On the Evaluation of the Shear Correction Factors: a Boundary Element Approach

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SUMMARY. Three popular approaches for the evaluation of the shear correction factors to be used in the Timoshenko beam theory are considered and compared in terms of results obtained. Two of the approaches use the shear stresses of the beam under constant shear as the main ingredients for the evaluation of shear correction factors, while the third uses the Saint-Venant flexure function. In the paper, new results in terms of shear correction factors are derived analytically for a semi-circular cross-section with the shear force acting along a direction perpendicular to the axis of symmetry. For the same cross-section results are provided numerically when the shear force acts parallel to the axis of symmetry. A convergence study is performed in order to assess the accuracy of the solution obtained. Finally, the results in terms of shear correction factors obtained by the three approaches are compared qualitatively and quantitatively showing that the differences can be significant and that the choice of the most appropriate shear correction factor may be relevant.

1 INTRODUCTION

Shear correction factors are needed in elementary beam theories whenever shear strains may affect significantly the required solution. The applications may be of static or dynamic nature and have been expounded in the most popular textbooks on strength of materials, structural mechanics and dynamics of structures. Already in the late 18 hundreds Lord Rayleigh introduced rotary inertia in an attempt to match theoretical and experimental frequencies in beams. However, it was S. Timoshenko [1] who first derived a differential equation combining the effects of shear strains and rotary inertia which provides much better results in the evaluation of the vibration frequencies of beams when the effects of shear deformation and rotary inertia are significant. Later, Timoshenko's beam theory has been applied to buckling analyses of compressed helical springs and rubber rods for vibration-free mountings, and laminated rubber bearings used in earthquake engineering for base isolation. Given this wide spectrum of applications, in a variety of fields ranging from statics to dynamics, it is unlikely that a single shear correction factor may satisfy all practical requirements. Even if the object of the exercise is the calculation of the frequencies of vibration of standard beams or columns the matching of the experimental frequencies may be a difficult problem. More accurate formulations for static and long wavelength dynamical applications have been derived in the second half of the last century. From the few available exact results it seems that one set of shear correction factors excels among the others when low-frequency long-wavelength problems must be tackled, namely the set developed separately at different times and with different methods by N.G. Stephen [2] and by J.R. Hutchinson [3].

However, two other sets of correction factors are worth considering because of the simplicity of the physical assumptions on which they are based and of the popularity that they have gained within the scientific community. One due to G.R. Cowper , [4], can be considered with good rights as the precursor of the Stephen-Hutchinson (S-H) set of shear correction factors. The other method of deriving shear correction factors has been known for a long time in mechanics and in engineering and is based on the principle of energy equivalence. The best set of shear correction factors obtained so far using the energy equivalence approach is due to J.D. Renton [5].

In this work we shall review three of the methods used in the literature to derive the shear correction factors that we consider well founded and will show a way to derive the ingredients needed for the actual calculation of the shear correction factors for those cross-sections for which those ingredients are not available analytically.

2 REVIEW OF THREE POPULAR APPROACHES

We shall review three of the most popular approaches for the evaluation of the shear correction factors. In the order we shall consider first the energy approach because it has been used for a long time, with approximate and exact distributions of shear stresses, and expressions for the shear correction factor can be found in classical textbooks on structural mechanics [6-7] and in research papers [5,8]. Second we consider the approach by G.R. Cowper based on the derivation of the Timoshenko beam theory from the three-dimensional theory of elasticity [4]. Its popularity is largely associated to the fact that it is mentioned in some classical books on structural mechanics [9]. The third approach that we shall review is due to N.G. Stephen [2] and to J.R. Hutchinson [3]. The same results obtained by N.G. Stephen were rediscovered some 20 years later by J.R Hutchinson starting from very different assumptions.

