Model calibration by interval analysis

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SUMMARY. A criterion for model admissibility is proposed and discussed for applications in the field of model updating in presence of uncertainties. The criterion is cast in the framework of interval algebra and precede the model calibration process. The concepts are applied to the case study of a railway bridge. Two models of the bridge with different complexity are implemented, analyzed for admissibility and then calibrated. The solutions are discussed to show how the interval approach can help in providing better insight into the modelling effectiveness and validity.

1 INTRODUCTION

Calibrated finite element (FE) models are used in a variety of situations that range from the response prediction to the damage evaluation of mechanical systems [1,2]. The results depend on the accuracy of the calibration that is checked against a set of experimental data. It is customary to consider amendable the FE model parameters and to assume "true" the experimental data.

Calibration involves optimization [3] and therefore a trade off between model complexity and number of parameters is one of the main concern of the problem. In fact, the increase of the model complexity is supposed to reduce the modelling errors at the expense of augmenting the model parameters. This process generally hampers the ill-conditioning of the problem as the amount of initial information (experimental data) could not prove sufficient to correct a large number of parameters.

Sub-structuring or model reduction are alternatives to overcome this inconvenient. However, both cases suffer from some problems. In the first case the amendable parts of the model should be known in advance; in the second case the condensed parameters could not result physically representative [4]. In addition, both the model and the experimental data are affected by errors that should be considered for a meaningful calibration.

In this work the model calibration problem is investigated taking into account the effects of the model complexity in presence of experimental and modelling errors. The problem, already addressed by the authors [5], is herein reconsidered and applied to a case study concerning the calibration of a railway bridge deck.

Model parameters and experimental measures are both assumed errors biased and are considered as uncertain quantities. These latter are mathematically represented through intervals and the problem of model calibration is formulated in the framework of interval algebra [6,7]. In this context it is possible to introduce a measure of the capability of the model to reproduce the observed data. This is termed *model admissibility* and is quantified by the degree of superposition between two *n*-dimensional intervals that stand respectively for the experimental data and their numerical counterparts.

The calibration process is divided in two steps: first a check of the model admissibility is performed and, if successful, the model calibration is carried out. The process is applied to two very different models of the same structure, according to a fixed set of experimental measures, to find out the influence of the model uncertainties on the solution.

The case study is a railway bridge deck. The deck is a simply supported grillage of precast and prestressed concrete beams for which modal data are known by an independent experimental campaign. Simple sectional models of the deck are considered together with grillage models with increasing number of parameters. Model calibration is performed for each modelling level and the results compared and discussed.

2 INCLUSION AND ADMISSIBILITY

In this paper it is supposed that the experimental response of a structure to an external excitation is measured and known at least with some uncertainty. The goal is to evaluate if a possible numerical model of the structure can be considered admissible in the sense that it should be capable to reproduce the experimental response within plausible physical ranges of the model parameters.

2.1 *Concepts of interval algebra*

An uncertain quantity is here represented by an interval number and the notation used in the text to distinguish intervals by standard crisp numbers is as follows (bold letters are reserved for vectors or matrices):

$$[x] = [x_{\text{inf}}, x_{\text{sup}}] = x_{\text{c}}+\Delta x\cdot[e_{\Delta}] \tag{1}$$

where the range of values attainable by $[x]$ are bounded by the infimum x_{inf} and supremum x_{sup} limits; alternatively $x_c = (x_{\text{inf}} + x_{\text{sup}})/2$, $[e_{\Delta}] = [-1,1]$ and $\Delta x = (x_{\text{sup}} - x_{\text{inf}})/2$ are respectively the central value, the unit interval and the uncertainty radius. A conventional number, or crisp quantity, corresponds to the degenerate interval $x_{\text{inf}} = x_{\text{sup}} = x_c$. Geometrically, crisp quantities are points in an Euclidean space, whereas intervals are boxes in the n-dimensional space that degenerate to rectangles and to segments respectively in the 2- and 1-dimensional spaces.

