On the Deformed Shape of an Elastic Cable under General Load Conditions

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Keywords: Cables, Catenary.

SUMMARY. The catenary problem for extensible cable is revisited in this paper. The deformed shape of the elastic cable is derived in a novel manner, extending the classic procedure for heavy cables with vertical loads to the case of generally oriented spatial loads. By writing the equilibrium equation of a cable element in vector form, the position vector in the deformed configuration has been written in closed vector form for the following three cases: distributed spatial loads, one point load generally oriented in the three-dimensional space, and finally in the case of many point loads; the final expression can be applied to engineering problems in a simple manner. Applications are proposed, in order to show the usefulness of the solutions which have been found.

1 INTRODUCTION

The circumstance that a chord under self weight is unable to maintain its rectilinear configuration in spite of the tension which can be applied at its ends, was known already in 1638 by Galileo Galilei; however, in Galilei opinion the configuration of the chord was parabolic, in accordance with the flight path of a projectile [1]. The mathematical treatment of the cable theory began in the latter half of the seventeenth century. The initial problem was to determine the equilibrium position of an inextensible string hanging between two points and subjected to various systems of loads. In particular the catenary problem means to find the equilibrium shape of the cable under self weight, supposed to be a parabola by Galilei. The non parabolic shape of the chord was noted by Jungius (1669) but he was not able to find out the real mathematical expression of the curve. The problem was proposed by Jakob Bernoulli in the *Acta Eruditorum* [2] and, after this date, was tackled and solved separately by Huygens, via pure geometric considerations, and by Leibniz (1691) [3-5] and Johann Bernoulli, Jakob's brother, via the integral calculus; all the three solutions were published in the *Acta* (1691). The name "catenary" it seems was due to Huygens, which use it in a letter to Leibniz in 1690. All the mentioned works did not take into account cable extensibility. Bernoulli brothers were the first which formulated the differential equation of equilibrium of the elastic cable, following the law postulated by Hooke (1675). Also Euler contributed to the study on the catenary, after a suggestion of Daniel Bernoulli; he uses the variation calculus, the "method of final causes", and showed that the equilibrium configuration is determined by the lowest position of the barycentre of mass, i.e. by the minimum of potential energy of the gravity forces [6,7].

Although the analytical expression of the solution for the elastic catenary is today well known and literature on cable structures [8] reports the subject, it seems interesting to re-examine the classic solution, in order to obtain more general expressions, as will be presented in this work. In particular, a novel procedure will be proposed, which follows that proposed in Irvine's book [8], but is extended to anyway oriented point or distributed forces; proposing a compact vector form of the solution. The proposed form, which differs considerably from other proposed solutions, is extended to the case of many point loads, maintaining a compact general expression, which makes it of easy application in engineering problems.

Finally, in order to show the usefulness of the proposed equations, two numerical applications are presented, with reference to cables with different load conditions. The first application shows the case of a heavy cable under the action of a transversal distributed load. In the example, the final shape is obtained. The second example proposes a method for the correct definition of the shape of a suspended bridge in the erection stages, based on the found equations derived herein.

2 SOLUTION FOR EXTENSIBLE CABLE WITH CONSTANT DISTRIBUTED LOADS

The catenary equation for extensible cables which follows Hooke's law will be derived in this paragraph. Although the subject is presented in a similar manner as the one reported by Irvine [8], the approach differs because the general case of distributed loads in x and y directions will be considered. In addition, the equations are compacted in a vector form, very suitable for numerical analysis. This form is amenable to application in three dimensional cases.

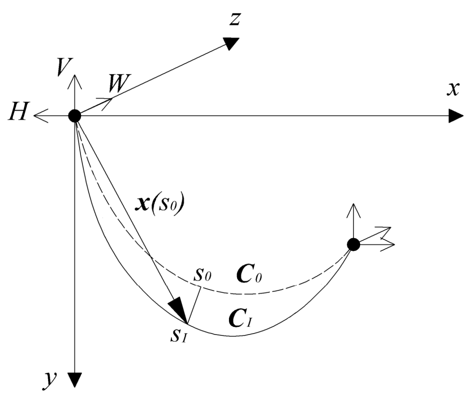

Figure 1: Un-stretched (C_0) and stretched (C_I) configurations of the elastic cable.