2.1 *The Energy Approach*

The energy approach is based on the equality of the elastic shear strain energy per unit length of beam in the Timoshenko beam theory and some accurate estimates of the same. In mathematical terms this can be expressed as follows:

$$\frac{1}{2}\cdot\int_A \left(\tau_{zx}\gamma_{zx}+\tau_{zy}\gamma_{zy}\right)dA=\frac{1}{2}\cdot T_y\,\frac{T_y}{k_y GA}\,, \tag{1}$$

This leads to the expression for the shear correction factor for symmetrical cross-sections:

$$k_y=\frac{T_y^2}{A\cdot\int_A\left(\tau_{zx}^2+\tau_{zy}^2\right)dA} \tag{2}$$

The shear stress distributions appearing in equation (2) have been derived in [6-7] from approximate theories [10] and in [5] from exact solutions [11]. Shear stress distributions evaluated through numerical analyses may be used as well, as it has been shown by the first author [8].

2.2 *Cowper's Approach*

G.R. Cowper [4] derived the Timoshenko beam equation from the three-dimensional theory of elasticity, obtaining an expression of the shear correction factor as a byproduct. This takes the following expression for symmetrical cross sections with shearing force applied along the axis of symmetry:

$$k_y = \frac{2\cdot(1+\nu)\cdot I_x}{\frac{\nu}{2}\cdot(I_y - I_x) - \frac{A}{I_x}\cdot\int_A y\cdot\left[\chi(x,y) + x^2 y\right]\cdot dA} \tag{3}$$

The function $\chi(x,y)$ appearing in equation (3) is the solution of a Neumann problem denoted as the "flexure problem" by A.E.H. Love ([11], page 332). Expressions for the flexure function are provided in analytical form by Love for some cross-sections and can be derived numerically for any cross-section by solving the flexure problem mentioned above.

2.3 *Stephen-Hutchinson's Approach*

In 1980 [2] N.G. Stephen derived an expression for the shear correction factor by equating the centerline curvature of a Timoshenko beam to the curvature of a beam subjected to uniform gravity loading. The formula produced by Stephen is structurally similar to that derived by Cowper and differs only by the doubling of the first term in the denominator and by the addition of a third term again in the denominator.

$$k_y = \frac{2(1+\nu)I_x}{\nu(I_y - I_x) - \frac{A}{I_x}\int_A y\left[\chi(x,y) + x^2 y\right]dA - \frac{\nu\cdot A}{2(1+\nu)I_x} I_S} \tag{4}$$

with

$$I_S = \int_A \left\{\frac{y^2 - x^2}{2}\cdot\left(\frac{\partial\chi}{\partial y} + \frac{\nu\cdot y^2}{2} + \frac{2-\nu}{2}x^2\right) + xy\left[\frac{\partial\chi}{\partial x} + (2+\nu)xy\right]\right\}dA \tag{5}$$

The similarity and the differences may be accounted for by considering that two static solutions are used in the derivation of the shear correction factor in the two cases and these solutions are complementary to each other in the sense that one of the two differs from the other only by some additional terms. The solution used by Cowper is the one provided by A.E.H. Love for the beam with constant shear and includes the corresponding shear stresses distribution over the cross-section. The solution used by Stephen is again provided by Love [11] (Chapter XVI, pp.349-364) for a beam under uniformly distributed transverse load ; this is characterized by the same transverse shear stresses distribution as before but with the inclusion of the stress components σ_x, σ_y, τ_{xy} not present in the previous solution.

In 2001 J. R. Hutchinson derived a new formulation for the Timoshenko beam equation, by using some appropriate kinematical assumption in the dynamic formulation of the Hellinger-

Reissner principle and the shear stress distribution derived by Love for the cantilever beam with constant shear. By equating the first different dominant terms of the characteristic equations of the classical and new theories he derived the following expression for the shear correction factor of symmetrical cross-sections:

$$k_y = -\frac{2\cdot(1+\nu)}{\dfrac{A}{I_x^2}\cdot C + \nu\cdot\left(1-\dfrac{I_y}{I_x}\right)} \tag{6}$$

where the constant C is provided in terms of the shear stress distribution for the beam with constant shear.

