Identification of non-conventional viscoelastic models for polymeric vibrating structures

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SUMMARY. In the field of structural vibrations, an appropriate rheological model should be accurate in fitting the experimental data on a wide interval of frequencies by means of a minimum number of parameters and, in particular, it should be able to reproduce the experimentally found behavior of the damping ratio ζ_n as a function of the natural angular frequency ω_n. In this study a non-integer order differential linear rheological model is considered, and its effectiveness in solving the above mentioned problem is discussed. This model, refferred to as the Fractional Double Kelvin model, combines the properties of both the Fractional Kelvin and Fractional Zener models, which are considered to be very effective in describing the viscoelastic dynamic behavior of mechanical structures made of polymers. An identification method of general validity for viscoelastic models is adopted, based on the concept of equivalent damping ratio and on the circle-fit technique. It is applied to the analysis of vibrating beams and plates of different sizes, made of polymeric materials such as Polyethylene, Polyvinyl-chloride and Delrin.

1 INTRODUCTION

When dealing with structural dynamic problems, it would be useful to obtain a viscoelastic model identification from vibration measurement data only. In this case, however, the direct identification of an optimal set of parameters from time or frequency domain measurements is a difficult task, especially if the structural dissipative contributions are low.

In this paper, an indirect approach is adopted, based on the concept of damping ratio ζ_n, focusing the attention on the behavior of ζ_n as a function of the modal frequency ω_n. It is well known that a modal parameter ζ_n can be analytically defined and experimentally estimated by considering a linear viscous dissipative model, based on a single Newton element. However this theoretical parameter shows a dependency on the modal frequency that in most cases dramatically fails in fitting the experimental data.

On the contrary, it was shown that a better agreement between theory and experiments can be achieved by means of non-integer order differential models, obtained by replacing the first derivative (Newton element) with a fractional derivative (Scott-Blair element) [1]. Extensive literature exists on the application of fractional calculus to viscoelasticity, since it yields to physically consistent stress-strain constitutive relations with a few parameters, good curve fitting properties and causal behavior [2-3-4].

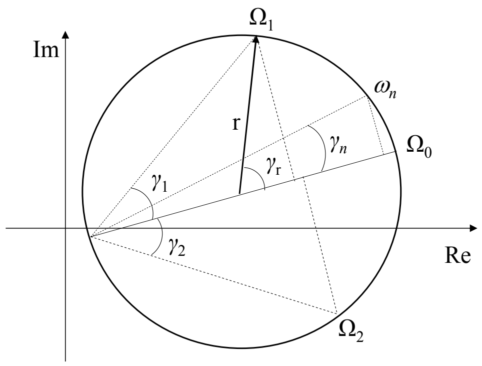
In particular, regarding the $\zeta_n = \zeta_n(\omega_n)$ behavior, it was experimentally observed that it may be non-monotonic, and consequently the Single Kelvin and Zener models (either integer or fractional order) may in some cases be not suitable for fitting the experimental data [5-6]. In this paper a more refined rheological model is thus considered, referred to as the Fractional Double Kelvin model, applied to the analysis of vibrating beams and plates of different sizes, made of polymeric materials such as Polyethylene, Polyvinyl-chloride and Delrin. Structural damping laws are not included in the analysis, since they can lead to non-causal behavior. Since in the case of fractional derivative models analytical expressions for ζ_n are usually difficult to find, a method of general validity for viscoelastic models was developed, introducing the concept of equivalent damping ratio applied to the circle-fit technique [7]. This identification method is based on the assumption that the Nyquist plot of the mobility for any mode *n* can be approximated by a circumference, which is still acceptable when considering fractional derivative models [8-9-10].

2 LINEAR VISCOELASTIC MODEL AND IDENTIFICATION TECHNIQUE

According to the circle-fit technique [7], the circular Nyquist plot of the Mobility for each vibration mode allows the experimental estimate of the related modal damping (Fig.1). As a consequence, the modal damping ratio ζ_n for the classical Integer Single Kelvin model can be evaluated by means of the expression:

$$\zeta_n = \frac{1}{2\omega_n}\left[\frac{\Omega_2^2 - \Omega_1^2}{\Omega_2 \tan(\gamma_2) + \Omega_1 \tan(\gamma_1)}\right] \tag{1}$$

where ω_n is the natural angular frequency and the other symbols refer to Fig.1.



