A procedure for the identification of concentrated damages on beams by free vibration tests

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SUMMARY. A novel procedure for the identification of multiple concentrated damages on a straight beam, based on the knowledge of the relevant eigen-mode explicit expressions, is presented. The special analytical structure of the direct problem solution allows for the determination of explicit expressions also for the inverse dynamic damage identification problem. Namely the damage intensity and the position of the concentrated cracks may be identified through the knowledge of the component of two vibration modes in the cross sections between two cracks and the corresponding frequencies.

1 INTRODUCTION

Structural health monitoring techinques are often based on dynamic response-based damage detection methods. Most of the dynamics-based structural health monitoring techniques rely on free vibration of beams with cracks which is a problem extensively studied in the last three decades [1-6]. Many crack models have been proposed in the literature to simulate the effect on the dynamic behaviour of beams [7]. The most widely adopted model is that based on a local flexibility induced by a transverse edge crack which is simulated by an internal hinge endowed with a rotational spring at the location of the crack [1]. According to this model, the beam is subjected to a slope discontinuity at the location of the crack.

A common approach to formulate the direct problem for the analysis of the free vibration of a beam with multiple cracks is to divide the beam into sub-beams with different modal displacement functions for each sub-beam. Therefore, in case of n-cracks in the beam, four boundary conditions and $4n$ continuity conditions have to be employed, and the eigenvalue equation of the problem is expressed by a $4(n+1)$ order determinant equated to zero [1]. Recent studies aimed at finding more efficient approaches able to reduce the order of the determinant down to $n+2$ [2]. However, the most interesting approach is that presented by Li [3] proposing a determinant of order two, avoiding the fulfillment of the continuity conditions at the crack locations by means of a recursive expression.

In this work an approach to study the free vibration of damaged beams based on modelling the cracks by means of distributions (generalised functions), such as the Dirac’s delta, is proposed. For the case of a beam with n cracks, the proposed approach leads to explicit expressions for the eigenmodes dependent on the intensities and positions of the damages and four integration constants, while the eigenvalue equation is obtained in explicit form by the evaluation of a fourth order determinant.

On the basis of the explicit expressions provided for the eigenmodes a novel multiple damage identification procedure can be set, once at least the first two eigen-modes, together with the respective eigenvalues, are given by free vibration experimental tests. By equating the values of the first experimental eigen-mode to the given analytical explicit expression a non-linear set of equations is obtained. A suitable increment of variables to be identified leads to a triangular structure for the latter set of equations. A convenient closed form solution in cascade for the damage intensities under the hypothesis that the concentrated damage locations coincide with the measurement positions is obtained. In case the concentrated damages lie in between two measurements, the second experimental eigen-mode allows their localization by means of a numerical procedure which is however decoupled for each damage.

2 THE DIRECT PROBLEM

The differential equation governing the free vibration of a multi-cracked beam may be written in the following form:

$$\left[E_o I_o \left(1-\sum_{i=1}^{n}\hat{\gamma}_i \delta(x-x_{oi})\right)u''(x,t)\right]'' + m\ddot{u}(x,t) = 0 \tag{1}$$

where $EI(x) = E_o I_o (1-\sum_{i=1}^{n}\hat{\gamma}_i \delta(x-x_{oi}))$ describes a flexural stiffness with n cracks which are represented as n Dirac delta distributions $\delta(x-x_{oi})$ in the flexural stiffness centred at cross-sections x_{oi}, $i=1,\ldots,n$. This model, already applied in static [8,9] and in stability [10], is equivalent to consider a straight beam with n massless elastic rotational springs. By considering the non-dimensional coordinate $\xi = x/L$, the differential Eq. (1) takes the following form:

$$\left[\left(1-\sum_{i=1}^{n}\gamma_i \delta(\xi-\xi_{oi})\right)u''(\xi,t)\right]'' + \frac{mL^4}{E_o I_o}\ddot{u}(\xi,t) = 0 \tag{2}$$

where the property $\delta(x-x_{oi}) = (1/L)\,\delta(\xi-\xi_{oi})$ of the Dirac's delta distribution has been exploited, and the dimensionless damage parameters $\gamma_i = \hat{\gamma}_i / L$ have been introduced.

The solution of Eq. (2), with the use of separation of variables, can be given the following form:

$$u(\xi,t) = y(t)\;\;\phi(\xi)\,. \tag{3}$$

Substitution of Eq. (3) in Eq. (2) yields to the following differential equation for modal displacements, that, after some simple algebraic manipulation, can be written in the form:

$$\left[\left(1-\sum_{i=1}^{n}\gamma_i \delta(\xi-\xi_{oi})\right)\phi''(\xi)\right]'' - \alpha^4\,\phi(\xi) = 0 \tag{4}$$

where the frequency parameter $\alpha^4 = \omega^2\, mL^4 \big/ E_o I_o$ has been introduced.

