Dynamic analysis of fractal antennas

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SUMMARY. The von Koch beam is one of the most well-known fractal structures, and it turns out to be particularly suitable to design fractal antennas. In this paper, its mechanical behaviour is investigated, by considering it as a hierarchical Eulero-Bernoulli framed-beam structure. Reductions of stiffness, mass and damping matrices lead to simple analytical recursive relationships, depending on the fractal dimension. A methodology to analyze damping of fractal antennas is proposed. Eventually, the results are exploited to perform a complete modal analysis and the response of the structure to the unit step function is considered.

1 INTRODUCTION

Fractal-shaped antennas have some unique characteristics that are linked to the geometrical properties of fractals [1-2]. Firstly, because the self-similarity property of fractals makes them specially suitable to design multi-frequency antennas [1]. Secondly, because the huge space-filling properties of some fractal shapes (i.e. the fractal dimension) help in the realization of small antennas to better take advantage of the surrounding space [2].

The properties of a von Koch curve [3], an archetype of fractal antennas, have recently been widely investigated: the existence of a homeomorphism between the closed real interval [0,1] and the von Koch curve has been proved in [4], while an analysis on the surface contained inside a Von Koch snowflake has been developed in [5]. On the other hand, to what concerns a von Koch beam, considered as a hierarchical Eulero-Bernoulli framed-beam structure, its static mechanical behaviour has been analyzed, both numerically [6], by means of a self-similarity postulate, and analytically [7], by means of simple recursive relationships on the strain energy and stiffness matrix. These results have been extended fot the computation of the mass matrix and to perform, consequently, a complete free-vibration analysis of the structure [8]. Thanks to matrix reduction [9-11], simple recursive scaling laws are provided.

In this paper, the analysis of the mechanical behaviour is extended to a generic von Koch beam i.e., with a generic indentation angle, and to the damped case. The paper is structured as follows: in Section 2, the von Koch beam construction is briefly recalled as well as the equation which describes how the fractal dimension of the structure varies as the indentation angle varies. Stiffness, mass and damping matrix scaling laws related to such structures are presented in Section 3. In Section 4, these results are exploited to perform a complete modal analysis and to evaluate the resonant and damped resonant frequencies of a von Koch cantilever beam. Eventually, the response of the structure to the unit step function is investigated by means of a Finite Element (FE) analysis.

2 GENERIC VON KOCH BEAM

The classical von Koch beam is generated starting from a line segment of length l_0 (called *the intiator*): at each step the middle third of each segment is removed and replaced by the other two sides of the equilateral triangle based on the removed segment (Fig. 1). In this case, an indentation angle θ=60° is taken into account and the fractal dimension of the structure is *D*=ln 4/ln 3.

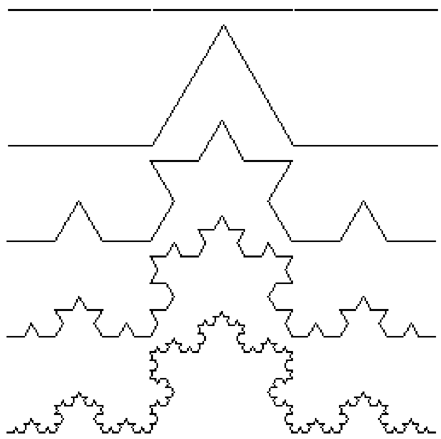

Figure 1: First four iterations in the classical von Koch beam generation.

The construction may be generalized to a generic von Koch beam i.e. with a generic value of θ, then considering a different fractal dimension according to the relationship:

$$D^* = -\frac{\ln 4}{\ln q} \tag{1}$$

where

$$q = \frac{1}{2(1+\cos\theta)} \tag{2}$$

The fractal dimension D^* is hence a monotonic increasing function of the angle θ, 0≤θ<90°.

