Wavelet-based Estimation of Fully Non-stationary Spectra and Applications to Seismic Engineering

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SUMMARY. First a method is presented to define a wavelet-based time-dependent spectrum for arbitrary non-stationary processes. Numerical results, assessed in terms of statistics depending on the spectral moments, prove satisfactory. Then the method is used to estimate the time-dependent spectrum of single-degree-of-freedom linear systems, in conjunction with an approximate analytical relation between the input and the output wavelet transform, already available in the literature. The validity of such relation is found to depend on the system and the input parameters. Interesting applications are feasible for linear systems with viscous dampers, subjected to seismic input.

1 INTRODUCTION

Non-stationary random processes, such as seismic motions or transient loads on structures, play a crucial role in many engineering applications. In a most general case, both the statistical moments and the frequency content of the process change in time. Earthquake records, for instance, decay in time and exhibit a time-dependent frequency composition due to the dispersion of the propagating seismic waves [1]. Capturing these features is critical in predicting the structural response, as shown by a number of studies in the recent decades [2,3].

The time-dependent frequency content of a signal cannot be captured by ordinary Fourier analysis since the Fourier transform provides only the average spectral composition of a signal. Then certain concepts of the traditional spectral theory have been adapted for analyzing non-stationary processes. For instance, in the Wigner-Ville method (WVM) a time-dependent spectrum at time t has been defined as the Fourier transform of an instantaneous correlation function, $R(\tau, t)$, which corresponds to the standard correlation function $R(\tau)$ with time lag τ centered at time t [4,5]. Alternatively, local information on the frequency content has been obtained by the so-called short-time Fourier transform (STFT), i.e. by applying the Fourier transform to "short windowed" data at various time instants [6]. In this context, a time-dependent spectrum is defined as the ensemble average of the squared amplitude of the STFT of the process. However there exist inherent limitations in the WVM and the STFT. The WVM cannot reflect the local behavior of the process at time t since the variable τ must be integrated over an infinite range to compute the Fourier transform of the instantaneous correlation function $R(\tau, t)$, due to the non-decaying nature of the harmonic waves. On the other hand, all the STFT Fourier coefficients feature the same frequency bandwidth, that is approximately $1/T$ if T is the width of the time window.

The inherent limitations of the STFT can be overcome by wavelet analysis [7,8]. The wavelet transform (WT) provides a time-frequency representation of a signal based on a double series of basis functions named 'wavelets', generated by scaling and shifting a single "mother" function.

Scaling allows the time duration of the wavelet to be adjusted according to the local frequency content of the signal. This allows to capture high- or low-frequency components with a significant reduction of the computational effort as compared to the STFT.

Recently, wavelet analysis has been applied to estimate non-stationary spectra [9]. Specifically, applications have concerned non-stationary processes belonging to the class of Priestley's oscillatory processes, for which an exact power spectrum is defined [10]. The method, which applies for arbitrary wavelet bases, has led to satisfactory results for both uniformly-modulated and fully non-stationary processes.

This paper investigates further, potential applications of the method proposed in ref. [9]. It will be shown that it provides a wavelet-based definition of time-dependent spectrum, with a consistent physical meaning, for arbitrary (non-oscillatory) non-stationary processes, that is for processes for which no exact spectrum is defined. Then the accuracy of the proposed wavelet-based spectrum will be assessed in terms of statistics depending on time-dependent spectral moments. As a second step, the method will be applied to estimate the time-dependent power spectrum of single-degree-of-freedom systems (SDOF), in conjunction with an analytical relation between the response spectrum and the mean-square value of the input WT based on a previous method [11]. The validity of such relation will be assessed and interesting applications will be found for linear systems equipped with dissipative devices and subjected to seismic input.

