Non classical modal parameters identification via dynamic response complexification

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SUMMARY. The paper aims to review and generalise the theoretical bases of an original identification method set by two of the authors. The method maps and analyzes the time domain response of linear dynamic systems in the complex plane. The mapping is obtained by adding the imaginary counterpart of the motion, provided by the Hilbert transform, to the real dynamic response. The formulation is specialised to deal with the problem of complex modes identification from quasi-harmonic test data. It is demonstrated that the complex plane representation turns out to be a natural framework to treat non classical damped system. An application to a reinforced concrete building tested in different structural conditions is presented.

1 INTRODUCTION

Experimental modal analysis is currently performed to identify the dynamic properties of systems and structures. Modal parameters and their changes are also valuable for damage assessment. Several methods are available on purpose [1]. Often, the dynamic response is mapped into the frequency domain to get the spectral properties of the system. This capability comes from the enlargement of the function space that accounts for the imaginary part of the dynamic response. Such a feature can be obtained also in the time domain by the help of the Hilbert transform [2].

The paper aims to review and generalise the theoretical bases of an original identification method [3], set by two of the authors, that provides for a time domain representation of linear non conservative dynamic systems in the complex plane. The Complex Plane Representation (CPR) is obtained by a complex function formed by the observed real motion and its imaginary counterpart provided by the Hilbert transform. The CPR turns out to be a natural framework to deal with non classical damped systems since it allows to arrive to the same proportionality relation valid for the modal components of classically damped systems.

Presently, the formulation is specialised to deal with the problem of complex modes identification from the analysis of the dynamic response close to resonance. The solution evolves mode by mode and relies upon the choice of a suitable reference degree of freedom (dof) of the system. A criterion for the best selection of the reference dof is given. Indices are also introduced to evaluate the quality of the estimates either global for the whole mode shape or local for the single dof. The method is applied to study the changes in the dynamic behaviour experienced by a reinforced concrete building tested in different structural configurations.

2 THEORETICAL BASIS

The Hilbert transform is the mathematical operator used to make complex the dynamic response of the system. This is done by adding the imaginary counterpart of the motion to the observed real response. In doing that some properties of the Hilbert transform are exploited. They are therefore shortly reported hereafter.

The complex dynamic response of the system *z(t)* is defined by:

$$z(t) = x(t) + i\,\tilde{x}(t), \tag{1}$$

where *x(t)* is a real time signal and $\tilde{x}(t)$ is the Hilbert transform of *x(t)* given by:

$$H\{x(t)\} = \tilde{x}(t) = P.V. \int_{-\infty}^{+\infty} \frac{x(\tau)}{\pi(t-\tau)} d\tau. \tag{2}$$

The Hilbert transform satisfies either the Energy Conservation property and the Orthogonality property that are respectively defined as follows:

$$\int_{-\infty}^{+\infty} [\tilde{x}(t)]^2 \, dt = \int_{-\infty}^{+\infty} [x(t)]^2 \, dt, \tag{3}$$

$$\int_{-\infty}^{+\infty} x(t)\tilde{x}(t) dt = 0. \tag{4}$$

Therefore, in view of equations (3) and (4), if $x(t) = x_1(t) + x_2(t)$ then:

$$\int_{-\infty}^{+\infty} \tilde{x}_1(t)\tilde{x}_2(t) dt = \int_{-\infty}^{+\infty} x_1(t) x_2(t) dt. \tag{5}$$

A further important property comes from the Bedrossian theorem and is often referred to as the Modulation property. The theorem states that if a time signal *x(t)* is made by the product between an oscillatory function *s(t)* and an envelope function *a(t)* and if both functions have non overlapping spectra *S(f)* and *A(f)*:

$$x(t) = a(t)s(t), \tag{6}$$

$$A(f) = 0 \quad \text{per} \quad |f| > f_s, \qquad S(f) = 0 \quad \text{per} \quad |f| < f_s, \tag{7}$$

then the action of the Hilbert transform is to leave the envelope unchanged and to transform only the oscillatory part:

$$\tilde{x}(t) = a(t)\tilde{s}(t), \tag{8}$$

In the case of linear dynamic systems, harmonic signals are of interest. In this case, the modulation property holds strictly for monochromatic stationary signals:

$$\begin{aligned} x(t) &= a \cdot \cos(2\pi f t) \\ \tilde{x}(t) &= H[a \cdot \cos(2\pi f t)] = a \cdot \sin(2\pi f t) \end{aligned} \tag{9}$$