$$C = -\nu\frac{I_x}{T_y}\int_A\left(\tau_{zy}y^2 - \tau_{zy}x^2 + 2\tau_{zx}xy\right)\cdot dA - 2(1+\nu)\frac{I_x^2}{T_y^2}\int_A\left(\tau_{zx}^2+\tau_{zy}^2\right)\cdot dA \tag{7}$$

It is clear that, as in the previous two approaches, the required shear stresses distribution may be provided analytically when exact or approximate closed form solutions are available or may be evaluated numerically by solving the appropriate Neumann problem as was shown in [8]. Soon after the publication of Hutchinson's paper, Stephen [12] proved that expressions (4) and (6) are equivalent.

3 ANALYTICAL RESULTS

Whenever an analytical solution for the *"flexure problem"* as defined by Love is available, it can be used for the evaluation of the shear correction factor according to the three approaches described in the previous section. One set of shear correction factors obtained in this way was provided by Cowper for his approach. For the same set of cross-sections analogous analytical results were provided by the first author in [8] also for the other two approaches, namely *Energy Approach* and *Stephen-Hutchinson Approach.* In what follows, just to show the procedure, the shear correction factors are evaluated for the semicircular cross-section shown in Figure 1.

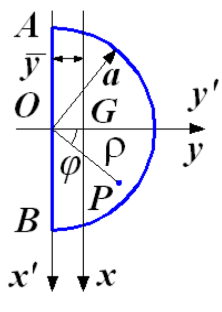

Figure 1: Cross section considered for the evaluation of shear correction factors.

The solution of the *"flexure problem"* derived in the way shown by I.S. Sokolnikoff, [13], may be written as follows in terms of shear stresses:

$$\tau_{zx} = A\rho^2 + 2A\rho^2 \cos^2 \varphi + 4B\rho \cos\varphi + \frac{1}{2}\frac{T_x}{I_y}\left(a^2 - \rho^2\right) +$$

$$+ \sum_{n=0}^{\infty} A_{2n+1}(2n+1)\rho^{2n}\left[\cos\varphi \cdot \cos(2n+1)\varphi + \sin\varphi \cdot \sin(2n+1)\varphi\right] \tag{8}$$

$$\tau_{zy} = -2A\rho^2 \sin\varphi \cdot \cos\varphi + \sum_{n=0}^{\infty} A_{2n+1}(2n+1)\rho^{2n}\left[\cos\varphi \cdot \sin(2n+1)\varphi - \sin\varphi \cdot \cos(2n+1)\varphi\right] \tag{9}$$

3.1 *The Energy Approach*

By substitution of the shear stresses given by equations (8) and (9) in expression (2) for the shear correction factor according to the energy approach, the following analytical result is found:

$$k_x = \frac{(1+\nu)^2}{\frac{7}{6} + \frac{7}{3}\cdot\nu + \left[\frac{4}{3} + \frac{64}{15\cdot\pi^2} - \frac{4096}{81\cdot\pi^4} - \frac{8192}{9\cdot\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n+1)\cdot(2n-1)^2(2n+3)^2}\right]\cdot\nu^2}, \tag{10}$$

3.2 *Stephen-Hutchinson's Approach*

When expressions (8) and (9) for the shear stresses are used in equation (7) for the evaluation of the constant C and then this is used in equation (6) , the following expression for the shear correction factor according to the Stephen-Hutchinson approach is found:

$$k_x = \frac{(1+\nu)^2}{\frac{7}{6} + \left(2 + \frac{64}{45\cdot\pi^2}\right)\cdot\nu + \left[\frac{2}{3} + \frac{64}{15\cdot\pi^2} - \frac{16^3}{9\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2(2n+3)^2}\right]\cdot\nu^2} \tag{11}$$

4 NUMERICAL RESULTS

The evaluation of the shear correction factor for the problem shown in Figure 1 according to Cowper's approach requires the evaluation of the flexure function $\chi(x,y)$ related to the function $\Phi(x,y)$ introduced in reference [8] by the following equality:

$$\chi(x,y) = -2(1+\nu)\frac{I_x}{T_y}\Phi(x,y) + (2+\nu)\cdot\left(\frac{1}{6}y^3 - \frac{1}{2}x^2 y\right) \tag{12}$$

