Buckling of thin-walled frames

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SUMMARY. We investigate the buckling of a 'Roorda' frame by means of a direct one-dimensional beam model. The frame is acted upon by a 'dead' load at the joint and is constrained there by an out-of-plane linear elastic spring. The possibility of warping constraints at the beam ends is also considered; the spring simulates the presence of braces in actual 3D frames. The numerical results are compared with those already obtained by the authors for an infinitely stiff spring.

1 INTRODUCTION. DIRECT ONE-DIMENSIONAL MODEL

In the first years of the 20th century a new problem arose in the stability of structures, namely the mixed buckling in bending and torsion of thin-walled members. The problem gained, and still retains, much importance due to the widespread applications of thin-walled elements in many fields of engineering, especially industrial (rack structures) and aero-spatial (frames supporting plates made up of thin elements). The first researches in this field are those in [1, 2], where the different role played by the shear centre and the centroid of the cross-sections was pointed out.

Afterwards, among the most significant works from the point of view of the physical and mathematical model we may quote [3], probably the first comprehensive book on the subject, and the well-known monograph [4], where a wide range of examples is presented. Beside these works, which in some sense constitute a starting point, there are a lot of papers dealing with the modelling of thin walled beams have been published in the last 50 years or so.

Some years ago, one of the authors introduced a direct one-dimensional beam model suitable for describing the behaviour of thin-walled beams with two axes of symmetry [5]. In a recent paper [6] a refinement of the previous model has been introduced by some of the authors, in order to describe the flexural-torsional buckling of beams with generic, non-symmetric cross-sections.

The aim of this contribution is to study the effect of lateral braces and warping constraints on the buckling of thin-walled framed structures on the base of that model. To this aim we consider a simple two-bar frame as an example of how more complex structures may behave when similarly loaded. Such a frame, usually called 'Roorda frame', exhibits interesting interactions between flexural and torsional modes occurring out of the plane where the frame originally lies.

The beams are considered constrained to the 'ground' with hinges allowing for the sole rotation with axis perpendicular to the plane of the frame. In order to simulate the qualitative behaviour of a 3D frame, a linear elastic spring orthogonal to the plane of the frame is attached to the joint where the two beams meet. Warping constraints are then considered and the stiffness of the spring is varied to account for different grades of lateral restraint. The results are compared with the ones obtained when the stiffness tends to infinity, reported in [7].

A 'dead' load acting at the joint and collinear to one of the beam axes is considered. We focus on the buckling of a frame composed of thin-walled beams with cross-sections exhibiting one axes of symmetry: indeed, such beams (e.g. channels) are of widespread use, a standard example being the so-called rack structures. Moreover, the coupling between flexural and torsional buckling, so important in these structures, is clearly put in evidence.

We summarise the model (details are in [6, 8]) and present the field equations for the problem at first. In the following sections, some numerical results illustrated by graphs will be presented.

The reference shape consists of plane cross-sections orthogonally attached either to the centroidal or to the shear centres axis, straight and parallel, directed along the x_1-axis of an orthogonal cartesian system with consistent right-handed unit vector basis $(\mathbf{i}_1,\ \mathbf{i}_2,\ \mathbf{i}_3)$. Strain measures are

$$
\begin{aligned}
&\mathbf{E} = \mathbf{R}^\top \mathbf{R}' = \chi_1 \mathbf{i}_2 \wedge \mathbf{i}_3 + \chi_2 \mathbf{i}_3 \wedge \mathbf{i}_1 + \chi_3 \mathbf{i}_1 \wedge \mathbf{i}_2,\\
&\mathbf{e}_o = \mathbf{R}^\top \mathbf{p}_o' - \mathbf{q}_o' = \varepsilon_1 \mathbf{i}_1 + \varepsilon_2 \mathbf{i}_2 + \varepsilon_3 \mathbf{i}_3,\\
&\mathbf{e}_c = \mathbf{R}^\top \mathbf{p}_c' - \mathbf{q}_c' = \mathbf{e}_o + \mathbf{E}\mathbf{c} = \varepsilon_{1c} \mathbf{i}_1 + \varepsilon_{2c} \mathbf{i}_2 + \varepsilon_{3c} \mathbf{i}_3 =\\
&\quad = (\varepsilon_1 + \chi_2 c_3 - \chi_3 c_2)\mathbf{i}_1 + (\varepsilon_2 - \chi_1 c_3)\mathbf{i}_2 + (\varepsilon_3 + \chi_1 c_2)\mathbf{i}_3,\\
&\alpha, \qquad\qquad \eta = \alpha',
\end{aligned}
\tag{1}
$$

