Patch of vorticity in motion

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Keywords: vortex patches, complex analysis, contour dynamics.

For the Special Session of Aimeta : *"The analytical approach to inviscid two dimensional vortex dynamics. Is it only an ancient issue?"*

SUMMARY How powerful modern-day computers could be, despite these times of CFD twilight, what the calculations of today are trying to solve are still the same equations which Euler put to paper more than two centuries ago. Now, a 260-somethings-years-old issue is undoubtedly old. Anyway, if he were to return for following a vortex on the plane, Euler also would maybe prefer ,to numerics and a lagrangian pursuit, a same attempt tried by analysis and eulerianly. This, in essence, is what essayed below.

1 INTRODUCTION

The contour dynamics algorithm, as devised by Zabusky, Hughes & Roberts in [1] (but see also [2]), is a simple-idea based and a fast numerical tool for the study of the dynamics of vortex patches in unbounded domains. Briefly, applied Green's theorem to Green function, the entire velocity field, for a given patch vorticity ω, is made there depend only on the shape marked by vorticity jumps and then determined by line integrals along it. Thus, the geometry of the patch boundary can be updated by integrating in time the contour velocity and, as consequence, the 2d dynamics of the entire flow field can be reduced, indeed, to easier 1d *contour dynamics*.

The extension of the algorithm to general multi-connected *bounded domains*, on the contrary, has not been straightforward. Only recently, Crowdy and Surana [3] have provided the integral formulation of a contour dynamics for complex domains bounded by arbitrary impermeable (fixed) boundaries. Their approach is based on the Green function which defines the Hamiltonian of point vortices in bounded domains, and on its transformation under conformal mapping, as devised by Masotti [4] for simply connected domains and generalized to multiply connected domains by Lin [5]. So far, as discussed in §1 of ref. [3], previous extensions of the contour algorithm were limited, in practice, to cases with straight or circular impermeable boundaries (see the examples worked out in [6], [7] and [8], for instance).

An alternative formulation for the problem of determining, on the basis of its contour data , the flow velocity of a vortex patch in unbounded domains exploited the concept of Schwarz function. It is described in §9.2 of ref. [2] and references therein.

A further method is here proposed, which, if it does not explicitly uses the Schwarz function concept, anyway, could be considered as inspired to it. It is here formulated for unbounded domains and for domains bounded by a single, arbitrarily shaped wall. A novel feature of the proposed method is that the bounding wall can be considered as movable and permeable. Moreover, the proposed method gains some computational efficiency by taking from Legras and Zeitlin [9] the idea of a *conformal dynamics* .

In addition to the general interest in enlarging the panorama of methods now available in vortex dynamics, the present formulation can be well suited for solving optimization and control problems (for instance, to find the shape of a vortex patch which stands in equilibrium in front of a given wall, as well the inverse problem, *viz.* to find the shape of a wall which holds a given 2-dimensional vortex in steady position). Moreover, the ability of the method in dealing with permeable and movable bounding walls makes possible to formulate the control problem of holding in equilibrium a vortex patch by wall motion or by wall blowing and suction.

The method here proposed, without resorting to other specific changes, shares with the Schwarz function formulation the drawback of being unable to follow the contour evolution when it becomes too far from a more or less rounded shape and filamentation phenomena occur. But for optimization and control purposes, the class of geometries which can be studied is quite large (wider than before at least [12], [13]).

The paper is organized this way: in §2 the present formulation is described for unbounded domains along with some details of implementation (§2.1 and §2.2). Section §3 faces the problem of the evolution of the patch in time, while §4 spots how the approach links to previous studies. In §5 the formulation is extended to bounded simply connected domains with movable and/or permeable walls and relative treatment follows (§5.1) . Concluding remarks are drawn in §6.