The standard Moore's algebra between intervals is defined by crisp operations between the interval limits x_{inf}, x_{sup} [5] and since intervals are also special kind of sets, both algebraic and set operations make sense. It is important to underline that in both cases the result of an interval evaluation is a new interval that includes all the possible results obtainable when every variable is made to vary arbitrarily within its limits. This aspect of interval computation is called "dependency" and its effect is to widen the uncertainty of the result as compared to the original uncertainty of the data.

A crisp function f of the variable x is represented by $f(x)$. Similarly, an interval function is denoted by $f([x])$ and can be defined as the natural extension of $f(x)$ provided that every single occurrence of the variable x is substituted by $[x]$ in $f(\cdot)$ and that $f([x, x]) = [f(x), f(x)]$. When the interval evaluation of $f([x])$ is the interval $[y]$ that contains all possible $y = f(x)$ values for any $x \in [x]$, the inclusion property holds for $f([x])$.

As a consequence of the dependency [6,7], standard interval algebra leads to overestimate the $[y]$ bounds. Different strategies have been proposed and can be adopted to mitigate the overbounding, however this leads to a lack of uniqueness in the $[y]$ computation. An example is provided below after the introduction of the admissibility concepts.

2.2 *Admissibility criterion*

The numerical model of a structure is endowed with some uncertainty due to errors related to model order and modelling assumption. In the paper, uncertainty is restricted to the model parameters [**k**] related to the structure stiffness. Generally, in the real world, also the experimental response is endowed with a certain amount of uncertainty. In this instance the crisp vector $\boldsymbol{\omega}_e$, that collects the measures, is substituted by the interval vector $[\boldsymbol{\omega}]_e$.

The admissibility conditions derive from the comparison between the output of the model $[\boldsymbol{\omega}] = [f([\mathbf{k}])]$ and the experimental measures $[\boldsymbol{\omega}]_e$ by way of the comparison function $[f([\mathbf{k}], [\boldsymbol{\omega}]_e)] = [\boldsymbol{\omega}] \cap [\boldsymbol{\omega}]_e$. In this context admissibility is strictly related to the inclusion property of the interval functions. The possible situations are summarized in equation (2) and in Figure 1:

$$\begin{array}{ll} \text{(a)} & [\omega]\cap[\omega]_e = [\omega]_e \\ \text{(b)} & [\omega]\cap[\omega]_e \neq [\omega]_e \\ \text{(c)} & [\omega]\cap[\omega]_e = \varnothing \end{array} \qquad (2)$$

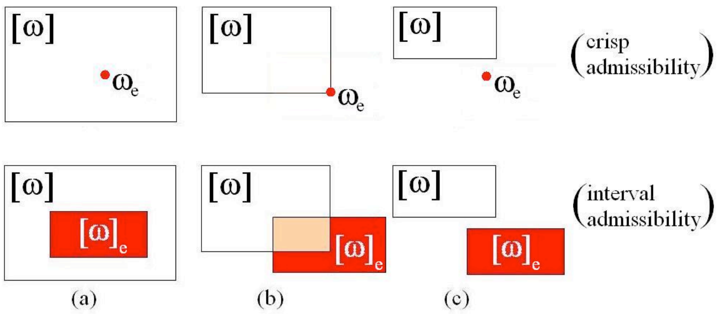


Figure 1. Inclusion/admissibility in the space of the measures

By virtue of the inclusion property, in the case (a), where a complete inclusion holds, the model is considered admissible. On the contrary, the case (c) indicates a non-admissible model because no intersection exists between experimental and numerical responses. The case (b) corresponds to an intermediate and indefinite situation. It can be considered identical to case (a) for crisp measures, but does not lead to a unique conclusion for interval measures. In this latter circumstance, the model can be considered admissible or not depending on the amount of intersection. In this work, the inclusion property in the strict sense is considered and therefore the case (b) corresponds to a non-admissible model.