2 WAVELET-BASED POWER SPECTRUM ESTIMATION

The continuous WT of an arbitrary stochastic process $f(t)$ is defined as

$$W(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \psi\left(\frac{t-b}{a}\right) f(t)\, dt \tag{1}$$

where $\psi(t)$ is the mother wavelet, a is the scale and b the shift parameter. If $f(t)$ belongs to the class of Priestley’s oscillatory processes [10], based on the localization properties of the wavelet functions, in ref. [9] the following expression has been found for the mean square value (m.s.v.) of the WT

$$E\left[W^2(a,b)\right] = 2\pi a \int_{-\infty}^{+\infty} \left|\Psi(a\omega)\right|^2 S(\omega,b)\, d\omega \tag{2}$$

where $\Psi(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \psi(t) e^{-i\omega t} dt$ is the Fourier transform of the mother wavelet and $S(\omega,b)$ is the time-dependent power spectral density (PSD) of the process $f(t)$. For $S(\omega,b)$ the following analytical form

$$S(\omega,b) = \sum_{j=1}^{n} c_j(b) \left|\Psi(a_j\omega)\right|^2 \tag{3}$$

has been proposed, where $c_j(b)$ are time-dependent constants and n is the number of scales used to estimate the sought PSD. The unknown constants $c_j(b)$ are computed by the set of equations

$$E\left[W^2\left(a_k,b\right)\right]=2\pi a_k\sum_{j=1}^{n}c_j\left(b\right)\int_{-\infty}^{+\infty}\left|\Psi\left(a_k\omega\right)\right|^2\left|\Psi\left(a_j\omega\right)\right|^2 d\omega \tag{4}$$

for k=1,2,…n. In ref. [9] results compared to exact spectrum have been found satisfactory. Also the method enjoys computational efficiency since the coefficient matrix of the solving system (4) is time independent and then shall be inverted only once.

It is now of interest to assess whether and, in which sense, the method proposed in ref. [9] can be extended to arbitrary (i.e., non-oscillatory) non-stationary processes. To this purpose first note that, for a given arbitrary non-stationary process $f(t)$, the continuous WT at a given scale a_k can be approximated as

$$W\left(a_k,b\right)=\frac{1}{\sqrt{a_k}}\int_{-\infty}^{+\infty}\psi\left(\frac{t-b}{a_k}\right)f\left(t\right)dt\approx\frac{1}{\sqrt{a_k}}\int_{b_k^-}^{b_k^+}\psi\left(\frac{t-b}{a_k}\right)f\left(t\right)dt \tag{5}$$

where $b_k^{\pm}=b\pm a_k\,d/2$, being d the support of the mother wavelet $\psi(t)$. From Eq.(5),

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{b_k^-}^{b_k^+}W\left(a_k,b\right)e^{-i\omega b}db&\approx\frac{1}{\sqrt{2\pi a_k}}\int_{b_k^-}^{b_k^+}\left[\int_{b_k^-}^{b_k^+}\psi\left(\frac{t-b}{a_k}\right)f\left(t\right)dt\right]e^{-i\omega b}db=\\&\approx\sqrt{\frac{a_k}{2\pi}}\int_{b_k^-}^{b_k^+}\left[\int_{-d/2}^{d/2}\psi\left(\tau\right)e^{ia\omega\tau}d\tau\right]f\left(t\right)e^{-i\omega t}dt\approx\sqrt{2\pi a_k}\,\Psi^*\left(a_k\omega\right)F_k\left(\omega,b\right)\end{aligned} \tag{6}$$

where $(t-b)/a_k=\tau$ and $F_k(\omega,b)$ is given by

$$F_k\left(\omega,b\right)=\frac{1}{\sqrt{2\pi}}\int_{b_k^-}^{b_k^+}f\left(t\right)e^{-i\omega t}dt\,. \tag{7}$$