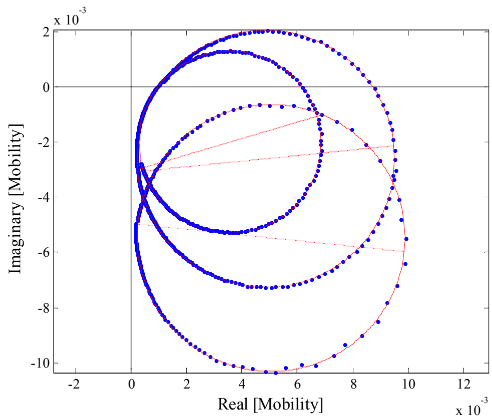


Figure1. Nyquist plot of Mobility. General scheme for mode *n* (left) and experimental plots (right).

The experimental estimates of ζ_n as a function of the natural frequency ω_n usually show a behavior which is very far from the straight increasing line passing through the origin predicted by the Integer Single Kelvin model [9-10]. As a consequence, in order to fit such experimental curves, more refined models are needed.

In the present study the Fractional Double Kelvin model is adopted, which can be considered as a generalization of both the Single Kelvin and the Zener models. Its analogical representation is shown in Fig.2, and its analytical expression in the frequency domain is given by:

$$E(\Omega) = \frac{\left[E_1 + C_1(\mathrm{i}\Omega)^{\alpha_1}\right]\left[E_2 + C_2(\mathrm{i}\Omega)^{\alpha_2}\right]}{E_1 + E_2 + C_1(\mathrm{i}\Omega)^{\alpha_1} + C_2(\mathrm{i}\Omega)^{\alpha_2}} \tag{2}$$

where α_1 and α_2 are non-integer or fractional derivative orders (values between 0 and 1).

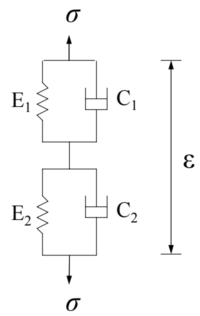


Figure 2. Viscoelastic Double Kelvin analogic model.

Equation 2 can be rewritten highlighting the static Young's modulus E_0 as well as the characteristic times τ_ε (related to the creep behavior) and τ_σ (related to the relaxation behavior), according to the definitions reported in Tab.1:

$$E(\Omega) = E_0 \frac{\left[1+(\mathrm{i}\tau_{\varepsilon 1}\Omega)^{\alpha_1}\right]\left[1+(\mathrm{i}\tau_{\varepsilon 2}\Omega)^{\alpha_2}\right]}{1+(\mathrm{i}\tau_{\sigma 1}\Omega)^{\alpha_1}+(\mathrm{i}\tau_{\sigma 2}\Omega)^{\alpha_2}} = E_0 \frac{N(\Omega)}{D(\Omega)} \tag{3}$$

which for $\alpha_1 = \alpha_2 = \alpha$ reduces to:

$$E(\Omega) = E_0 \frac{\left[1+(\mathrm{i}\tau_{\varepsilon 1}\Omega)^{\alpha}\right]\left[1+(\mathrm{i}\tau_{\varepsilon 2}\Omega)^{\alpha}\right]}{1+(\mathrm{i}\tau_{\sigma}\Omega)^{\alpha}} \tag{4}$$

The relations among the viscoelastic parameters in the Fractional Double Kelvin model, as well as in the Fractional Single Kelvin and Zener models, are reported in Tab.1. Clearly for $\alpha = 1$ these fractional models correspond to the more conventional integer order ones.

Table 1: parameter relations for the viscoelastic models.