Eq. (4), by performing double differentiation with respect to ξ of the first term containing the Dirac's delta distribution, and after simple algebra, may be given the following form:

$$\phi^{IV}(\xi) - \alpha^4 \phi(\xi) = B(\xi) \tag{5}$$

where the function $B(\xi)$ collects all the terms with the Dirac's deltas and their derivatives as follows:

$$B(\xi) = \left[\sum_{i=1}^{n}\gamma_i \phi^{IV}(\xi)\,\delta(\xi-\xi_{oi}) + 2\sum_{i=1}^{n}\gamma_i \phi'''(\xi)\,\delta'(\xi-\xi_{oi}) + \sum_{i=1}^{n}\gamma_i \phi''(\xi)\,\delta''(\xi-\xi_{oi})\right]. \tag{6}$$

The general explicit solution of Eq. (5) has been derived in [11] by making use of generalised functions and may be written as follows:

$$\begin{aligned}\phi(\xi) = & C_1\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_i\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]U(\xi-\xi_{oi})+\sin\alpha\xi\right\}+ \\ & +C_2\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\upsilon_i\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]U(\xi-\xi_{oi})+\cos\alpha\xi\right\}+ \\ & +C_3\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\zeta_i\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]U(\xi-\xi_{oi})+\sinh\alpha\xi\right\}+ \\ & +C_4\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\eta_i\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]U(\xi-\xi_{oi})+\cosh\alpha\xi\right\}\end{aligned} \tag{7}$$

where: $U_i\left(\xi-\xi_{oi}\right)$ represents the Heaviside unit step function, $\lambda_i=\gamma_i/(1-A\gamma_i)$, with A an arbitrary constant, are the damage intensity parameters related to γ_i, and the terms $\mu_i,\nu_i,\zeta_i,\eta_i$ are given by the following expressions:

$$\begin{aligned}\mu_j &= \frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\mu_i\left[-\sin\alpha\left(\xi_{oj}-\xi_{oi}\right)+\sinh\alpha\left(\xi_{oj}-\xi_{oi}\right)\right]-\alpha^2\sin\alpha\xi_{oj} \\ \upsilon_j &= \frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\upsilon_i\left[-\sin\alpha\left(\xi_{oj}-\xi_{oi}\right)+\sinh\alpha\left(\xi_{oj}-\xi_{oi}\right)\right]-\alpha^2\cos\alpha\xi_{oj} \\ \zeta_j &= \frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\zeta_i\left[-\sin\alpha\left(\xi_{oj}-\xi_{oi}\right)+\sinh\alpha\left(\xi_{oj}-\xi_{oi}\right)\right]+\alpha^2\sinh\alpha\xi_{oj} \\ \eta_j &= \frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\eta_i\left[-\sin\alpha\left(\xi_{oj}-\xi_{oi}\right)+\sinh\alpha\left(\xi_{oj}-\xi_{oi}\right)\right]+\alpha^2\cosh\alpha\xi_{oj}\quad .\end{aligned} \tag{8}$$

The solution given by Eq. (7) can also be expressed in the following more compact form:

$$\begin{aligned}\phi(\xi) = & C_1\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_iS_i(\alpha,\xi)U_i(\xi)+\sin\alpha\xi\right\}+C_2\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\upsilon_iS_i(\alpha,\xi)U_i(\xi)+\cos\alpha\xi\right\}+ \\ & +C_3\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\zeta_iS_i(\alpha,\xi)U_i(\xi)+\sinh\alpha\xi\right\}+C_4\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\eta_iS_i(\alpha,\xi)U_i(\xi)+\cosh\alpha\xi\right\}\end{aligned} \tag{9}$$

where the compact notation $U_i(\xi)=U_i\left(\xi-\xi_{oi}\right)$ has ben adopted and, for simplicity, the following function $S_i(\alpha,\xi)=\left[\sin\alpha\left(\xi-\xi_{oi}\right)+\sinh\alpha\left(\xi-\xi_{oi}\right)\right]$ has been defined.

It is worth noting that the solution expressed by the Eqs. (7) or (9) is valid for the overall beam and for any number and positions of cracks, furthermore, it preserves the same analytical structure of the undamaged beam. Eqs. (7) or (9) can also be used for analysing the non-linear dynamic behaviour of multi-damaged beams with closing cracks [12] as well as for deriving the dynamic stiffness matrix of the multi-cracked beam with open cracks.

3 THE INVERSE CRACK IDENTIFICATION PROCEDURE

It is worth noting that, due to the analytical structure of Eq. (7), the values of the *m*-th mode shape $\phi_m(\xi)$, except for the integration constants C_1,C_2,C_3,C_4, depend only on the damages that are located at positions $\xi_i<\xi$.

Considering the *n*+1 beam segments individuated by the cracked cross-sections, as reported in Figure 1, the solution for each segment, expressed for the overall beam by Eqs. (7) or (9), may be specialised, for each segment, as follows:

$$
\begin{aligned}
&\phi(\xi) = C_1 \sin\alpha\xi + C_2 \cos\alpha\xi + C_3 \sinh\alpha\xi + C_4 \cosh\alpha\xi = \phi^{(0)}(\xi) && 0 < \xi \le \xi_{o1} \\
&\phi(\xi) = \phi^{(0)}(\xi) + \frac{\lambda_1}{2\alpha} S_1(\alpha,\xi)\left[C_1\mu_1 + C_2\upsilon_1 + C_3\zeta_1 + C_4\eta_1\right] && \xi_{o1} < \xi \le \xi_{o2} \\
&\phi(\xi) = \phi^{(0)}(\xi) + \frac{\lambda_1}{2\alpha} S_1(\alpha,\xi)\left[C_1\mu_1 + C_2\upsilon_1 + C_3\zeta_1 + C_4\eta_1\right] + \ldots \\
&\qquad \ldots + \frac{\lambda_k}{2\alpha} S_k(\alpha,\xi)\left[C_1\mu_k + C_2\upsilon_k + C_3\zeta_k + C_4\eta_k\right] && \xi_{o(k-1)} < \xi \le \xi_{ok}
\end{aligned} \tag{10}
$$