With reference to Fig. 1, the equilibrium of the cable in the stretched state C_I, up to abscissa s_0, leads to the equation:

$$T(s_0)\frac{d\boldsymbol{x}}{ds_I} = \boldsymbol{R} - \boldsymbol{p}s_0 \tag{1}$$

Vector $\boldsymbol{R} = [H, V, W]^T$ collects the reaction forces at the left end of the cable, whereas $\boldsymbol{x}(s_0) = [x(s_0), y(s_0), z(s_0)]^T$ and $\boldsymbol{p} = \left[p_x, p_y, p_z\right]^T$. The cable tension $T(s_0)$ is obtained by taking the modulus of the two sides of Eq. (1):

$$T(s_0) = \|\boldsymbol{R} - \boldsymbol{p}s_0\| \tag{2}$$

in which the norm operator $\|\cdot\|$ has been introduced. The Hooke's law states:

$$T(s_0) = EA_0 \frac{ds_I - ds_0}{ds_0} = EA_0\left(\frac{ds_I}{ds_0} - 1\right) \tag{3}$$

Taking into account the Hooke's law, Eq. (3), and using the chain rule of differentiation $d\boldsymbol{x}/ds_0 = (d\boldsymbol{x}/ds_I)(ds_I/ds_0)$, the following equation is obtained:

$$\frac{d\boldsymbol{x}}{ds_0} = \frac{\boldsymbol{R} - \boldsymbol{p}s_0}{T(s_0)}\left(\frac{T(s_0)}{EA_0} + 1\right) \tag{4}$$

After integration Eq. (4) becomes:

$$\boldsymbol{x}(s_0) = \int \frac{\boldsymbol{R} - \boldsymbol{p}s_0}{T(s_0)}\left(\frac{T(s_0)}{EA_0} + 1\right) ds_0 = \frac{1}{EA_0}\boldsymbol{R}s_0 - \frac{1}{2EA_0}\boldsymbol{p}s_0^2 + \int \frac{\boldsymbol{R} - \boldsymbol{p}s_0}{\|\boldsymbol{R} - \boldsymbol{p}s_0\|} ds_0 \tag{5}$$

In order to solve the integral $I = \int \frac{\boldsymbol{R} - \boldsymbol{p}s_0}{\|\boldsymbol{R} - \boldsymbol{p}s_0\|} ds_0$, one can perform the following steps:

Rewrite I in the following form:

$$I = \int \frac{\boldsymbol{R}\|\boldsymbol{p}\|^2 - \boldsymbol{p}\|\boldsymbol{p}\|^2 s_0}{\|\boldsymbol{p}\|^2 \|\boldsymbol{R} - \boldsymbol{p}s_0\|} ds_0 \tag{6}$$

Take into account the equality $\|\mathbf{p}\|^2 = \mathbf{p}^T\mathbf{p}$, so that:

$$I = \int \frac{\boldsymbol{R}(\boldsymbol{p}^T\boldsymbol{p}) - \boldsymbol{p}(\boldsymbol{p}^T\boldsymbol{R}) + \boldsymbol{p}(\boldsymbol{R}^T\boldsymbol{p}) - \boldsymbol{p}(\boldsymbol{p}^T\boldsymbol{p})s_0}{\|\boldsymbol{p}\|^2 \|\boldsymbol{R} - \boldsymbol{p}s_0\|} ds_0 \tag{7}$$

Divide the latter integral in two fractional parts:

$$I = \int \left(\frac{\boldsymbol{R}(\boldsymbol{p}^T\boldsymbol{p}) - \boldsymbol{p}(\boldsymbol{p}^T\boldsymbol{R})}{\|\boldsymbol{p}\|^2 \|\boldsymbol{R} - \boldsymbol{p}s_0\|} + \frac{\boldsymbol{p}(\boldsymbol{R}^T\boldsymbol{p}) - \boldsymbol{p}(\boldsymbol{p}^T\boldsymbol{p})s_0}{\|\boldsymbol{p}\|^2 \|\boldsymbol{R} - \boldsymbol{p}s_0\|}\right) ds_0 \tag{8}$$

