A theoretical and numerical approach to the interaction between buckling and resonance instabilities in discrete and continuous mechanical systems

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SUMMARY. The paper deals with the interaction between buckling and resonance instabilities of mechanical systems. Taking into account the effect of geometric nonlinearity in the equations of motion through the geometric stiffness matrix, the problem is reduced to a generalized eigenproblem where both the loading multiplier and the natural frequency of the system are unknown. According to this approach, all the forms of instabilities intermediate between those of pure buckling and pure forced resonance can be investigated. Numerous examples including discrete mechanical systems and continuous mechanical systems, such as oscillating deflected beams subjected to a compressive axial load and oscillating beams subjected to lateral-torsional buckling, are analyzed. The proposed results provide a new insight in the interpretation of coupled phenomena such as flutter instability of long-span or high-rise structures.

1 INTRODUCTION

Buckling, resonance and flutter are the main forms of instability of the elastic equilibrium of structural systems. In buckling instability, by removing the hypothesis of small displacements so that the deformed structural configuration can be distinguished from the undeformed one, it is possible to show that the solution of an elastic problem can represent a condition of stable, neutral or unstable equilibrium, depending on the magnitude of the applied load. As is well-known, buckling is usually observed in slender structural elements subjected to a compressive stress field, such as columns of buildings, machine shafts, struts of trusses, thin arches and shells. In some cases, elastic instability can also take place for special loading and geometrical conditions, as for example in the lateral torsional buckling of slender beams [1].

The phenomenon of resonance is also particularly important in structural engineering. It represents a form of dynamic instability, which occurs when an external periodic frequency matches one of the natural frequencies of vibration of the mechanical system. In this case, therefore, structural design deals with the determination of such dangerous natural frequencies according to modal analysis [2].

Finally, the phenomenon of flutter is a form of aeroelastic instability observed in long-span or high-rise structures subjected to wind loads, such as towers, tall buildings [3] and suspended [4–6] or cable-stayed bridges [7]. In this case, the instability is attributed to motion-induced or self-excited forces, which are loads induced or influenced by the deformation of the structure itself [8, 9]. Forces originated in this way modify the initial deformation of the mechanical system which, consequently, leads to modified forces, and so on. This feed-back mechanism can give rise to an amplification effect on the initial deformation, leading to premature failure of the structure. The well-known dramatic Tacoma Narrows bridge disaster of 1940 is a famous example of this catastrophic interaction, and it is still very much in the public eye today. In this field, which is not at present completely understood, aeroelastic instability is often considered as the result of the interaction between buckling (static) and resonance (dynamic) instabilities. However, only a few theoretical formulations have been proposed for modelling aerodynamic forces and, in most investigations, empirical models are set up in which the parameters related to the fluid-structure interaction are established by experiments [10].

In the present contribution, we deal with the phenomenon of interaction between buckling and resonance instabilities. A state-of-the-art survey of the existing Literature shows that this problem has been mainly addressed in the field of multi-parameter stability theory [11–16], where the conditions for stability of a mechanical system are studied with reference to a perturbation of the problem parameters. In our treatment, we will adopt the classical notions of vibration and stability of conservative systems. In the presence of several input parameters, the graph of natural frequencies of vibration in the parameter space represents a boundary of the stability domain. The analysis of such graphs is the main focus of the present paper, where we consider the influence of stationary loads on the natural frequencies of vibration of elastic conservative discrete or continuous mechanical systems. Analysis of these stability boundaries -and therefore the study of the interaction between the aforementioned elementary forms of instability- allows changing design parameters in order to either modify the natural frequencies of the system, or to increase the values of the critical buckling loads. Special focus will be given to the role played by the geometric stiffness matrix, which contributes to the reduction of the elastic stiffness matrix due to the effect of the geometric nonlinearity. According to the proposed formulation, we will demonstrate that the interaction between buckling and resonance leads to a generalized eigenvalue problem where both the buckling loads and the natural frequencies of the system are unknown and represent the eigenvalues. This approach will permit to inspect all the forms of structural instability intermediate between pure buckling and pure resonance.