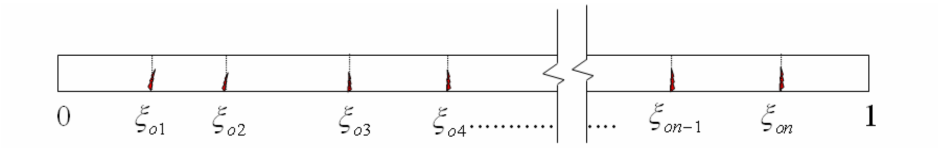

Figure 1: The beam segments between the ends and the cracked cross-sections.

From the observation of the analytical structure of the explicit solution emerges that it possesses a triangular structure with reference to the intensity and positions of the damages, except for the integration constants whose values are influenced by the boundary conditions and by the damage intensities and positions as it has been shown in [11]. For the *m*-th mode shape $\phi_m(\xi)$, the triangular structure of the Eq. (7), also maintained by its derivatives, can be highlighted as follows:

$$
\begin{aligned}
&\phi_m\left(\hat{\xi}_o\right) = \phi_m\left(C_1, C_2, C_3, C_4\right) && 0 < \hat{\xi}_o \le \xi_{o1} \\
&\phi_m\left(\hat{\xi}_{o1}\right) = \phi_m\left(C_1, C_2, C_3, C_4, \boxed{\xi_{o1}}, \boxed{\lambda_1}\right) && \xi_{o1} < \hat{\xi}_{o1} \le \xi_{o2} \\
&\phi_m\left(\hat{\xi}_{o2}\right) = \phi_m\left(C_1, C_2, C_3, C_4, \boxed{\xi_{o1}, \xi_{o2}}, \boxed{\lambda_1, \lambda_2}\right) && \xi_{o2} < \hat{\xi}_{o2} \le \xi_{o3} \\
&\phi_m\left(\hat{\xi}_{ok}\right) = \phi_m\left(C_1, C_2, C_3, C_4, \boxed{\xi_{o1}, \ldots, \xi_{ok}}, \boxed{\lambda_1, \ldots, \lambda_k}\right) && \xi_{ok} < \hat{\xi}_{ok} \le \xi_{ok+1}
\end{aligned} \tag{11}
$$

According to Eq. (11), the generic mode shape $\phi_m\left(\hat{\xi}_o\right)$ evaluated at abscissa $\hat{\xi}_o$ located between the abscissa 0 and the first cracked cross section ($0 < \hat{\xi}_o \le \xi_{o1}$) depends only on the integration constants C_1, C_2, C_3, C_4. The value of the mode shape $\phi_m\left(\hat{\xi}_{ok}\right)$ evaluated at the generic abscissa $\hat{\xi}_{ok}$ situated between the cracks *k* and *k*+1 ($\xi_{ok} < \hat{\xi}_{ok} \le \xi_{ok+1}$), besides the integration constants, depends only on the previous damages, i.e. the damages collocated at positions $\xi_{o1}, \ldots, \xi_{ok} < \hat{\xi}_{ok}$.

This particular analytical structure of the solution suggests two procedures for solving the inverse problem based on dynamic tests which are described and discussed in the following. The first approach accounts for position of the sensors coincident with the cracked cross-sections and, measuring one vibration mode, leads to explicit expressions of the damage intensities to be identified. The second procedure is able to provide both the positions and the intensities of the damages, based on the measurements of two vibration modes, by means of a numerical procedure. In particular the case of a multi-cracked free-free beam is treated and the reliability of the proposed procedures will be verified by means of finite element numerical simulations on damaged beams modelled by means of two-dimensional shell elements.

3.1 *Evaluation of the damage intensity with measurements at the cracked cross-sections*

The previously described analytical structure of the solution leads to an identification procedure which provides explicit expressions of the damage intensities as a function of the values of a mode shape (or its derivatives) at the damaged cross-sections, assuming that the measurements are taken at the damage positions. According to Eq. (11), the values of the *m*-th mode shape $\phi_m(\xi_{ok})$ at the abscissa ξ_{ok} of the *k*-th crack depends linearly on the integration constants C_1, C_2, C_3, C_4 and on the previous damages only $\lambda_1, ..., \lambda_{k-1}$ located at the abscissae $\xi_{o1}, \xi_{o2}, ..., \xi_{ok-1}$ as follows:

$$\phi_m(\xi_{ok}) = \phi_m(C_1, C_2, C_3, C_4, \lambda_1, ..., \lambda_{k-1}) . \tag{12}$$

If the beam is restrained by means of perfect constraints only two constants are needed to represent the corresponding mode shape and Eq. (12) can be written as:

$$\phi_m(\xi_{ok}) = \phi_m(C_1, C_2, \lambda_1, ..., \lambda_{k-1}) . \tag{13}$$

Therefore in order to evaluate the n+2 unknowns (n damage intensities and 2 integration constants) the corresponding conditions can be obtained by equating the experimental and the theoretical results at n+2 cross sections. In order to take advantage of the triangular scheme of the solution, firstly it is convenient to identify the integration constants C_1, C_2, and afterwards the successive measurements will provide explicit expressions for each damage intensity as a function of the previously evaluated unknowns.