Resort to Lagrange rule $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}^T\mathbf{c}) - \mathbf{c}(\mathbf{a}^T\mathbf{b})$ (where $\times$ is the cross product), so that:

$$I=\int\left(\frac{-\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{R})}{\|\boldsymbol{p}\|^2\|\boldsymbol{R}-\boldsymbol{p}s_0\|}+\boldsymbol{p}\frac{(\boldsymbol{R}-\boldsymbol{p}s_0)^T\boldsymbol{p}}{\|\boldsymbol{p}\|^2\|\boldsymbol{R}-\boldsymbol{p}s_0\|}\right)ds_0 \tag{9}$$

Multiply and divide the first addend by the quantity $\left[\|\boldsymbol{R}-\boldsymbol{p}s_0\|\|\boldsymbol{p}\|-\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)\right]/\|\boldsymbol{p}\|$:

$$\begin{aligned}I=\int\Bigg(&\frac{-\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{R})}{\|\boldsymbol{p}\|^3}\frac{\|\boldsymbol{R}-\boldsymbol{p}s_0\|\|\boldsymbol{p}\|-\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|}\frac{\|\boldsymbol{p}\|}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|\|\boldsymbol{p}\|-\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}+\\&+\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2}\frac{(\boldsymbol{R}-\boldsymbol{p}s_0)^T\boldsymbol{p}}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|}\Bigg)ds_0\end{aligned} \tag{10}$$

which, after some manipulation, gives:

$$\begin{aligned}I=\int\Bigg(&\frac{-\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{R})}{\|\boldsymbol{p}\|^3}\left(\frac{\boldsymbol{p}^T\boldsymbol{p}}{\|\boldsymbol{p}\|}-\frac{\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|}\right)\frac{\|\boldsymbol{p}\|}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|\|\boldsymbol{p}\|-\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}+\\&+\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2}\frac{(\boldsymbol{R}-\boldsymbol{p}s_0)^T\boldsymbol{p}}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|}\Bigg)ds_0\end{aligned} \tag{11}$$

Recall that for a vector function $\mathbf{v}(s)$ the following equation holds:

$$\frac{d}{ds}\|\mathbf{v}\|=\frac{\mathbf{v}^T}{\|\mathbf{v}\|}\frac{d\mathbf{v}}{ds} \tag{12}$$

so that assuming $\mathbf{v}(s_0)=\mathbf{R}-\mathbf{p}s_0$, the quantity in the brackets in the first addend of the integral is written as:

$$\left(-\frac{\boldsymbol{p}^T}{\|\boldsymbol{p}\|}\frac{d}{ds_0}(\boldsymbol{R}-\boldsymbol{p}s_0)+\frac{(\boldsymbol{R}-\boldsymbol{p}s_0)^T}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|}\frac{d}{ds_0}(\boldsymbol{R}-\boldsymbol{p}s_0)\right)=\frac{d}{ds_0}\left(\|\boldsymbol{R}-\boldsymbol{p}s_0\|-\frac{\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}{\|\boldsymbol{p}\|}\right) \tag{13}$$

whereas, for the second addend:

$$\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2}\frac{(\boldsymbol{R}-\boldsymbol{p}s_0)^T\boldsymbol{p}}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|}=-\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2}\frac{d}{ds_0}\|\boldsymbol{R}-\boldsymbol{p}s_0\| \tag{14}$$

So operating, the integral I becomes:

$$I = \frac{-\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{R})}{\|\boldsymbol{p}\|^3}\int\left\{\frac{\|\boldsymbol{p}\|}{\|\boldsymbol{R}-\boldsymbol{p}s_0\|\|\boldsymbol{p}\|-\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}\frac{d}{ds_0}\left(\|\boldsymbol{R}-\boldsymbol{p}s_0\|-\frac{\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}{\|\boldsymbol{p}\|}\right)\right\}ds_0 + \\ -\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2}\int\frac{d}{ds_0}\|\boldsymbol{R}-\boldsymbol{p}s_0\|ds_0 \tag{15}$$

which admit the following solution:

$$I = \frac{-\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{R})}{\|\boldsymbol{p}\|^3}\log\left(\|\boldsymbol{R}-\boldsymbol{p}s_0\|-\frac{\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}{\|\boldsymbol{p}\|}\right)-\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2}\|\boldsymbol{R}-\boldsymbol{p}s_0\|+\boldsymbol{c} \tag{16}$$