Finally, we will discuss on the possibility to apply the proposed approach to the analysis of flutter instability of cable-suspended bridges. This will represent a novelty of our approach with respect to the models available in the Literature. In fact, the use of the geometric stiffness matrix may provide the proper link between the multi-parameter stability theory, typical of rational mechanics, and the bridge engineering approach.

2 DISCRETE MECHANICAL SYSTEMS

Let us consider the mechanical system with two degrees of freedom shown in Fig.1, which consists of three rigid rods on four supports, of which the central ones are assumed to be elastically compliant with rigidity k. A mass m is placed in correspondence of the intermediate hinges and the system is loaded by a horizontal axial force N.

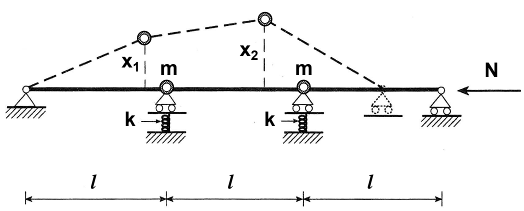

Figure 1: Scheme of the two-degrees of freedom system analyzed.

Assuming the vertical displacements x_1 and x_2 of the elastic hinges as the generalized coordinates, the linearized total potential energy, W, and the linearized kinetic energy, T, of the whole system are given by ($x_1/l < 1/10$ and $x_2/l < 1/10$):

$$
\begin{aligned}
W(x_1,x_2) =& \frac{1}{2}k\left(x_1^2+x_2^2\right) - Nl\left[3-\cos\left(\arcsin\frac{x_1}{l}\right)\right.\\
&\left.-\cos\left(\arcsin\frac{x_2}{l}\right)-\cos\left(\arcsin\frac{x_2-x_1}{l}\right)\right]\\
\cong& \frac{1}{2}k\left(x_1^2+x_2^2\right)-\frac{N}{l}\left(x_1^2+x_2^2-x_1x_2\right),\\
T(\dot{x_1},\dot{x_2}) \cong& \frac{1}{2}m\dot{x_1}^2+\frac{1}{2}m\dot{x_1}^2.
\end{aligned} \tag{1}
$$

The Lagrange's equations yield the following matrix form:

$$
\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}-N\begin{bmatrix} \frac{2}{l} & -\frac{1}{l} \\ -\frac{1}{l} & \frac{2}{l} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{2}
$$

Looking for the solution to Eq.(2) in the general form $\{q\} = \{q_0\}\mathrm{e}^{\mathrm{i}\omega t}$, where ω denotes the natural angular frequency of the system, we obtain the following equation:

$$
\left(-\omega^2[M]+[K]-N[K_g]\right)\{q_0\}=\{0\}, \tag{3}
$$

where $[M]$, $[K]$ and $[K_g]$ denote, respectively, the mass matrix, the elastic stiffness matrix and the geometric stiffness matrix of the mechanical system.

A nontrivial solution to Eq.(3) exists if and only if the determinant of the resultant coefficient matrix of the vector $\{q_0\}$ is equal to zero. This yields the following generalized eigenvalue problem:

$$
\det\left([K]-N[K_g]-\omega^2[M]\right)=0, \tag{4}
$$

where N and ω^2 are the eigenvalues of the system. For this example, Eq.(4) provides the following relationships between the eigenvalues:

$$
\omega^2=\frac{k}{m}-3\frac{N}{ml}, \tag{5a}
$$

$$
\omega^2=\frac{k}{m}-\frac{N}{ml}. \tag{5b}
$$

As limit cases, if $m = 0$, we obtain the Eulerian buckling loads $N_1 = kl/3$ and $N_2 = kl$, whereas, if $N = 0$, then we obtain the natural frequencies of the system $\omega_1 = \omega_2 = \sqrt{k/m}$.