That is, $F_k(\omega,b)$ is the Fourier transform as applied to the portion of the process $f(t)$ spanning the interval $\left[b_k^-,b_k^+\right]$, centred at $t=b$ and whose length, $b_k^+-b_k^-=a_k d/2$, depends on the scale a_k. Similarly it can be stated that, for any b belonging to the interval $\left[b_k^-,b_k^+\right]$, the WT $W(a,b)$ can be obtained by taking the inverse Fourier transform of the r.h.s. of Eq.(6), i.e.

$$W\left(a_k,b\right)\approx\sqrt{a_k}\int_{-\infty}^{+\infty}\Psi^*\left(a_k\omega\right)F_k\left(\omega,b\right)e^{+i\omega b}d\omega\,. \tag{8}$$

In fact recognize that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{b_k^-}^{b_k^+}W\left(a_k,b\right)e^{-i\omega b}db\right]e^{+i\omega\bar{b}}d\omega=\frac{1}{2\pi}\int_{b_k^-}^{b_k^+}W\left(a_k,b\right)\left[\int_{-\infty}^{+\infty}e^{i\omega\left(\bar{b}-b\right)}d\omega\right]db$$

$$=\int_{b^-}^{b^+}W\left(a_k,b\right)\delta\left(\bar{b}-b\right)db=W\left(a_k,\bar{b}\right) \tag{9}$$

Then, based on Eq.(6) and Eq.(8) the identity

$$\int_{b^-}^{b^+}W^2\left(a_k,b\right)db=2\pi a_k\int_{-\infty}^{\infty}\left|\Psi\left(a_k\omega\right)\right|^2\left|F_k\left(\omega,b\right)\right|^2 d\omega, \tag{10}$$

can be set, which can be readily derived (details are omitted for brevity) as the well-known Parseval's identity [10]. Taking the expectation of both sides of Eq.(10) and assuming that the m.s.v. of $W(a_k,b)$ is constant over the time support $a_k d$ of the wavelet at scale a_k, lead to

$$\int_{b^-}^{b^+}E\left[W^2\left(a_k,b\right)\right]db=E\left[W^2\left(a_k,b\right)\right]=2\pi a_k\int_{-\infty}^{\infty}\left|\Psi\left(a_k\omega\right)\right|^2\hat{S}\left(\omega,b\right)d\omega, \tag{11}$$

where

$$\hat{S}\left(\omega,b\right)=\frac{E\left[\left|F_k\left(\omega,b\right)\right|^2\right]}{a_k d}. \tag{12}$$

is a time-dependent spectrum describing the frequency content of the portion of the process $f(t)$ spanning the interval $\left[b_k^-,b_k^+\right]$ centred at $t=b$. Eq.(11) is formally equivalent to Eq.(2): this suggests that: (i) when the method in ref. [9] is applied to arbitrary non-stationary processes, the obtained wavelet-based spectrum (3) corresponds to an estimate of the time-dependent spectrum in Eq.(12); (ii) in this context, therefore, Eq.(3) can be consistently taken as a *wavelet-based definition* of the time-dependent frequency content of an arbitrary non-stationary process.

Since no exact spectrum can be defined for an arbitrary process, the accuracy of the power spectrum estimate (3) will be performed in terms of time-dependent statistics depending on the spectral moments m_i [13], such as the instantaneous rate of zero up-crossings (IU), the instantaneous rate of peaks occurrence (IP), the instantaneous bandwidth (IB) parameter given respectively by

$$\Omega\left(b\right)=\sqrt{\frac{m_2}{m_0}}\ ;\qquad N\left(b\right)=\frac{1}{2\pi}\sqrt{\frac{m_4}{m_2}}\ ;\qquad \varepsilon\left(b\right)=\sqrt{1-\frac{m_2^2}{m_0 m_4}}\ . \tag{13a-c}$$