When *a(t)* is a slowly varying function and *s(t)* is quasi-harmonic depending on a slowly varying frequency $f(t) = (d\theta/dt)/2\pi$):

$$x(t) = a(t)\cos[\theta(t)] \tag{10}$$

it is possible to show that, under condition (7), the Hilbert transform of (10) is [4]:

$$\tilde{x}(t) = a(t)\sin[\theta(t)] \tag{11}$$

In those cases in which at least one of the two functions *a(t)* or *s(t)* has not compact support, or their spectra are overlapped, the Bedrossian theorem states that equation (8) can be retained valid only within an error related to the amount of spectra overlapping. Defined by:

$$E = \int_{-\infty}^{+\infty} |A(f)||S(f)|\, df \Big/ \int_{-\infty}^{+\infty} |A(f)|^2\, df, \tag{12}$$

the normalized measure of the overlapping, equation (8) is generalized as:

$$\tilde{x}(t) = a(t)\sin[\theta(t)] + \Delta(t), \tag{11'}$$

where the norm of the error function *Δ(t)* vanishes as *E* goes to zero.

3 COMPLEX PLANE REPRESENTATION

3.1 *Harmonic excitation, proportional damping*

A linear and non conservative system endowed with proportional damping and excited by a harmonic force:

$$F(t) = \overline{F}\cos(2\pi f_k t), \tag{13}$$

is initially considered. If f_k corresponds to the *k*-th natural frequency then the *k*-th mode dominates the motion and the stationary resonant response at dof *h* is:

$$q_h(t) = A_{hk}\sin(2\pi f_k t + \theta_0), \quad h = 1,2,....,N \text{ dofs}. \tag{14}$$

In equation (14) A_{hk} is the response amplitude (with sign) ad θ_0 is 0 for displacement or acceleration and is $\pi/2$ for velocities. If the response q_r at dof *r* is selected as the reference measure, the following simple proportional relationship between q_r and the others modal components q_h exists:

$$q_h(t) = \psi_{hk} q_r(t), \quad h = 1,2,....,N; \qquad \psi_{hk} = \frac{A_{hk}}{A_{rk}}, \quad h = 1,2,....,N, \tag{15}$$

Where $[\Psi_{\mathbf{k}}]^r$ is the k-th real mode shape, scaled such that its r-th entry is equal to one.

3.2 *Harmonic excitation, non proportional damping*

In the case of non proportional damping the general form of the response becomes:

$$q_h(t) = A_{hk} \sin(2\pi f_k t) + B_{hk} \cos(2\pi f_k t), \tag{16}$$

and the relationship of simple proportionality can be recovered reverting to the complex plane representation of $q_h(t)$:

$$\begin{aligned} z_h(t) &= A_{hk} \sin(2\pi f_k t) + B_{hk} \cos(2\pi f_k t) - iA_{hk} \cos(2\pi f_k t) + iB_{hk} \sin(2\pi f_k t) = \\ &(A_{hk} + iB_{hk})[\sin(2\pi f_k t) - i\cos(2\pi f_k t)]. \end{aligned} \tag{17}$$

Using the Hilbert properties (3) to (5), equation (17) can be rewritten through a generalized relationship between complex quantities:

$$z_h(t) = \psi_{hk} z_r(t) = (\xi_{hk} + \mathrm{i}\eta_{hk}) z_r(t), \quad \xi_{hk} = \frac{A_{hk} A_{rk} + B_{hk} B_{rk}}{A^2{}_{rk} + B^2{}_{rk}}, \quad \eta_{hk} = \frac{A_{hk} B_{rk} - B_{hk} A_{rk}}{A^2{}_{rk} + B^2{}_{rk}} \tag{18}$$

Where $[\Psi_{\mathbf{k}}]^r$ is the k-th complex mode shape, scaled such that its r-th entry is equal to one.