In conclusion, the comparison function allows to characterize the model by checking its admissibility with respect to the known measures. An important consequence is that if the model ensues non-admissible, then it is necessary to revise the modelling options adopted. It is also observed that the conditions stated by equation (2) can be used as well within model updating methods. In [5] the authors developed the INTIM interval updating method, in which the inclusion property and the intersection criterion are used in place of conventional objective functions.

2.3 *Inclusion property*

The presented approach relies upon the inclusion property. For a quite general kind of interval functions this property is guaranteed by the inclusion theorem [7]. However, the literature [8] points out that the inclusion is satisfied at the expenses of large overbound, in the model response, due to uncertainty propagation. Severe overbounding can lead to cases in which standard interval analysis turns out to be useless from an engineering point of view. This topic is currently under investigation and computational alternatives are proposed to perform interval computations capable to limit overbounding problems, yet preserving the inclusion property.

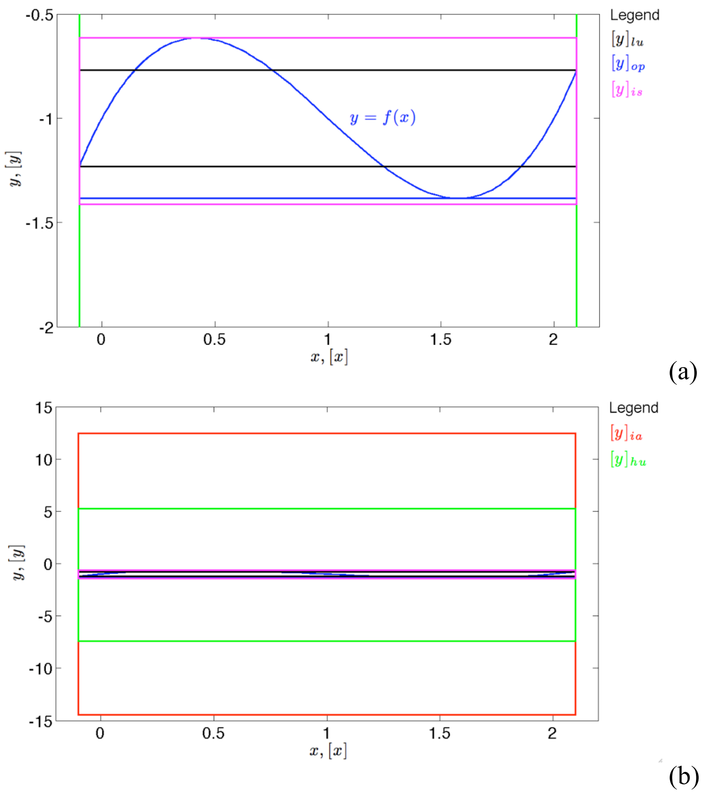


Figure 2. Computational alternatives to interval-include a sample function.

An example is shown in Figure 2, where different interval evaluations of the same sample function are shown. In order to appreciate the results, it should be observed in advance that Figure 2a is the same as Figure 2b, but zoomed along the y axis (x axis unchanged). The crisp function $y = f(x)$ should be interval evaluated providing for inclusion. The function is non-monotonic and has one maximum $y_{\max}$ and one minimum $y_{\min}$ in $[x] = [-0.1, 2.1]$ where it must be evaluated. The exact (not overbounded) interval solution would be $[y] = [y_{\max}, y_{\min}]$.

The function domain is properly selected to show that for non-monotonic functions the computation of $[y]$ is not trivial. In fact, when $f(x)$ is monotonic the solution is simply given by $[y]_{lu} = [f(x_{\inf}), f(x_{\sup})]$ that clearly underestimates and does not include the true solution, Figure 2a. Such solutions cannot be used in the admissibility criterion.