The values of the integration constants can be easily identified considering one measurement of the first vibration mode at a position that precedes the first damage cross section $\hat{\xi}_o < \xi_{o1}$, and a further measurement corresponding to the first cracked cross-section ξ_{o1}. By equating the experimental values with the theoretical expression, corresponding to the measured first frequency parameter α_1^{ex}, the following linear system of two equations is obtained:

$$\phi_1^{ex}(\hat{\xi}_o) = \phi_1^{th}(\alpha_1^{ex}, C_1, C_2) \qquad , \qquad \phi_1^{ex}(\xi_{o1}) = \phi_1^{th}(\alpha_1^{ex}, C_1, C_2) \tag{14}$$

In Eq. (14) the only unknowns are the integration constants C_1, C_2 which can be evaluated in closed form. Once the integration constants have been evaluated, the measurement at the second cracked cross-section, according to Eq. (13) written at abscissa ξ_{o2}, leads to the evaluation of the first damage parameter λ_1, and so on. For the evaluation of the last damage intensity λ_n a further measurement located at a position that follows the final damaged cross-section is needed.

3.2 *Evaluation of both the position and the intensity of damages with measurements between cracked cross-sections*

If the damage positions are not known the beam may be instrumented with several sensors to verify the presence of damage between two measurement points. Assuming perfect constraint conditions, the value of the *m*-th mode shape $\phi_m(\hat{\xi}_{ok})$ at the generic abscissa $\hat{\xi}_{ok}$, where the sensor is located, depends linearly on two integration constants (C_1, C_2) and on the previous damage intensities ($\lambda_1, ..., \lambda_k$) and nonlinearly on the corresponding locations represented by the abscissae $\xi_{o1}, \xi_{o2}, ..., \xi_{ok}$.

Once the integration constants have been identified as in the procedure outlined in section 3.1, the n damage intensities and the corresponding positions can be obtained by equating the experimental measurements and the theoretical expressions of two vibration modes *m,s,* for each segment of the beam between two subsequent cracks, as follows:

$$\begin{aligned}\phi_m^{ex}\left(\hat{\xi}_{ok}\right)&=\phi_m^{th}\left(C_1,C_2,\xi_{o1},...,\xi_{ok},\lambda_1,...,\lambda_k\right)\\ \phi_s^{ex}\left(\hat{\xi}_{ok}\right)&=\phi_s^{th}\left(C_1,C_2,\xi_{o1},...,\xi_{ok},\lambda_1,...,\lambda_k\right)\end{aligned}\qquad \xi_{ok}<\hat{\xi}_{ok}\le\xi_{ok+1}\tag{15}$$

and solving with respect to λ_k , ξ_{ok} the nonlinear sistem (15) by means of a suitable numerical procedure. Also in this case, in which the damage positions are not known a priori, in order to take advantage of the triangular scheme of the solution, it is convenient to evaluate, first, the integration constants and than proceed to the evaluation of the damage intensity and location, crack by crack, starting from the first crack.

4 IDENTIFICATION OF MULTIPLE CRACKS ALONG A FREE-FREE BEAM

In this section the closed-form solution presented in Eq. (9) is adopted to treat the case of a Free-Free Euler-Bernoulli beam and the inverse problem will be formulated.

For a Free-Free beam, the boundary conditions at the left and right ends may be written as

$$\phi''(0)=0;\qquad \phi'''(0)=0;\qquad \phi''(1)=0;\qquad \phi'''(1)=0;\tag{16}$$

by means of the conditions (16), the following expressions of the four integration constants for the *k*-th vibration mode can be derived:

$$C_1=C;\qquad C_2=\vartheta_k C;\qquad C_3=C;\qquad C_4=\vartheta_k C\ .\tag{17}$$

Therefore the explicit expression of the generic vibration mode of a multi-cracked free-free beam may be written as.

$$\begin{aligned}\phi_k\left(\xi\right)=C\Bigg\{&\left[\frac{1}{2\alpha_k}\sum_{i=1}^{n}\lambda_i\left(\mu_i+\zeta_i\right)S_i\left(\alpha_k,\xi\right)U_i\left(\xi\right)+\sin\alpha_k\xi+\sinh\alpha_k\xi\right]\\ &+\vartheta_k\left[\frac{1}{2\alpha_k}\sum_{i=1}^{n}\lambda_i\left(\upsilon_i+\eta_i\right)S_i\left(\alpha_k,\xi\right)U_i\left(\xi\right)+\cos\alpha_k\xi+\cosh\alpha_k\xi\right]\Bigg\}\ .\end{aligned}\tag{18}$$