3.3 *Sweep excitation*

Consider now an excitation given by a quasi-harmonic forcing function with frequency changing linearly in time $[0,\Delta T]$ in the range $[f_0, f_0+\Delta f]$:

$$F(t) = \overline{F} \cos[\theta(t)], \qquad \text{with } \theta(t) = 2\pi\left(f_0 t + \frac{1}{2}\frac{\Delta f}{\Delta T} t^2\right) \tag{19}$$

If $\Delta f/\Delta T \ll 1$, in view of equation (17) the system response at dof h is assumed as:

$$q_h(t) = A_{hk}(t) \sin[\theta(t)] + B_{hk}(t) \sin[\theta(t)], \tag{20}$$

where the instantaneous frequency of the response coincides with excitation frequency and the instantaneous envelopes $A_{hk}(t)$ e $B_{hk}(t)$ equates time by time those of equation (17) valid for a constant frequency $f=f(t)=f_0+(\Delta f/\Delta T)t$. Considered that $\Delta(t)$ in equation (11') is negligible when $\Delta f/\Delta T \ll 1$, it is assumed that at time t_k, where the instantaneous frequency $f(t)$ coincides with the k-th resonant frequency f_k, equation (18) holds true also in the present case.

4 PARAMETERS ESTIMATE

4.1 *Frequency and damping*

The representation (1) in the complex plane of the dynamic response *x(t)* allows for a straightforward estimation of both the instantaneous envelope and frequency by means of equation (10) and (11) [5]:

$$a(t) = | z(t) |, \qquad f(t) = \frac{1}{2\pi} \frac{\angle[z(t)]}{dt}. \tag{21}$$

4.2 *Mode shapes*

The estimate of the mode shapes needs further development that is given below. The bases of the method were initially set in [3]; here a generalization is provided. In order to simulate actual experimental conditions a white noise $e_h(t)$ is added to $q_h(t)$ that is assumed to be known in a discrete set of *m* time instant t_j spaced with constant time step Δt:

$$q_h^*(t_j) = q_h(t_j) + e_h(t_j) = A_{hk} \sin(2\pi f_k t_j) + B_{hk} \sin(2\pi f_k t_j) + e_{hj}, \qquad \rho_h^2 = \frac{1}{m} \sum_{j=1}^{m} [e_{hj}]^2. \tag{22}$$

Using the complex plane representation (18) for the discrete signal (22) the *k*-th mode shape is identified by minimizing the following error function that measures the deviation from the proportionality in the complex plane between any dof *h* and the reference dof *r*:

$$\varepsilon_h = \Sigma_{j=1}^{m} \left| z_h^*(t_j) - (\xi_{hk} + i\eta_{hk}) z_r^*(t_j) \right|^2. \tag{23}$$

Taking the derivative of ε_h with respect to ξ_{hk} and η_{hk} and introducing the Hilbert properties (3), (4) and (5) one gets the following two uncoupled equations that provide the estimates ξ'_{hk} and η'_{hk} to the real and imaginary part of the *h*-th component of the *k*-th mode shape:

$$\xi'_{hk} = \frac{\sum_{j=1}^{m} q_h^* t(j) q_r^* t(j)}{\sum_{j=1}^{m} q_r^{*2} t(j)}, \quad \eta'_{hk} = -\frac{\sum_{j=1}^{m} q_h^* t(j) \tilde{q}_r^* t(j)}{\sum_{j=1}^{m} q_r^{*2} t(j)},. \tag{24a}$$

$$\xi'_{hk} = \frac{A_{hk} A_{rk} + B_{hk} B_{rk}}{A^2{}_{rk} + B^2{}_{rk} + 2\dfrac{\rho_r^2}{A^2{}_{rk} + B^2{}_{rk}}}, \quad \eta'_{hk} = \frac{A_{hk} B_{rk} - B_{hk} A_{rk}}{A^2{}_{rk} + B^2{}_{rk} + 2\dfrac{\rho_r^2}{A^2{}_{rk} + B^2{}_{rk}}}. \tag{24b}$$

Equation (24a) is used for computation, whereas equation (24b) is useful for theoretical considerations. Equation (24b) is the same as (18) modified to account for noise and shows that the estimated parameters coincide with the exact ones when the reference measurement is noise free. In order to get the best estimate an appropriate choice of the reference dof is hence required.