Model	E_0	$\tau_{\varepsilon 1}^{\alpha_1}$	$\tau_{\varepsilon 2}^{\alpha_2}$	τ_{σ}^{α}	$\tau_{\sigma 1}^{\alpha_1}$	$\tau_{\sigma 2}^{\alpha_2}$
Single Kelvin $C_2 = 0, \quad E_2 = \infty$	E_1	$\frac{C_1}{E_1}$	0	0	0	0
Zener $C_2 = 0$	$\frac{E_1 E_2}{E_1 + E_2}$	$\frac{C_1}{E_1}$	0	$\frac{C_1}{E_1 + E_2}$	$\frac{C_1}{E_1 + E_2}$	0
Double Kelvin	$\frac{E_1 E_2}{E_1 + E_2}$	$\frac{C_1}{E_1}$	$\frac{C_2}{E_2}$	$\frac{C_1 + C_2}{E_1 + E_2}$	$\frac{C_1}{E_1 + E_2}$	$\frac{C_2}{E_1 + E_2}$

At this stage a definition of general validity is needed for ζ_n, in order to create a link between the experimental damping estimates of Eq.1 and the viscoelastic parameters reported in Tab.1. Let us consider, for example, a homogeneous Kirchhoff plate. Its (homogeneous) equilibrium equation in the Laplace domain can be written as:

$$\rho_h s^2 w(\mathbf{x},s) + \frac{Eh^3}{12(1-\nu^2)} \nabla^4 w(\mathbf{x},s) = 0, \quad \nabla^4(\cdot) = \frac{\partial^4}{\partial x^4}(\cdot) + 2\frac{\partial^4}{\partial x^2 \partial y^2}(\cdot) + \frac{\partial^4}{\partial y^4}(\cdot) \tag{5}$$

where $\mathbf{x}$ denotes the spatial coordinates, ρ_h the mass per unit area of the plate and ν the Poisson's ratio [11].

The expression of the Young's modulus for the Fractional Double Kelvin model (Eq.3) can be rewritten in the Laplace domain, and then introduced in Eq.5 in place of E. Multiplying both sides of Eq.5 for $D(s)$ (Eq.3) yields:

$$\rho_h s^2 \left[\tau_{\sigma_1}^{\alpha_1} s^{\alpha_1} + \tau_{\sigma_2}^{\alpha_2} s^{\alpha_2} + 1\right] w(\mathbf{x},s) + \frac{E_0 h^3}{12(1-\nu^2)} \left[\tau_{\varepsilon 1}^{\alpha_1} \tau_{\varepsilon 2}^{\alpha_2} s^{\alpha_1+\alpha_2} + \tau_{\varepsilon 1}^{\alpha_1} s^{\alpha_1} + \tau_{\varepsilon 2}^{\alpha_2} s^{\alpha_2} + 1\right] \nabla^4 w(\mathbf{x},s) = 0 \tag{6}$$

By separating the variables, i.e. introducing $w(\mathbf{x},s) = W(\mathbf{x})Q(s)$, Eq.6 can be easily decoupled (for a rigorous approach to this decoupling problem, the reader is referred to [12-13-14]). More in detail, Eq.6 can be rewritten separating the $\mathbf{x}$-dependent part from the s-dependent part in the form:

$$\frac{E_0 h^3}{12(1-\nu^2)\rho_h} \cdot \frac{\nabla^4 W(\mathbf{x})}{W(\mathbf{x})} = -\frac{s^2 D(s)}{N(s)} = \kappa = \text{constant} \tag{7}$$

Introducing the modal natural frequency $\kappa = \omega_n^2$, Eq.7 yields the homogeneous characteristic equation associated with the n-th mode:

$$s^2 \left[\tau_{\sigma_1}^{\alpha_1} s^{\alpha_1} + \tau_{\sigma_2}^{\alpha_2} s^{\alpha_2} + 1\right] + \omega_n^2 \left[\tau_{\varepsilon 1}^{\alpha_1} \tau_{\varepsilon 2}^{\alpha_2} s^{\alpha_1+\alpha_2} + \tau_{\varepsilon 1}^{\alpha_1} s^{\alpha_1} + \tau_{\varepsilon 2}^{\alpha_2} s^{\alpha_2} + 1\right] = 0 \tag{8}$$

Note that in Eq.8 the geometry and boundary conditions of the vibrating structure are influential on ω_n only. In the cases of practical interest this equation gives a couple of complex conjugate roots and a real negative one.