where $\boldsymbol{c}$ is the integration constant. Due to the extensibility of the cable the curvilinear abscissa in the stretched configuration can be evaluated by means of Eq. (3), which allows us to write:

$$s_I(s_0) = \int_0^{s_0}\left(\frac{T(s)}{EA_0}+1\right)ds \tag{17}$$

It is interesting to note that the vector method allows us to easily analyse three-dimensional cases. The constant vector $\boldsymbol{c}$ can be determined by imposing the boundary condition in $s_0 = 0$, so that Eq. (5) can be finally rewritten as follows:

$$\boldsymbol{x}(s_0) = \frac{1}{EA_0}\boldsymbol{R}s_0 - \frac{1}{2EA_0}\boldsymbol{p}s_0^2 + \\ -\left(\frac{\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{R})}{\|\boldsymbol{p}\|^3}\right)\log\left\{\left[\|\boldsymbol{R}-\boldsymbol{p}s_0\|-\frac{\boldsymbol{p}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}{\|\boldsymbol{p}\|}\right]\left[\|\boldsymbol{R}\|-\frac{\boldsymbol{p}^T\boldsymbol{R}}{\|\boldsymbol{p}\|}\right]^{-1}\right\}+ \\ -\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2}\left(\|\boldsymbol{R}-\boldsymbol{p}s_0\|-\|\boldsymbol{R}\|\right) \tag{18}$$

The unknown vector $\boldsymbol{R}$ must evaluated imposing the coordinates of the end point of the cable, namely in $s_0 = L_0$, by employing a numerical method such as the Newton-Raphson. Once $\boldsymbol{R}$ has been calculated, the elastic catenary can be written explicitly. In a non dimensional form, the Eq. (18) can be written as follows:

$$\boldsymbol{x}(\hat{s}_0) = \hat{\boldsymbol{R}}\hat{s}_0 - \frac{1}{2}\hat{\boldsymbol{p}}\hat{s}_0^2 + \\ -\left(\frac{\hat{\boldsymbol{p}}\times(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{R}})}{\|\hat{\boldsymbol{p}}\|^3}\right)\log\left\{\left[\|\hat{\boldsymbol{R}}-\hat{\boldsymbol{p}}\hat{s}_0\|-\frac{\hat{\boldsymbol{p}}^T(\boldsymbol{R}-\boldsymbol{p}s_0)}{\|\hat{\boldsymbol{p}}\|}\right]\left[\|\hat{\boldsymbol{R}}\|-\frac{\hat{\boldsymbol{p}}^T\hat{\boldsymbol{R}}}{\|\hat{\boldsymbol{p}}\|}\right]^{-1}\right\}+ \\ -\frac{\hat{\boldsymbol{p}}}{\|\hat{\boldsymbol{p}}\|^2}\left(\|\hat{\boldsymbol{R}}-\hat{\boldsymbol{p}}\hat{s}_0\|-\|\hat{\boldsymbol{R}}\|\right) \tag{19}$$

where the following positions have been made:

$$\hat{\boldsymbol{x}} = \boldsymbol{x} / L_0; \quad \hat{s}_0 = s_0 / L_0; \quad \hat{\boldsymbol{R}} \equiv [\hat{H}, \hat{V}, \hat{W}] = \boldsymbol{R} / EA_0; \quad \hat{\boldsymbol{p}} = \boldsymbol{p} L_0 / EA_0 \tag{20}$$

3 SOLUTION FOR EXTENSIBLE CABLE WITH POINT LOADS

The effect of concentrated loads $\boldsymbol{P}_i$ is introduced by redefinition of the equilibrium equation as follows:

$$T(s_0) \frac{d\boldsymbol{x}}{ds_I} = \boldsymbol{R} - \sum_i \boldsymbol{P}_i U(s_0 - \overline{s}_{0,i}) - \boldsymbol{p} s_0 \tag{21}$$

The score identifies the abscissa of the load application point. In the previous equation, the unitary step function U has been introduced. Thus, the same procedure previously described can be directly extended to the case of concentrated loads.