The criterion adopted to select the best dof is to minimize the noise to signal ratio in the reference dof. The best dof to be used as reference is dof s for which:

$$E_s = \min\left(E_r\right) \qquad \text{with } E_r = \Sigma_{h=1}^{N} \varepsilon_{hr},\ r = 1,2,....N. \tag{25}$$

It is noted that the index E_r offers also a global evaluation of the accuracy of the estimates of the relevant mode shape. A local evaluation referred to the single dof is obtained as:

$$\sigma_h = \frac{1}{N}\sum_{\mathrm{r}=1}^{\mathrm{N}} (\psi_h^r - \psi_h^s)^2, \tag{26}$$

considered that in ideal condition ψ^r and ψ^s should be equal at dof h, the variance of the estimates among all dofs can be considered as a measure of the stability of the identification.

5 CASE STUDY

The CPR (Complex Plane Representation) method developed above has been used to identify the modal parameters of a building subjected to a large experimental campaign devoted to analyze changes in the modal behaviour for different structural conditions. A small sample set of results is reported with the purpose of illustrating the method capability.

5.1 *Building and instrumentation*

The building under test, Figure 1, is the former place of the Municipality of Vagli di Sotto (LU). The building has two storeys (total avg height 6.7m) and a rectangular plan of 27.25m (longitudinal or X direction) by 13.60m (transversal or Y direction). The structure is composed by plane resisting r/c frame in the transversal direction. Secondary beams in the longitudinal facade connect the main frames. The floors and the roof are of tile-lintel type and allow for rigid floor assumption.

The dynamic response has been recorded using unidirectional piezoelectric accelerometers. Four accelerometers per floor have been used. The sensor placement is shown in Figure 2 by oriented arrows. Dynamic range, linearity range and resolution of the instruments are respectively: ±5g, [0.3 4000] Hz and 10 μg. The acquisition has been performed via a multichannel analyzer endowed with a 24 bit A/D card and 124dB dynamics. The forced vibration have been obtained through a dynamic actuator capable to deliver a maximum force of 4180 kN at 30 Hz.

5.2 *Demolition stages and dynamic tests*

The building should be dismantled and it was decided to proceed with a controlled demolition. The demolition stages were aimed at studying the effects of the infillings on the overall dynamic behaviour of the building. The masonry panels were removed in 11 different stages starting from the finished building (stage 0) and ending at the bare frames (stage 11) where all the infillings were removed. The sequence of demolition is given in Figure 2. Stages 1 to 4 refer to ground floor whereas stages 5 to 11 refer to first floor. The broken masonry was left on the floor in order to preserve the overall mass of the building.

It was performed a testing session for each demolition stage. The testing session is composed by a preliminary sweep test (f_{sampl} = 100Hz) and by a number of subsequent harmonic tests (f_{sampl} =

200Hz) at resonant frequencies identified during the sweep test. Both the forced response and the free decay of the oscillations are recorded. During the sweep test the frequency of the forcing function varies linearly with a rate of 0.02 Hz/sec in the range [0 12] Hz. During the harmonic test the frequency of the forcing function is kept constant.

Figure 1: General view of the building.