The function $\Phi(x,y)$ is evaluated numerically as shown in reference [8] by application of the Boundary Elements Method while equation (12) provides the needed flexure function. By introducing the results in expression (3) for the shear correction factor according to Cowper's approach, a numerical integration scheme provides a set of numerical values depending on Poisson's ratio ν. Polynomial interpolation by least squares error minimization provided the following analytical expression for the shear correction factor according to Cowper's approach.

$$k_x = \frac{(1+\nu)^2}{1.1672 + 2.2392 \cdot \nu + 1.0719 \cdot \nu^2} \tag{13}$$

The same method has been used for the evaluation of the shear correction factors according to each of the above approaches considered for the cross-section shown in Figure 1 when the shear force is parallel to the *y* axis. The following interpolated expressions were found:

Energy Approach
$$k_y = \frac{1 + 2.465 \cdot \nu + 2.534 \cdot \nu^2}{1.321 + 3.162 \cdot \nu + 5.127 \cdot \nu^2} \tag{14}$$

Cowper's Approach
$$k_y = \frac{1 + 0.857 \cdot \nu - 0.141 \cdot \nu^2}{1.318 + 1.081 \cdot \nu - 0.179 \cdot \nu^2} \tag{15}$$

Stephen-Hutchinson's Approach
$$k_y = \frac{1 + 152.643 \cdot \nu - 96.580 \cdot \nu^2}{1.319 + 201.240 \cdot \nu - 150.062 \cdot \nu^2} \tag{16}$$

5 CONVERGENCE STUDIES

It is obvious that the accuracy of the evaluated shear correction factors according to any of the approaches considered depends on the accuracy of the ingredients required for its evaluation, that is the shear stresses τ_{zx} and τ_{zy} for the *Energy* and *Stephen-Hutchinson's* approaches and the f*lexure function* $\chi(x, y)$ in the case of *Cowper's* approach. When these quantities are evaluated numerically, as in the cases considered in section 3, a convergence study is necessary for an assessment of the accuracy achieved. The convergence study has been conducted in energy terms with reference to the dimensionless shear elastic energy per unit length of beam defined as follows:

$$E = \frac{A}{T_y^2} \int_A \left(\tau_{zx}^2 + \tau_{zy}^2 \right) dA \tag{17}$$

With reference to the second problem considered in section 4, the boundary element analysis has been conducted using isoparametric linear elements. The length of the elements was constant in the linear and in the circular parts of the boundary, possibly slightly different from one part to the other. The minimum number of boundary elements considered over a radius and over a quarter of circumference was 10 and this number was doubled iteratively so that in each analysis this number was 10, 20, 40 and 80.

5.1 *Exact energy estimate*

The energy defined above was calculated for each analysis and an estimate of the exact value was obtained by using the formula [18]:

$$E^{exact} = \frac{E_{(40)}^2 - E_{(80)}E_{(20)}}{2E_{(40)} - E_{(80)} - E_{(20)}} \tag{18}$$

The change in energy as the mesh is refined is shown in Figure 2 where it can be seen how the calculated energy approaches the estimated exact value as the number of elements increases. In terms of error relative to the estimate, the results shown in Figure 3 are obtained. It may be seen that when the mesh is sufficiently refined the accuracy that is obtained is independent of Poisson's ratio.

5.2 *Convergence properties of the shear correction factors*

The convergence properties of the BEM solution are reflected also on the shear correction factors. With reference to the shear correction factor evaluated according to the energy approach, the variation of the results as a function of the mesh refinement is shown in Figure 4.

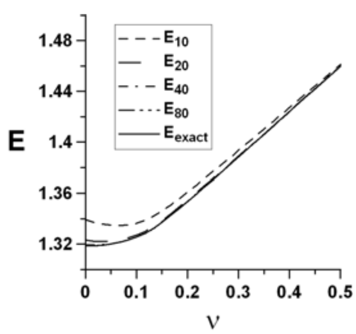

Figure 2:Energy as a function of mesh size.