Since the real negative part of the complex conjugate roots is responsible for the oscillation decay, a general definition for an equivalent ζ_n can be given as the absolute value of the real part of the complex conjugate roots of the characteristic equation, divided by ω_n, i.e.:

$$\zeta_n \triangleq \frac{\left|\mathrm{Re}(s_{2,3})\right|}{\omega_n} \tag{9}$$

It should be noted that definition 9 is generally coherent with the equivalent ζ_n evaluation via circle-fit technique (with the assumption $E_1 < E_2$, $C_2 < C_1$, $\tau_{\varepsilon 2} << \tau_{\varepsilon 1}$ and $\tau_{\sigma 2} << \tau_{\sigma 1}$, valid for most polymeric materials), making it possible to experimentally validate the theoretical model.

The ζ_n experimental estimates can be plotted with respect to the natural frequency ω_n, and then compared with the theoretical curves, computed according to different viscoelastic models (Eqs.3-4). A least square procedure for minimizing the difference between experimental and analytical curves can finally be adopted for optimal parameter identification.

Figures 3-4 highlight the qualitative behavior of the equivalent damping ratio ζ_n as a function of the natural frequency ω_n according to definition 9. The curves were computed assuming realistic values for the viscoelastic parameters, with $E_1 < E_2$, $C_2 < C_1$, $\tau_{\varepsilon 2} << \tau_{\varepsilon 1}$ and $\tau_{\sigma 2} << \tau_{\sigma 1}$.

In Fig.3 the Double Kelvin model is compared with the Single Kelvin and Zener ones, integer orders (left, $\alpha_1 = \alpha_2 = 1.0$) and fractional derivative orders (right, $\alpha_1 = \alpha_2 = 0.5$). Both the Single Kelvin and Zener models yield a $\zeta_n(\omega_n)$ monotonic behavior, while the Double Kelvin yields a minimum. In the medium-high frequency range, the Integer Double Kelvin model yields a $\zeta_n(\omega_n)$ plot curve which is asymptotic to a straight line (i.e. the Integer Single Kelvin model line). In the fractional derivative case the Single Kelvin $\zeta_n(\omega_n)$ plot is not a straight line.

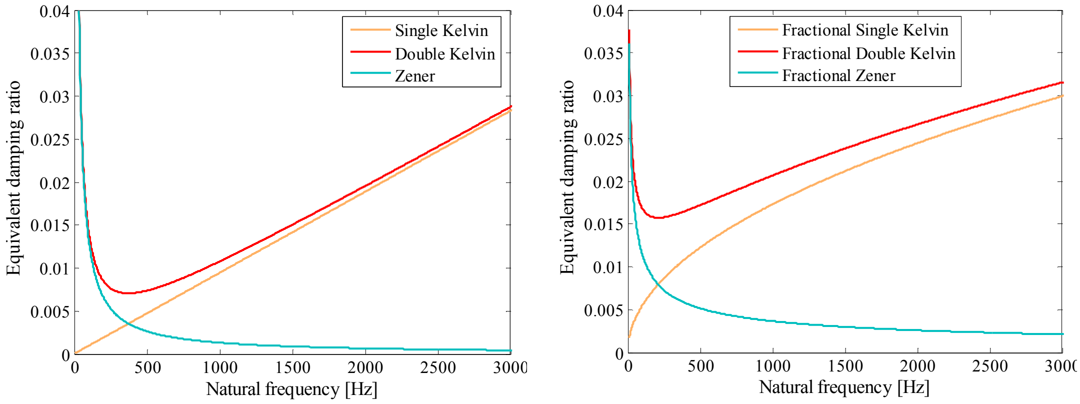


Figure 3. Behavior of ζ_n as a function of ω_n. Integer order models (left) and fractional ones (right).