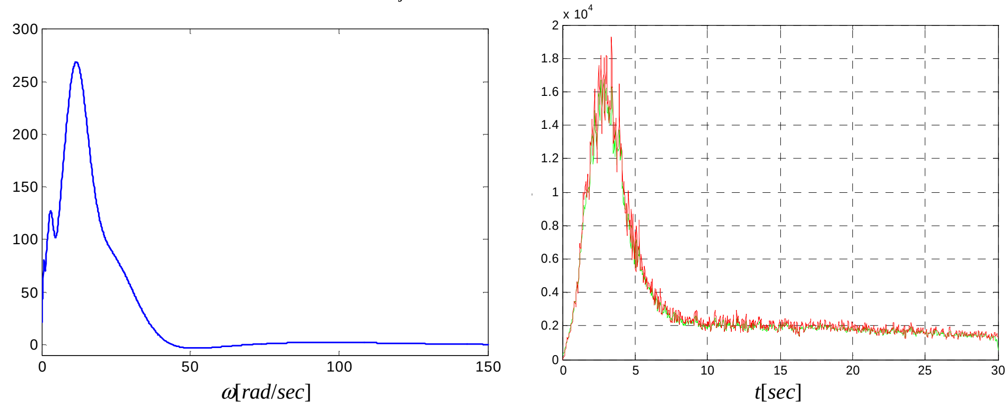
Also, the probability distribution that the process remains below a given level x in the time interval (0, T) and the largest peak estimate cast within the Vanmarcke formulation

$$p_T(x)=\exp\left[-\int_0^T \alpha(\tau)\,d\tau\right] \quad \text{for} \quad \alpha(b)=\frac{\Omega}{\pi}\exp\left(-\frac{x^2}{2m_0}\right); \quad E(x_1)=\int_0^1 p_T^{-1}(u)\,du \, , \qquad (14\text{-}15)$$

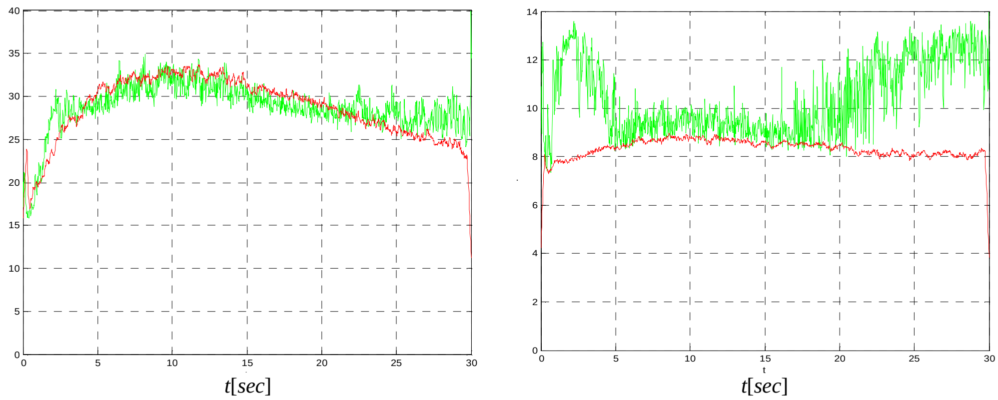
will be considered.

2.1 *Numerical results*

As a study case, consider the fully non-stationary process generated according to the method in ref. [12] and modeled based on the El Centro earthquake. The m.s.v. of the WT is computed over 200 samples of the process of length T = 30 *sec* and the proposed time-dependent spectrum (3) is built by solving a set of Eqs.(4) where $a_j = \sigma^j$, for $\sigma = 2$ and j = 1,2….8.