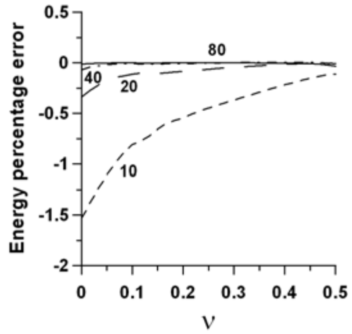

Figure 3: Energy relative error.

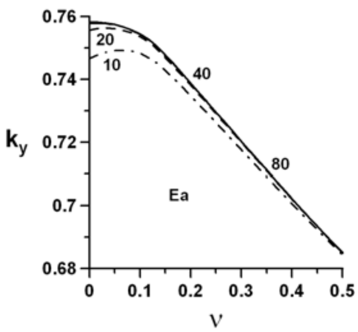

Figure 4:Energy approach

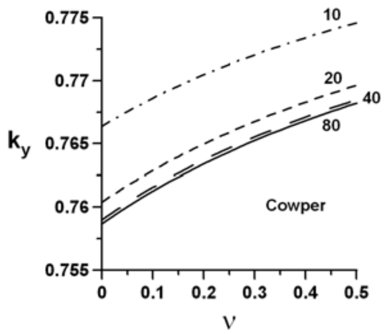

Figure 5: Cowper's approach.

It may be seen that there is little difference in the results when using 40 or 80 elements on a radius and on a quarter of circle. Also, the mesh refinement appears to be more significant for small values of Poisson's ratio while for values close to 0.5 even a coarse mesh produces good results. In the case of the shear correction factor evaluated according to Cowper's approach, the situation is quite different. As it may be seen from the results shown in Figure 5 the rate of convergence appears to be independent of Poisson's ratio and a rather fine mesh is required to obtain accurate results. The different behaviour in the two cases is due to the fact that the *energy approach* relies on the shear stresses for the definition of the shear correction factor while *Cowper's approach* makes use of the *flexure function* $\chi(x, y)$. The behaviour described is confirmed by the results shown in Figure 6 relative to the shear correction factor evaluated according to *Stephen-Hutchinson's* approach.

5.3 *Comparison of results from three different approaches*

An overall view of the behavior of the shear correction factor evaluated according to the three approaches considered is shown in Figure 7. The three approaches provide the same value of the shear correction factor for zero Poisson's ratio and increasingly diverging values as the Poisson's ratio increases. While the shear correction factor remains nearly independent of ν according to *Cowper's approach*, it increases as ν increases according to *Stephen-Hutchinson's approach* and decreases according to the *energy approach*. The maximum difference between any couple of shear correction factors is calculated in percentage terms by dividing the maximum difference by the average value of the couple of values. The results are shown in Table 1 where it can be seen that the difference between *Cowper's* and the *Energy approach* is of the order of 11%, that between Cowper's and Stephen-Hutchinson's approaches is of the order of 7% while the difference between the Energy approach and Stephen's -Hutchinson's approach is of the order of 19%.

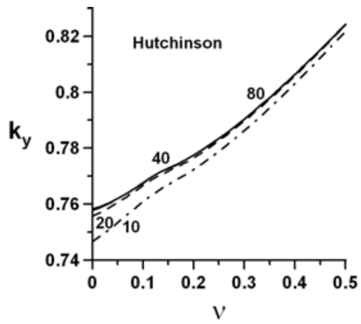

Figure 6: Stephen-Hutchinson's approach.

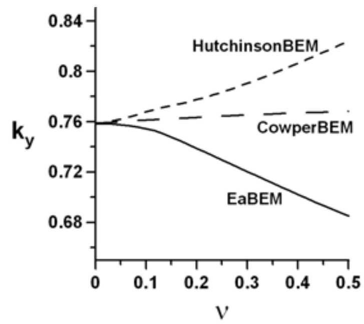

Figure 7: Comparison of shear correction factors.