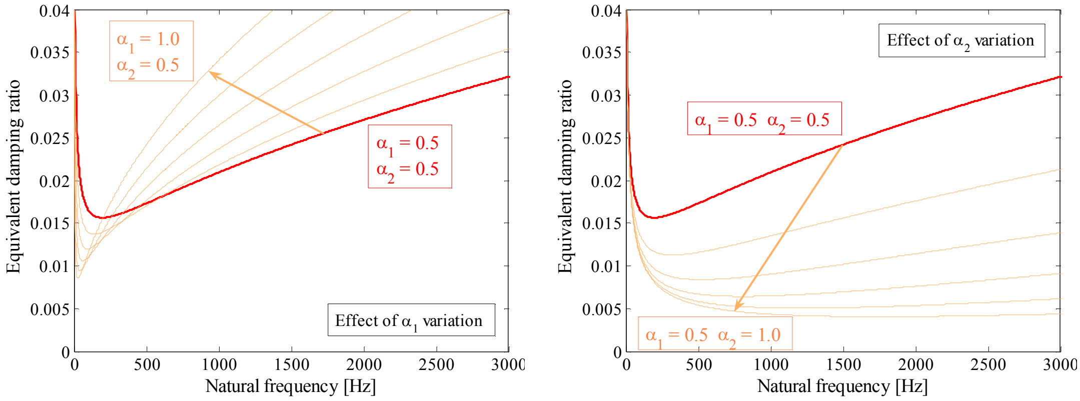


Figure 4. Fractional Double Kelvin model. Variation of α_1 from 0.5 to 1.0
(left, steps $\Delta\alpha_1 = 0.1$, $\alpha_2 = 0.5$) and variation of α_2 from 0.5 to 1.0 (right, steps $\Delta\alpha_2 = 0.1$, $\alpha_1 = 0.5$).

Figure 4 shows the effect due to two different values α_1 and α_2 of fractional derivative in the Double Kelvin model. Recalling the assumption $E_1 < E_2$, $C_2 < C_1$, $\tau_{\varepsilon 2} << \tau_{\varepsilon 1}$ and $\tau_{\sigma 2} << \tau_{\sigma 1}$, the effect of raising α_1 from 0.5 to 1.0 (keeping constant $\alpha_2 = 0.5$) is shown on the left, while the effect of raising α_2 from 0.5 to 1.0 (keeping constant $\alpha_1 = 0.5$) is shown on the right. Moreover, it was shown [5-6] that the primary effect of parameters $\tau_{\varepsilon 1}$ and $\tau_{\varepsilon 2}$ is on the slopes of the low and high frequency branches of the $\zeta_n(\omega_n)$ plot curve respectively. Regarding $\tau_{\sigma 1}$ and $\tau_{\sigma 2}$, they are functions of $\tau_{\varepsilon 1}$ and $\tau_{\varepsilon 2}$ respectively, and both of them are also a function of the ratio E_1/E_2. A variation of this latter parameter produces a shift in the magnitude of the $\zeta_n(\omega_n)$ plot curve.

3 NUMERICAL RESULTS

The proposed technique was applied to the analysis of clamped-free beams in plane flexural vibration and plates clamped on one edge, made of Polyethylene (PE, density $\rho = 964$ Kg×m^{-3}) Polyvinyl-chloride (PVC, $\rho = 1425$ Kg×m^{-3}) and Delrin ($\rho = 1437$ Kg×m^{-3}).

Tests were made on specimens of different size, as reported in Tab.2 (plates L × b and beams with thickness h_1 and h_2, cut at 5 different lengths L), using the experimental setup shown in Fig.5. The system was excited by a suspended shaker with random excitation, and acceleration responses were evaluated by means of 3 ICP piezoelectric accelerometers. Frequency response functions (inertances) were estimated on the interval 0-4000 Hz with resolution $\Delta f = 0.3$ Hz, H1 technique [7], 50 averages, Hanning window (beams and plates) and on the interval 0-1250 Hz with resolution $\Delta f = 0.1$ Hz, H1 technique, 20 averages, Hanning window (plates only). During the tests, the average temperature was 28°C ± 1°C.