The straightforward application of the interval algebra rules would lead to the interval $[y]_{ia}$, Figure 2b, with excessive overbounding. This aspect turns out to be explosive in some circumstances and should therefore be controlled during interval evaluations. An alternative to reduce the excess of overbound is given by $[y]_{hu}$ that correspond to a partitioning of $[x]$ and reassembling of the results [9]. Theoretically, the best interval inclusion (no overbounding) can be obtained by solving two global optimization problems to get $[y]_{\text{op}} = [\min(f(x))_{[x]}, \max(f(x))_{[x]}]$,

Figure 2a. However this approach is presently unfeasible from a computational point of view. Different strategies can be devised to obtain inclusive intervals with limited overbounding. One of these $[y]_{is}$ is presently under consideration by the authors since it can provide an overbounding of order 1%. This strategy is applied to the case study reported below. In particular it is used in conjunction to methods [10] for the interval evaluation of frequencies.

Finally, it is worth noticing that the inclusion property, no matter on how the overbounding is, holds for all of the methods above and allows to apply the admissibility criterion (2). This latter however can be compromised and made useless for large overbounding.

3 CASE STUDY

The above concepts are illustrated through their application to the Sinello railway bridge located between Termoli and Vasto in the southern Italy, Figure 3. The central values and the uncertainty intervals of both the modal parameters and the mechanical parameters are available from previous experimental investigations. The masses are assumed known crisp quantities, therefore only the stiffnesses need to be tuned until the numerical frequencies include the experimental ones. For simplicity, only the elastic modulus is adjusted since it acts as a scale factor for the stiffnesses.

The bridge is a four-spans post-tensioned concrete railway bridge with a service life of about 30 years. The bridge deck has the static scheme of a simply supported grillage and is representative of many Italian railway bridges. The grillage, with the span length of 20 m, is composed by 5 longitudinal beams and 4 transversal beams and rests upon steel supports at the top of the piers.

Figure 3. Longitudinal view of the bridge.

The dynamic behaviour of the bridge deck was identified under service conditions. To this end the response to the trains transit was recorded and processed. Several dynamic tests were performed for different train speeds and weights. The vertical motion of the deck was recorded using 8 accelerometers arranged to capture the global bending and torsion modes of the deck.

The results of the data processing show that the first two frequencies, i.e. the first bending mode and the first torsion mode of the deck, vary in the ranges $[\omega_1]_e = [7.22,7.29]$ Hz and $[\omega_2]_e = [8.53,8.76]$ Hz. It is worth noticing that the said intervals are relatively narrow, therefore both frequencies are characterized by small uncertainty, 0.5% and 1.3% respectively. In a crisp setting the corresponding values would be the central values of the above ranges, i.e. $\omega_{1c,e}$ = 7.25 Hz and $\omega_{2c,e}$ = 8.64 Hz.

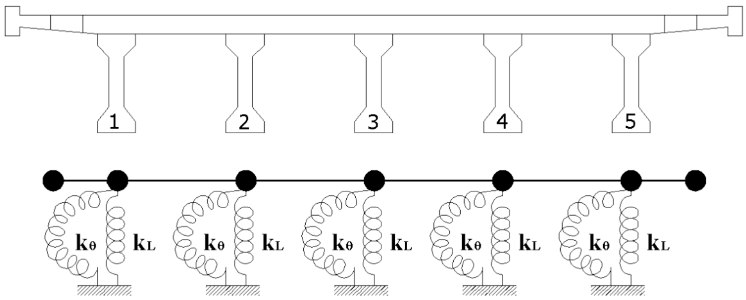


Figure 4. Transverse section and sectional model (Model I) of the bridge deck

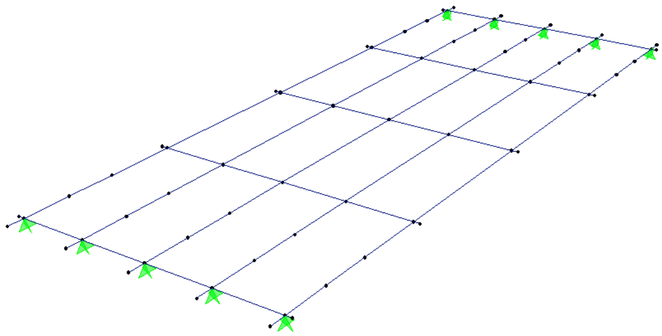
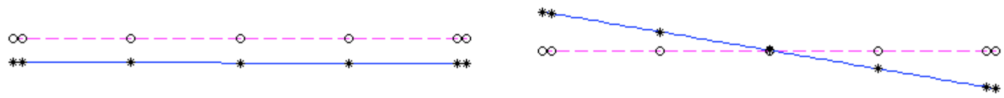


