

# On the approximation of 3D hyperbolic boundary integral equations

A. Temponi, A. Salvadori, F. Mordenti, A. Carini, P. Pelizzari

*CeSiA - Centro di studio e ricerca di sismologia applicata e dinamica strutturale,*

*DICATA - Dipartimento di Ingegneria Civile, Architettura, Territorio e Ambiente*

*Università di Brescia, via Branze 43, 25123 Brescia, Italy*

*E-mail: alessandro.temponi@ing.unibs.it, alberto.salvadori@ing.unibs.it*

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**SUMMARY.** The present note summarizes some new results for hyperbolic problems involving 3D scalar fields modeled by integral equations extensively reported in [1]. Classical approximation schemes as well as recently published energetic weak forms are considered; algorithms for the numerical solution are formulated adopting polynomial shape functions of arbitrary degree (in space and time) on trapezoidal flat tessellations of polynomial domains. Analytical integrations are performed both in space and time for Lebesgue integrals working in a local coordinate system; for singular integrals, both a limit to the boundary as well as the finite part of Hadamard approach have been pursued..

## 1 INTRODUCTION

Modeling hyperbolic problems by means of boundary integral equations (BIEs) and approximating their solution through boundary element methods (BEM) is firmly established in the academic community as well as in industry. Such methods have been successfully used for decades in the propagation and scattering of acoustics, electromagnetic [2] and elastic waves [3]. Several modern research and applications topics are dealt with them: see [1] for a short review.

The integral formulation of the *scalar wave problem* can be formulated (see e.g. [3, 4]) stemming from Graffi's [5] generalization of steady state reciprocity theorem to dynamics. Under the hypothesis of vanishing initial conditions and no external body forces, the boundary integral representation (BIR) of the primal field  $u$  in the interior of the open domain  $\Omega$  at time  $t$  reads:

$$u(\mathbf{x}, t) = \int_{\Sigma} G_{uu}(\mathbf{r}, t - \tau) p(\mathbf{y}, \tau) d\Sigma_{\tau, y} - \int_{\Sigma} G_{up}(\mathbf{r}, \mathbf{l}(\mathbf{y}), t - \tau) u(\mathbf{y}, \tau) d\Sigma_{\tau, y} \quad (1)$$

Here,  $\Sigma$  is the lateral boundary  $\Sigma = (0, T) \times \Gamma$  and  $\mathbf{r} = \mathbf{x} - \mathbf{y}$  stands for the vector that joins point  $\mathbf{y}$  to  $\mathbf{x}$ . Identity (1) is based on Green's functions (also called kernels)  $G_{uu}$  and  $G_{up}$ . An additional integral equation can be provided by the application of the co-normal derivative operator to identity (1): the BIR of the dual field  $p(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$  on a surface of normal  $\mathbf{n}(\mathbf{x})$  in the interior of the domain, i.e.  $\{t, \mathbf{x}\} \in \Sigma$  turns out to be:

$$p(\mathbf{x}, t) = \int_{\Sigma} G_{pu}(\mathbf{r}, \mathbf{n}(\mathbf{x}), t - \tau) p(\mathbf{y}, \tau) d\Sigma_{\tau, y} - \int_{\Sigma} G_{pp}(\mathbf{r}, \mathbf{n}(\mathbf{x}), \mathbf{l}(\mathbf{y}), t - \tau) u(\mathbf{y}, \tau) d\Sigma_{\tau, y} \quad (2)$$

Such a BIR involves Green's functions  $G_{pu}$  and  $G_{pp}$ .

A set of two BIEs can be derived from BIRs (1) (thus obtaining the so-called "primal equation") and (2) (thus obtaining the so-called "dual equation") by performing the space boundary limit  $\Omega \ni \mathbf{x} \rightarrow \mathbf{x} \in \Gamma$ . In the limit process, after integration in time, singularities of Green's functions are

triggered off: their singularity-orders show to be equivalent to the steady state case<sup>1</sup>. Assuming smooth boundaries, after imposing the fulfilment of the primal equation on Dirichlet boundary  $\Gamma_u$  and of the dual equation on Neumann boundary  $\Gamma_p$ , the following linear boundary integral problem (omitting the arguments of Green's functions for paucity of space) comes out:

$$\int_{\Gamma_u} \int_0^t G_{uu} p(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} - \int_{\Gamma_p} \int_0^t G_{up} u(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} = f^u(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma_u, t \in (0, T) \quad (3)$$

$$\int_{\Gamma_u} \int_0^t G_{pu} p(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} - \int_{\Gamma_p} \int_0^t G_{pp} u(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} = f^p(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma_p, t \in (0, T) \quad (4)$$