3.1 *One point load*

For a singular point load, if $s_0 < \overline{s}_0$ the solution is the same found for cables with only distributed load, Eq. (18). For $s_0 \geq \overline{s}_0$, the solution is:

$$\begin{aligned} \boldsymbol{x}(s_0) = {} & \frac{1}{EA_0} \boldsymbol{R} s_0 - \frac{1}{EA_0} \boldsymbol{P}(s_0 - \overline{s}_0) - \frac{1}{2EA_0} \boldsymbol{p} s_0^2 + \\ & - \left(\frac{\boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{R})}{\|\boldsymbol{p}\|^3} \right) \log \left\{ \left[\|\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} s_0\| - \frac{\boldsymbol{p}^T (\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} s_0)}{\|\boldsymbol{p}\|} \right] \left[\|\boldsymbol{R}\| - \frac{\boldsymbol{p}^T \boldsymbol{R}}{\|\boldsymbol{p}\|} \right]^{-1} \right\} + \\ & - \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2} \left(\|\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} s_0\| - \|\boldsymbol{R}\| \right) + \boldsymbol{c} \end{aligned} \tag{22}$$

The constant vector $\boldsymbol{c}$ can be defined by imposition of the continuity in $\overline{s}_0$, so that the solution for the cable with one point load and distributed loads, for $s_0 \geq \overline{s}_0$ is written as follows:

$$\begin{aligned} \boldsymbol{x}(s_0) = {} & \frac{1}{EA_0} \boldsymbol{R} s_0 - \frac{1}{EA_0} \boldsymbol{P}(s_0 - \overline{s}_0) - \frac{1}{2EA_0} \boldsymbol{p} s_0^2 - \left(\frac{\boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{R})}{\|\boldsymbol{p}\|^3} \right) \cdot \\ & \cdot \log \left\{ \frac{\left[\|\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} s_0\| - \frac{\boldsymbol{p}^T (\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} s_0)}{\|\boldsymbol{p}\|} \right] \left[\|\boldsymbol{R} - \boldsymbol{p} \overline{s}_0\| - \frac{\boldsymbol{p}^T (\boldsymbol{R} - \boldsymbol{p} \overline{s}_0)}{\|\boldsymbol{p}\|} \right]}{\left[\|\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} \overline{s}_0\| - \frac{\boldsymbol{p}^T (\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} \overline{s}_0)}{\|\boldsymbol{p}\|} \right] \left[\|\boldsymbol{R}\| - \frac{\boldsymbol{p}^T \boldsymbol{R}}{\|\boldsymbol{p}\|} \right]} \right\} + \\ & - \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|^2} \left(\|\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} s_0\| - \|\boldsymbol{R}\| + \|\boldsymbol{R} - \boldsymbol{p} \overline{s}_0\| - \|\boldsymbol{R} - \boldsymbol{P} - \boldsymbol{p} \overline{s}_0\| \right) \end{aligned} \tag{23}$$

The vector $\boldsymbol{R}$ is derived by enforcing cable end position at $s_0 = L_0$. A formal simplification can be introduced by defining the operator:

$$\Omega(\bullet) = \left\| \boldsymbol{R} - (\bullet) \right\| - \frac{\boldsymbol{p}^T \left(\boldsymbol{R} - (\bullet) \right)}{\left\| \boldsymbol{p} \right\|} \tag{24}$$

and denoting with $\Re(s_0)$ the integral function of $U(s_0)$, so that the solution is rewritten in the whole domain $0 \le s_0 \le L_0$ as follows:

$$\begin{aligned} \boldsymbol{x}(s_0) = {} & \frac{1}{EA_0}\boldsymbol{R}s_0 - \frac{1}{EA_0}\boldsymbol{P}\Re(s_0 - \overline{s}_0) - \frac{1}{2EA_0}\boldsymbol{p}s_0^2 + \\ & -\left(\frac{\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{R})}{\left\|\boldsymbol{p}\right\|^3} \right) \log\left\{ \frac{\Omega(\boldsymbol{P}U(s_0-\overline{s}_0)+\boldsymbol{p}s_0)\Omega(\boldsymbol{p}\overline{s}_0)}{\Omega(\boldsymbol{P}U(s_0-\overline{s}_0)+\boldsymbol{p}\overline{s}_0)\Omega(\boldsymbol{0})} \right\} + \\ & -\frac{\boldsymbol{p}}{\left\|\boldsymbol{p}\right\|^2}\left(\left\|\boldsymbol{R}-\boldsymbol{P}U(s_0-\overline{s}_0)-\boldsymbol{p}s_0\right\| - \left\|\boldsymbol{R}\right\| + \left\|\boldsymbol{R}-\boldsymbol{p}\overline{s}_0\right\| - \left\|\boldsymbol{R}-\boldsymbol{P}U(s_0-\overline{s}_0)-\boldsymbol{p}\overline{s}_0\right\| \right) \end{aligned} \tag{25}$$