3 STIFFNESS, MASS AND DAMPING MATRICES

The static analysis of a von Koch beam has been widely investigated in [7]. Since at each iteration n the number of nodes (and hence of the degrees of freedom, d.o.f.'s) grows exponentially as 2^{2n+1}, the dimensions of the stiffness and mass matrices increase. In [7-8] it has been proved that, by reducing matrices with respect to the same set of nodes (henceforth called master, as the related d.o.f.'s), particular scaling laws, depending on the fractal dimension, emerge after different iterations of the structure . Starting from the results on the strain energy, obtained by considering a classical von Koch cantilever beam subjected to three different loading conditions [7-12], the reduced stiffness matrix $[K]_n$ of a generic von Koch beam can be written as:

$$[K]_n = (4q)^{1-n}\,\frac{k}{l_0^3}\,[\overline{K}]_n = \left(\frac{l_n}{l_0}\right)^{D^*-1}\frac{4qk}{l_0^3}\,[\overline{K}]_n\,, \qquad n>1 \tag{3}$$

where l_0 is the length of the *initiator* (n=0), $l_n=q^n\ l_0$ is the length of each rectilinear beam constituting the structure at the n-th step, k is the beam rigidity, i.e. the product of the Young's modulus E of the material times the moment of inertia I of the cross-section with respect to the neutral axis, and $[\overline{K}]_n$ is the dimensionless stiffness matrix, which converges to finite values after approximately six iterations. The stiffness matrix $[K]_n$ in Eq. (3) scales asymptotically as $(4q)^{-n}$, exactly as the total length L_n. For n tending to infinity, the structural stiffness trivially tends to zero and the structure becomes infinitely compliant.

On the other hand, taking into account the real distribution of the masses over the beam, the following recursive relationship of the mass matrix $[M]_n$ is satisfied [8]:

$$[M]_n = (4q)^{n-1}\,\frac{ml_0}{420}\,[\overline{M}]_n = \left(\frac{l_n}{l_0}\right)^{1-D^*}\frac{ml_0}{1680q}\,[\overline{M}]_n\,, \qquad n>1 \tag{4}$$

$m=\rho A$ being the mass per unit length, where ρ is the material density and A is the area of the cross-section, and $[\overline{M}]_n$ the dimensionless mass matrix, which converges to finite values after a few iterations. Eq. (4) represents the counterpart of Eq. (3): while each term of the stiffness matrix tends to vanish (by scaling asymptotically as $(4q)^{-n}$) as the number of iterations n increases, the coefficients of the mass matrix diverge (by scaling as $(4q)^n$).

The validity of Eqs. (3) and (4) for a generic angle θ different from 60°, which has been implicitly assumed so far, will be proved in the next section by considering the related modal analysis.

Eventually, let us consider the special case in which the symmetric damping matrix $[C]_n$ is a linear combination of the matrices $[M]_n$ and $[K]_n$, namely when:

$$[C]_n = \alpha_n\,[M]_n + \beta_n\,[K]_n\,, \qquad n>1, \tag{5}$$

where α_n and β_n are real constants. This damping model is also known as "proportional damping" or "Rayleigh damping". Modes of proportionally damped systems preserve the simplicity of the real normal modes as in the undamped case. It follows from Eqs. (3) and (4) that, if α_n and β_n are not supposed to vary at each iteration, i.e. $\alpha_n = \alpha$ and $\beta_n = \beta$, the damping matrix scales asymptotically exactly as the mass matrix. In Section 4, it will be shown that this condition is physically unacceptable and suitable scaling laws for the two constants will be provided.

Observe that stiffness and mass matrices (and consequently the proportional damping one) remain finite as n increases only if the beam rigidity k and the mass per unit length m scale as $(4q)^n$ and $(4q)^{-n}$, respectively.

4 MODAL ANALYSIS

By means of Guyan's reduction [9] the choice of reducing stiffness and mass matrices with respect just to the extreme nodes is sufficient for the investigation of the first two vibrating frequencies related to a von Koch cantilever beam [8]. By increasing the number $N/3$ of master nodes, as to the five of the first order von Koch beam, the number of modes which can be accurately analyzed increases to seven [11]. Note that self-similarity of the structure can be exploited to simplify calculations [6].