Dividing Eqs.(5a) and (5b) by ω_1^2, we obtain the following nondimensional relationships between the eigenvalues:

$$
\left(\frac{\omega}{\omega_1}\right)^2=1-\left(\frac{N}{N_1}\right), \tag{6a}
$$

$$
\left(\frac{\omega}{\omega_1}\right)^2=1-\frac{N_1}{N_2}\left(\frac{N}{N_1}\right). \tag{6b}
$$

A graphical representation of Eqs.(6a) and (6b) is provided in Fig.2. Both the frequencies are decreasing functions of the compressive axial load. Starting from $N = 0$, bifurcation of the equilibrium would correspond to pure resonance instability. Entering the diagram with a value of the nondimensional compressive axial force in the range $0 < N/N_1 < 1$, the coordinates of the points of the two curves provide the critical eigenfrequencies of the mechanical system leading to bifurcation. Axial forces higher than N_1 in the range $1 < N/N_1 < N_2/N_1$ can only be experienced if an additional constraint is introduced into the system.

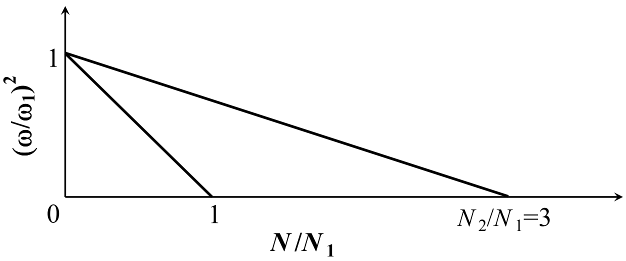

Figure 2: Nondimensional frequencies vs. nondimensional axial forces for the two-degrees of freedom system in Fig.1.

3 CONTINUOUS MECHANICAL SYSTEMS

3.1 Oscillations of deflected beams under compressive axial loads

Let us consider a slender elastic beam of constant cross-section, inextensible and not deformable in shear, though deformable in bending, constrained at one end by a hinge and at the other by a roller support, loaded by an axial force, N. In this case, with the purpose of analyzing the free flexural oscillations of the beam, the differential equation of the elastic line with second-order effects can be written by replacing the distributed load with the force of inertia:

$$EI\frac{\partial^4 v}{\partial z^4} + N\frac{\partial^2 v}{\partial z^2} = -\mu\frac{\partial^2 v}{\partial t^2}, \tag{7}$$

where EI denotes the flexural rigidity of the beam and μ is its linear density (mass per unit length). Equation (7) can be rewritten in the following form:

$$\frac{\partial^4 v}{\partial z^4} + \beta^2\frac{\partial^2 v}{\partial z^2} = -\frac{\mu}{EI}\frac{\partial^2 v}{\partial t^2}, \tag{8}$$

where we have set $\beta^2 = N/EI$.

Equation (8) is an equation with separable variables, the solution being represented as the product of two different functions, each one depending on a single variable:

$$v(z,t) = \eta(z)f(t). \tag{9}$$

Introducing Eq.(9) into Eq.(8), leads:

$$\frac{\mathrm{d}^4\eta}{\mathrm{d}z^4}f + \beta^2\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2}f + \frac{\mu}{EI}\eta\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = 0. \tag{10}$$

Dividing Eq.(10) by the product ηf, we find:

$$-\frac{\dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2}}{f} = \frac{EI}{\mu}\frac{\dfrac{\mathrm{d}^4\eta}{\mathrm{d}z^4} + \beta^2\dfrac{\mathrm{d}^2\eta}{\mathrm{d}z^2}}{\eta} = \omega^2, \tag{11}$$

where ω^2 represents a positive constant, the left and the right hand-sides of Eq.(11) being at the most functions of the time t and the coordinate z, respectively. From Eq.(11) there follow two ordinary differential equations:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + \omega^2 f = 0, \tag{12a}$$

$$\frac{\mathrm{d}^4 \eta}{\mathrm{d}z^4} + \beta^2 \frac{\mathrm{d}^2 \eta}{\mathrm{d}z^2} - \alpha^4 \eta = 0, \tag{12b}$$