3.2 *Many point loads*

The general case with many point loads, arbitrarily oriented, and constant distributed load, is now considered. Assume that the point loads P_j $(j = 1,..,i,..,N)$ are applied at points $\overline{s}_{0,j}$. The following non dimensional equation governs the final shape of the cable, in the interval $\hat{\overline{s}}_{0,i} \le \hat{s}_0 \le \hat{\overline{s}}_{0,i+1}$:

$$\begin{aligned} \hat{\boldsymbol{x}}(\hat{s}_0) = {} & \hat{\boldsymbol{R}}\hat{s}_0 - \sum_{j=1}^{i}\hat{\boldsymbol{P}}_j\Re(\hat{s}_0 - \hat{\overline{s}}_{0,j}) - \frac{1}{2}\hat{\boldsymbol{p}}\hat{s}_0^2 + \\ & -\left(\frac{\hat{\boldsymbol{p}}\times(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{R}})}{\left\|\hat{\boldsymbol{p}}\right\|^3} \right) \log\left\{ \frac{\Omega(\sum_{j=1}^{i}\hat{\boldsymbol{P}}_j + \boldsymbol{p}\hat{s}_0)}{\Omega(\boldsymbol{0})} \prod_{j=1}^{i}\left(\frac{\Omega(\sum_{l=1}^{j-1}\hat{\boldsymbol{P}}_l + \hat{\boldsymbol{p}}\hat{\overline{s}}_{0,j})}{\Omega(\sum_{l=1}^{j}\hat{\boldsymbol{P}}_l + \boldsymbol{p}\hat{\overline{s}}_{0,j})} \right) \right\} + \\ & -\frac{\hat{\boldsymbol{p}}}{\left\|\hat{\boldsymbol{p}}\right\|^2}\left[\left\| \hat{\boldsymbol{R}} - \sum_{j=1}^{i}\hat{\boldsymbol{P}}_j - \hat{\boldsymbol{p}}\hat{s}_0 \right\| - \left\|\hat{\boldsymbol{R}}\right\| + \sum_{j=1}^{i}\left(\left\| \hat{\boldsymbol{R}} - \sum_{l=1}^{j-1}\hat{\boldsymbol{P}}_l - \hat{\boldsymbol{p}}\hat{\overline{s}}_{0,j} \right\| - \left\| \hat{\boldsymbol{R}} - \sum_{l=1}^{j}\hat{\boldsymbol{P}}_l - \hat{\boldsymbol{p}}\hat{\overline{s}}_{0,j} \right\| \right) \right] \end{aligned} \tag{26}$$

The solution in the first cable segment, before the first point load, is given by Eq. (19), whereas the solution in the last cable segment, after the last point load, is still given by Eq. (26) assuming $i = N$. It is clearly inexact to use the above equations for one load per time and sum finally the partial solutions.

4 APPLICATIONS

Some examples are presented in this paragraph, in order to show the applicability of the proposed equations.