where: o is the centroid, c is the shear centre and $\mathbf{c} = c - o = c_2\mathbf{i}_2 + c_3\mathbf{i}_3$; $\mathbf{q}_o(x_1)$, $\mathbf{q}_c(x_1)$ are the placement of the axes in the reference shape, $\mathbf{p}_o(x_1, t)$, $\mathbf{p}_c(x_1, t)$ those in the present shape; $\mathbf{R}(x_1, t)$ is the cross-sections rotation; and $\alpha(x_1, t)$ is the scalar coarse descriptor of warping. The skew tensor $\mathbf{E}$ provides the curvature of the axes, the vectors $\mathbf{e}_o,\ \mathbf{e}_c$ compare the tangents to the axes in the present and reference shape. The torsion curvature is χ_1, the bending curvatures are $\chi_2,\ \chi_3$; the wedge product $\wedge$ between vectors provides skew tensors; ε_1 is the elongation of the centroidal axis, $\varepsilon_2,\ \varepsilon_3$ are the shearing strains between this axis and the cross-sections. The displacement of the centroidal axis and the rotation are decomposed as

$$
\mathbf{u} = \mathbf{p}_o - \mathbf{q}_o = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3, \qquad \mathbf{R} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 \tag{2}
$$

where the rotation $\mathbf{R}_1$ (amplitude φ_1) is around $\mathbf{i}_1$; $\mathbf{R}_2$ (amplitude φ_2) is around $\mathbf{R}_1\mathbf{i}_2$; $\mathbf{R}_3$ (amplitude φ_3) is around $\mathbf{R}_2\mathbf{R}_1\mathbf{i}_3$.

The external power is linear in the velocities with respect to the shear centre, the internal power P^i is linear in the same velocities and their x_1-derivatives. The balance of power and a pull-back [6, 8] yield the balance of force and torque in the reference shape with respect to c, the auxiliary equations for bi-shear and bi-moment and the reduced internal power:

$$
\begin{aligned}
&\mathbf{s}' + \mathbf{E}\mathbf{s} + \mathbf{a} = \mathbf{0}, \qquad && \mathbf{S}' + \mathbf{E}\mathbf{S} - \mathbf{S}\mathbf{E} + (\mathbf{q}_c' + \mathbf{e}_c) \wedge \mathbf{s} + \mathbf{A} = \mathbf{0},\\
&\tau = \beta + \mu', && P^i = \int_0^l \left(\mathbf{s} \cdot \dot{\mathbf{e}}_c + \mathbf{S} \cdot \dot{\mathbf{E}} + \tau\omega + \mu\omega' \right) \ ;
\end{aligned}
\tag{3}
$$

the vectors $\mathbf{a}$, $\mathbf{s}$ are bulk and contact forces; the skew tensors $\mathbf{A}$, $\mathbf{S}$ are bulk and contact couples; the scalar β is the bulk action spending power on warping; the scalars μ, τ are the bi-moment and bi-shear ([3]), respectively, decomposed as:

$$
\mathbf{s} = Q_1\mathbf{i}_1 + Q_2\mathbf{i}_2 + Q_3\mathbf{i}_3, \qquad \mathbf{S} = S_1\mathbf{i}_2 \wedge \mathbf{i}_3 + S_2\mathbf{i}_3 \wedge \mathbf{i}_1 + S_3\mathbf{i}_1 \wedge \mathbf{i}_2. \tag{4}
$$