with $\alpha = \sqrt[4]{\mu\omega^2/(EI)}$. Whereas Eq.(12a) is the well-known equation of the harmonic oscillator, Eq.(12b) has the following complete integral

$$\eta(z) = C\mathrm{e}^{\lambda_1 z} + D\mathrm{e}^{\lambda_2 z} + E\mathrm{e}^{-\lambda_1 z} + F\mathrm{e}^{-\lambda_2 z}, \tag{13}$$

where λ_1 and λ_2 are functions of α and β:

$$\lambda_{1,2} = \sqrt{\frac{-\beta^2 \pm \sqrt{\beta^4 + 4\alpha^4}}{2}}. \tag{14}$$

As in the modal analysis, the constants C, D, E and F can be determined by imposing the boundary conditions. Hence, for a given value of β, the parameters ω and α can be determined by solving a generalized eigenvalue problem resulting from the imposition of the boundary conditions. From the mathematical point of view, this eigenvalue problem is analogous to that shown for the discrete systems. On the other hand, since we are considering a continuous mechanical system having infinite degrees of freedom, we shall obtain an infinite number of eigenvalues ω_i and α_i, just as also an infinite number of eigenfunctions f_i and η_i. The complete integral of the differential equation (7) may therefore be given the following form, according to the Principle of Superposition:

$$v(z,t) = \sum_{i=1}^{\infty} \eta_i(z) f_i(t). \tag{15}$$

It is important to remark that the eigenfunctions η_i are still orthonormal functions, as in the classical modal analysis. As regards the boundary conditions, let us consider as an example a beam supported at both ends, of length l:

$$\left\{ \begin{array}{ll} \eta(0) & = 0, \\ \eta''(0) & = 0, \\ \eta(l) & = 0, \\ \eta''(l) & = 0, \end{array} \right. \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_1^2 & \lambda_2^2 \\ \mathrm{e}^{\lambda_1 l} & \mathrm{e}^{\lambda_2 l} & \mathrm{e}^{-\lambda_1 l} & \mathrm{e}^{-\lambda_2 l} \\ \lambda_1^2\mathrm{e}^{\lambda_1 l} & \lambda_2^2\mathrm{e}^{\lambda_2 l} & \lambda_1^2\mathrm{e}^{-\lambda_1 l} & \lambda_2^2\mathrm{e}^{-\lambda_2 l} \end{bmatrix} \left\{ \begin{array}{c} C \\ D \\ E \\ F \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}, \tag{16}$$

For a nontrivial solution to the system in Eq.(16), the determinant of the coefficient matrix has to vanish. The resulting eigenequation permits, for each given value of the parameter β, to determine the eigenvalues α_i of the system. Finally, the corresponding natural eigenfrequencies ω_i can be obtained.

As an illustrative example, the first three nondimensional frequencies of a simply supported beam are reported in Fig.3 as functions of the applied nondimensional axial force. Parameters ω_i and N_i denote, respectively, the i-th frequency of the system determined according to modal analysis and the i-th buckling load determined according to the Euler's formula. In close analogy with the discrete mechanical systems, the curves in the $(\omega/\omega_1)^2$ vs. N/N_1 plane are represented by straight lines. Also in this case, the coordinates of the points along these lines provide the critical conditions leading to the system instability in terms of frequency of the excitation and magnitude of the applied compressive axial force.