4.1 *Heavy inclined elastic cable under constant wind flow along z*

The cable un-deformed length is $L_0 = 460\,m$; the cable is suspended in two points, whose coordinates are $\boldsymbol{x}_0 = [0,0,0]^T$ and $\boldsymbol{x}_L = [400,150,0]^T\ m$; in non dimensional form: $\hat{\boldsymbol{x}}_0 = [0,0,0]^T$ and $\hat{\boldsymbol{x}}_L = [0.869565,\ 0.326087, 0]^T$. The stiffness is $EA_0 = 3.86\cdot 10^8\ N$, the self weight is $p_y = 144.43\,N/m$. The wind load, neglecting turbulence, is defined considering only a constant velocity $U_w = 30\,m/\sec$; the air density is assumed to be $\rho_w = 1.25\,Kg/m^3$, the cable diameter is $b = 22\,cm$ and the drag coefficient is $C_D = 1.2$. The wind load is $p_z = 0.5\rho_w C_D U_w^2 = 148.50\,N/m$; for simplicity, a wind acting in the *z* direction has been considered. Therefore, the distributed non-dimensional load is $\hat{\boldsymbol{p}} = 10^{-4}[0,\ 1.71952,\ 1.76801]^T$. The solution for the reaction vector is $\hat{\boldsymbol{R}} = 10^{-4}[1.54976, 1.4846, 0.92892]^T$. Fig. 2 show the deformed shape of the cable.

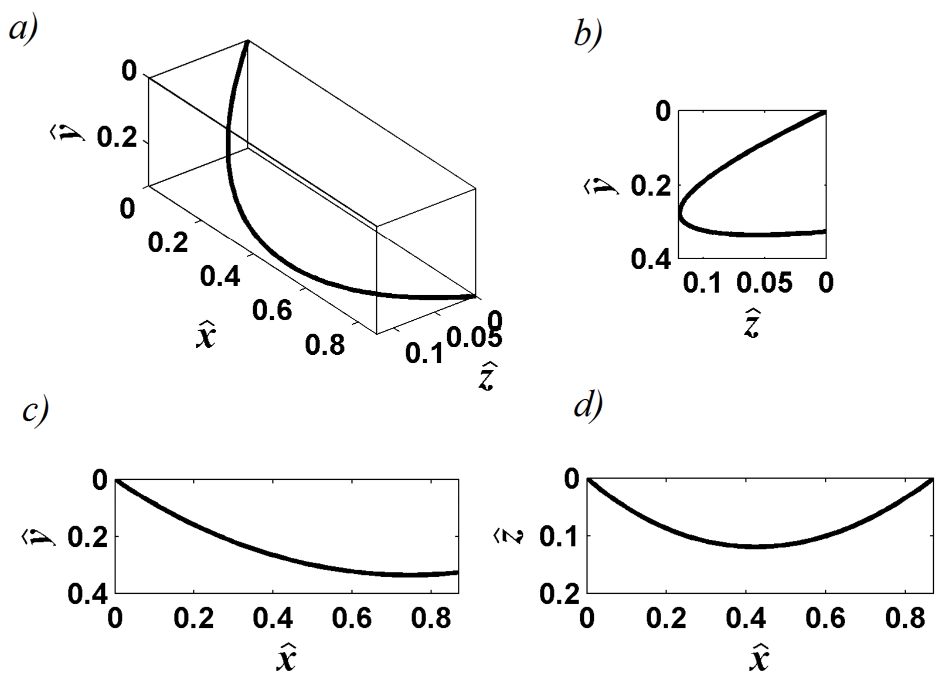

Figure 2. Deformed shape of an inclined cable under self weight and constant along *z* wind.

4.2 *Final shape of a suspended bridge*

As an example the cables of the designed Messina strait bridge is considered. The four main cables are constituted by 44352 steel wires each of 5.38 mm diameter. Each cable has a cross section area of $A_0^{(c)} = 1.00825\,m^2$. The total area of the four cables is $A_0 = 4.03299\,m^2$. Assume the steel density to be $\rho_s = 7850\,Kg/m^3$, and the Young modulus is $E = 2\cdot 10^{11}\,N/m^2$ (in reality, the elastic modulus of the cable is slightly lower than the elastic modulus of the material). The weight per unit length of the cable is $p_y = 310{,}575\,KN/m$. The total weight of the bridge deck is $705000\,KN$ and the total weight of the hangers is about $258{,}641\,KN$. Thus the sum of weight of deck and hangers is equal to $705258{,}641\,KN$.