2 VORTEX PATCHES IN UNBOUNDED DOMAINS

We consider the unbounded 2D motion of an inviscid fluid taking place on the complex z-plane ($z = x + \mathrm{i}\,y$). Let D_i be a simply connected vortex patch, with vorticity ω bounded by ∂, and let D_e be the irrotational region external to it and which extends to infinity. As shown in fig.[1], the region D_i is conformally mapped inside the unit circle of the transformed ζ-plane and the region D_e is mapped outside the unit circle of the transformed λ-plane. Such mappings existing according to the Riemann mapping theorem , they can be expressed as Theodorsen-Garrick [10] transformations: D_i is mapped inside the unit circle of the ζ-plane by

$$z(\zeta) - z_o = \zeta \exp \sum_{n=0}^{\infty} a_n \zeta^n \tag{1}$$

while D_e is mapped outside the unit circle of the λ-plane by

$$z(\lambda) - z_o = \lambda \exp \sum_{n=0}^{\infty} b_n \lambda^{-n}. \tag{2}$$

where z_o is a point inside the patch. Once the series are truncated at a proper large value $n = N$, the coefficients a_n and b_n can conveniently be determined according to the iterative process proposed by Ives [11].

In D_e the motion is irrotational and a complex potential w_e can be defined. For regularity reasons, the complex velocity $\frac{\mathrm{d}w_e}{\mathrm{d}z}$ has to be a holomorphic function of z in D_e. Since $z(\lambda)$ in (2) is holomorphic outside the unit circle of the λ-plane, the complex velocity can, in general, be expressed by the series

$$u_e - \mathrm{i}\, v_e = \frac{\mathrm{d}w_e}{\mathrm{d}z} = \sum_{j=0}^{\infty} c_j \lambda^{-j}. \tag{3}$$

In D_i the motion has constant vorticity ω and the complex velocity can, in general, be written as

$$u_i - \mathrm{i}\, v_i = -\mathrm{i}\,\frac{\omega}{2} z^{\star} + \frac{\mathrm{d}w_i}{\mathrm{d}z}, \tag{4}$$

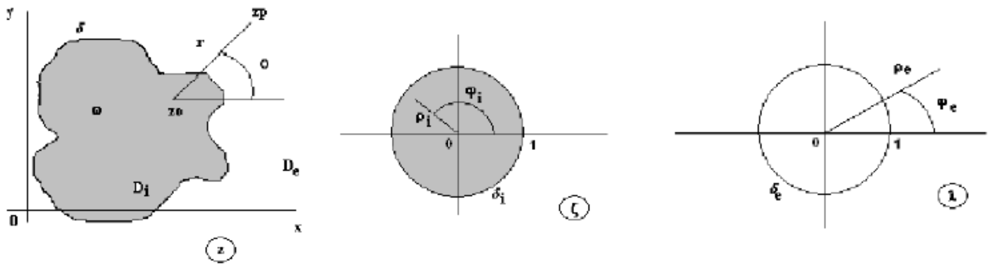


Figure 1: The physical plane , the interior and the exterior mapping.

with $\frac{\mathrm{d}w_i}{\mathrm{d}z}$ analytic and holomorphic in D_i and with * denoting complex conjugation. For the same regularity reason as above, that inner complex velocity can, in general, be expressed as a function of ζ by

$$u_i - \mathrm{i}\, v_i = -\mathrm{i}\frac{\omega}{2}[z(\zeta)]^\star + \sum_{j=0}^{\infty} d_j \zeta^j. \tag{5}$$

Let it be $\zeta = \rho_i \exp(\mathrm{i}\varphi_i)$ and $\lambda = \rho_e \exp(\mathrm{i}\varphi_e)$. According to the mappings (1) and (2), each patch contour point z_∂ is the image of a point of the unit circle of the ζ-plane and of a point of the unit circle of the λ-plane, that is,

$$z_\partial - z_o = \sum_{n=0}^{N-1} a_n \exp(\mathrm{i}\, n\, \varphi_{\partial i}) = \sum_{n=0}^{N-1} b_n \exp(-\mathrm{i}\, n\, \varphi_{\partial e}) \tag{6}$$

which establishes an implicit relationship $F(\varphi_{\partial i}, \varphi_{\partial e}) = 0$ among the anomalies of the two unit circles. In principle, by making explicit $\varphi_{\partial i}$ or $\varphi_{\partial e}$, the relationship $\varphi_{\partial i} = \varphi_{\partial i}(\varphi_{\partial e})$ and its inverse $\varphi_{\partial e} = \varphi_{\partial e}(\varphi_{\partial i})$ can be found.