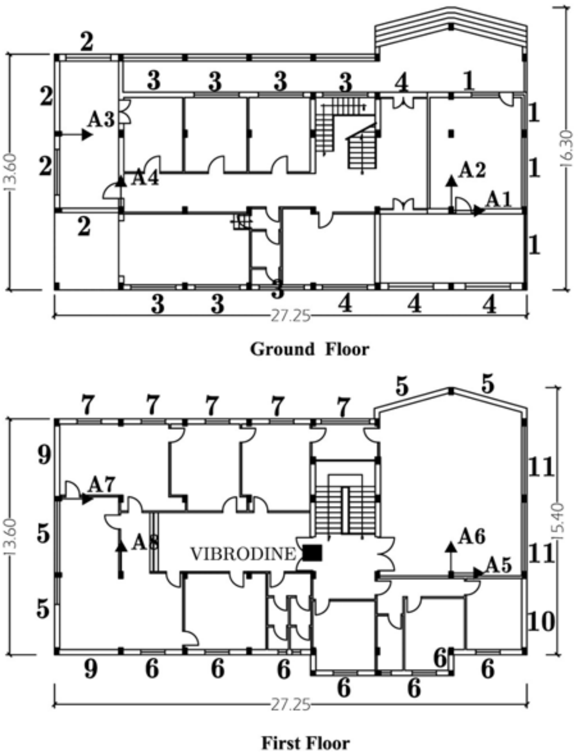
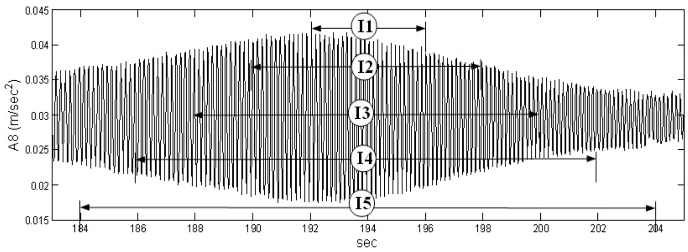


Figure 2: Plan view (numbers = demolition sequence; arrows = accelerometers)

5.3 *Identification of the modal parameters*

The identification has been preceded by a sensitivity analysis aimed at evaluating changes in the modal parameters as a function of the time window selected for the analysis during a sweep test. A sample time history (accelerometer no. 8) is provided in Figure 3. The results refer to transversal excitation and to the first frequency for different time window around $t = 194$ sec, close to resonance. The changes in the mode shapes are measured by the MAC [6], whereas the changes in the frequency are given in percent. All the changes are computed with respect to the shortest interval I1. It is concluded that the identified mode shapes and frequencies do not depend on the time window processed as long as it encompasses the time instant at resonance.



Interval	MAC	Δf (%)
I2	1.000	0.998
I3	0.999	0.996
I4	0.999	0.998
I5	0.998	0.997

changes with respect to I1

Figure 3: Time intervals used for the sensitivity analysis and results

The modal model of the building has been identified for all the structural states corresponding to the different demolition stages except for states 4 and 5 for which sweep tests are unavailable due to technical problems.

The comparison between the frequencies obtained with the CPR method and conventional methods operating on harmonic tests is shown in Figure 4. In both cases, as expected, the first frequency decreases progressively as long as the demolition of the masonry panels advances. The major changes happen when the infillings of the ground floor are removed or when infillings without openings are removed. In any case, an excellent matching is observed between the two set of results; however, the CPR results show more regular and monotonic than those obtained with different identification methods.

The tracking of changes in the mode shapes is given in Figure 5 according to a sample case concerning the first mode. Changes are quantified by means of the MAC index (complex evaluated) that is computed state by state with respect to the mode shape of state 0 (building in its original configuration). In the same figure, superimposed to the MAC value, the “complexity index” I_c is also reported. The I_c is defined as:

$$I_c = 1 - \frac{\left\langle \mathrm{Re}(\Psi), \mathrm{Re}(\Psi) \right\rangle}{\left\langle \Psi, \Psi \right\rangle}, \tag{27}$$

where Ψ is the generic eigenvector and $\mathrm{Re}(\Psi)$ is its real part. Both indices ranges from 0 to 1; MAC tends to zero for uncorrelated modes and I_c tends to zero for real evaluated modes. From the figure it appears that strong changes in MAC are associated to as many strong changes in I_c.