The suspended weight, taking into account a distance of $L = 3300\ m$ between the supports at the same level, is $\bar{p}_y = 213715\,N/m$; this distributed load is applied on the horizontal projection of the cable and can be reported to the curvilinear abscissa through the following expression:

$$p_{y,d} = \bar{p}_y \frac{dx}{ds} \tag{27}$$

where the $x(s_0)$ is still to be determined. We know a priori that in the stretched configuration the sag in the midpoint has to be $d = 300\ m$. The deck is divided into N segments of equal length $\Delta x = L/N$, so to simulate the construction stages, each of which is represented by a point load of intensity $\bar{p}_y \Delta x$. Curvilinear abscissas s_i correspondent to the $N+1$ x_i values are required, and concentrated loads are applied at points $(s_i + s_{i+1})/2$. The coordinates of the end point are: $\boldsymbol{x}_L = [3300, 0, 0]^T\ m$. The problem can be formulated by means of Eq. (26) in which L_0 and $\boldsymbol{R}$ are unknown quantities. Two conditions can be imposed: $\boldsymbol{x}(L_0, \boldsymbol{R}) = \boldsymbol{x}_L$ and $\boldsymbol{x}(L_0/2, \boldsymbol{R}) = [3300/2,\ 300, 0]^T\ m$. Besides, the following equations can be written: $x(s_i) = i\,\Delta x$.

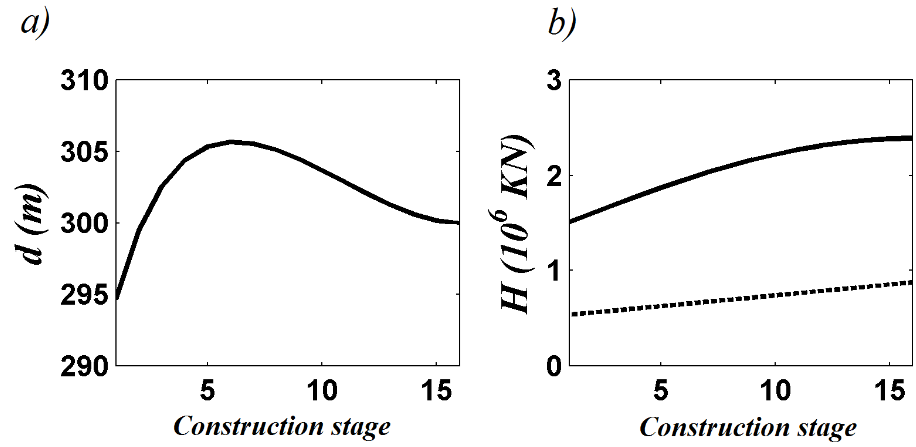

Figure 3: Sag and tension versus construction stages: a) sag; b) horizontal component of cable tension H (continuous line) and vertical component V (dashed line).

Due to the planarity of the problem, the whole equations are not required. The equations have to be solved contemporarily, by means of methods like Newton-Raphson, to obtain the quantities L_0, *H*, *V*, s_i (i=2,..,N). For the sake of the example, it has been divided the deck into N=31 parts, so that the deck segment length is $\Delta x = 106.452\, m$. The final sag of 300 m is obtained by an initial length $L_0 = 3361.32\, m$, and the cable tension components in the final stage are: $H = 2.39091 \cdot 10^9\, N$, $V = 8.74600 \cdot 10^8\, N$. If we adopt a parabolic approximation of the catenary, the horizontal component of cable tension is $H_p = 2.37896 \cdot 10^9\, N$. In the initial stage, when they are only the cables the cable tension components are: $H = 1.46406 \cdot 10^9\, N$, $V = 5.21970 \cdot 10^8\, N$. The parabolic approximation gives $H_p = 1.45191 \cdot 10^9\, N$. Fig. 3a shows the sag variation in the 16 considered construction stages, starting from the mid-span to the piers. The initial value is 291.181 m; the sag increases to a maximum value of 10.996 m when the middle span segments are placed and decreases of 5.654 m to reach the final value of 300 m. Fig. 3b shows the variation for horizontal and vertical components of cable tension.

5 CONCLUSIONS

A model for obtaining the deformed shape of an elastic cable under general load conditions has been proposed. The model differs from other approaches because it allows to take into account distributed loads generally oriented and not only self-load; besides the model accounts for point loads, also generally oriented. The solution is given in a vector form, which reduces to that contained in classic literature when only vertical loads act on the cable. Particularly useful is the solution for many point loads, which is also given in a closed compact form. The possibility offered by the method suggest as a further study, the application to the mechanics of spatial cable systems.

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