It is worth noting that, due to the analytical structure of Eq. (18), the values $\phi_k\left(\xi\right)$ depend only on the damages that are located at positions $\xi_i<\xi$, namely:

$$\begin{aligned}&\phi_k\left(\xi\right)=C\left[\left(\sin\alpha_k\xi+\sinh\alpha_k\xi\right)+\vartheta_k\left(\cos\alpha_k\xi+\cosh\alpha_k\xi\right)\right] && 0<\xi\le\xi_{o1}\\ &\phi_k\left(\xi\right)=C\Bigg\{\left[\frac{1}{2\alpha_k}\lambda_1\left(\mu_1+\zeta_1\right)S_1\left(\alpha_k,\xi\right)+\sin\alpha_k\xi+\sinh\alpha_k\xi\right]\\ &\qquad+\vartheta_k\left[\frac{1}{2\alpha_k}\lambda_1\left(\upsilon_1+\eta_1\right)S_1\left(\alpha_k,\xi\right)+\cos\alpha_k\xi+\cosh\alpha_k\xi\right]\Bigg\} && \xi_{o1}<\xi\le\xi_{o2}\\ &\phi_k\left(\xi\right)=C\Bigg\{\left[\frac{1}{2\alpha_k}\sum_{i=1}^{j-1}\lambda_i\left(\mu_i+\zeta_i\right)S_i\left(\alpha_k,\xi\right)+\sin\alpha_k\xi+\sinh\alpha_k\xi\right]\\ &\qquad+\vartheta_k\left[\frac{1}{2\alpha_k}\sum_{i=1}^{j-1}\lambda_i\left(\upsilon_i+\eta_i\right)S_i\left(\alpha_k,\xi\right)+\cos\alpha_k\xi+\cosh\alpha_k\xi\right]\Bigg\} && \xi_{o\,j-1}<\xi\le\xi_{oj}\ .\end{aligned}\tag{19}$$

According to the procedure described in section 3.1, it is possible to identify the presence and quantify the damage at the cross-sections where the measurement sensors are located. The procedure is based on the knowledge from experimental tests of at least a frequency parameter α_m^{ex} and the corresponding mode $\phi_m^{ex}(\xi)$. In the following, reference to the first frequency α_1^{ex} and the corresponding vibration mode $\phi_1^{ex}(\xi)$ is made. It is assumed that at the abscissae $\hat{\xi}_o$ and $\hat{\xi}_{n+1}$, where the first and the last sensors are located, no damage occurs, while the other *n* instruments are located at the cracked cross-sections, $\xi_{o1},...,\xi_{on}$, as depicted in Figure 2.

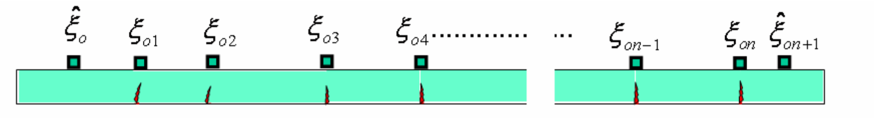

Figure 2: Crack and sensor positions along the beam span.

Employing the first two measurements of the first eigen-mode $\phi_1^{ex}(\hat{\xi}_o)$, $\phi_1^{ex}(\xi_{o1})$, under the assumption that $\hat{\xi}_o < \xi_{o1}$, and by equating the experimental and the theoretical results, according to Eq. (17), the following system of equations is obtained:

$$\begin{aligned} \phi_1^{ex}\left(\hat{\xi}_o\right) &= C\left[\vartheta_1\left(\sin\alpha_1^{ex}\hat{\xi}_o + \sinh\alpha_1^{ex}\hat{\xi}_o\right) + \left(\cos\alpha_1^{ex}\hat{\xi}_o + \cosh\alpha_1^{ex}\hat{\xi}_o\right)\right] \\ \phi_1^{ex}\left(\xi_{o1}\right) &= C\left[\vartheta_1\left(\sin\alpha_1^{ex}\xi_{o1} + \sinh\alpha_1^{ex}\xi_{o1}\right) + \left(\cos\alpha_1^{ex}\xi_{o1} + \cosh\alpha_1^{ex}\xi_{o1}\right)\right] \end{aligned} \tag{20}$$

from which the following values of the constants C and ϑ_1 are derived in explicit form:

$$\begin{aligned} \vartheta_1 &= \frac{\phi_1^{ex}(\hat{\xi}_o)[\sin(\alpha_1^{ex}\xi_{o1}) + \sinh(\alpha_1^{ex}\xi_{o1})] - \phi_1^{ex}(\xi_{o1})[\sin(\alpha_1^{ex}\hat{\xi}_o) + \sinh(\alpha_1^{ex}\hat{\xi}_o)]}{\phi_1^{ex}(\xi_{o1})\left[\cos(\alpha_1^{ex}\hat{\xi}_o) + \cosh(\alpha_1^{ex}\hat{\xi}_o)\right] - \phi_1^{ex}(\hat{\xi}_o)\left[\cos(\alpha_1^{ex}\xi_{o1}) + \cosh(\alpha_1^{ex}\xi_{o1})\right]} \\ C &= \frac{\phi_1^{ex}(\hat{\xi}_o)}{\left[\sin(\alpha_1^{ex}\hat{\xi}_o) + \sinh(\alpha_1^{ex}\hat{\xi}_o)\right] + \vartheta_1\left[\cos(\alpha_1^{ex}\hat{\xi}_o) + \cosh(\alpha_1^{ex}\hat{\xi}_o)\right]} \quad . \end{aligned} \tag{21}$$