The trend shown in Figure 7 and expressed quantitatively in terms of maximum differences in Table 1 is confirmed by the behavior observed in many other cross-sections, as is shown in reference [8].

Table 1: Maximum differences between couples of shear correction factors /average value

Approaches	Energy	Cowper's	Stephen-Hutchinson's
Energy	0.00	0.11	0.19
Cowper's	0.11	0.00	0.07
Stephen-Hutchinson's	0.19	0.07	0.00

6 CONCLUSIONS

In this paper three of the most popular approaches for the evaluation of the shear correction factor to be used in the Timoshenko beam theory have been considered. The idea behind each approach is quite clear and the results are quite different, apart for values of Poisson's ratio close to zero whereby the three approaches converge to the same value. This should provide a clue of where the differences come from in order to eventually find a way to improve the coefficient even more. The energy approach is essentially a static one and cannot be expected to produce good results for dynamic problems. Cowper's approach is of a dynamic nature and one should expect to obtain better results from it when applied to a dynamics problem. However, as it was clearly evidenced in the seminal paper by Cowper [4], some terms were neglected in deriving the Timoshenko beam theory from the three-dimensional theory of dynamic elasticity, namely the direct stresses σ_x and σ_y. Stephen's contribution [2] was an attempt to improve Cowper's results by somewhat accounting of those missing terms. His strategy, although stemming from a static approach, succeeded in improving Cowper's formula by keeping the same fractional structure, the same numerator, adding a third term in the denominator and doubling the second one. Two decades later J.R. Hutchinson, by using a dynamical variations approach, arrived at a different formula which N.G. Stephen immediately proved to be equivalent to his own, [12]. It is interesting to realize that Hutchinson did not include the direct stresses σ_x and σ_y in his treatment, but his kinematical assumptions were much wider than the usual ones in elementary beam theory, considering terms that account for the deformation of the cross-section within its plane. The overall result was to find an expression for the shear correction factor equivalent to the one derived by Stephen. This analysis leads to the conclusion that for dynamic analysis the most appropriate value for the shear correction factor should be the one provided by the Stephen-Hutchinson formula. Comparisons with a few analytical results available in the literature shown by J.R. Hutchinson [3] and by the present authors [12] appear to confirm this insight, at least with reference to the evaluation of the low natural frequencies of elastic beams.

The shear correction factors produced in the literature are generally derived by introducing exact or approximate expressions for the shear stresses or for some other related quantities into some general expressions as those shown in the present paper for the three considered approaches. It has been shown that the required ingredients for the evaluation of the shear correction factors can also be derived by numerically solving some Neumann problem over the cross-section, as was done in detail in reference [8]. New shear correction factors have been derived analytically according to the energy approach and the Stephen-Hutchinson formula for the semicircular cross-section with shear force acting perpendicularly to the axis of symmetry. The corresponding coefficient according to Cowper's approach has been derived numerically, although an analytical formula has been provided by fractional polynomial curve fitting using the least squares method. Similar formulae have been derived, according to the three approaches considered, when the shear force acts along the symmetry axis.

However, numerical methods must be used carefully and the accuracy of the solution obtained must be checked. In the present work the convergence of the numerical procedure was checked in energy terms and on each of the evaluated shear correction factors.

Finally, the results obtained according to the three approaches considered where compared against each other in a qualitative as well as in a quantitative way. The difference appeared to be an increasing function of Poisson's coefficient and significant in engineering terms for large values of such coefficient. Therefore, the appropriate choice of the shear correction factor may be an important issue.

While the Stephen-Hutchinson formula for the evaluation of the shear correction factor may appear as the most appropriate so far for dynamic analyses, there is not enough evidence that it might be as good when applied to static problems. The energy approach in such cases appears to be just as sound as the Stephen-Hutchinson one and so far it is not clear which one would be preferable. Besides, an investigation performed by the present authors, [12], seems to be pointing towards another set of shear correction factors.

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