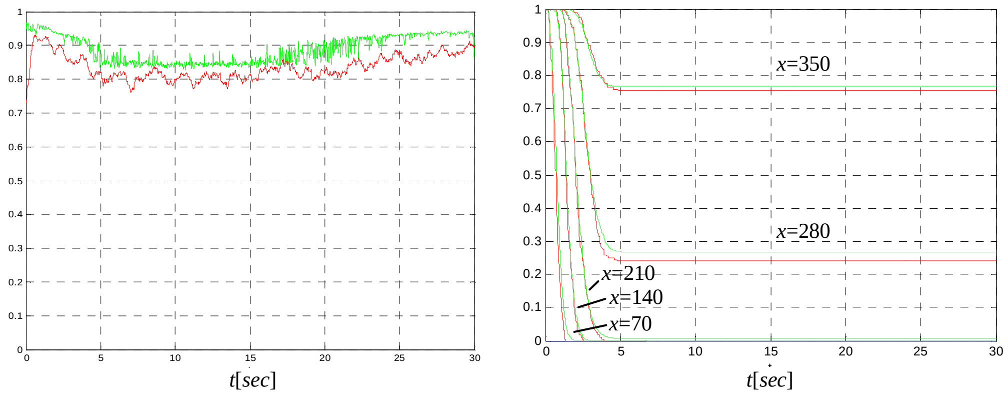


Figures 1-2: Time-dependent power spectrum at t = 1.5 sec and m.s.v.

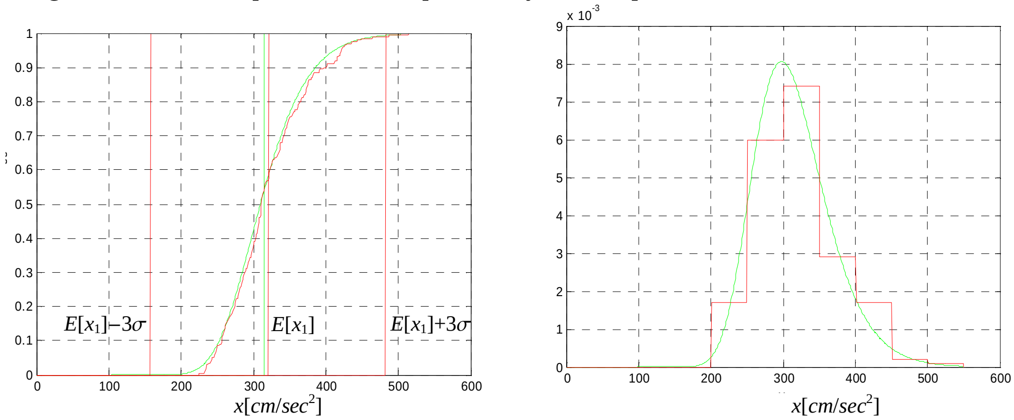


Figures 3-4: Time-dependent IU and IP.

As an example, Figure 1shows the time-dependent power spectrum (3) at t = 1.25 sec. Figure 2 through 8 shows the m.s.v. and the statistics (13)-(15) for the entire duration of the process, computed based on the spectral moments of the power spectrum (3) (green line) and based on the 200 generated samples (red line). A substantial good agreement is found for all the statistics depending on the 2nd order spectral moments, while a certain loss of accuracy is encountered for the statistics depending on the 4th spectral moments, more sensitive to the actual shape of the spectrum. However, errors can be considered within engineering margins.



Figures 5-6: Time-dependent IB and probability that the process remains below a level *x*.



Figures 7-8: Largest peak estimate and probability density that the process remain below a level *x*.

3 SDOF SYSTEMS

Potential applications of the method proposed in ref. [9] for the estimation of the power spectrum of SDOF systems are investigated. Consider the motion equation of a SDOF in the form

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = f(t) \qquad (16)$$

where $f(t)$ is a fully non-stationary process, ω_0 is the natural frequency and ζ is the damping coefficient. Apply the WT to both sides of the previous equation. It yields [11]

$$\frac{\partial^2 W_x(a,b)}{\partial b^2} + 2\zeta\omega_0 \frac{\partial W_x(a,b)}{\partial b} + \omega_0^2 W_x(a,b) = W_f(a,b) \qquad (17)$$

where $W_x(a,b)$ and $W_f(a,b)$ are the continuous WT of the input and the output process respectively, defined by Eq.(1). Assume that the wavelet basis is a Littlewood-Paley basis, whose wavelet functions at different scales feature a non-overlapping and stepped frequency content

$$\Psi(\omega) = \frac{1}{\sqrt{2(\sigma-1)\pi}} \qquad \pi \le \omega \le \sigma\pi$$
$$0 \qquad\qquad \textit{otherwise} \tag{18}$$