By equating the internal and external flow velocities at the patch contour, eqs.(3) and (5), truncated at a large value $j = J$, yield

$$\sum_{j=0}^{J-1} c_j \exp(-\mathrm{i}\, j\, \varphi_{\partial e}) = -\mathrm{i}\frac{\omega}{2}[z(\varphi_{\partial i}(\varphi_{\partial e}))]^\star + \sum_{j=0}^{J-1} d_j \exp[\mathrm{i}\, j\, \varphi_{\partial i}(\varphi_{\partial e})] \tag{7}$$

and

$$-\mathrm{i}\frac{\omega}{2}[z(\varphi_{\partial i})]^\star + \sum_{j=0}^{J-1} d_j \exp[\mathrm{i}\, j\, \varphi_{\partial i}] = \sum_{j=0}^{J-1} c_j \exp(-\mathrm{i}\, j\, \varphi_{\partial e}(\varphi_{\partial i})) \tag{8}$$

which will allow the coefficients c_j, d_j to be determined.

2.1 *The functions $\varphi_{\partial i}(\varphi_{\partial e})$ and $\varphi_{\partial e}(\varphi_{\partial i})$*

The above idea has been here implemented in a practical way for vortex patches belonging to the so called *star shaped* class. Let it be $\vartheta_\partial = \arg(z_\partial - z_o)$, the patch is said to be star shaped if there exists a z_o such that $r_\partial = |z_\partial - z_o|$ is a single-valued function of ϑ_∂ (in few words, all z_∂ boundary points must result "visible" from z_o). Not incidentally, such a class coincides with the class of shapes for which the Theodersen-Garrick mappings (1) and (2) can be carried out.

The function $\varphi_{\partial i}(\varphi_{\partial e})$ and its inverse $\varphi_{\partial e}(\varphi_{\partial i})$ are determined by means of a numerical procedure based on periodic splines. From $\log(z_\partial - z_o) = \log r_\partial + \mathrm{i}\,\vartheta_\partial$ and from eq. (1), one gets

$$\vartheta_\partial(\varphi_{\partial i}) = \varphi_{\partial i} + \mathrm{Im}\left[\sum_{n=0}^{N-1} a_n \exp(\mathrm{i}\, n\, \varphi_{\partial i})\right]. \tag{9}$$

Placed $\delta = \varphi_{\partial i} - \vartheta_\partial(\varphi_{\partial i})$, a vector of values $(\vartheta_\partial(\varphi_{\partial i}), \delta)$ can be built for a discrete set of values of $\varphi_{\partial i}$ and a continuous representation $\delta = \delta(\vartheta_\partial)$ can be obtained by a periodic (cubic) spline. From eq. (2) one gets

$$\vartheta_\partial(\varphi_{\partial e}) = \varphi_{\partial e} + \mathrm{Im}\left[\sum_{n=0}^{N-1} b_n \exp(-\mathrm{i}\, n\, \varphi_{\partial e})\right]. \tag{10}$$

and the function $\varphi_{\partial i}(\varphi_{\partial e})$ is given by

$$\varphi_{\partial i} = \delta(\vartheta_\partial(\varphi_{\partial e})) + \vartheta_\partial(\varphi_{\partial e}). \tag{11}$$

The same procedure is followed by inverting the role of eqs. (10) and (11) to get the inverse function $\varphi_{\partial e}(\varphi_{\partial i})$.

2.2 *The flow velocity*

Once the c_j and d_j coefficients on the right-hand sides of eqs. (3) and (5) are computed, the entire flow velocity field, inside and outside the patch, is determined at once. These c_j and d_j coefficients are numerically computed by means of a fixed point iterative process based on the condition that the inner and outer flow velocities, as expressed by eqs. (7) and (8), have to match on the contour. Put $c_j = c_{rj} + \mathrm{i}\, c_{ij}$ and $d_j = d_{rj} + \mathrm{i}\, d_{ij}$, the imaginary part of eq. (7) yields

$$\begin{aligned} &\sum_{j=0}^{J-1}[c_{ij}\cos(j\,\varphi_{\partial e}) - c_{rj}\sin(j\,\varphi_{\partial e})] = \\ &\sum_{j=0}^{J-1}[d_{ij}\cos(j\,\varphi_{\partial i}(\varphi_{\partial e})) + d_{rj}\sin(j\,\varphi_{\partial i}(\varphi_{\partial e}))] - \frac{\omega}{2}\mathrm{Im}[\mathrm{i}\, z_\partial^\star(\varphi_{\partial e})] \end{aligned} \tag{12}$$

