parameter defining the level of noise. The objective function G^U is a function of the random variables ω_i^e and therefore the ratio *E/m* correspondent to its absolute minimum is a random variable itself with mean value $\langle E/m \rangle$ and standard deviation $\sigma_{E/m}$. The sensitivity of the identification procedure to experimental noise is studied by means of the normalized average mean error (AME) and the normalized average standard deviation (ASD) of the ratio *E/m*, defined as $AME_{E/m}=[\langle E/m\rangle - (E/m)^*]/(E/m)^*$; $ASD_{E/m}=\sigma_{E/m}/(E/m)^*$, where $(E/m)^*$ is the exact value of *E/m*. The normalized values of AME and ASD are respectively measures of the distance between the identified value and actual one and scatter of the identified value around the mean value.

Figure 7 (a) and (b) show respectively the normalized values of AME and ASD as functions of the number of samples for the random variables ω_i^e with a small level of noise ε =0.05. For a small number of samples the results are not representative of the error due to experimental noise affecting the identification procedure, but, for a large number of experimental tests, the average mean error for *E/m* approaches a constant value different from zero, i.e. the identification procedure is affected by a bias error [13] when the data are contaminated by experimental noise.

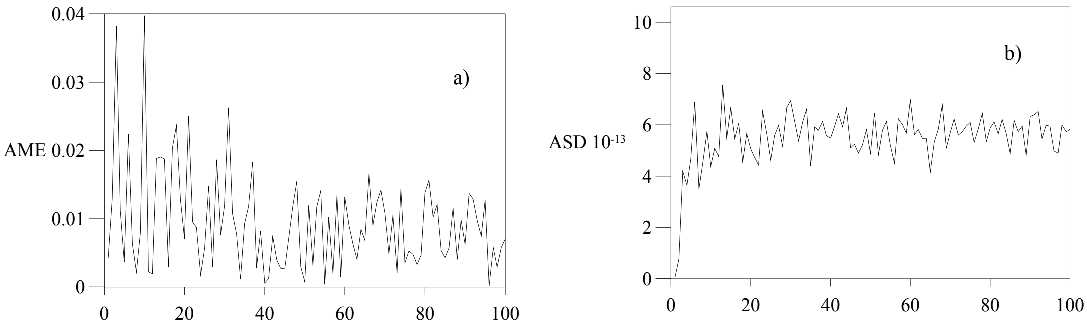

Figure 7 (a) AME and (b) ASD of *E/m* versus n° of experimental tests

4.3 *Damaged frame*

The identification procedure that will be used is based on the response comparison. An optimal estimate of the damage parameters is obtained by minimizing the objective functions:

$$G_{D1}(l,r)=\sum_i\left(\frac{\Delta\omega_i^{D1}(l,r)}{\omega_i^U(l,r)}-\frac{\Delta\omega_{ei}}{\omega_{ei}^U}\right)^2 \quad G_{D2}(l,r)=\sum_i\left(\frac{\Delta\omega_i^{D2}(l,r)}{\omega_i^U(l,r)}-\frac{\Delta\omega_{ei}}{\omega_{ei}^U}\right)^2 . \tag{9}$$

These are defined as the sum of the squares of the differences between analytical $\Delta\omega_i(k,s)$ and experimental $\Delta\omega_{ei}$ values of the variation of frequencies between the undamaged and damaged state, normalized with respect to the related frequencies of the undamaged arch ω_i^U and ω_{ei}^U.

The damage parameters l and r are obtained in two phases, by successively seeking two distinct minima. First, for each possible discrete damage position l, each function is minimized with respect to r providing

$$\widetilde{G}_{D1}(l) = \min_r G_{D1}(l,r)\,,\ \widetilde{G}_{D2}(l) = \min_r G_{D2}(l,r) \tag{10}$$

If the damage is located in the transverse beam, the solution to the inverse problem is given by the minimum of $\widetilde{G}_{D1}(l)$ with respect to l, otherwise, for damage on one column, the solution will be provided by the minimum of $\widetilde{G}_{D1}(l)$ with respect to l. Only the function related to the correct position of the damage will exhibit one global minimum in which the value of the function is almost zero and therefore, the solution of the inverse problem exists and is unique.