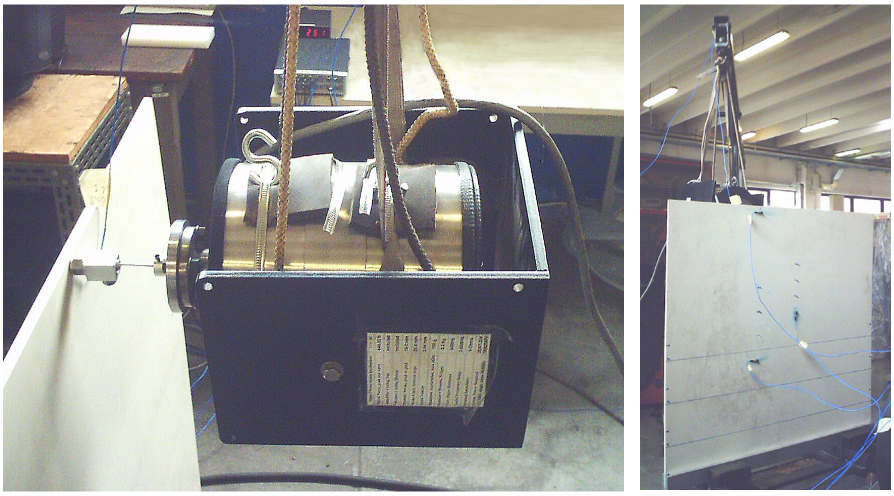


Figure 5. Experimental testing setup: suspended shaker (left) and plate (right).

Figures 6-8 show the plots of the experimentally estimated damping ratios with respect to the natural frequency. The analytical curves due to the Fractional Double Kelvin model (black solid lines) are superimposed on the experimental data plots. Table 3 refers to the colors adopted in Figs.6-8, related to specimen sizes. The identified parameters of the Fractional Double Kelvin model are reported in Tab.4.

Table 2: size of specimens.

Beams										
Material: **PE**										
Width b = 100 mm										
Thickness	h_1 = 25.0 mm					h_2 = 10.5 mm				
Length L	L_1	L_2	L_3	L_4	L_5	L_1	L_2	L_3	L_4	L_5
[mm]	952	852	748	400	200	952	852	746	400	200
Material: **PVC**										
Width b = 100 mm										
Thickness	h_1 = 26.5 mm					h_2 = 10.5 mm				
Length L	L_1	L_2	L_3	L_4	L_5	L_1	L_2	L_3	L_4	L_5
[mm]	971	871	744	400	200	968	868	769	400	200
Material: **Delrin**										
Width b = 100 mm										
Thickness	h_1 = 25.0 mm					h_2 = 10.5 mm				
Length L	L_1	L_2	L_3	L_4	L_5	L_1	L_2	L_3	L_4	L_5
[mm]	951	851	752	400	200	951	851	755	400	200

Plates			
Material	**PE**	**PVC**	**Delrin**
Thickness h [mm]	10	10	10
Width b [mm]	1000	1000	1000
Length L [mm]	946	969	951

Table 3: Meaning of colors in Figs.6-8.

Beam (color = length of specimen)						**Plate**	
length	L_1	L_2	L_3	L_4	L_5	size	L × b
color	*red*	*magenta*	*cyan*	*blue*	*green*	color	*black*

Table 4: Material estimated parameters.