Equations (1), $(3)_4$, (4) imply

$$
P^i = \int_0^l \left[Q_1\dot{\varepsilon}_1 + Q_2\dot{\varepsilon}_{2c} + Q_3\dot{\varepsilon}_{3c} + S_1\dot{\chi}_1 + (S_2 + c_3Q_1)\dot{\chi}_2 + (S_3 - c_2Q_1)\dot{\chi}_3 + \tau\omega + \mu\omega' \right] ; \tag{5}
$$

the normal force Q_1 acts at o, the shearing forces Q_2, Q_3 at c; S_1 is the twisting couple, while $M_2 = S_2 + c_3Q_1$, $M_3 = S_3 - c_2Q_1$ are the bending torques, evaluated with respect to the centroid.

If ξ is a constant [3, 9, 10, 11, 5] the inner constraints

$$\alpha = \xi\chi_1, \qquad \xi \in \mathbb{R}, \qquad \eta = \xi\chi_1', \qquad \mathbf{e}_o = \varepsilon_1\mathbf{q}_o' = \varepsilon_1\mathbf{e}_1, \qquad \varepsilon_2 = \varepsilon_3 = 0 \tag{6}$$

hold: cross-sections and shear axis do not remain normal ($\varepsilon_{2c} \neq 0$, $\varepsilon_{3c} \neq 0$, equations (1)).

If the beam is homogeneous and elastic, the material response depends on $\mathbf{e}$, $\mathbf{E}$, α, η and inner constraints imply reactive contact actions [14]. Then the normal force, the bending torques and the bi-moment are entirely active, while the shearing forces and the bi-shear have a reactive part; the reactive twisting torque S_{1r} contains the bi-shear, see also [3]. We make the standard assumption that the shearing force and the bi-shear depend only on the shearing strain; thus, the constraint 6 imply that they are purely constraint reactions. Thus, some actions are entirely active, others reactive; only the twisting torque has both components [6, 8].

We adopt non-linear hyperelastic constitutive relations [12, 13]:

$$\begin{aligned}
Q_{1a} = Q_1 &= a\varepsilon_1 + \frac{1}{2}d\chi_1^2, \\
S_{1a} &= (k + d\varepsilon_1 + f_2\chi_2 + f_3\chi_3 + g\eta)\chi_1, \\
M_{2a} = M_2 &= b_2\chi_2 + \frac{1}{2}f_2\chi_1^2, \\
M_{3a} = M_3 &= b_3\chi_3 + \frac{1}{2}f_3\chi_1^2, \\
\mu_a = \mu &= h\eta + \frac{1}{2}g\chi_1^2\,.
\end{aligned} \tag{7}$$

The factors $a, b_j\ (j=2,3), k, h$ are the extension, bending, torsion, warping stiffnesses, respectively; the d, f_j $(j = 2, 3)$, g express the couplings between extension and torsion, bending and torsion, warping and torsion, respectively [14, 15]. If the bulk action β vanishes, we obtain [6, 8]

$$\begin{aligned}
\tau &= h\xi\chi_1'' + g\chi_1\chi_1', \\
S_1 &= (k + d\varepsilon_1 + f_2\chi_2 + f_3\chi_3)\chi_1 - h\xi^2\chi_1'' + c_3Q_2 - c_2Q_3
\end{aligned} \tag{8}$$

Comparing equation $(\mathbf{V}.1.10)_3$ in [3] with ours one has

$$\begin{aligned}
a &= EA, \qquad & b_j &= EI_j\ (j = 2,3), \qquad & k &= GI_c, \\
d &= EI_d, \qquad & f_j &= EI_{f_j}\ (j = 2,3), \qquad & h\xi^2 &= EI_\omega\ ;
\end{aligned} \tag{9}$$

E, G are the moduli in extension and shear; A is the cross-section area, I_j $(j = 2, 3)$ the centroidal principal moments of inertia; I_c is the torsion factor; I_d is the polar inertia with respect to c; I_ω is the warping inertia (second moment of the sectorial coordinate with respect to the area) and $I_{f_2} = \int_A x_3r^2$, $I_{f_3} = \int_A x_2r^2$, with x_j $(j = 2, 3)$ the coordinates of a point with respect to the centroid and r its distance from the shear centre.