As an example, the procedure is applied to the same cases as those of Figure 5, considering the frequencies obtained by the Wittrick and Williams algorithm as pseudo-experimental data. The objective function $\widetilde{G}_{D1}(l)$ for l=0.15 related to a damage in the column is reported in Figure 8 a) and b), where in a) the summation is extended to three frequencies only, which is the minimum number necessary to have a unique solution, and in b) to four frequencies. As already remarked in paragraph 2.2, nevertheless the global minimum is unique, other local minima appear in the vicinity of multiple intersections between curves $r(l)$. Furthermore, the absolute value of the objective function at global minima is close to zero, which can bring difficulties in experimental cases where modelling and experimental errors are unavoidable. Analogous results are found for a damage located on the beam at l=0.35, for which the objective function is reported in Figure 9 a-b, where the summation is extended to 3 a) and to 4 b) frequencies too. Here, a double minimum very close to zero appears in the vicinity of l=0.1 because of the multiple close intersections of curves $r(l)$, as can be seen from Figure 5 b).

The study of the pseudo experimental case has shown that the minimum number of frequencies required may not be sufficient to assure the exact solution of the inverse problem in presence of experimental errors, due to the existence of relative minima very close to the global one.

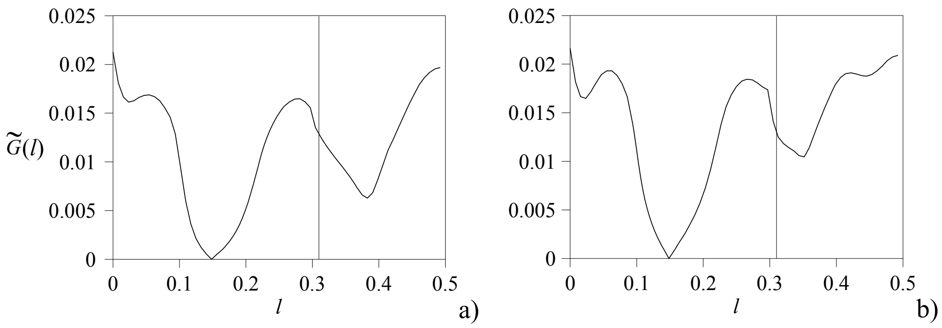

Figure 8 Objective functions for l=0.15, i=1-3 a), i=1-4 b)

5 CONCLUSIONS

An identification procedure for damage parameters based on the response comparison has been applied to an elastic frame. An optimal estimate of the location and intensity of damage is obtained by minimizing an appropriate objective function which is defined as the sum of the squares of the differences between the analytical and measured variations of frequencies, evaluated by means of the Wittrick and Williams algorithm, in the undamaged and the damaged state. Considerations are made on the minimum number of frequencies required to assure the exact solution of the inverse problem.

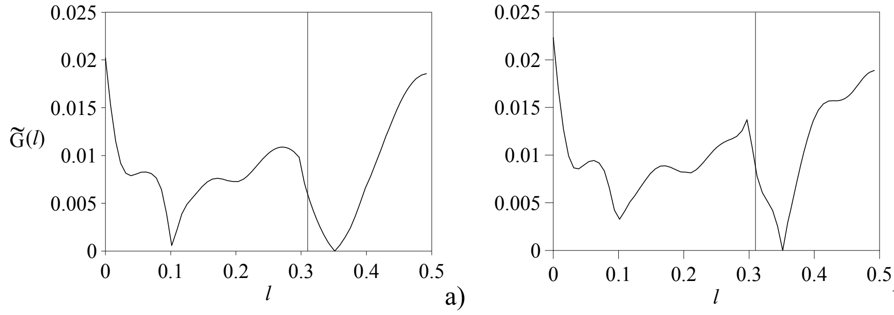

Figure 9 Objective functions for l=0.35, i=1-3 a), i=1-4 b)

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