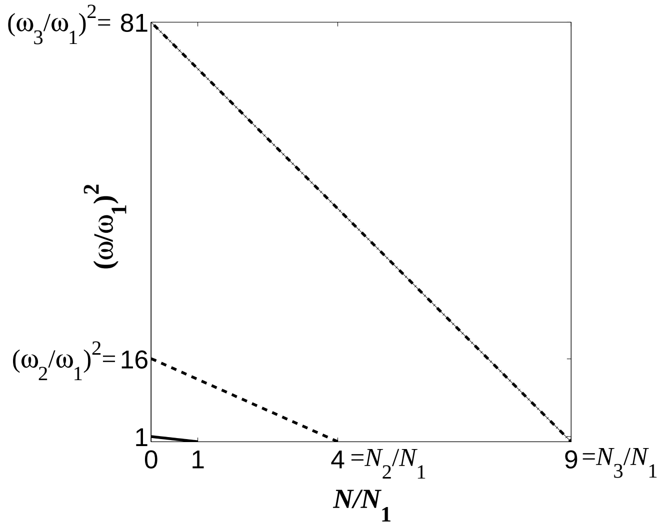

Figure 3: Nondimensional frequencies vs. nondimensional axial force for a simply supported beam.

3.2 Oscillations and lateral-torsional buckling of beams

Let us consider a beam of thin rectangular cross-section, constrained at the ends so that rotation about the longitudinal axis Z is prevented. Let this beam be subjected to uniform bending by means of the application at the ends of two moments m contained in the plane YZ of greater flexural rigidity.

Considering a deformed configuration of the beam, with deflection thereof in the XZ plane of smaller flexural rigidity, and simultaneous torsion about the Z axis, bending-torsional out-of-plane vibrations of the beam are described by the following partial differential equations:

$$\begin{aligned} EI_y \frac{\partial^4 u}{\partial z^4} + m \frac{\partial^2 \varphi_z}{\partial z^2} &= -\mu \frac{\partial^2 u}{\partial t^2}, \\ -GI_t \frac{\partial^2 \varphi_z}{\partial z^2} + m \frac{\partial^2 u}{\partial z^2} &= -\mu \rho^2 \frac{\partial^2 \varphi_z}{\partial t^2}, \end{aligned} \tag{17}$$

where $u(z,t)$ and $\varphi_z(z,t)$ are, respectively, the out-of-plane deflection and the twist angle of the beam cross-section; EI_y and GI_t are the bending and torsional rigidities; μ is the mass of the beam per unit length, and $\rho = \sqrt{I_P/A}$ is the polar radius of inertia of the beam cross-section.

A solution to the system (17) can be found in the following variable-separable form [17]:

$$\begin{aligned} u(z,t) &= U(t)\eta(z), \\ \varphi_z(z,t) &= \Phi(t)\psi(z), \end{aligned} \tag{18}$$

where the functions $\eta(z)$ and $\psi(z)$ are such that the boundary conditions $\eta(0) = \eta(l) = \eta^{''}(0) = \eta^{''}(l) = \psi(0) = \psi(l) = 0$ are satisfied. According to Bolotin [17], we can assume $\eta(z) = \psi(z) = \sin \dfrac{n\pi z}{l}$, with n being a natural number. In this case, we obtain the following matrix form:

$$\begin{bmatrix} \mu & 0 \\ 0 & \mu\rho^2 \end{bmatrix} \begin{Bmatrix} \ddot{U} \\ \ddot{\Phi} \end{Bmatrix} + \begin{bmatrix} EI_y \dfrac{n^4\pi^4}{l^4} & 0 \\ 0 & GI_t \dfrac{n^2\pi^2}{l^2} \end{bmatrix} \begin{Bmatrix} U \\ \Phi \end{Bmatrix} - m \begin{bmatrix} 0 & \dfrac{n^2\pi^2}{l^2} \\ \dfrac{n^2\pi^2}{l^2} & 0 \end{bmatrix} \begin{Bmatrix} U \\ \Phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19}$$

which can be symbolically rewritten as:

$$[M]\{\ddot{q}\} + [K]\{q\} - m\,[K_g]\{q\} = \{0\}, \tag{20}$$

where $\{q\} = (U, \Phi)^{\mathsf{T}}$. The mass matrix, $[M]$, the elastic stiffness matrix $[K]$, and the geometric stiffness matrix $[K_g]$ in Eq.(20) can be defined in comparison with Eq.(19). Looking for a general solution in the form $\{q\} = \{q_0\}\mathrm{e}^{\mathrm{i}\omega t}$, we obtain:

$$\left([K] - m\,[K_g] - \omega^2\,[M]\right)\{q_0\} = \{0\}. \tag{21}$$

A nontrivial solution to Eq.(21) exists if and only if the determinant of the resultant coefficient matrix of the vector $\{q_0\}$ vanishes. This yields the following generalized eigenvalue problem:

$$\det\left([K] - m\,[K_g] - \omega^2\,[M]\right) = 0, \tag{22}$$

where m and ω^2 are the eigenvalues of the system.