Scalars  $f^i$ ,  $i = u, p$ , that gather all data are the following:

$$\begin{aligned} f^u(\mathbf{x}, t) &= \frac{1}{2} \bar{u}(\mathbf{x}, t) - \int_{\Gamma_p} \int_0^t G_{uu} \bar{p}(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} + \int_{\Gamma_u} \int_0^t G_{up} \bar{u}(\mathbf{y}, \tau) d\tau d\Gamma_{(y)}, \\ &\quad \mathbf{x} \in \Gamma_u, t \in (0, T) \\ f^p(\mathbf{x}, t) &= \frac{1}{2} \bar{p}(\mathbf{x}, t) - \int_{\Gamma_p} \int_0^t G_{pu} \bar{p}(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} + \int_{\Gamma_u} \int_0^t G_{pp} \bar{u}(\mathbf{y}, \tau) d\tau d\Gamma_{(y)}, \\ &\quad \mathbf{x} \in \Gamma_p, t \in (0, T) \end{aligned}$$

The above integral problem (3,4) can be solved by means different approximation schemes: the *space-time collocation BEM* (see e.g. [3]) requires the fulfillment of the primal BIE (3), after discretization, onto a set of collocation nodes,  $\{t_h, \mathbf{x}_i\} \in \Sigma \stackrel{def}{=} (0, T) \times \Gamma$ . Just like in the elliptic case, the mathematical analysis of the collocation approach, for which stabilized versions have been recently proposed [6], lags seriously behind the practical experience. The convergence and stability analysis of hyperbolic BIEs is rooted on *variational* methods proposed in several fashions: the convolutive variational method [7], the variational formulation in extended sense [8], the energetic formulation [9, 10].

The *symmetric Galerkin in space and collocation in time BEM* requires the application of a standard Galerkin scheme in space on the two equations (3,4), after discretization, collocated thereafter onto a set of nodes in time,  $\{t_h\} \in (0, T)$ .

*Energetic formulations* are rooted in the principle of virtual power. A “velocity equation” for  $\mathbf{x} \in \Gamma_u$ ,  $t \in (0, T)$  can be obtained from equation (3) by derivation in time:

$$\frac{\partial}{\partial t} \left[ \int_{\Gamma_u} \int_0^t G_{uu} p(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} - \int_{\Gamma_p} \int_0^t G_{up} u(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} \right] = \frac{1}{2} \dot{\bar{u}}(\mathbf{x}, t) + \frac{\partial}{\partial t} g^u(\mathbf{x}, t) \quad (5)$$

where:

$$g^u(\mathbf{x}, t) = - \int_{\Gamma_p} \int_0^t G_{uu} \bar{p}(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} + \int_{\Gamma_u} \int_0^t G_{up} \bar{u}(\mathbf{y}, \tau) d\tau d\Gamma_{(y)}$$

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<sup>1</sup>Kernel  $G_{uu}$  shows an integrable singularity (named “weak”); kernel  $G_{up}$  and  $G_{pu}$  present a strong singularity  $O(r^{-2})$ ; kernel  $G_{pp}$  is usually named hypersingular, because it shows a singularity (of  $O(r^{-3})$ ) greater than the dimension of the integral.