while the real part of eq. (8) gives

$$\begin{aligned} &\sum_{j=0}^{J-1}[d_{rj}\cos(j\,\varphi_{\partial i}) - d_{ij}\sin(j\,\varphi_{\partial i})] = \\ &\sum_{j=0}^{J-1}[c_{rj}\cos(j\,\varphi_{\partial e}(\varphi_{\partial i})) + c_{ij}\sin(j\,\varphi_{\partial e}(\varphi_{\partial i}))] + \frac{\omega}{2}\mathrm{Re}[\mathrm{i}\, z_\partial^\star(\varphi_{\partial i})]. \end{aligned} \tag{13}$$

An initial set of J values are guessed for d_j (typically, $d_j = 0$). By dividing the unit circle of the λ-plane into $2J$ equispaced intervals, the right-hand side of eq. (12) can be evaluated for $2J$ values of $\varphi_{\partial e}$ and the FFT algorithm can be used to compute J values of c_j. Let now the unit circle of the ζ-plane be divided into $2J$ equispaced intervals, the right-hand side of eq. (13) can be evaluated for $2J$ values of $\varphi_{\partial i}$ and the FFT algorithm provides a new set of J values of d_j. The process is repeated until, in a prescribed range, the convergence is achieved.

Eq. (12), it has to be noted, does not provide c_{r0} and eq. (13) does not provide d_{i0} as well. Their values can be retrieved by knowing the complex velocity q_∞ at infinity in the physical plane. According to eqs. (2) and (3), in fact, it is $c_{r0} + \mathrm{i}\, c_{i0} = q_\infty$; it follows that $c_{r0} = \mathrm{Re}(q_\infty)$, while d_{i0} has to be such that $c_{i0} = \mathrm{Im}(q_\infty)$. By enforcing eq. (12) for $\varphi_{\partial i} = 0$, it follows that

$$
\begin{aligned}
d_{i0} = & -\sum_{j=1}^{J-1} d_{ij} + \mathrm{Im}(q_\infty) + \\
& + \left\{ \sum_{j=1}^{J-1} [c_{ij} \cos(j\,\varphi_{\partial e}) - c_{rj} \sin(j\,\varphi_{\partial e})] + \frac{\omega}{2} \mathrm{Im}[\mathrm{i}\, z_\partial^\star] \right\}_{\varphi_{\partial i}=0} .
\end{aligned}
$$

3 THE CONTOUR EVOLUTION AS CONFORMAL MAPS DYNAMICS

Owing to the material character of vorticity, the evolution in time of the vortex patch coincides with the advection of its contour. So, in principle, it can be carried out by numerically integrating in time its velocity $\frac{\mathrm{d}z_\partial}{\mathrm{d}t}$. Such a procedure is not convenient neither simple; actually, at each time step the updated geometry should undergo the entire process above explained, including the laborious and time consuming Ives's [11] iterative computation of the new set of a_n and b_n coefficients of the mappings (1) and (2).

Following the ideas proposed by Legras and Zeitlin [9], a more robust and fast procedure can be implemented which avoids the iterative process by computing, in closed form, the time derivatives $\dot{a}_n$ and $\dot{b}_n$ of the mappings coefficients. These are integrated in time and the updated countour is then obtained as the new map from the unit circle of both the ζ- and the λ-plane.

Let us first take into consideration the mapping (2). The coordinate z_p of a fluid particle depends on time through its transformed coordinate $\lambda(t)$ and through the time-dependence of the coefficients $b_n(t)$. The Lagrangian derivative of $\log(z_p - z_o)$ is then

$$
\frac{1}{z_p - z_o} \frac{\mathrm{d}(z_p - z_o)}{\mathrm{d}t} = h(\lambda) \frac{\dot{\lambda}}{\lambda} + \sum_{n=0}^{N-1} \dot{b}_n \lambda^{-n}
$$

with $h(\lambda) = 1 - \sum_{n=1}^{N-1} n\, b_n \lambda^{-n}$. For the points lying on the contour, enforcing $|\lambda| = 1$, it is