Equating the measurement at the second cracked cross-section $\phi_1^{ex}(\xi_{o2})$ at ξ_2 to the theoretical expression $\phi_1^{th}(\xi_{o2})$ of the first mode leads to the following equation:

$$\begin{aligned} \phi_1^{ex}\left(\xi_{o2}\right) = C\Bigg\{\vartheta_1\left[\frac{1}{2\alpha_1}\lambda_1\left(\mu_1+\zeta_1\right)S_1\left(\alpha_1^{ex},\xi_{o2}\right) + \sin\alpha_1^{ex}\xi_{o2} + \sinh\alpha_1^{ex}\xi_{o2}\right] \\ \left[\frac{1}{2\alpha_1}\lambda_1\left(\upsilon_1+\eta_1\right)S_1\left(\alpha_1^{ex},\xi_{o2}\right) + \cos\alpha_1^{ex}\xi_{o2} + \cosh\alpha_1^{ex}\xi_{o2}\right]\Bigg\} \end{aligned} \tag{22}$$

in which the only unknown is the extent of first damage λ_1 present at ξ_{o1}. Once the first damage has been identified, the second damage intensity λ_2 can be obtained by means of the further measurement $\phi_1^{ex}(\xi_{o3})$ at ξ_{o3} and so on. The intensity λ_i of the generic damage can be written explicitly as follows:

$$\begin{aligned} \lambda_i = & \frac{[\phi_1^{ex}(\xi_{oi+1})] - C[\sin(\alpha_1^{ex}\xi_{oi+1}) + \sinh(\alpha_1^{ex}\xi_{oi+1}) + \vartheta(\cos(\alpha_1^{ex}\xi_{oi+1}) + \cosh(\alpha_1^{ex}\xi_{oi+1}))]}{C[\mu_i + \varsigma_i + \vartheta(\nu_i + \eta_i)][\sin(\alpha_1^{ex}(\xi_{oi+1} - \xi_{oi})) + \sinh(\alpha_1^{ex}(\xi_{oi+1} - \xi_{oi}))]} + \\ & - \frac{\sum_{j=1}^{i-1}\lambda_j C[\mu_i + \varsigma_i + \vartheta(\nu_i + \eta_i)][\sin(\alpha_1^{ex}(\xi_{oi+1} - \xi_{oi})) + \sinh(\alpha_1^{ex}(\xi_{oi+1} - \xi_{oi}))]}{C[\mu_i + \varsigma_i + \vartheta(\nu_i + \eta_i)][\sin(\alpha_1^{ex}(\xi_{oi+1} - \xi_{oi})) + \sinh(\alpha_1^{ex}(\xi_{oi+1} - \xi_{oi}))]} \quad . \end{aligned} \tag{23}$$

If there is no crack at the cross-section ξ_{oi}, the identified damage parameter λ_i given by Eq. (23) will be zero, indicating the absence of damage.

On the other hand, in the case the measurement positions are not coincident with the damage locations ξ_{oi} (i.e. the sensor are placed at abscissae $\hat{\xi}_{oi}$ between damages $\xi_{oi} < \hat{\xi}_{oi} < \xi_{oi+1}$), both intensity and damage position are unknown. The measurements of the second frequency and the corresponding eigen-mode provide the sufficient additional data, and, according to the procedure in section 3.2, Eq. (15) for the case under study, where the first and the second vibration modes are given by experimental data, can be written as follows:

$$\begin{aligned}\phi_1^{ex}\left(\hat{\xi}_{oi}\right)=C\Bigg\{&\left[\frac{1}{2\alpha_1^{ex}}\sum_{i=1}^{j-1}\lambda_i\left(\mu_i+\zeta_i\right)S_i\left(\alpha_1^{ex},\hat{\xi}_{oi}\right)+\sin\alpha_1^{ex}\hat{\xi}_{oi}+\sinh\alpha_1^{ex}\hat{\xi}_{oi}\right]\\&+\vartheta_1\left[\frac{1}{2\alpha_1^{ex}}\sum_{i=1}^{j-1}\lambda_i\left(\upsilon_i+\eta_i\right)S_i\left(\alpha_1^{ex},\hat{\xi}_{oi}\right)+\cos\alpha_1^{ex}\hat{\xi}_{oi}+\cosh\alpha_1^{ex}\hat{\xi}_{oi}\right]\Bigg\}\\ \phi_2^{ex}\left(\hat{\xi}_{oi}\right)=C\Bigg\{&\left[\frac{1}{2\alpha_2^{ex}}\sum_{i=1}^{j-1}\lambda_i\left(\mu_i+\zeta_i\right)S_i\left(\alpha_2^{ex},\hat{\xi}_{oi}\right)+\sin\alpha_2^{ex}\hat{\xi}_{oi}+\sinh\alpha_2^{ex}\hat{\xi}_{oi}\right]\\&+\vartheta_k\left[\frac{1}{2\alpha_2^{ex}}\sum_{i=1}^{j-1}\lambda_i\left(\upsilon_i+\eta_i\right)S_i\left(\alpha_2^{ex},\hat{\xi}_{oi}\right)+\cos\alpha_2^{ex}\hat{\xi}_{oi}+\cosh\alpha_2^{ex}\hat{\xi}_{oi}\right]\Bigg\}\end{aligned}\qquad \xi_{oi}<\hat{\xi}_{oi}\le\xi_{oi+1}\ . \quad (24)$$