From the internal energy definition (see [1] for details) a bilinear form  $\mathcal{A}_E$  can be envisaged with an energy meaning:

$$\begin{aligned} \mathcal{A}_E\left(\begin{pmatrix} p \\ u \end{pmatrix}, \begin{pmatrix} \varsigma \\ \dot{\nu} \end{pmatrix}\right) &= \left\langle \mathcal{L} \begin{bmatrix} p \\ u \end{bmatrix}, \begin{pmatrix} \varsigma \\ \dot{\nu} \end{pmatrix} \right\rangle \quad (6) \\ &= \int_{\Gamma_u} \int_0^T \frac{\partial}{\partial t} \left[ \int_{\Gamma_u} \int_0^t G_{uu} p(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} - \int_{\Gamma_p} \int_0^t G_{up} u(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} \right] \varsigma(\mathbf{x}, t) dt d\Gamma_{u(x)} \\ &+ \int_{\Gamma_p} \int_0^T \left[ \int_{\Gamma_u} \int_0^t G_{pu} p(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} - \int_{\Gamma_p} \int_0^t G_{pp} u(\mathbf{y}, \tau) d\tau d\Gamma_{(y)} \right] \dot{\nu}(\mathbf{x}, t) dt d\Gamma_{p(x)} \end{aligned}$$

and the energetic weak form of problem (4-5) reads:

$$\text{given } \begin{pmatrix} \dot{f}_u \\ \dot{f}_p \end{pmatrix} \text{ find } \begin{pmatrix} p \\ u \end{pmatrix} \in W \text{ s.t. } \mathcal{A}_E\left(\begin{pmatrix} p \\ u \end{pmatrix}, \begin{pmatrix} \varsigma \\ \dot{\nu} \end{pmatrix}\right) = \left\langle \begin{pmatrix} \dot{f}_u \\ \dot{f}_p \end{pmatrix}, \begin{pmatrix} \varsigma \\ \dot{\nu} \end{pmatrix} \right\rangle \quad \forall \begin{pmatrix} \varsigma \\ \dot{\nu} \end{pmatrix} \in \widetilde{W} \quad (7)$$

After the discretization process, collocation as well as variational formulations turn out to require the evaluation of “integrals” of the form:

$$\int_{\Gamma_s} \int_0^t G_{rs}(\mathbf{x}, \mathbf{y}, t - \tau) \boldsymbol{\psi}(\tau) \otimes \boldsymbol{\phi}(\mathbf{y}) d\Gamma_{(y)} d\tau, \quad t \in (0, T) \quad \mathbf{r}, \mathbf{s} = \mathbf{u}, \mathbf{p} \quad (8)$$

## 2 SHAPE FUNCTIONS

Discretization over the lateral boundary  $\Sigma = (0, T) \times \Gamma$  is achieved through space-time polynomial shape functions on a trapezoidal tassellation of  $\Gamma$  and a non uniform decomposition of segment  $(0, T)$ . Collect in matrix  $\boldsymbol{\xi}^u$  all shape functions for the discrete approximation  $\hat{u}(\mathbf{y}, \tau)$  of the Dirichlet field  $u(\mathbf{y}, \tau)$  over the lateral boundary  $\Sigma_u = (0, T) \times \Gamma_u$ . Matrix  $\boldsymbol{\xi}^p$  for the discrete approximation  $\hat{p}(\mathbf{y}, \tau)$  of Neumann field  $p(\mathbf{y}, \tau)$  is defined analogously over the lateral boundary  $\Sigma_p = (0, T) \times \Gamma_p$ . Accordingly,

$$\hat{u}(\mathbf{y}, \tau) = \boldsymbol{\xi}^u(\mathbf{y}, \tau) \cdot \hat{\mathbf{u}}, \quad \hat{p}(\mathbf{y}, \tau) = \boldsymbol{\xi}^p(\mathbf{y}, \tau) \cdot \hat{\mathbf{p}} \quad (9)$$

In the former equation: i)  $\hat{\mathbf{u}}$  and  $\hat{\mathbf{p}}$  are matrices of unknown nodal values; ii) scalar product is defined as the usual Frobenius product; iii) shape functions are taken of tensor product form; they read:

$$\boldsymbol{\xi}^u(\mathbf{y}, \tau) = \boldsymbol{\Psi}(\tau) \otimes \boldsymbol{\Phi}^u(\mathbf{y}) \quad \boldsymbol{\xi}^p(\mathbf{y}, \tau) = \boldsymbol{\Psi}(\tau) \otimes \boldsymbol{\Phi}^p(\mathbf{y}) \quad (10)$$