The WT $W_f(a,b)$ in the r.h.s. of Eq.(17), at a given scale a, may be then thought of as a stochastic process of the time parameter b, whose frequency content spans non-overlapping frequency bands $[\pi/a, \sigma\pi/a]$. For the WT $W_f(a,b)$ then holds the following harmonic approximation [11]

$$\begin{aligned} W_f(a,b) &\approx V(b)\,\mathrm{Sin}(\omega_1 b+\phi_1)\,\mathrm{Sin}(\omega_2 b+\phi_2) = \\ &= \frac{V_j(b)}{2}\left\{\mathrm{Cos}\left[(\omega_{1j}-\omega_{2j})b+\phi_{1j}-\phi_{2j}\right]-\mathrm{Cos}\left[(\omega_{1j}+\omega_{2j})b+\phi_{1j}+\phi_{2j}\right]\right\} \end{aligned} \tag{19}$$

where $\omega_1=(\sigma-1)\pi/2$, $\omega_2=(\sigma+1)\pi/2$; ϕ_1 and ϕ_2 are random variables uniformly-distributed over the interval $[0,2\pi]$; $V(b)$ is an envelope process with time-dependent m.s.v. given by squaring and averaging both sides of Eq.(19)

$$4E\left[W_f^2(a,b)\right] = E\left[V^2(b)\right] \tag{20}$$

Based on harmonic balance over adjacent time intervals and neglecting the effects of the initial conditions, in ref. [11] the following relation

$$E\left[W_x^{\,2}(a_j,b)\right] = \frac{1}{8}E\left[V_j^{\,2}(b)\right]\left\{\frac{1}{k_{1j}}+\frac{1}{k_{2j}}\right\} \tag{21}$$

has been provided between the m.s.v. of $W_x(a,b)$ and $W_f(a,b)$, where k_{1j} and k_{2j} are

$$k_{1,2j} = \left[\omega_0^2-(\omega_{1j}\pm\omega_{2j})^2\right]^2+\left[2\zeta_0\omega_0(\omega_{1j}\pm\omega_{2j})\right]^2 \tag{22}$$

Based on Eq.(20) and Eq.(21), the time-dependent power spectrum (3) of the response reads

$$S_x(b,\omega) = \sum_{j=1}^{n} c_j(b)\left|\hat{\Psi}(\omega a_j)\right|^2 = \sum_{j=1}^{n}\frac{E\left[W_f^{\,2}(a_j,b)\right]}{k_j Q_{j,j}}\left|\hat{\Psi}(\omega a_j)\right|^2 , \tag{23}$$

where

$$Q_{j,j} = 2\pi a_j\int_{-\infty}^{+\infty}\left|\hat{\Psi}(\omega a_j)\right|^4 d\omega; \quad k_j = \frac{1}{2}E\left[V_j^{\,2}(b)\right]\left\{\frac{1}{k_{1j}}+\frac{1}{k_{2j}}\right\}. \tag{24}$$