$$
\mathrm{Re}\left[\frac{\dot{\lambda}}{\lambda}\right] = 0 \tag{14}
$$

and hence

$$
\mathrm{Re}\left[\frac{1}{h(\lambda)} \sum_{n=0}^{N-1} \dot{b}_n \lambda^{-n}\right]_{|\lambda|=1} = \mathrm{Re}\left[\frac{((u_e - \mathrm{i}\, v_e)^\star - \dot{z}_o)}{(z_\partial \ - z_o) h(\lambda)}\right]_{|\lambda|=1},
$$

with z_∂ and $(u_e - \mathrm{i}\, v_e)$ given by eqs. (2) and (3), respectively, and $\dot{z}_o = \mathrm{i}\frac{\omega}{2} z_o + d_0^\star$. The left-hand side is the real part of a function of λ which is holomorphic outside the unit circle. Let us call such a function $H(\lambda)$. The definition of its imaginary part, once $\mathrm{Re}(H(\lambda))$ is known on the unit circle from the right-hand side of above equation, states a classical problem of analysis, already solved in several manners. In the general spirit of the numerical procedures here adopted, we solve the problem by expressing $H(\lambda)$ as

$$H(\lambda) = \sum_{n=0}^{N-1} e_n \lambda^{-n} \tag{15}$$

and by computing the e_n coefficients by means of the FFT algorithm applied to its known real part. Once $H(\lambda)$ has been determined, it can be computed

$$\sum_{n=0}^{N-1} \dot{b}_n \lambda^{-n} = h(\lambda) H(\lambda) \tag{16}$$

and, finally, the $\dot{b}_n$ coefficients can be found by applying again the FFT algorithm either to the real or imaginary part of the right-hand side as evaluated on the unit circle.

The procedure for computing the derivatives $\dot{a}_n$ of the coefficients of the internal mapping (1) follows the same guidelines, with the trivial variation that the unknown functions to be determined are holomorphic inside instead of outside the unit circle.

4 THE RELANTIONSHIP WITH THE SCHWARZ FUNCTION METHOD

The present treatment is similar to the Schwarz function method, as described, for instance, in [2]. In that method, the equations (3) and (5) are replaced, respectively, by functions of the physical complex coordinate z :

$$u_e - \mathrm{i}\, v_e = -\mathrm{i}\frac{\omega}{2} G(z) \tag{17}$$

and

$$u_i - \mathrm{i}\, v_i = \mathrm{i}\frac{\omega}{2} [F(z) - z^\star] \tag{18}$$

with

$$G(z) = \sum_{j=0}^{\infty} g_n z^{-n}, \qquad F(z) = \sum_{j=0}^{\infty} f_n z^n$$

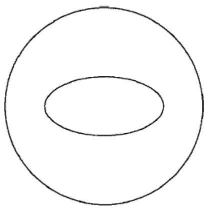
and where $\Phi(z) = F(z) + G(z)$ is the Schwarz function of the patch contour, expressed as a Laurent series converging in an annulus containing the contour. Like for the method here presented, the procedure then consists on determining the series coefficients.

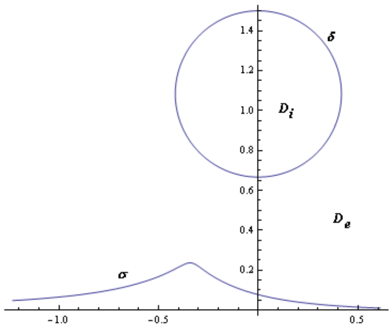
But unlike the above splitting of the Schwarz function, in the present method the two pieces F(z) and G(z) are expressed in two different parametrizations, one outside the unit circle of the λ-parameter-plane, the other inside the unit circle of the ζ-parameter-plane. Some advantage is gained since the solution is determined on the entire flow field and it is not limited to an annulus. Moreover, the class of possible contour geometries which are contained in an annulus where the Laurent expansion of their Schwarz functions converges is, quite reasonably, smaller then the class of star shaped contours, for which the present method can be carried out. For instance, with a smart choice of mappings to propitiate convergence (in the case, some appropriate Joukowski maps preparatory to (1) and (2)), all 6 Riccardi's vortices [13] can be recovered and then, even if far from circular, coul be made enter the method as well.