where  $\boldsymbol{\Psi}(\tau) = \{\Psi_m(\tau), (m = 1, 2, \dots, N_T)\}$  is a basis of a space of finite elements on the time interval, and  $\boldsymbol{\Phi}(\mathbf{y}) = \{\phi_n(\mathbf{y}), (n = 1, 2, \dots, N_\Gamma)\}$  is a basis of a space of finite elements on boundary  $\Gamma$ ; iv) tensor product  $\otimes : \mathbb{R}^{N_T} \times \mathbb{R}^{N_\Gamma} \rightarrow \mathbb{R}^{N_T \times N_\Gamma}$  is defined as:  $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$  for all  $\mathbf{c} \in \mathbb{R}^{N_\Gamma}$ .

### 2.1 Definition and representation of shape functions in time

Shape functions  $\Psi_m(\tau)$  are defined over a decomposition of the time axis with nodes  $\{t_m, (m = 1, 2, \dots, N_T)\}$  and  $t_0 = 0$ . Let  $T_k$  be the generic  $(N_k + 1)$ -nodes interval of the decomposition and

chose over  $T_k$  a local basis  $\boldsymbol{\psi}(\tau) = \{\psi_j(\tau), (j = 1, 2, \dots, N_k + 1)\}$  where  $\psi_j(\tau)$  is a polynomial (usually lagrangian) of degree  $N_k$  with compact support  $T_k$ . Collect in set  $\mathcal{T}_m = \{T_k\}_{k=1}^{N_{\mathcal{T}_m}}$  the (two at most, i.e.  $N_{\mathcal{T}_m} \leq 2$ ) intervals sharing vertex  $t_m$ ; then  $\Psi_m$  is defined as:

$$\Psi_m(\tau) = \begin{cases} \psi_{\omega(k,m)}(\tau) & \tau \in T_m \\ 0 & \tau \notin T_m \end{cases} \quad (11)$$

where the index  $\omega(k, m)$  selects the local basis function on  $T_k$  such that  $\psi_{\omega}(t_m) = 1$ . By construction,  $\Psi_m(\tau) \in C^0(\mathbb{R})$ , and its compact support is  $\mathcal{T}_m$ . Collecting into vector  $\mathbf{b}_{\omega(k,m)}$  suitable constants and defining with  $\boldsymbol{\tau} = \{1, \tau, \tau^2, \dots, \tau^{N_k}\}$ , shape function  $\psi_{\omega(k,m)}(\tau)$  in definition (11) reads:

$$\psi_{\omega(k,m)}(\tau) = \chi_{T_k}(\tau) \mathbf{b}_{\omega(k,m)} \cdot \boldsymbol{\tau} \quad (12)$$

where  $\chi_{T_k}(\tau)$  is the characteristic function of interval  $T_k$ .

## 2.2 Definition and representation of shape functions in space

Let  $\Gamma_h$  be a flat tessellation of boundary  $\Gamma$ ,  $Q_j \subset \Gamma_h$  its generic trapezoidal and  $\mathbf{a}_n$  a generic node of  $\Gamma_h$ . Collect in set  $\mathcal{Q}_n := \{Q_j \text{ s.t. } \mathbf{a}_n \in Q_j\}$  all panels of  $\Gamma_h$  sharing node  $\mathbf{a}_n$  (see Fig. 1). Choose over  $Q_j$  a local (lagrangian) basis  $\boldsymbol{\varphi}_j := \{\varphi_1, \varphi_2, \dots, \varphi_M\}$  and denote with  $\varphi_{\omega(j,n)}$ ,  $1 \leq \omega \leq M$  the unique element of  $\boldsymbol{\varphi}_j$  such that  $\varphi_{\omega(j,n)}(\mathbf{a}_n) = 1$ . Define shape function  $\phi_n(\mathbf{x})$  (see Fig. 1) as a piecewise continuous function over  $\Gamma_h$  whose value is zero outside  $\mathcal{Q}_n$ , as follows:

$$\phi_n \in C^0(\Gamma_h) \quad \text{supp}(\phi_n) = \mathcal{Q}_n \quad \phi_n|_{Q_j} = \varphi_{\omega(j,n)} \quad (13)$$

A suitable choice of an orthogonal cartesian coordinate system<sup>2</sup> allows an effective representation for  $\varphi_{\omega(j,n)}(\mathbf{y})$ .