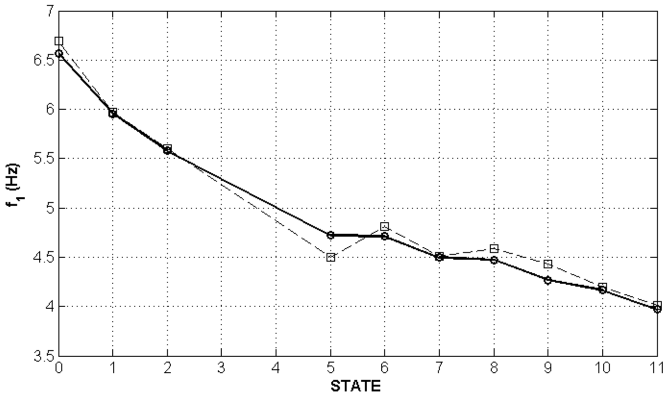


Figure 4: Variation of the first frequency vs. demolition stages.
Comparison between cpr method (sweep tests) and conventional methods (harmonic tests).

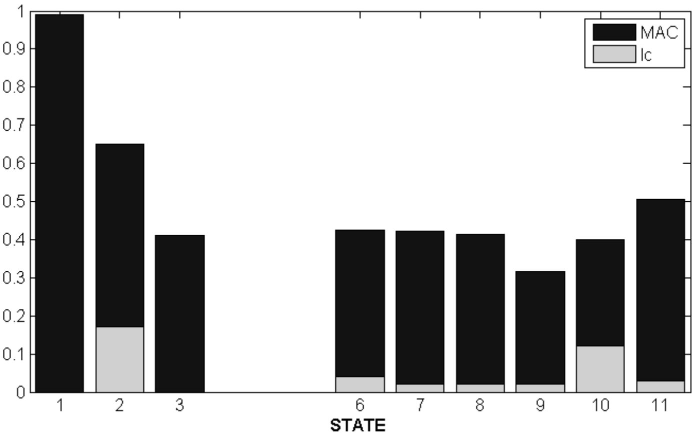


Figure 5: Variation of MAC index of mode 1 vs. different demolition stages
Index of mode complexity I_c superimposed.

6 CONCLUSIONS

It has been presented the CPR Complex Plane Representation Method for the identification of the modal parameters of systems endowed with non proportional damping. The method works in the time domain and uses the Hilbert transform to construct the imaginary counterpart of the system dynamics. The identification of the complex modes is decoupled in two independent equations each of which related to the real or to the imaginary part alone.

The results show that the proposed formulation, in spite of its theoretical and practical simplicity, leads to a very precise and reliable identification of the modal parameters. In particular, the phase shifts of the mode shape components, hard to estimate using standard techniques, can be identified as well. Two indices have introduced to evaluate the quality of the estimates of both the eigenvectors and the modal coordinates.

The performance of the method has been demonstrated against experimental data coming from a reinforced concrete building subjected to a controlled demolition of the masonry infillings. After each demolition step, a sequence of harmonic and sweep tests has been carried out in order to identify changes in the modal parameters. Harmonic tests have been processed with standard techniques and sweep tests have been analysed using the CPR method. The comparison of the results show excellent agreement between the two.

References

[1] *DAMAS 2007- VII International Conference on Damage Assessment of Structures*, Torino, Italy, 25-27 June, (2007).

[2] Hahn, S.L., *Hilbert Transforms in Signal Processing*, Artech House, Norwood MA (1996).

[3] Spina D., Valente C., "A Representation of the Harmonic Response in the Complex Plane for Modal Parameters Identification", in *Proc. XV IMAC*, Orlando, Florida (1997).

[4] Spina D., "Nonlinear system free vibrations analysis for mechanical parameters identification" (in Italian), *PhD Thesis University of Rome "La Sapienza"* (1994).

[5] Thrane, N., *The Hilbert Transform*, Bruel & Kjaer Technical Review No. 3 (1984).

[6] Ewins, D.J., *Modal Testing: Theory, Practice and Applications*, 2nd Edition. Research Study Press (2000).