Material	α_1	α_2	$\tau_{\varepsilon 1}$ [s]	$\tau_{\varepsilon 2}$ [s]	τ_{σ} [s]	$\tau_{\sigma 1}$ [s]	$\tau_{\sigma 2}$ [s]
PE	**0.45**	**0.45**	**10**	$\mathbf{2.0 \times 10^{-8}}$	**1.23**	–	–
PVC	**0.4**	**0.4**	**35**	$\mathbf{0.6 \times 10^{-8}}$	**7.00**	–	–
Delrin	**0.5**	**0.3**	**30**	$\mathbf{0.6 \times 10^{-8}}$	–	**8.33**	$\mathbf{4.33 \times 10^{-9}}$

4 DISCUSSION

Examination of the experimentally estimated damping ratios in Figs.6-8 suggests the following remarks:

- different specimen shapes and measurement points lead to ζ_n values converging on a single curve, depending on the material properties only (and not on the geometry). The dispersion, mainly due to measurement errors, may also be partially due to the temperature effect (varying from 27 to 29°C);

- the curves related to different materials show different behaviors in magnitude and shape;
- in the medium-high frequency region the data show a moderately increasing trend with respect to frequency, which in the frequency range examined is linear (or almost linear) for PE and PVC, but it is clearly non-linear for Delrin (with decreasing slope);
- in the low frequency range the data show a rather sharp decreasing trend, leading to a minimum;
- the proposed fractional double Kelvin model was able to accurately fit the experimentally found behavior of ζ_n for all of the different materials under analysis;
- in the case of PE and PVC a single value for the fractional derivative order ($\alpha_1 = \alpha_2 = \alpha$) seems to be sufficient for fitting the data. On the contrary, Delrin required the adoption of two different orders α_1, α_2 to be able to reproduce the data behavior.

It is important to point out that the Fractional Double Kelvin model was adopted for fitting the presented experimental data after having recognized the inefficiency of simpler models. The Single Kelvin and Zener models (both integer or fractional order) do not seem to be able to reproduce the experimentally found dependency of ζ_n to ω_n, giving a monotonic trend (Fig.3), while the Integer Double Kelvin model lacks in flexibility [5]. As a consequence, a more refined model is needed, e.g. the Fractional Double Kelvin model.

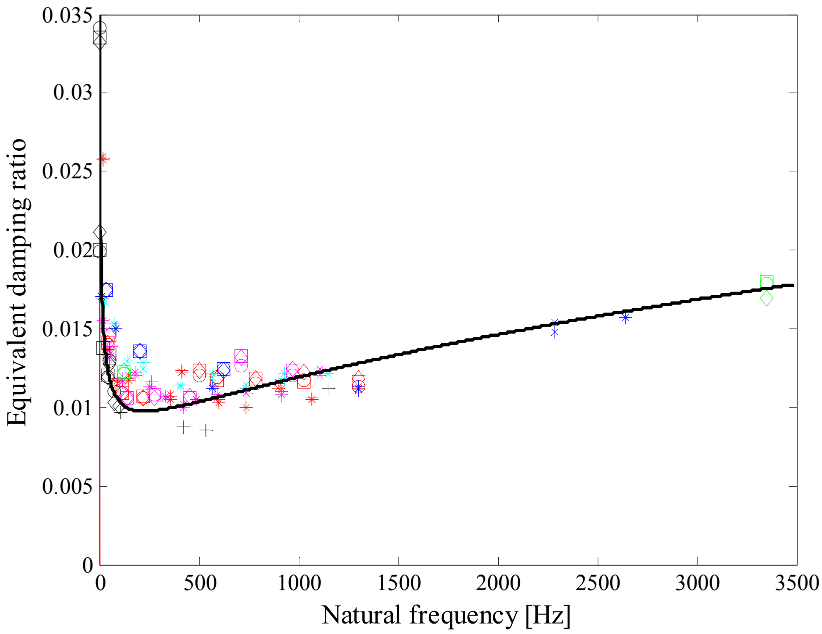


Figure 6. Equivalent damping ratio ζ_n as a function of ω_n. Material: PE.