Finally, as below shown, the present formulation can be extended to domains confined by permeable and movable walls.

5 CONFINED DOMAIN

Let the flow region be a 2D simply connected domain bounded by a wall which can be a closed line (fig.[2]) or extend to infinity (as here considered). With reference to fig.[3] and with the same notation as for the unbounded case, D_i denotes a simply connected vortex patch, ∂ its boundary and D_e the potential flow region bounded by ∂ and the wall σ.





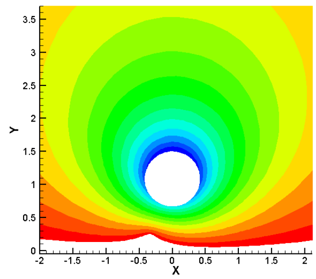


Figure 2: The doubly-connected case: Kirchhoff vortex in a circular Penning trap [8], Figure 3: Rankine vortex on a Ringleb snow cornice, Figure 4: Streamlines about last case.

As well as above, D_i is mapped by eq. (1) into the unit disk of the complex ζ-plane. The domain D_e being now doubly connected, it is mapped onto an annulus of the complex λ-plane bounded by the unit circle and by a circle of radius $1/R$, with $R < 1$. As consequence of the Riemann mapping theorem, such mapping exists for a unique value of $R(t)$ at a time t.

The procedure to define the mapping function and the value of R is inspired to [11]. It can be seen as a chain of mappings. First, the Möbius mapping

$$\mu = \frac{\alpha\, z + \beta}{\gamma\, z + 1} \tag{19}$$

maps σ onto a closed line of the μ-plane, with the flow field mapped inside it. The parameters (α, β, γ) are such that the centroids of $\sigma$and ∂ are close to the origin of the μ-plane (for that, a preliminary translation in the z-plane could be needed). The Theodorsen-Garrick mapping

$$\mu = \nu \exp \sum_{n=0}^{\infty} b_n \nu^n \tag{20}$$

maps the flow field into the unit disk of the ν-plane. The patch contour ∂ results as mapped onto an interior closed line. Finally, the Garrick mapping

$$\nu = \lambda\, R\, \exp\left[\sum_{n=0}^{\infty}(-c_n + \mathrm{i}\, d_n)(R^2\lambda)^n + \sum_{n=0}^{\infty}(c_n + \mathrm{i}\, d_n)\lambda^{-n}\right], \tag{21}$$

with real (c_n, d_n), maps D_e into the annulus of the λ-plane which is bounded inside by the unit circle (preimage of ∂) and outside by the circle of radius $1/R$ (preimage of σ). Once the series have been truncated at a large value $n = N$, the same iterative procedure as in [11] is used to determine the values of R and of the coefficients b_n, c_n, d_n.

Let the unit circle of the ζ-plane be $\zeta_\partial = \exp(\mathrm{i}\,\varphi_{\partial i})$ and that of the λ-plane $\lambda_\partial = \exp(\mathrm{i}\,\varphi_{\partial e})$. Again, the condition $z(\zeta_\partial) = z(\lambda_\partial)$ establishes the implicit relationship between $\varphi_{\partial i}$ and $\varphi_{\partial e}$. With obvious mappings variations, the same as above method is used to numerically define the functions $\varphi_{\partial i}(\varphi_{\partial e})$ and $\varphi_{\partial e}(\varphi_{\partial i})$

5.1 *The flow velocity*

In the external domain D_e, the complex flow velocity can be expressed as $u_e - \mathrm{i}\,v_e = \frac{\mathrm{d}w_e}{\mathrm{d}z}$ and, inside the D_i patch region, as $u_i - \mathrm{i}\,v_i = \frac{\mathrm{d}w_i}{\mathrm{d}z} - \mathrm{i}\,\frac{\omega}{2}\,z^\star$, with w_e, w_i being complex flow potentials. For the external flow is regular, the quantity $\frac{\mathrm{d}w_e}{\mathrm{d}\lambda}\,\lambda$ can be expressed as a Laurent series which converges in the annulus of the λ plane $1 \leq |\lambda| \leq 1/R$, that is

$$\frac{\mathrm{d}w_e}{\mathrm{d}\lambda}\lambda = \sum_{j=0}^{\infty} e_j\,\lambda^{-j} + \sum_{j=1}^{\infty} f_j (R\lambda)^j \tag{22}$$

while the internal flow regularity allows $\frac{\mathrm{d}w_i}{\mathrm{d}\zeta}\,\zeta$ to be written as a positive power series converging in the unit disk of the ζ-plane, that is

$$\frac{\mathrm{d}w_i}{\mathrm{d}\zeta}\zeta = \sum_{j=1}^{\infty} g_j\,\zeta^j. \tag{23}$$