As limit cases, if $\mu = 0$, then we obtain the critical bending moments given by the Prandtl's formula:

$$m_{nc} = \frac{n\pi}{l}\sqrt{EI_y GI_t}, \tag{23}$$

whereas, if $m = 0$, then we obtain the natural flexural and torsional eigenfrequencies of the beam:

$$\omega_n^{\text{flex}} = \left(\frac{n\pi}{l}\right)^2 \sqrt{\frac{EI_y}{\mu}}, \quad \omega_n^{\text{tors}} = \frac{n\pi}{\rho l}\sqrt{\frac{GI_t}{\mu}}. \tag{24}$$

Considering a rectangular beam with a depth to span ratio of $1/3$ and with a thickness to depth ratio of $1/10$, the evolution of the first two flexural and torsional eigenfrequencies of the system are shown in Fig.4 as functions of the applied bending moment. In this case, the curves in the nondimensional plane $(\omega/\omega_1^{\text{flex}})^2$ vs. m/m_{1c} are no longer straight lines. This fact can be ascribed to the coupling between torsional and flexural vibrations of the beam. Moreover, when m is increased from zero (pure resonance instability) up to the critical bending moment computed according to the Parandtl's formula (pure buckling instability), m_{1c}, we note that the resonance frequencies related to flexural oscillations progressively decrease from ω_1^{flex} down to zero in correspondence of the critical bending moments given by the Prandtl's formula. Conversely, the resonance frequencies related to torsional oscillations increase. From the mathematical point of view, this is the result of the fact that the sum of the squares of the two eigenfrequencies for a given value of n is constant when the applied moment m is varied.

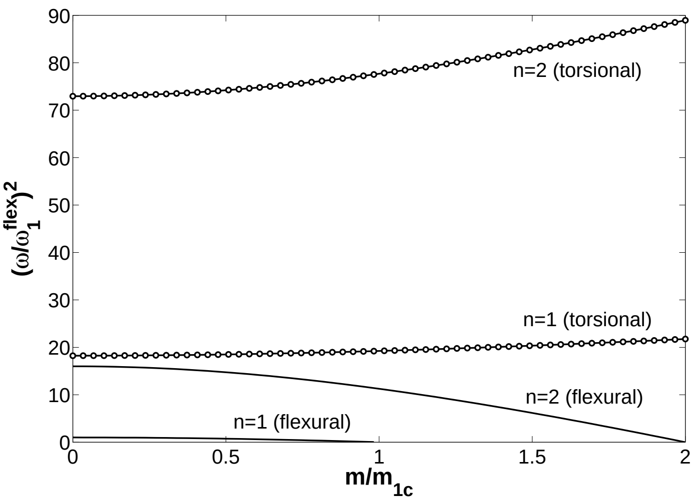

Figure 4: Nondimensional flexural and torsional frequencies vs. nondimensional bending moments.

4 DISCUSSION AND CONCLUSIONS

The problems of elastic instability (buckling) and dynamic instability (resonance) have been the subject of extensive investigation and have received a large attention from the structural mechanics community. Nonetheless, the study of the interaction between these elementary forms of instability is still an open point.

The phenomenon of flutter instability of the Tacoma Narrows Bridge occurred on November 7, 1940, can be reinterpreted as the result of such a catastrophic interaction. Shortly after construction, it was discovered that the bridge would sway and buckle dangerously in windy conditions. This resonance was flexural, meaning the bridge buckled along its length, with the roadbed alternately raised and depressed in certain locations. However, the failure of the bridge occurred when a never-before-seen twisting mode occurred.