In ref. [11] the validity of the input-output relation (21) has not been discussed. This can be pursued, however, by noting that Eq.(21) can be readily derived if the Duhamel integral is manipulated as follows

$$
\begin{aligned}
W_x\left(a_j,b\right) &= \int_{-\infty}^{b} h\left(b-\tau\right) W_f\left(a_j,\tau\right) d\tau \approx \\
&\approx \frac{V_j\left(b\right)}{2}\int_{-\infty}^{b} h\left(b-\tau\right)\left\{\mathrm{Cos}\left[\left(\omega_{1j}-\omega_{2j}\right)\tau+\phi_{1j}-\phi_{2j}\right]-\mathrm{Cos}\left[\left(\omega_{1j}+\omega_{2j}\right)\tau+\phi_{1j}+\phi_{2j}\right]\right\} d\tau
\end{aligned}
\tag{25}
$$

and the effects of the initial conditions are neglected. Based on Eq.(25), therefore, it can be stated that Eq.(21) holds true only whenever the impulse response function $h(t)$ decays rapidly with respect to the modulating function $V(t)$. Therefore, in general Eq.(21) holds true depending on the system parameters and on the input process as well.

To investigate the range of validity of Eq.(21), the error measure

$$
\varepsilon = \frac{1}{\Gamma}\int_{\Gamma} \frac{\left| E\left[\tilde{W}_x^2\left(a,b\right)\right] - E\left[W_x^2\left(a,b\right)\right]\right|}{E\left[\max_b\left\{W_x^2\left(a,b\right)\right\}\right]} db \tag{26}
$$

is introduced, where $\tilde{W}\left(a,b\right)$ is the WT computed based on Eq.(21). Figure 9 shows the error measure (26) for a variety of periods T and damping coefficients ζ. The error map features a resonance peak at the scale value a that corresponds to a WT $W(a,b)$ whose stepped frequency band, see Eq.(18), include the natural frequency of the SDOF system. However it is evident that, as ζ increases, for any period T and for any scale value a, the error decreases significantly. This suggests the potential use of Eq.(21) for structural systems equipped with viscous dampers.

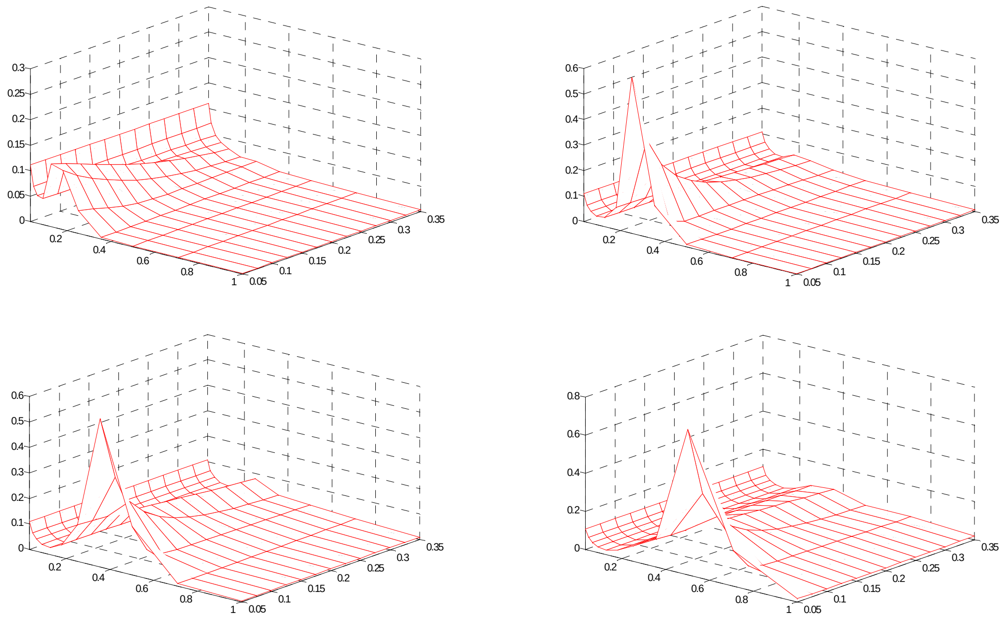


Figure 9: Error measure (26) for different system parameters.