Let $\tilde{u}, \tilde{v}$ denote the normal and tangential components, respectively, of the flow velocity at the ∂ patch boundary. The condition on ∂

$$\tilde{u}_e - \mathrm{i}\,\tilde{v}_e = \tilde{u}_i - \mathrm{i}\,\tilde{v}_i, \tag{24}$$

yields the equation:

$$\left(\frac{\mathrm{d}w_e}{\mathrm{d}\lambda}\,\frac{\lambda}{|\mathrm{d}z/\mathrm{d}\lambda|}\right) = \left(\frac{\mathrm{d}w_i}{\mathrm{d}\zeta}\,\frac{\zeta}{|\mathrm{d}z/\mathrm{d}\zeta|}\right) - \mathrm{i}\,\frac{\omega}{2} z_\partial^\star Z, \tag{25}$$

with Z expressed either as $Z = \zeta\frac{\mathrm{d}z}{\mathrm{d}\zeta}/|\frac{\mathrm{d}z}{\mathrm{d}\zeta}|_{\zeta=\exp(\mathrm{i}\varphi_{\partial i})}$ or $Z = \lambda\frac{\mathrm{d}z}{\mathrm{d}\lambda}/|\frac{\mathrm{d}z}{\mathrm{d}\lambda}|_{\lambda=\exp(\mathrm{i}\varphi_{\partial e})}$ and with the left- and right-hand sides computed for values of $\varphi_{\partial i}$ or $\varphi_{\partial e}$, respectively, such that $\varphi_{\partial i} = \varphi_{\partial i}(\varphi_{\partial e})$ and $\varphi_{\partial e} = \varphi_{\partial e}(\varphi_{\partial i})$, equivalently.

This complex equation (25) can be rearranged in the form of the two real equations:

$$\mathrm{Im}\left(\frac{\mathrm{d}w_e}{\mathrm{d}\lambda}\,\lambda\right)_{\varphi_{\partial e}} = \mathrm{Im}\left\{\left|\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right|_{\varphi_{\partial e}}\left[\left(\frac{\mathrm{d}w_i}{\mathrm{d}\zeta}\,\frac{\zeta}{|\mathrm{d}z/\mathrm{d}\zeta|}\right)_{\varphi_{\partial i}(\varphi_{\partial e})} - \left(\mathrm{i}\,\frac{\omega}{2} z_\partial^\star Z\right)_{\varphi_{\partial e}}\right]\right\}, \tag{26}$$

$$\mathrm{Re}\left(\frac{\mathrm{d}w_i}{\mathrm{d}\zeta}\,\zeta\right)_{\varphi_{\partial i}} = \mathrm{Re}\left\{\left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|_{\varphi_{\partial i}}\left[\left(\frac{\mathrm{d}w_e}{\mathrm{d}\lambda}\,\frac{\lambda}{|\mathrm{d}z/\mathrm{d}\lambda|}\right)_{\varphi_{\partial e}(\varphi_{\partial i})} + \left(\mathrm{i}\,\frac{\omega}{2} z_\partial^\star Z\right)_{\varphi_{\partial i}}\right]\right\}. \tag{27}$$

Moreover, if $\tilde{u}_\sigma$ is the normal component of the flow velocity at the σ bounding wall, it is

$$\mathrm{Re}\left(\frac{\mathrm{d}w_e}{\mathrm{d}\lambda}\,\lambda\right)_{\lambda=1/R\,\exp(\mathrm{i}\varphi_\sigma)} = \frac{\tilde{u}_\sigma}{R}\left|\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right|_{\lambda=1/R\,\exp(\mathrm{i}\varphi_\sigma)}. \tag{28}$$

The eqs. (26), (27) and (28) allow the coefficients of the series (22) and (23) to be computed. As above, a fixed point iteration process can be used.