Figure 5. Grillage FE model (Model II) of the bridge deck

In situ NDT tests were also performed to confirm the design assumptions for the elastic modulus $E_d = 3.60 \cdot 10^7$ kN/m^2. The measured values are well above E_d and fall in the range $[E]_e = [3.78,4.10] \cdot 10^7$ kN/m^2 with central value $E_{c,e} = 3.94 \cdot 10^7$ kN/m^2 and relative uncertainty $\Delta\% = 100 \cdot \Delta E/E_{c,e} = 4\%$. If the uncertainties of ω and E are compared, it is seen that $\Delta E\%$ is about an order of magnitude greater than $\Delta\omega_1\%$.

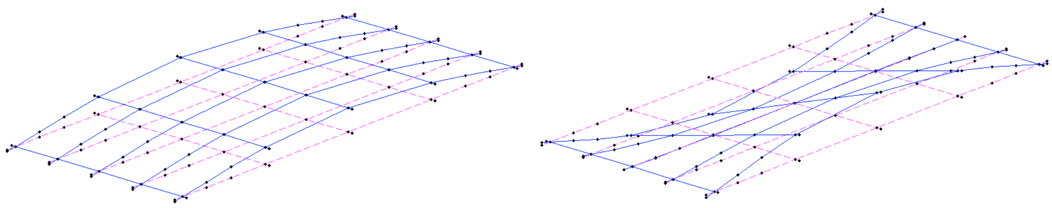
Two different modelling options are used to simulate the modal behaviour of the bridge: one is strongly condensed, Figure 4, and the other is of conventional type, Figure 5. The condensed model, Model I, is the simplest model capable to account for the fundamental deck modes (bending and torsion). It is a sectional model coherent with the hypothesis of rigid body motion of the transversal section. The longitudinal beams act as condensed masses and springs with appropriate bending k_L and torsional k_θ stiffness. Assuming a sinusoidal deformed shape of the beam, the Ritz method gives: $k_L = \pi^4 EI/(2L^3)$, $k_\theta = \pi^2 GJ/(2L)$, where E is the Young modulus, G is the shear modulus, I is the flexural moment of inertia, J is the torsional inertia constant and L is the beam span length. The lumped masses are accordingly calculated as $m = \mu L/2$, being μ the mass density per unit length. The conventional model, Model II, is a detailed FE grillage whose geometry closely reproduces even minor changes in the beams and deck geometry.

The modal response of Models I and II is first crisp evaluated using the design value E_d for the elastic modulus, Figures 6, 7. The comparison of the results gives the initial discrepancy between the two models. It is observed that both models share the same phenomenological aspects of the first two mode shapes, yet Model I results slightly less stiff than the Model II as can be appreciated by the frequencies listed in Table 1.



(a) (b)

Figure 6: Model I –1st mode (a) and 2nd mode (b)



(a) (b)

Figure 7: Model II –1st mode (a) and 2nd mode (b)

	Model I	Model II	Δ%
ω_1 [Hz]	6.66	6.94	4%
ω_2 [Hz]	7.89	8.37	6%

Table 1: Crisp frequencies for $E = E_d$.