The nonlinear sistem of equations (24) has to be solved with respect to ξ_{oi}, appearing in the terms $\mu_i, \varsigma_i, \upsilon_i, \eta_i$, and to the intensity λ_i, by means of a numerical procedure, for each damage in cascade starting from the first damage.

5 APPLICATION

The damage identification procedure proposed has been tested against a finite element model of a cracked Free-Free beam and the results are briefly summarised in this section.

A steel beam, free at its both ends, of lenght L=1.2 m with a rectangular cross-section (b=19.8 mm, h=12.2 mm) has been considered for FEM simulation by making use of the code Sap2000®. The beam is subjected to 3 damages concentrated at the abscissae $\xi_{o1}=0.2$, $\xi_{o2}=0.5$, $\xi_{o3}=0.7$, whose intensities $\lambda_1=0.0175, \lambda_2=0.0334, \lambda_3=0.1586$, are related, as described in [10], to the ratios of the crack depth to the cross-section heigth $d_1/h=0.375$, $d_2/h=0.5$, $d_3/h=0.75$, respectively. For this simulation, 11 measurements placed as in Figure 3 has been considered available and 3 of them are coincident with the damage positions according to the hypothesis in section 3.1. The beam has been modelled by means of shell elements and the results of the FEM modal analysis (natural frequencies and vibration modes) has been considered for the identification procedure.

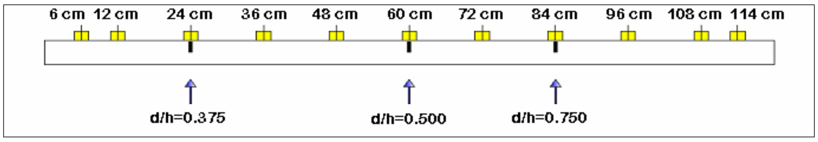

Figure 3: Crack and sensor positions along the beam span.

According to the explicit solution proposed in Eq. (23), by making use of the first natural frequency and vibration mode, the damage intensity parameters λ_i can be evaluated for each measurement position and the results are reported in Figure 4 in terms of the ratio d_i/h showing a maximum error of 7.8%. Eq. (23) provides zero values for the damage intensities for those measurements where the damage is absent.

The same steel beam has been then considered subjected to 2 damages at $\xi_{o1}=0.15$, $\xi_{o2}=0.45$ whose intensities $\lambda_1=0.0334, \lambda_2=0.1586$, correspond to $d_1/h=0.5$, $d_2/h=0.75$, respectively. The measurements are placed as in Figure 5 and they are not coincident with the damage positions according to the hypothesis in section 3.2. The first and the second natural frequencies and vibration modes of the beam, obtained by the FEM modal analysis, have been considered for the identification procedure.

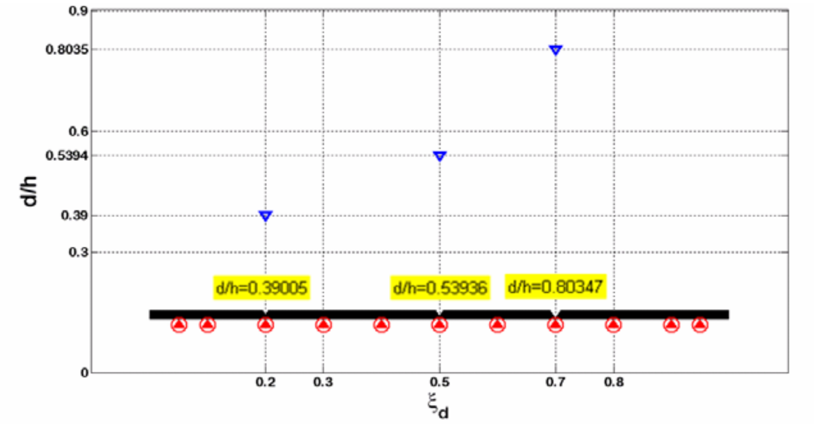

Figure 4: Identified damage intensity for each measurement position

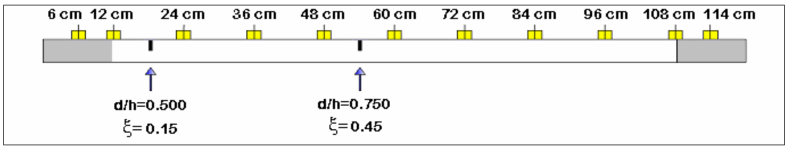

Figure 5: Crack and sensor positions along the beam span.

The system of Eqs. (24) has been solved numerically for each segment of the beam between two measurements in order to identify the damage intensity and position of each damage separately.