4.1 *Free vibration motion*

Once the stiffness and mass matrices are known (Eqs. (3) and (4)), the governing differential equation of motion of a von Koch beam, in its free natural vibration, can be written as:

$$[M]_n \{\ddot{\delta}\}_n + [K]_n \{\delta\}_n = \{0\}, \tag{6}$$

$\{\delta\}_n$ and $\{\ddot{\delta}\}_n$ being the vectors of nodal displacements and the corresponding accelerations, respectively, at the iteration n. In order to investigate the free oscillation of the system, let us suppose that the generalized coordinates vary harmonically in time t as:

$$\{\delta\}_n = \{\delta_0\}_n \sin \omega_n t \,, \tag{7}$$

where angular frequencies ω_n and maximum amplitudes $\{\delta_0\}_n$ are to be determined via the eigenvalue problem:

$$\left([K]_n - \omega_n^2 [M]_n\right)\{\delta_0\}_n = 0 \,. \tag{8}$$

By solving Eq. (8), the following natural frequency scaling law is obtained [8]:

$$\omega_{i,n} = (4q)^{1-n} a_{i,n}^{(\omega)} \omega_{i,1} = 4q\left(\frac{l_n}{l_0}\right)^{D^*-1} a_{i,n}^{(\omega)} \omega_{i,1} \,, \tag{9}$$

where the first subscript refers to the mode, while the second one refers to the von Koch beam iteration ($\omega_{1,n}$, for instance, is the fundamental frequency related to the n-th order iteration). Fundamental frequency behaviour of a von Koch cantilever beam is plotted in Fig. 2, while the coefficients $a_{i,n}^{(\omega)}$ related to the first three natural frequencies are reported in Table 1: if four decimal digits are taken into account, convergence is expected after nearly six iterations.

Note that the $T_{i,n}$ period scaling law is trivially recovered by inverting Eq. (9). On the other hand, inserting it into Eq. (8) yields:

$$\{\delta_0\}_{i,n} = \sqrt{4q}\left(\frac{l_n}{l_0}\right)^{\frac{D^*-1}{2}} \left[a_{i,n}^{(\delta)}\right]\{\delta_0\}_{i,1} \,, \qquad i=1,\ldots,N \tag{10}$$

the modes having been opportunely normalized with respect to the mass. $[a_{i,n}^{(\delta)}]$ is the $N\times N$

diagonal matrix of the normalized eigenvector coefficients.

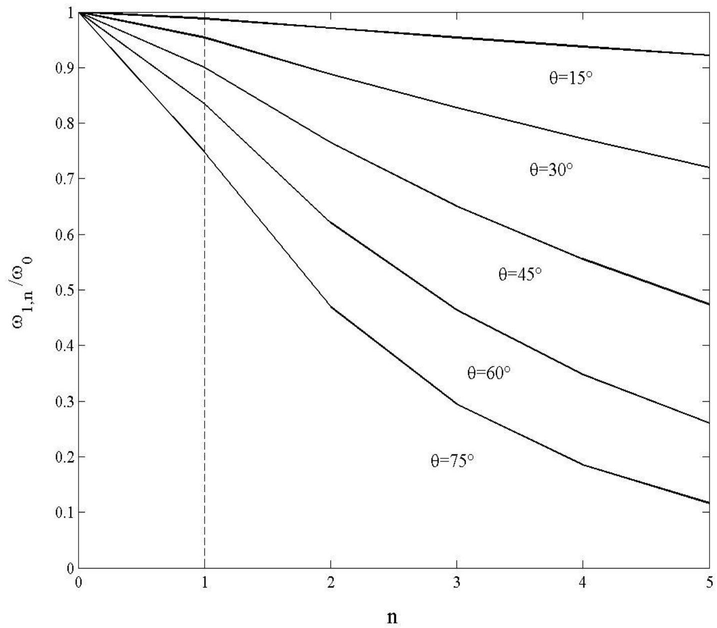

Figure 2: Dimensionless natural frequencies vs. iteration *n*, for different indentation angles θ.