Let  $\mathcal{L} \equiv \{y_1, y_2, y_3\}$  define a local coordinate system such that: i) a vertex of trapezoidal  $Q_j$  is the origin; ii) the plane  $y_1 = 0$  contains  $Q_j$ ; iii) plane  $y_2 = 0$  contains one base of trapezoidal  $Q_j$ ; iv) plane  $y_3 = 0$  is orthogonal to the parallel sides of  $Q_j$ . In  $\mathcal{L}$ ,  $Q_j$  is defined by:

$$Q_j := \{\mathbf{y} \in \mathbb{R}^3 \text{ s.t. } y_1 = 0; 0 \leq y_2 \leq \bar{y}_2; ay_2 \leq y_3 \leq by_2 + \bar{y}_3\}$$

where  $a$  and  $b$  denote the slopes of the two slanted sides of  $Q_j$  (see Fig. 1) and  $\bar{y}_3$  the intersection with axis  $y_3$ . Shape function  $\varphi_{\omega(j,n)}(\mathbf{y})$ , denoting with  $\mathbf{d} = \mathbf{y} - \mathbf{x}$ ,  $r = \|\mathbf{d}\|$  and using the binomial expansion rule, can be straightforwardly rearranged in the form:

$$\varphi_{\omega(j,n)}(\mathbf{d}) = \mathbf{a}_{\omega(j,n)}^{(3)\top} \mathbf{X}^{(3)} \mathbf{d}_3 \otimes \mathbf{d}_2 \mathbf{X}^{(2)\top} \mathbf{a}_{\omega(j,n)}^{(2)} \quad (14)$$

where:

$$\mathbf{d}_\alpha = \{1, d_\alpha, d_\alpha^2, \dots, d_\alpha^{N_\omega}\}, \quad \mathbf{X}_{il}^{(\alpha)} = \binom{l-1}{i-1} x_\alpha^{(l-i)} \quad i, l = 1, 2, \dots, 1 + N_\omega; \quad \alpha = 2, 3$$

<sup>2</sup>The choice of an orthogonal coordinate system is arbitrary because the jacobian is unit and no distortions are introduced with regard to the Hadamard's finite part.

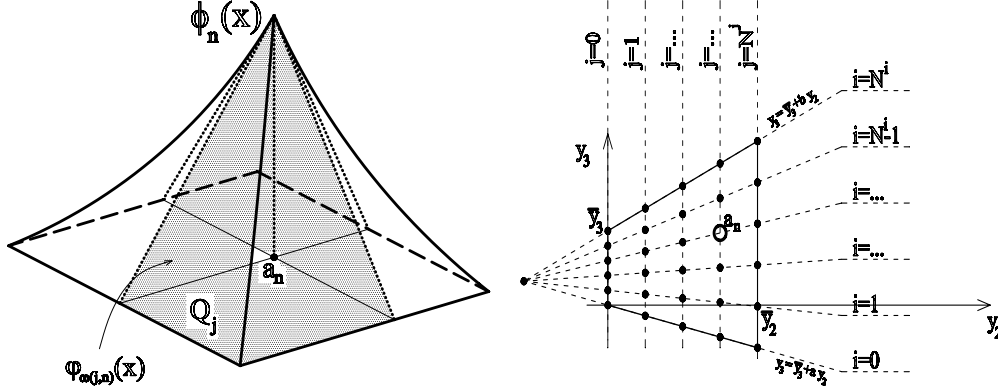


Figure 1: Shape functions in space. a) Local  $\varphi_{\omega(j,n)}(\mathbf{x})$  and global  $\phi_n(\mathbf{x})$  shape functions.; b) Local coordinate system  $\mathcal{L}$  and shape function definition.

Here  $N_\omega$  stands for the degree of polynomial shape functions  $\varphi_{\omega(j,n)}$  and  $\mathbf{a}_{\omega(j,n)}^{(2)}$ ,  $\mathbf{a}_{\omega(j,n)}^{(3)}$  are suitable vectors of constants.