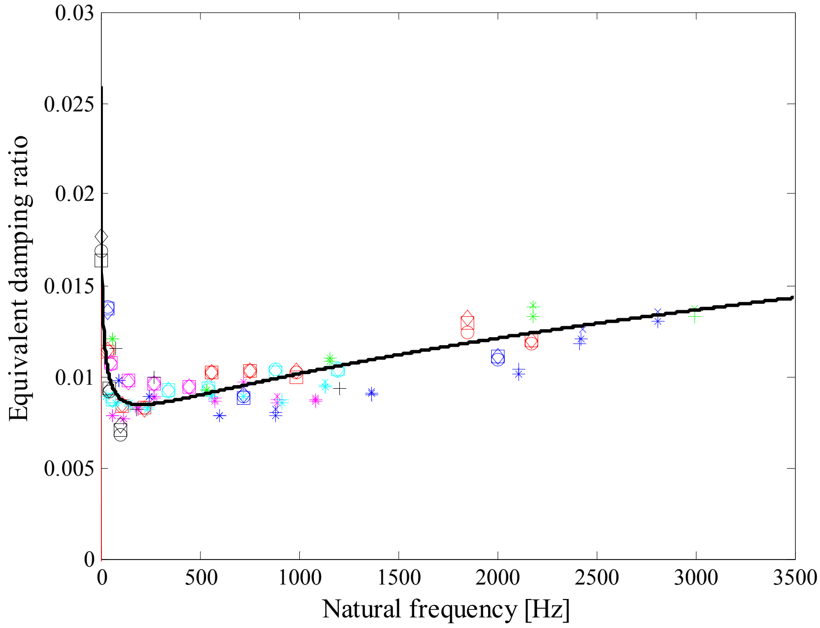


Figure 7. Equivalent damping ratio ζ_n as a function of ω_n. Material: PVC.

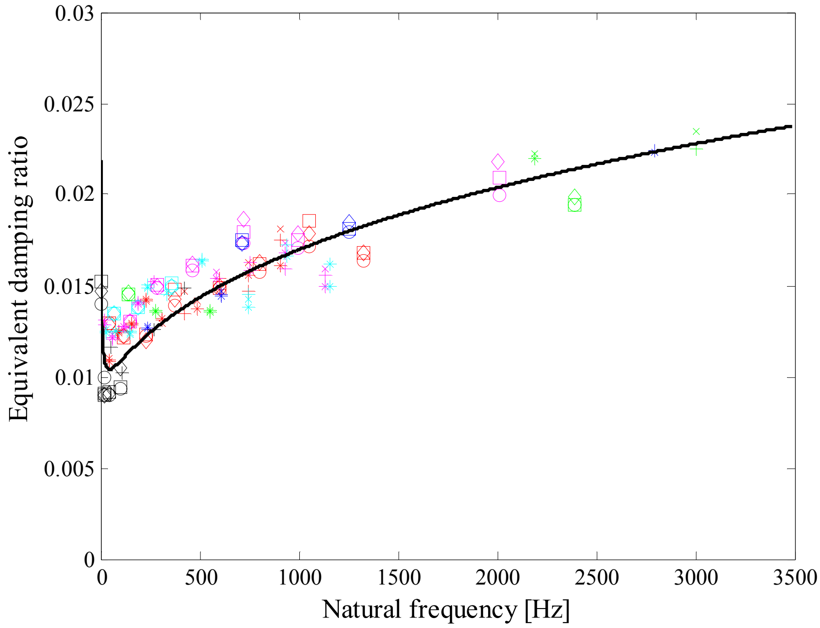


Figure 8. Equivalent damping ratio ζ_n as a function of ω_n. Material: Delrin.

5 CONCLUSIONS

In this paper, the circle-fit technique was applied for the experimental evaluation of an equivalent modal damping ratio, adopting a method valid for any linear viscoelastic model. Plane flexural vibrations of clamped-free beams and vibrations of plates clamped at one edge were considered, with specimens made of viscoelastic materials such as Polyethylene, Polyvinyl Chloride and Delrin. To fit the experimentally found equivalent modal damping ratio, the Fractional Double Kelvin model was adopted. The accuracy of the proposed model was discussed and proved in comparison with simpler viscoelastic models, namely the Simple Kelvin and Zener models.

The presented procedure and rheological model are suitable for possible future application to the analysis of vibrating viscoelastic structures with more complicated shapes than beams and plates, being possible their implementation in the finite element method [12-13-14].

Acknowledgments

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