Angle	Iteration *n*	1	2	3	4	5
15°	$a_{1,n}^{(\omega)}$	1.0000	0.9994	0.9990	0.9989	0.9989
	$a_{2,n}^{(\omega)}$	1.0000	1.0004	1.0008	1.0010	1.0011
	$a_{3,n}^{(\omega)}$	1.0000	0.9632	0.9618	0.9615	0.9614
30°	$a_{1,n}^{(\omega)}$	1.0000	0.9975	0.9964	0.9961	0.9961
	$a_{2,n}^{(\omega)}$	1.0000	1.0152	1.0182	1.0188	1.0193
	$a_{3,n}^{(\omega)}$	1.0000	0.9678	0.9636	0.9625	0.9622
45°	$a_{1,n}^{(\omega)}$	1.0000	0.9949	0.9926	0.9919	0.9918
	$a_{2,n}^{(\omega)}$	1.0000	1.0434	1.0515	1.0532	1.0536
	$a_{3,n}^{(\omega)}$	1.0000	0.9842	0.9811	0.9797	0.9793
60°	$a_{1,n}^{(\omega)}$	1.0000	0.9932	0.9899	0.9986	0.9884
	$a_{2,n}^{(\omega)}$	1.0000	1.0905	1.1084	1.1118	1.1125
	$a_{3,n}^{(\omega)}$	1.0000	1.0225	1.0253	1.0235	1.0228
75°	$a_{1,n}^{(\omega)}$	1.0000	0.9979	0.9934	0.9920	0.9917
	$a_{2,n}^{(\omega)}$	1.0000	1.1667	1.2047	1.2132	1.2152
	$a_{3,n}^{(\omega)}$	1.0000	1.0922	1.1138	1.1157	1.1157

Table 1: Coefficients $a_{i,n}^{(\omega)}$ related to the natural frequency scaling laws (Eq. (10)).

The physical soundness of the scaling laws provided by Eqs. (9) and (10) is supported by introducing the Rayleigh's quotient:

$$\omega_{i,n}^2 = \frac{\{\delta_0\}_{i,n}^T [K]_n \{\delta_0\}_{i,n}}{\{\delta_0\}_{i,n}^T [M]_n \{\delta_0\}_{i,n}}, \qquad i=1,\ldots,N \tag{11}$$

which consistently scales as $(4q)^{-2n}$.

4.2 *Forced damped motion*

The response of a general viscously damped system represents a considerably more difficult problem, due to the coupling introduced by damping. Also in this case, the differential equations of motion may be written in the matrix form:

$$[M]_n \{\ddot{\delta}\}_n + [C]_n \{\dot{\delta}\}_n + [K]_n \{\delta\}_n = \{F\}_n, \tag{12}$$

where $\{F\}_n$ is the reduced vector of the applied forces.

Let us now introduce the modal matrix $[\Delta]_n$ (i.e., the matrix whose columns are the normalized eigenvectors provided by Eq. (10)) and the transformation

$$\{\delta(t)\}_n = [\Delta]_n \{\eta(t)\}_n, \tag{13}$$

$\{\eta(t)\}_n$ being the normal coordinates. Inserting Eq. (13) into Eq. (12) yields

$$\{\ddot{\eta}\}_n + [c]_n \{\dot{\eta}\}_n + [\omega^2]_n \{\eta\}_n = \{Q\}_n \tag{14}$$

where $[\omega^2]_n$ is the $N\times N$ diagonal matrix of the natural angular frequencies, $\{Q\}_n$ is the N modal force vector and $[c]_n$ is a $N\times N$ symmetric matrix, generally non-diagonal. In the proportional damping case (Eq. (5)), $[c]_n$ does indeed become diagonal (let us remember that Eq. (5) is a sufficient but not necessary condition to get $[c]_n$ diagonal, [13]):

$$[c]_n = \alpha_n [I]_n + \beta_n [\omega^2]_n = \begin{bmatrix} 2\zeta_{1,n}\omega_{1,n} & 0 & \ldots & 0 \\ 0 & 2\zeta_{2,n}\omega_{2,n} & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & 2\zeta_{N,n}\omega_{N,n} \end{bmatrix}, \tag{15}$$

where $[I]$ is the $N\times N$ diagonal unit matrix and $\zeta_{i,n}$ is the modal damping factor

$$\zeta_{i,n} = \frac{1}{2}\left(\frac{\alpha_n}{\omega_{i,n}} + \beta_n \omega_{i,n}\right), \tag{16}$$

so that Eq. (14) reduces to an independent set of equations:

$$\ddot{\eta}_{i,n} + 2\zeta_{i,n}\omega_{i,n}\dot{\eta}_{i,n} + \omega_{i,n}^2\eta_{i,n} = Q_{i,n}\,. \qquad i=1,\ldots,N \tag{17}$$