4 MODEL CALIBRATION

Model calibration is here intended as the tuning of the parameters such that the model response matches, in the sense of a prescribed criterion, some experimental measures. In a deterministic setting this means to find the parameters that minimize the distance between the experimental and the numerical outcomes. On the contrary, in an interval setting, this means to find the parameters that allow the numerical outcomes to include the experimental ones with the minimum degree of uncertainty. This is equivalent to say that both the central value and the interval radius are to be optimised. Accordingly, the interval calibration is performed in two steps. In the first step the central value of the parameters is kept fixed and an admissibility interval is searched increasing only the radius until a complete inclusion of the whole experimental response is achieved. Once find the admissibility interval a second step is preformed to find, inside of it, an optimal interval in the sense that the inclusion is obtained with a minimum radius and a different central value.

In the following the calibrations steps are applied to both Models I and II with the purpose to discuss how the above process can be help at evaluating the admissibility of the model and the range of validity. For simplicity only the elastic modulus of the longitudinal beams is subjected to calibration. This assumption does not affect the qualitative aspects of the results, in fact the joint treatment of more parameters is only a problem of computational type.

Two aspects are important to be underlined: first, that uncertainty is a variable of the problem and affect both the bridge response and the bridge mechanical properties; second, that the soundness of the identified parameters can be checked against the measured values of the mechanical properties.

4.1 *Step 1 – Admissibility interval*

The measured elastic modulus $[E]_e$ is assumed as the initial estimate and inserted in Model I and II. The comparison is initially carried out using the crisp values for the experimentally evaluated frequencies, $\omega_{1c,e} = 7.25$ Hz and $\omega_{2c,e} = 8.64$ Hz. From Table 2 it is seen that the numerical frequencies computed by Model II include $\omega_{1c,e}$ and $\omega_{2c,e}$; hence $[E]_{II} = [E]_e$. The same does not happen for Model I that is not capable to include $\omega_{1c,e}$ and $\omega_{2c,e}$ with $[E]_e$. It should be concluded that Model I does not match the requirements of admissibility. Nevertheless it is instructive to analyse what happens if the interval of the parameters is widened. The relative uncertainty is increased to 6%, 8% and 10%. Only in this latter case the numerical frequencies computed by Model I include the experimental ones. The set of parameter values for which Model I is inclusive, or admissible, is $[E]_I=[3.546, 4.334]\cdot 10^7$ kN/m^2.

In conclusion, even if referred to different uncertainty levels, both Models I and II satisfy the inclusion (2) and can be considered admissible. However, Model II appears physically grounded, whereas Model I needs excess of uncertainty to be inclusive. This result seems to suggest an apparent obvious conclusion: the solution converges towards the experimental behaviour as the degree of model refinement increases, i.e. uncertainty reduces and intervals shrink to crisp values. However a definite sentence can be given only after the second calibration step.

<table>
<tr><td>Model I</td><td></td><td colspan="3">[ω_1] (Hz)</td><td colspan="3">[ω_2] (Hz)</td></tr>
<tr><td>$[E]/10^7$ (kN/m^2)</td><td>Δ%</td><td>num.</td><td>exp.</td><td>inclusion</td><td>num.</td><td>exp.</td><td>inclusion</td></tr>
<tr><td>[3.782, 4.098]</td><td>4%</td><td>[6.83,7.11]</td><td rowspan="4">7.25</td><td>no</td><td>[8.09,8.42]</td><td rowspan="4">8.64</td><td>no</td></tr>
<tr><td>[3.704, 4.176]</td><td>6%</td><td>[6.76,7.18]</td><td>no</td><td>[8.01,8.50]</td><td>no</td></tr>
<tr><td>[3.625, 4.255]</td><td>8%</td><td>[6.68,7.24]</td><td>no</td><td>[7.92,8.58]</td><td>no</td></tr>
<tr><td>[3.546, 4.334]</td><td>10%</td><td>[6.61,7.31]</td><td>yes</td><td>[7.83,8.66]</td><td>yes</td></tr>
</table>

Model II		[ω_1] (Hz)			[ω_2] (Hz)		
$[E]/10^7$ (kN/m^2)	Δ%	num.	exp.	inclusion	num.	exp.	inclusion
[3.782, 4.098]	4%	[7.02, 7.49]	7.25	yes	[8.63, 8.89]	8.64	yes

Table 2: Results of model calibration – step 1.