### 3 MAIN RESULT

By means of (12, 14), integrals (8) can be recast in the form:

$$\begin{aligned} \sum_{j,k} \mathbf{b}_{\omega(k,m)} \cdot \left[ \int_{Q_j} \varphi_{\omega(j,n)}(\mathbf{y}) \int_0^{t_h} G_{rs}(\mathbf{x}, \mathbf{y}; t_h - \tau) \boldsymbol{\tau} \chi_{T_k}(\tau) d\tau d\Gamma_y \right] \\ = \sum_{j,k} \mathbf{b}_{\omega(k,m)} \cdot \mathbf{f}_{\omega(j,n)}^{rs}(\mathbf{x}, t_h, T_k) \end{aligned} \quad (15)$$

having denoted with  $\mathbf{f}_{\omega(j,n)}^{rs}(\mathbf{x}, t_h, T_k)$  the vector of components:

$$\mathbf{f}_{\omega(j,n)_l}^{rs}(\mathbf{x}, t_h, T_k) = \mathbf{a}_{\omega(j,n)}^{(2)\top} \mathbf{X}^{(2)} \mathbb{K}_l^{rs}(\mathbf{x}, t_h, T_k) \mathbf{X}^{(3)\top} \mathbf{a}_{\omega(j,n)}^{(3)} \quad (16)$$

where  $l = 0, 1, 2, \dots, N_k$  is the degree of the monomial  $\tau^l$  in vector  $\boldsymbol{\tau}$  and  $\mathbb{K}^{rs} \in \mathbb{R}^{1+N_k} \times \mathbb{R}^{(1+N_\omega) \times (1+N_\omega)}$  is the matrix:

$$\mathbb{K}^{rs}(\mathbf{x}, t_h, T_k) = \int_{-x_2}^{\bar{y}_2 - x_2} \int_{a_{d_2 + K_{a,0}}}^{b_{d_2 + K_{b, \bar{y}_3}}} \mathbf{g}^{rs}(\mathbf{d}, t_h, T_k) \otimes \mathbf{d}_2 \otimes \mathbf{d}_3 dd_3 dd_2 \quad (17)$$

which collects integration in space of the ‘‘time integral’’ vector  $\mathbf{g}^{rs}$ :

$$\mathbf{g}^{rs}(\mathbf{d}, t_h, T_k) = \int_0^{t_h} G_{rs}(\mathbf{d}; t_h - \tau) \boldsymbol{\tau} \chi_{T_k}(\tau) d\tau \quad (18)$$

In definition (17)  $K_{a,0} = a x_2 - x_3 = 0$  and  $K_{b, \bar{y}_3} = b x_2 - x_3 + \bar{y}_3 = 0$  denote the equations of the two slanted sides of  $Q_j$  (see Fig. 1).

Vector  $\mathbf{g}^{rs}$  has a distributional nature, due to the definition of Green's function  $G_{rs}$ . As a consequence, evaluation of (18) has to be performed as a limit process within a suitable functional space, which appears to be  $C_0^2$ . The basic distributions:

$$\left\langle \frac{\partial^n \mathbf{H}\left(t - \tau - \frac{r}{c}\right)}{\partial \tau^n}, \tau^l \chi_{T_k}(\tau) \right\rangle \quad n = 1, 2, 3 \quad (19)$$

are required to evaluate  $\mathbf{g}_l^{rs}(\mathbf{d}, t, T_k)$ . It can be proved - see [1] for detail - that:

$$\left\langle \frac{\partial}{\partial \tau} \delta\left(t - \tau - \frac{r}{c}\right), \tau \mathbf{H}(\tau) \mathbf{H}(t_k - \tau) \right\rangle = \quad (20)$$

$$\begin{aligned} & \mathbf{H}\left(t - \frac{r}{c}\right) \mathbf{H}\left(t_k - t + \frac{r}{c}\right) - t_k \delta\left(t - t_k - \frac{r}{c}\right) \\ & \left\langle \frac{\partial^2}{\partial \tau^2} \delta\left(t - \tau - \frac{r}{c}\right), \tau \mathbf{H}(\tau) \mathbf{H}(t_k - \tau) \right\rangle = \quad (21) \end{aligned}$$

$$\delta\left(t - \frac{r}{c}\right) - \delta\left(t - t_k - \frac{r}{c}\right) - t_k \frac{\partial}{\partial t} \delta\left(t - t_k - \frac{r}{c}\right)$$