It follows from Eqs. (9) and (16) that, if α_n and β_n do not vary, $\zeta_{i,n}$ scales asymptotically as $(4q)^{n-1}$: in such a case, the von Koch beam tends to become an over-damped system for each mode, a physically unacceptable condition. The problem of computation of Rayleigh damping coefficients has been faced by several authors (see, for a deeper analysis, [14-15]). The easiest practice consists in assuming a constant damping ratio ζ for all significant modes. On the other hand, since it is generally observed that ζ_i increases with increasing the mode order, it is not difficult to describe the Rayleigh damping by choosing $\zeta_i = \zeta$ (n fixed) for two modes in Eq. (16) and solving the corresponding damping coefficients α_n and β_n. Considering the first two frequencies, yields:

$$\alpha_n = \frac{2\zeta\omega_{1,n}\omega_{2,n}}{\omega_{1,n}+\omega_{2,n}} = (4q)^{1-n}\,\frac{2\zeta\, a_{1,1}^{(\omega)} a_{2,1}^{(\omega)}\omega_{1,1}\omega_{2,1}}{a_{1,1}^{(\omega)}\omega_{1,1} + a_{2,1}^{(\omega)}\omega_{2,1}}\,, \tag{18a}$$

$$\beta_n = \frac{2\zeta}{\omega_{1,n}+\omega_{2,n}} = (4q)^{n-1}\,\frac{2\zeta}{a_{1,1}^{(\omega)}\omega_{1,1} + a_{2,1}^{(\omega)}\omega_{2,1}}\,. \tag{18b}$$

Once ζ_i is evaluated, the damped frequency related the i-th mode clearly reads

$$(\omega_\mathrm{d})_{i,n} = \omega_{i,n}\left(1-\zeta_i^2\right)^{1/2} = (4q)^{1-n} a_{2,1}^{(\omega)}\omega_{i,1}\left(1-\zeta_i^2\right)^{1/2}, \tag{19}$$

Let us now investigate the damped response of a generic von Koch cantilever beam to a transversal unit step function $F=u(t)$, where $u(t)=0$ for $t<0$ and $u(t)=1$ for $t>0$, applied at the free end (Fig. 3). Null initial conditions are assumed. By means of Eqs. (18a,b), values of α_n and β_n for all significant modes have been evaluated, starting from a damping coefficient ζ equal to 0.05. LUSAS® code has been used to perform FE simulations.

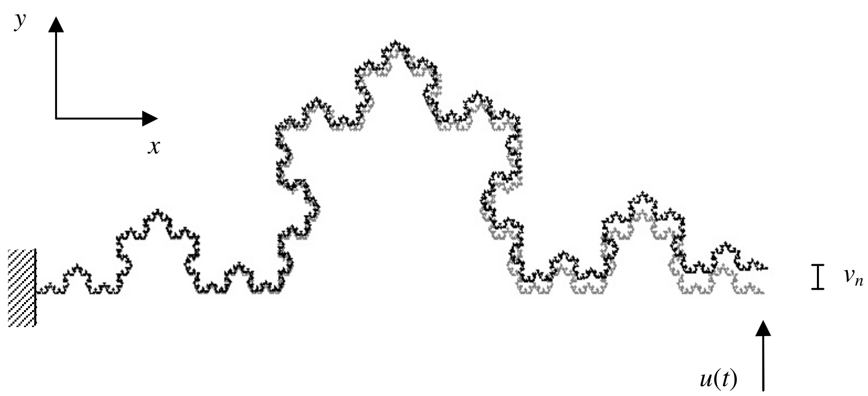

Figure 3: Von Koch cantilever beam subjected to a unit step transversal force at the free end.

First of all, let us turn our attention to a specified von Koch cantilever beam, with a fixed angle θ, and let us consider the time history of the transversal displacement at the free end for different iterations, v_n. Results are presented in Fig. 4: as the iteration n increases, the frequency of oscillation decreases, nevertheless its amplitude increases. Furthermore, the structure becomes more compliant, in perfect agreement with the analysis performed in [7]. Note that the steady-state response is reached earlier by lower order von Koch structures.

Let us now compare the step response between generic von Koch beams, with n fixed (Fig. 5, n=5): the frequency of oscillation decreases as the indentation angle θ increases, since lower natural frequencies correspond to higher values of θ [16] (Fig. 2). The steady-state response is reached earlier by smaller indentation angle structures, which also result to be stiffer.