4.2 *Step 2 – Interval calibration*

The admissibility intervals $[E]_I$ and $[E]_{II}$ are the smallest intervals that satisfy the inclusion (2). However they have been evaluated under the condition of fixed central value. The central value is a position of mathematical convenience, and the related point in the interval has no particular physical property to be preferred to other points of the interval. Any point belonging to an interval has the same chance to be physically representative of the structure properties.

Therefore, it is plausible to search subintervals of $[E]_{\mathrm{I}}$ and $[E]_{\mathrm{II}}$ for which the inclusion (2) is still verified. These sub-intervals will be those endowed with the minimum uncertainty level. Of course, in doing that the central value should be let free to move inside $[E]$ in order to accommodate the position of the sub-intervals. An algorithm based on the bisection rule is used on purpose [4].

The results of this step are given in Table 3. Here, for generality, both the crisp $\omega_{1c,e}$, $\omega_{2c,e}$ and the interval evaluated $[\omega_1]_e = [7.22,7.29]$ Hz, $[\omega_2]_e = [8.53,8.76]$ Hz experimental frequencies are considered. When $\omega_{1c,e}$ and $\omega_{2c,e}$ are used, the relative uncertainty on the parameters can be reduced to $\Delta\% = 1\%$ for Model II and $\Delta\% = 2\%$ for Model I as compared to 4% and 10% of step 1.

On the contrary, when $[\omega_1]_e$ and $[\omega_2]_e$ are used, then the relative uncertainty on the parameters slightly increases, as expected: $\Delta\% = 3.3\%$ for Model II and $\Delta\% = 2.8\%$ for Model I; but surprisingly, Model I now performs better than Model II. This results shows that Model I is anyway robust particularly in the presence of uncertainties and can be used, with some cautions, either for qualitative and for quantitative simulations.

Model I		$[\omega_1]$ (Hz)		$[\omega_2]$ (Hz)	
$[E]/10^7$ (kN/m^2)	Δ%	Num.	exp . crisp	num	exp. crisp
[4.260, 4.320]	2%	[7.24, 7.30]	7.25	[8.59, 8.65]	8.64
		Num.	exp . unc.	num	exp. unc.
[4.200, 4.440]	2.8%	[7.20, 7.40]	[7.22,7.29]	[8.52,8.77]	[8.53,8.76]

Model II		$[\omega_1]$ (Hz)		$[\omega_2]$ (Hz)	
$[E]/10^7$ (kN/m^2)	Δ%	Num.	exp . crisp	num	exp. crisp
[3.826, 3.934]	1 %	[7.15, 7.25]	7.25	[8.63, 8.75]	8.64
		Num.	exp . unc.	num	exp. unc.
[3.733, 3.987]	3.3%	[7.06, 7.30]	[7.22,7.29]	[8.52, 8.81]	[8.53,8.76]

Table 3: Results of model calibration – step 2.

5 CONCLUSIONS

In this paper an admissibility criterion based on interval analysis concepts is used to evaluate the soundness of numerical models and to calibrate them against experimental data. Mathematically, admissibility is made equivalent to the inclusion property of interval functions. In this respect, strategies should be devised to limit the overbounding due to interval computations.

The criterion is applied to the case study of a railway bridge. Two models of the bridge with different complexity are considered and compared. Model I is made by rigid bodies with lumped masses and stiffnesses, Model II is a conventional FE grillage model.

The model calibration is performed in two steps. First the admissibility interval of the model parameters is sought for; next this interval is optimised and reduced to the interval of minimum radius of uncertainty. The calibration process shows that Model II is generally preferable and satisfies admissibility with lower uncertainty with crisp data. Model I should compensate the model simplicity with larger uncertainty to be admissible, but shows robust and performs better if uncertain data are considered.

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