3.1 *Numerical results for systems with viscous dampers*

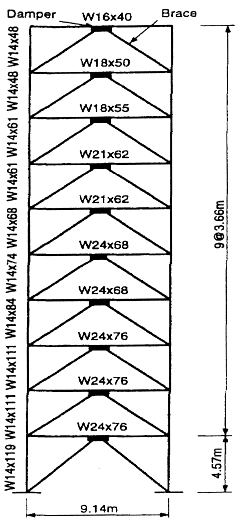
Consider the steel MRF presented in ref. [14], subjected to an ground motion oscillatory process $f(t) = A(t)f_0(t)$, with amplitude modulating function and PSD of the stationary part given by

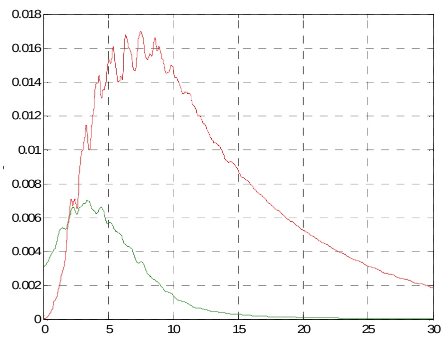
$$A(t) = \frac{e^{-0.25t} - e^{-0.5t}}{0.25}; \qquad S_{f_0 f_0}(\omega) = \left(\frac{2\zeta_s \omega}{\omega_s}\right)^2 \left\{ \left[1 - \left(\frac{\omega}{\omega_s}\right)^2\right]^2 + \left(\frac{2\zeta_s \omega}{\omega_s}\right)^2 \right\}^{-1}, \tag{27}$$

where ω_s=20 rad/sec and ζ_s=0.40. Following ref. [14], a classic damping is assumed and the response according to the 1st mode is considered. A comparison is made between the responses obtained for the following system parameters:

	T_1 [sec]	ω_1 [rad/sec]	ζ_1[%]
MRF	2.28	2.76	50
VS1	2.40	2.62	25
VS2	2.43	2.59	2

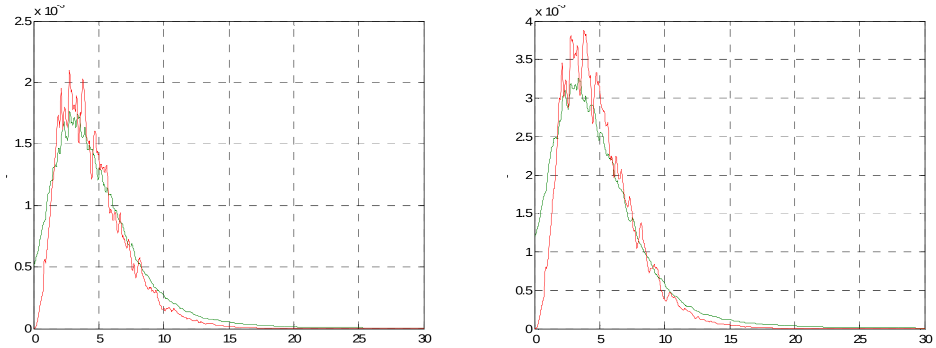
Table 1: System parameters [14]: MRF = system without viscous dampers; VS1-2 = system with two different sets of viscous dampers.





Figures 10-11: Steel MRF and time-dependent m.s.v. of the response without viscous dampers (system MRF in Table 1).

Digital simulation is performed by generating 200 samples of the ground motion. Figure 11 through 13 show the m.s.v. of the 1st mode response computed as spectral moment m_0 of the response power spectrum (23) (green line), as compared to the m.s.v. of the response computed over the 200 samples. As expected, very accurate results are obtained as the damping level increases.



Figures 12-13: Time-dependent m.s.v. of the response of systems VS1 and VS2.

4 CONCLUSIONS

A wavelet-based time-dependent spectrum has been proposed for non-stationary processes. Statistics depending on the spectral moments substantiate its validity. Applications to linear systems, in conjunction with a specific relation between input and output WT, have been found satisfactory for systems with viscous dampers under seismic excitations.

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