In particular, the solutions of Eq. (24a) and (24b) in terms of damage intensity λ_i are plotted against the damage position ξ_{oi} for each interval; the intersection point between the two solutions provides the correct damage intensity and position. The results are shown in Figure 6 in terms of λ_i and of the ratio d_i / h showing a maximum error of 2.6 % for the identified position and 7.0% for the identified intensity. For those intervals where the damage is absent, there is no intersection point between the solutions of Eqs. (24a) and (24b).

6 CONCLUSIONS

In this work a model for the Euler-Bernoulli beam with multiple concentrated cracks based on the generalised functions (distributions) have been adopted. Closed form solutions for the vibration modes have been presented in terms of intensities and positions of the damages and dependent on four integration constants to be determined by the standard boundary conditions. The latter solutions can be efficaciously employed to set an identification procedure for beams with multiple damages based on measurements of the mode shapes by vibration tests. In particular the advantage of the proposed procedure consists in a sequential identification of single damages either coincident with the position of the experimental measurements or lying in the beam intervals between two experimental measurements.

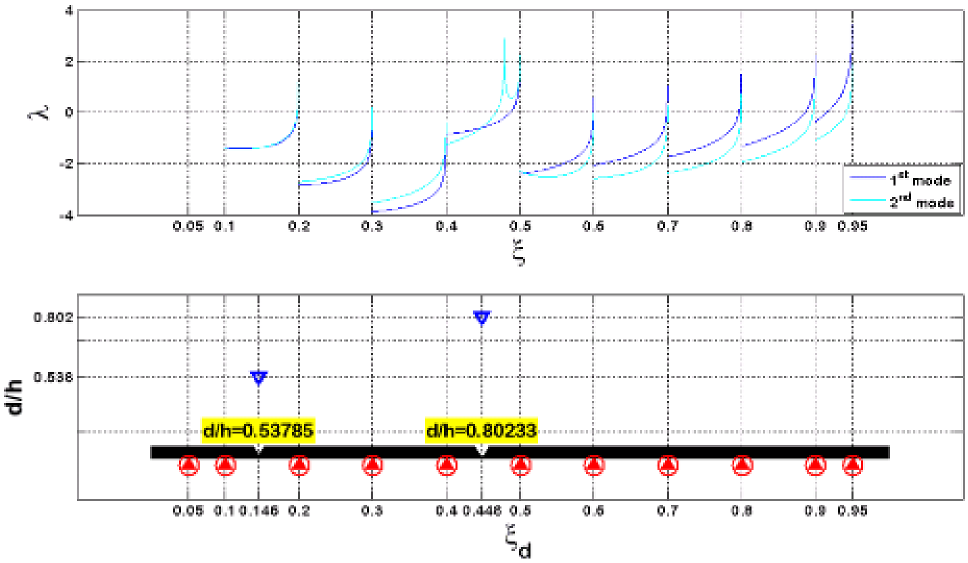

Figure 6: Identified damage intensities between each measurement position

References

[1] P.F. Rizos, N. Aspragatos and A.D. Dimarogonas, "Identification of crack location and magnitude in a cantilever beam from vibration modes", *J. Sound and Vibration*, **138**, 381-388, (1990)

[2] E.I. Shifrin and R. Ruotolo, "Natural frequencies of a beam with an arbitrary number of cracks", *J. Sound and Vibration*, **223**, 409-423, (1999).

[3] Q.S. Li, "Free vibration analysis of non-uniform beams with an arbitrary number of cracks and concentrated masses", *J. Sound and Vibration*, **252**, 509-525, (2001).

[4] Y. Narkis, "Identification of crack location in vibrating simply supported beams", *J. Sound and Vibration*, **172**, 549-558, (1994).

[5] A. Morassi, "Identification of a crack in a rod based on changes in a pair of natural frequencies", *J. Sound and Vibration*, **242**, 577-596, (2001).

[6] M. Dilena and A. Morassi, "The use of antiresonances for crack detection in beams", *J. Sound and Vibration*, **276**, 195–214, (2004).

[7] A.D. Dimarogonas, "Vibration of cracked structure: a state of the art review", *Eng. Fract. Mech.*, **55**, 831-857, (1996).

[8] B. Biondi, S. Caddemi, "Closed form solutions of Euler-Bernoulli beams with singularities", *International Journal of Solids and Structures*, **42**, 3027-3044, (2005).

[9] B. Biondi, S. Caddemi, "Euler-Bernoulli beams with multiple singularities in the flexural stiffness European", *Journal of Mechanics A/Solids*, **26**(5), 789-809, (2007).

[10] S. Caddemi, I. Caliò, "Exact solution of the multi-cracked Euler-Bernoulli column", *International Journal of Solids and Structures*, **45**(16), 1332-1351, (2008).

[11] S. Caddemi, I. Caliò, "Exact solution of the multi-cracked Euler-Bernoulli column", *Journal of Sound and Vibrations*, in press, (2009).

[12] S. Caddemi, I. Caliò, M.Marletta, "Exact solution of the multi-cracked Euler-Bernoulli column", *International Journal of Nonlinear Mechanics*, submitted to, (2009).