2 BUCKLING IN A ROORDA FRAME

Results for a 'Roorda frame' made of I-beams as well as of channels constrained by an out-of-plane rigid pin at the joint are in [7]. There is an in-plane, flexural (Euler-like) buckling mode and another flexural-torsional: one of the bars undergoes torsion, the other bends out of the plane. Here we study a Roorda frame composed of channels constrained by an out-of-plane linear elastic spring.

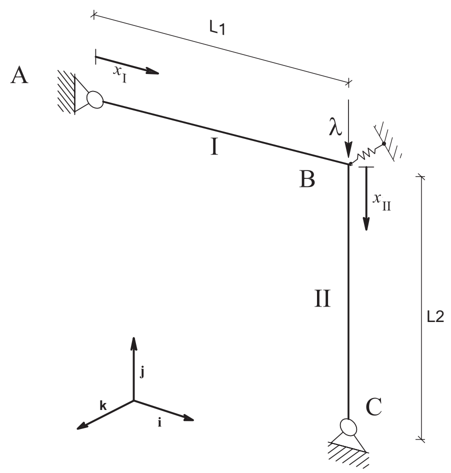

Figure 1: Two-bar Roorda frame.

The bars AB ('beam') and BC ('column') are hinged to the 'ground' in A and C and clamped at the common joint B, Figure 1. A 'dead' load of magnitude λ acts at B and we may apply standard techniques [13, 12]. A global basis and local abscissas are indicated; the subscripts I, II distinguish the beam and the column, respectively. We imagine various warping constraints at A, B, C.

The fundamental equilibrium path is, provided $\mathbf{I}$ is the identity,

$$\begin{gathered}
\mathbf{u}_{\mathrm{I}}^{\mathrm{f}}=\mathbf{0},\qquad \mathbf{R}_{\mathrm{I}}^{\mathrm{f}}=\mathbf{I},\qquad \alpha_{\mathrm{I}}^{\mathrm{f}}=0,\qquad \mathbf{e}_{\mathrm{I}}^{\mathrm{f}}=\mathbf{0},\qquad \mathbf{E}_{\mathrm{I}}^{\mathrm{f}}=\mathbf{0},\qquad \eta_{\mathrm{I}}^{\mathrm{f}}=0,\\
\mathbf{s}_{\mathrm{I}}^{\mathrm{f}}=\mathbf{0},\qquad \mathbf{S}_{\mathrm{I}}^{\mathrm{f}}=\mathbf{0},\qquad \tau_{\mathrm{I}}^{\mathrm{f}}=0,\qquad \mu_{\mathrm{I}}^{\mathrm{f}}=0,\\
\mathbf{u}_{\mathrm{II}}^{\mathrm{f}}=-\frac{\lambda}{a}x_1\mathbf{i}_1,\qquad \mathbf{R}_{\mathrm{II}}^{\mathrm{f}}=\mathbf{I},\qquad \alpha_{\mathrm{II}}^{\mathrm{f}}=0,\qquad \mathbf{e}_{\mathrm{II}}^{\mathrm{f}}=-\frac{\lambda}{a}\mathbf{i}_1,\qquad \mathbf{E}_{\mathrm{II}}^{\mathrm{f}}=\mathbf{0},\qquad \eta_{\mathrm{II}}^{\mathrm{f}}=0,\\
\mathbf{s}_{\mathrm{II}}^{\mathrm{f}}=-\lambda\mathbf{i}_1,\qquad \mathbf{S}_{\mathrm{II}}^{\mathrm{f}}=\mathbf{0},\qquad \tau_{\mathrm{II}}^{\mathrm{f}}=0,\qquad \mu_{\mathrm{II}}^{\mathrm{f}}=0\,.
\end{gathered}\tag{10}$$