Analogous results have been obtained by applying both a longitudinal unit step function and a unit impulse function. In the former case, the steady axial displacement at the free end is nearly a order of magnitude less than in the previous case [7], while in the latter the steady response is simply v_n=0. By summarizing, more damped responses are expected by either decreasing n or θ: these results, together with the multi-frequency analysis performed in [1-2], could be particularly useful in fractal antenna design.

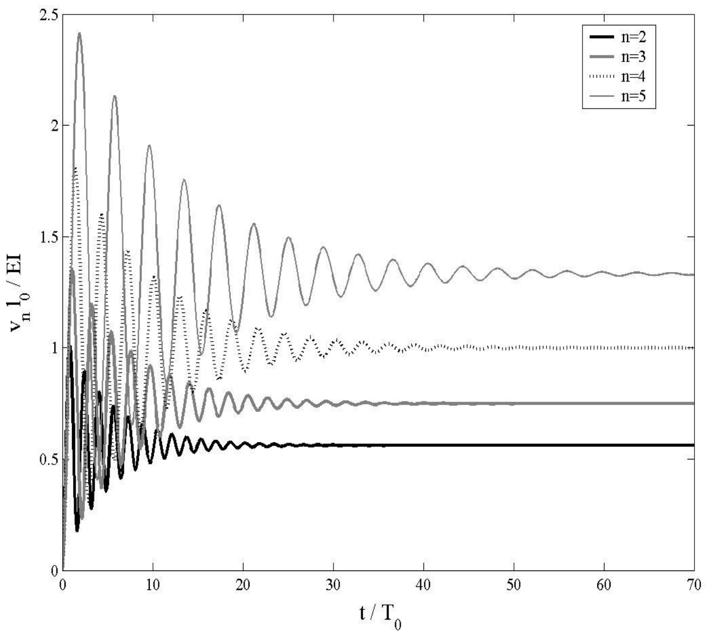

Figure 4: Von Koch cantilever beam subjected to a unit step transversal force: dimensionless transversal displacement vs. normalized time, for different iterations n (θ=60°).

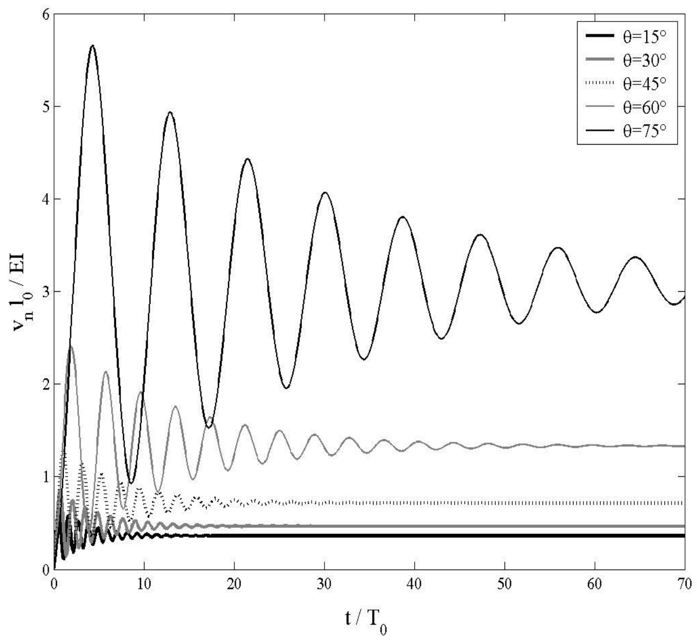

Figure 5: Von Koch cantilever beam subjected to a unit step transversal force: dimensionless transversal displacement vs. normalized time, for different indentation angles θ (*n*=5).

5 CONCLUSIONS

The mechanical behaviour of a damped von Koch beam has been investigated in this paper. Simple recursive scaling laws are obtained, by keeping fixed at each iteration *n* the number of master nodes, to which stiffness, mass and proportional damping matrices are reduced. Eventually, the forced damped response of the structure to the unit step function for different iterations and indentation angles is analyzed: a methodology to investigate damping of fractal antennas is proposed.

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