Consider  $\mathbf{x} \notin \overline{Q_j}$ . By means a suitable recursive relationship - which comes out from the binomial expansion rule -,  $\mathbb{K}^{rs}(\mathbf{x}, t, T_k)$  is reduced to the combination of a set of two basic integrals:

$$\int_{-a-x_2}^{a-x_2} d_2^{\bar{h}} \int_{-b-x_3}^{b-x_3} r^p d_3^{\bar{h}} dd_3 dd_2 \Big|_{d_1=-x_1} \quad ; \quad \int_{-a-x_2}^{a-x_2} d_2^{\bar{h}} \int_{-b-x_3}^{b-x_3} \frac{d_3^{\bar{h}}}{r^{2l+1}} dd_3 dd_2 \Big|_{d_1=-x_1} \quad (22)$$

with  $h, \bar{h}, p \in \mathbb{N}_0$  and  $l = 0, 1, 2$ .

Algebraic manipulations lead from (22) to the following tabular expression for  $\mathbb{K}_l^{rs}(\mathbf{x}, t, T_k)$  in the local coordinate system  $\mathcal{L}$ :

$$\mathbb{K}_l^{rs}(\mathbf{x}, t, T_k) = -\frac{\alpha}{4\pi} \widehat{\mathbb{K}}_m^{rs}(\mathbf{d}, t, T_k) \Big|_{\substack{\theta=b, \eta=\bar{y}_3 \\ \theta=a, \eta=0}} \Big|_{\substack{d_2=\bar{y}_2-x_2 \\ d_2=-x_2}} \Big|_{d_1=-x_1} \quad (23)$$

with:

$$\widehat{\mathbb{K}}_l^{rs}(\mathbf{d}, t, T_k) = \mathbb{L}_l^{rs} \log(\zeta_2 + r) + \mathbb{A}_l^{rs} \operatorname{arctanh} \frac{d_3}{r} + \mathbb{I}_l^{rs} I_{\square}^{-3}(\mathbf{d}) + \mathbb{R}_l^{rs} r + \mathbb{S}_l^{rs} \frac{1}{r} + \mathbb{P}_l^{rs} \quad (24)$$

where: I)  $I_{\square}^{-3}(\mathbf{d})$  is the Lebesgue integral of the function  $\frac{1}{r^3}$  over  $Q_j$ ; II)  $\mathbb{L}_l^{rs}, \mathbb{A}_l^{rs}, \mathbb{I}_l^{rs}, \mathbb{R}_l^{rs}, \mathbb{S}_l^{rs}, \mathbb{P}_l^{rs}$  are matrices of the same order of  $\mathbb{K}_l^{rs}$  detailed in [1].

#### 4 CONCLUDING REMARKS

The present note aims at providing a closed form for analytical integrations involved in 3D space-time hyperbolic BIEs. In it, analytical integrations have been performed for equations (3)-(4) in their discrete form; both the singular and the regular part have been considered, so that the closed form is obtained as a function of the collocation point. The outcomes presented in [1] are exhaustive for the collocation approach as well as for the post-process reconstruction of primal and dual fields. The

results show that space-time variational formulations may take advantage on the proposed methodology as well. With reference to the outer integral in space, the approach pursued in [11] maps the road to show the mutual cancelation of the singular terms in the outer integration process. Besides accuracy, the availability of the closed form for the approximated primal and dual fields entails the possibility of analytical manipulations - see e.g. [12] - which are hardly possible with alternative approaches. Closed forms obtained are amenable to extension to dynamic fracture mechanics. The use of fast integral operator techniques [13] seems to be possible in a way similar to steady state problems. However, further investigations are required to assess conditions for their robustness and accuracy. Broader impacts may result from the present note: obtained results may have influence on extremely modern and stimulating applications, e.g. [14] but they need to be extended in order to comply with very promising techniques for time marching schemes. Furthermore, on the educational side, introductory courses in the boundary integral equations may benefit from the closed form for equations (3)-(4), which may lighten the effort of the numerical approximation of hyperbolic problems via BEM to inexpert auditors.

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