The bifurcated path is written in terms of differences

$$\begin{gathered}
\mathbf{u}_{\mathrm{I}}^{\mathrm{b}}=\mathbf{u},\qquad \mathbf{R}_{\mathrm{I}}^{\mathrm{b}}=\mathbf{R}+\mathbf{I},\qquad \alpha_{\mathrm{I}}^{\mathrm{b}}=\alpha,\qquad \mathbf{e}_{\mathrm{I}}^{\mathrm{b}}=\mathbf{e},\qquad \mathbf{E}_{\mathrm{I}}^{\mathrm{b}}=\mathbf{E},\qquad \eta_{\mathrm{I}}^{\mathrm{b}}=\eta,\\
\mathbf{s}_{\mathrm{I}}^{\mathrm{b}}=\mathbf{s},\qquad \mathbf{S}_{\mathrm{I}}^{\mathrm{b}}=\mathbf{S},\qquad \tau_{\mathrm{I}}^{\mathrm{b}}=\tau,\qquad \mu_{\mathrm{I}}^{\mathrm{b}}=\mu\\
\mathbf{u}_{\mathrm{II}}^{\mathrm{b}}=\mathbf{u}-\frac{\lambda}{a}x_1\mathbf{i}_1,\qquad \mathbf{R}_{\mathrm{II}}^{\mathrm{b}}=\mathbf{R}+\mathbf{I},\qquad \alpha_{\mathrm{II}}^{\mathrm{b}}=\alpha,\\
\mathbf{e}_{\mathrm{II}}^{\mathrm{b}}=\mathbf{e}-\frac{\lambda}{a}\mathbf{i}_1,\qquad \mathbf{E}_{\mathrm{II}}^{\mathrm{b}}=\mathbf{E},\qquad \eta_{\mathrm{II}}^{\mathrm{b}}=\eta,\\
\mathbf{s}_{\mathrm{II}}^{\mathrm{b}}=\mathbf{s}-\lambda\mathbf{i}_1,\qquad \mathbf{S}_{\mathrm{II}}^{\mathrm{b}}=\mathbf{S},\qquad \tau_{\mathrm{II}}^{\mathrm{b}}=\tau,\qquad \mu_{\mathrm{II}}^{\mathrm{b}}=\mu
\end{gathered}\tag{11}$$

A static perturbation provides the first-order field equations in terms of displacement components

$$
\begin{aligned}
b_3 u_2'''' &= 0, \\
b_2 u_3'''' &= 0, \quad \text{(AB)} \\
h\xi^2 \varphi_1'''' - c\varphi_1'' &= 0, \\
b_3 u_2'''' + \lambda \frac{a-\lambda}{a}\left(u_2'' - c_3 \varphi_1''\right) &= 0, \\
b_2 u_3'''' + \lambda \frac{a-\lambda}{a}\left(u_3'' + c_2 \varphi_1''\right) &= 0, \quad \text{(BC)} \\
h\xi^2 \varphi_1'''' + \frac{d\lambda - ac}{a}\varphi_1'' + \lambda \frac{a}{a-\lambda}\left(c_2 u_3'' - c_3 u_2''\right) &= 0\,.
\end{aligned}
\tag{12}
$$

Equations (12) plus boundary conditions constitute an eigenvalue problem providing the critical values λ_c and the mode shapes u_{2c}, u_{3c}, φ_{1c}. The effect of different spring stiffnesses as well as

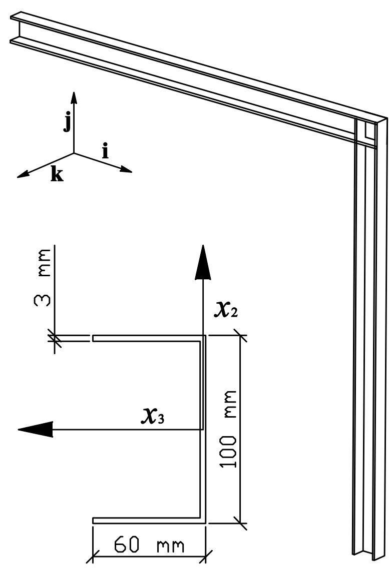

Figure 2: Three-dimensional view.