Let us consider eq. (28). Its right-hand side is defined by the motion $\dot{z}_\sigma$ of the wall and, if this is porous, by the suction/blowing distribution $\frac{\mathrm{d}\dot{m}}{\mathrm{d}s}$ modelled along its surface. In fact, it is

$$\tilde{u}_\sigma = \mathrm{Re}\left(\dot{z}_\sigma^\star \, R \, \lambda \, \frac{\mathrm{d}z/\mathrm{d}\lambda}{|\mathrm{d}z/\mathrm{d}\lambda|}\right) + \frac{\mathrm{d}\dot{m}}{\mathrm{d}s}.$$

Instead, the left-hand side of (28) results as

$$\mathrm{Re}\left(\frac{\mathrm{d}w_e}{\mathrm{d}\lambda}\,\lambda\right)_{\lambda=1/R\ \exp(\mathrm{i}\varphi_\sigma)} = \sum_{j=0}^{J}[A_j\cos(j\varphi_\sigma) + B_j\sin(j\varphi_\sigma)]$$

with

$$A_j = e_{rj}R^j + f_{rj} \qquad \text{and} \qquad B_j = e_{ij}R^j - f_{ij} \tag{29}$$

and with $e_{rj} + \mathrm{i}\, e_{ij} = e_j$, $f_{rj} + \mathrm{i}\, f_{ij} = f_j$. Thus, J coefficients A_j, B_j can be, once and for all, calculated. As above, the FFT algorithm is used to this purpose.

The iteration process is applied to eqs. (26) and (27). First, a set of J values for the g_j coefficients is guessed, allowing the right-hand side of eq. (26) to be evaluated at $2J$ equispaced intervals of the unit circle of the λ-plane. Since the left-hand side can be written as

$$\mathrm{Im}\left(\frac{\mathrm{d}w_e}{\mathrm{d}\lambda}\,\lambda\right)_{\varphi_{\partial e}} = \sum_{j=0}^{J}[C_j\cos(j\varphi_{\partial e}) + D_j\sin(j\varphi_{\partial e})]$$

with

$$C_j = e_{ij} + f_{ij}R^j \qquad \text{and} \qquad D_j = -e_{rj} + f_{rj}R^j,$$

J values of C_j, D_j can be found by FFT algorithm, and finally, together with eqs.(29), a set of coefficients e_j, f_j can be determined. Being then possible, with these findings, to evaluate the right-hand side of eq. (27) and being

$$\mathrm{Re}\left(\frac{\mathrm{d}w_i}{\mathrm{d}\zeta}\,\zeta\right)_{\varphi_{\partial i}} = \sum_{j=0}^{J}\mathrm{Re}[g_j\exp(\mathrm{i}\,j\,\varphi_{\partial i})],$$

a new set of J values for the g_j coefficients can be computed by FFT. The process, to end, is repeated until convergence is achieved under a certain tolerance (an example of solution is made in fig.[4]).

6 CONCLUSIONS

The paper gives both the theory and a practical way of implementation for the problem of determining the 2-dimensional motions of a flow in presence of a patch of vorticity. Unlike contour dynamics [1], the treatment stands on more purely analytical grounds by making systematic use of complex analysis tools. In many aspects, the approach is of clear aeronautical derivation for what it inherits from airfoil design experience in exploiting conformal maps [11].

The unknown to solve for is the flow velocity of a nonviscous incompressible fluid filling the entire (xy) -plane of complex variable z = x + i y. It is chosen to look directly for the solution of the instantaneous boundary-value problem for the complex potential around the given vortex. To help its deduction, the shape of this vortex is previously transformed into the unit circles of two parametric and distinct planes. The theory indeed develops on the basis of Riemann's fundamental theorem of conformal mappings and then goes along with considerations on the analiticity properties of those latter. While enforcing the matching of external and internal flows on the patch boundary, it is exploited an equation already known in the field of interface dynamics [9] to compute the time evolution of the parameters of the mappings, put in the role of dynamical variables of the problem. The possibility of the presence as of a solid wall as of another boundary in pair with the vortex can be handled. An example is visited and represented in figures.

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