A Report to the Federal Works Agency [18] excluded the phenomenon of pure forced resonance as the actual reason of instability: "*...it is very improbable that resonance with alternating vortices plays an important role in the oscillations of suspension bridges. First, it was found that there is no sharp correlation between wind velocity and oscillation frequency such as is required in case of resonance with vortices whose frequency depends on the wind velocity...*". A new theory for the interpretation of these complex aerodynamic instabilities was developed by Scanlan [19, 20] and then elaborated by various researchers [21–23]. Basically, the so-called *flutter theory* considers the following equation of motion for the mechanical system in the finite element framework [7, 22]:

$$[M]\left\{\ddot{\delta}\right\} + [C]\left\{\dot{\delta}\right\} + [K]\left\{\delta\right\} = \{F\}_{mi} + \{F\}_{md}\,, \tag{25}$$

where $\{F\}_{mi}$ and $\{F\}_{md}$ are, respectively, the motion-independent wind force vector and the motion-dependent aeroelastic force vector. A special attention is given to the structural damping, which is included in the equations of motion through the damping matrix $[C]$. The motion-dependent force vector is then put in relationship with the nodal displacements of the system, $\{\delta\}$, and the nodal velocities, $\{\dot{\delta}\}$, according to the *flutter derivative matrices*, $[K^*]$ and $[C^*]$, that are empirically determined in the wind tunnel by using section models of the bridge. As a result, the problem becomes highly nonlinear, and the flutter velocity, U_{cr}, and the flutter frequency, ω_{cr}, can be determined from the following eigenproblem:

$$\det\left([K] - \frac{1}{2}\rho U_{cr}^2 [K^*] - \omega_{cr}^2 [M] + \omega_{cr} [C] - \frac{1}{2}\rho U_{cr}\omega_{cr} [C^*]\right) = 0, \tag{26}$$

where ρ is the air density and $1/2\rho U_{cr}^2$ is the wind pressure.

It is important to remark that the eigenproblem resulting from the classical approach to flutter instability shares most of the features of the generalized eigenproblem that we have analyzed in the present study. In fact, in both cases, two eigenvalues have to be determined from the eigenequation. However, the flutter theory gives prominence to the role played by the structural damping, although this is generally less than 1% (see e.g. [22]). Moreover, the value of mechanical damping seems to represent a sort of free parameter in the model. In fact, this parameter is usually assumed, like in [22], rather than experimentally evaluated. Hence, the classical approaches to flutter instability used in the field of bridge engineering *do not provide a single pair of critical frequency and wind speed, but rather a number of pairs, each one for a given value of the assumed structural damping*. This suggests that also flutter instability can be mathematically treated in the framework of multi-parameter stability theory, where the stability boundary of the graph relating the flutter frequency to the critical wind speed is of main concern.

The problem of flutter instability can therefore be reconsidered according to a completely different route. The complex wind-structure interaction can be approximated by the sum of the action of a stationary load, given by the wind pressure acting on the bridge deck, and the action of wind gusts, characterized by their specific frequency. Under such conditions, the interaction between buckling and resonance would apply as for the mechanical systems addressed in the present paper. The problem is now finding the boundary of the stability domain in the graph relating the natural frequencies to the applied stationary wind pressure. Therefore, it is the combination of the wind speed and of the frequency of the wind gusts which is responsible for the bridge instability and not their single values. Moreover, according to this approach, the geometric stiffness matrix has a preeminent role and the mechanical damping can be neglected, as usually done in most of structural engineering applications. On this line, the collapse of the Tacoma Narrows bridge can be considered as the result of the interaction between buckling (related to the wind pressure proportional to the square of the wind velocity) and resonance (caused by the frequency of the wind gusts). Thus, this would give a new explanation on why the Tacoma Narrows bridge failure took place under moderate wind velocities (wind pressure lower than the critical buckling load) and wind gusts frequencies different from the natural frequencies of the bridge.

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