of warping constraints at the beam ends on the critical loads has been investigated numerically by means of the commercial COMSOL Multiphysics code. The beams composing the frame have cross-sections exhibiting one axis of symmetry; in particular we choose a channel (U-shape) with outer dimensions 100 mm (web) and 60 mm (flanges) and uniform thickness of 3 mm. By assuming the local coordinate systems as in Figure 2, the geometric and inertia quantities of the cross-section are obtained by standard calculations and well-known tables, e.g. [4, 16]

- U100:
  $a = 642\ mm^2 E$; $b_3 = 1054726\ mm^4 E$; $h\xi^2 = 446086956\ mm^6 E$; $k = 1875\ mm^4 G$; $d = 2370581\ mm^4 E$; $b_2 = 236653\ mm^4 E$; $c3 = -41\ mm$; $c2 = 0\ mm$;

Let Young's and shear modulus be $E = 206\ GPa$, $G = 79\ GPa$.

In 3D frames actual hinges may be assumed as cylindrical and the out-of-plane movement of nodes is constrained by braces. Therefore we consider B to be constrained along the unit vector $\mathbf{k}$ and the hinges in A, C to allow the sole rotation around $\mathbf{k}$:

$$\begin{gathered}\mathbf{u} = \mathbf{0}, \quad \varphi_1 = 0, \quad \varphi_2 = 0, \quad M_3 = 0 \qquad \text{in A and C} \\ \mathbf{u}_{\mathrm{I}} = \mathbf{u}_{\mathrm{II}}, \quad \mathbf{R}_{\mathrm{I}} = \mathbf{R}_{\mathrm{II}}, \\ \mathbf{S}_{\mathrm{I}} = \mathbf{S}_{\mathrm{II}}, \quad \mathbf{u}_{\mathrm{I}} \cdot \mathbf{k} = \frac{r}{s}, \quad (\mathbf{I} - \mathbf{k} \otimes \mathbf{k})(\mathbf{s}_{\mathrm{I}} - \mathbf{s}_{\mathrm{II}}) = \mathbf{0} \qquad \text{in B}\end{gathered} \tag{13}$$

where r is the constraint reaction of the spring at the joint B and s is the spring stiffness. Because

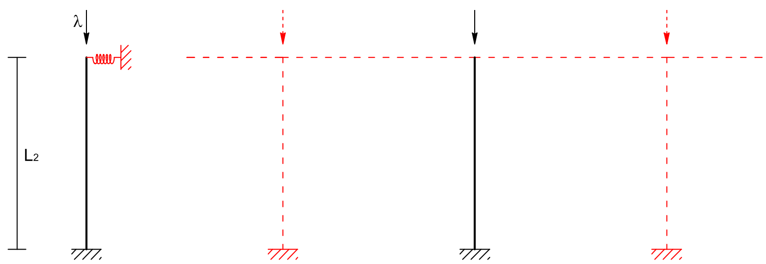

Figure 3: The spring simulating an actual frame.

of the phenomenon we wish to model, on the basis of standard results of the mechanics of structures we assume, see Figure 3, that

$$r = R\frac{12EI_2}{L_2^3} \tag{14}$$

with R a scalar multiplier. By varying the value of R in (14) we represent the effect of combination of braces of different length in actual three-dimensional frames on the considered frame. To investigate the influence of warping on buckling, additional constraints have been assumed: once all three ends A, B, C are supposed free to warp (case a.), then all three are supposed constrained against warping (case b.). This equals to the boundary conditions

$$\begin{array}{lll} \text{case a.} & \mu = 0 & \text{in A, B and C;} \\ \text{case b.} & \alpha = 0 & \text{in A, B and C.} \end{array}$$

We consider frames in which both the length of the beam and of the column equals 2 000 mm; results have been obtained for each of the boundary cases a.–b. above. Two buckling modes are possible, depicted in Figure 4 for the case a. above: the first, denoted by the subscript 1, is characterized by the fact that the buckled axes remain in-plane both for the beam and the column; the beam undergoes Euler-like buckling about the axis of greater inertia, while the column undergoes bending about the axis of greater inertia and torsion. It is remarkable, and it was to be inferred, that this mode is not affected by the variation of the braces stiffness multiplier R. The second buckling mode, denoted by the subscript 2, is out-of-plane: both the beam and the column undergo bending about the axis of smaller inertia and torsion; the buckling modes are easily seen to be influenced by the value of the braces stiffness multiplier R. It is easy to see, and the relevant picture is not broght

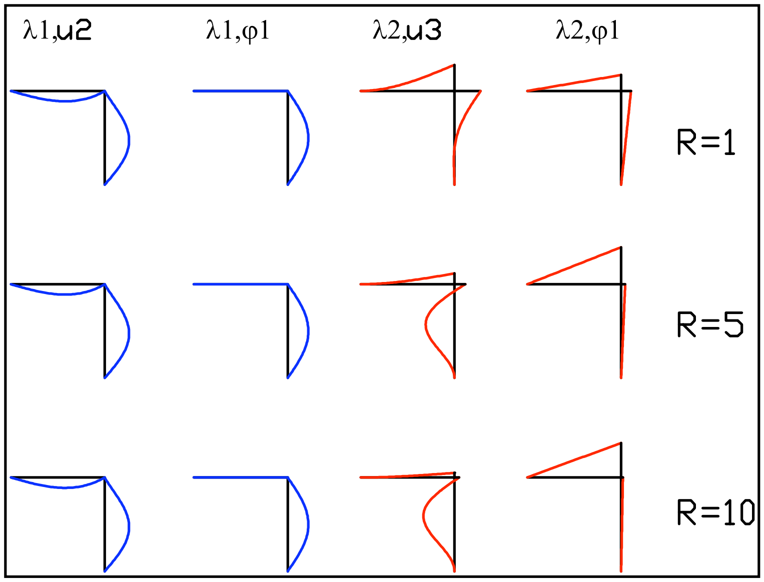

Figure 4: Buckling modes for various braces stiffness multiplier.

here for the sake of space, that similar buckling modes exist for the case b. above, the only difference being in the zero slope of the torsion mode due to the constrained warping at the beam ends.

R	λ_1	λ_2
0.05	95374.13128	64937.97421
0.1	95374.13128	70563.47893
0.5	95374.13127	113209.959
1	95374.13128	158011.8016
2	95374.13128	208359.3581
5	95374.13128	236020.6425
10	95374.13128	242016.9179
50	95374.13128	245842.6284
100	95374.13128	246269.4108

Table 1. Buckling load for the frame, all ends free to warp.

When all ends are free to warp (boundary case a. above) the buckling load is shown in Table 1; as already remarked, the in-plane buckling load λ_1 is not affected by the braces stiffness multiplier, which, on the contrary, clearly affects the out-of-plane buckling load λ_2: the stiffer the brace, the greater the value of the critical load multiplier, until it reaches the same value obtained by the authors in the case of null out-of-plane displacement imposed, see [7].

The numerical data contained in Table 1 are arranged in a graph in Figure 5. It is apparent how, for the considered geometry, the out-of-plane critical load becomes quickly dominant when the brace stiffness increases (i.e., roughly speaking, for moderately short lateral braces) so that the lowest critical load is that of the in-plane buckling mode. This is, of course, of importance in the

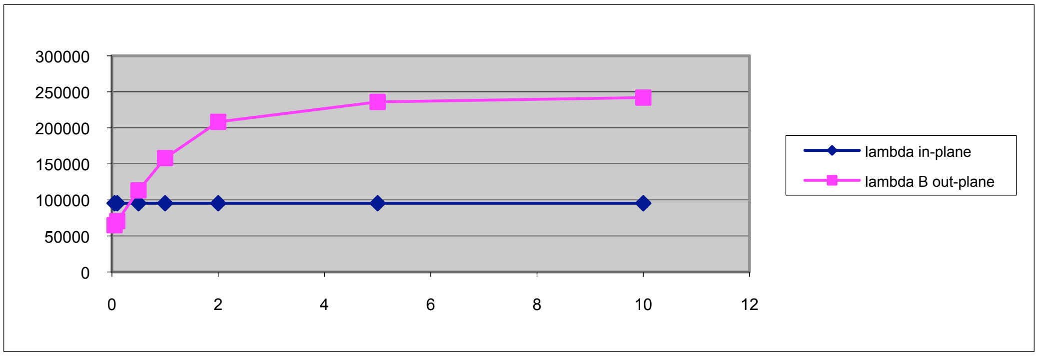

Figure 5: Critical loads vs. braces stiffness multiplier, warping free.

applications, since it implies that design against buckling can be limited to design against standard Euler buckling and flexural-torsional buckling of single elements (the column, in this case). On the other hand, it is apparent that for modest brace stiffness (i.e., roughly speaking, for 'long' lateral braces) the first critical load attained is that of out-of-plane buckling, which is strongly mixed. This is also of importance in the applications.

R	λ_1	λ_2
0.05	244115.3233	65215.35557
0.1	244115.3233	70840.68069
0.5	244115.3235	113489.0777
1	244115.3238	158301.5494
2	244115.3244	208658.2766
5	244115.3268	236285.505
10	244115.3363	242267.0749
50	244115.3122	246082.1201
100	244115.3143	246507.6497

Table 2. Buckling load for the frame, all ends constrained against warping.

When all ends are constrained against warping (boundary case b. above) the buckling load is shown in Table 2; again, the in-plane buckling load λ_1 is not affected by the braces stiffness multiplier, while the opposite happens for the out-of-plane buckling load λ_2. Again, the stiffer the brace, the greater the critical load, until it reaches the same value obtained by the authors in the case of null out-of-plane displacement imposed [7]. Remark how for this system constrained against warping, hence stiffer with respect to the preceding one, the critical loads are higher than the corrisponding ones for the system without warping constraints, in favour of security. This effect is more remarkable for the in-plane buckling.

The numerical data contained in Table 2 are arranged in a graph in Figure 6. For the considered geometry and the range of the brace stiffnes multiplier in the figure, the out-of-plane critical load is always lower than the in-plane one, irrespective of the stiffness of the lateral braces. This is, of course, of importance in the applications, since it implies that design against buckling cannot be

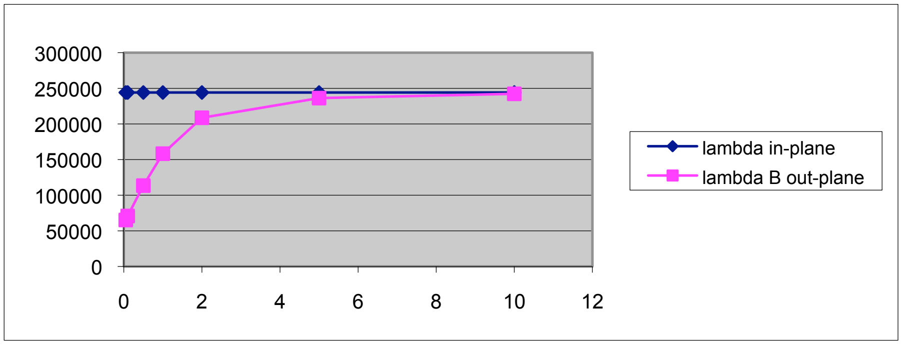

Figure 6: Critical loads vs. braces stiffness multiplier, warping constrained.

limited to design against standard Euler buckling.

3 FINAL REMARKS

In this contribution we have described qualitatively and quantitatively by a direct one-dimensional beam model and standard static perturbation techniques the buckling of a plane Roorda frame constrained against lateral buckling by a linear elastic spring simulating the effect of actual lateral braces. We have evaluated both the buckling modes and the critical loads for various lateral restraints and warping constraints at the frame ends and joint for a frame composed of members with channel sections, showing that for the considered geometry the lateral mixed buckling remains very important and the stiffness of lateral braces as well as warping constraints play a very important role.

Further developments of the present study are in due course and will investigate the effect of the length ratio between beam and column, as well as of other constraints at the beam ends and of different possibilities of lateral restraint.

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