Multi-parameter perturbation methods for the eigensolution sensitivity in discrete systems exhibiting multiple frequency veering

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SUMMARY. The linear dynamics of many structural systems is characterized by multiple internal resonances. Such systems may exhibit a high sensitivity of the eigenproperties with respect to a set of significant mechanical parameters. This condition is recognized as the source of relevant phenomena, as frequency veering and mode localization or hybridization. The leading idea of the present work consists in systematically treating nearly-resonant Hamiltonian systems as perturbations of a reference, unknown a priori, resonant system. Given a nearly-resonant experimental system, a multi-parameter perturbation method is presented in order to, first, identify in the parameter space a close resonant system (*inverse problem*), and, second, use the identified resonant system as suited initial point to approximate the eigensolution of the systems originated by a generic multi-parameter perturbation (*direct problem*). The conditions of existence and uniqueness of the inverse problem solution, as well as the subsequent validity of the perturbation-based sensitivity analysis are discussed.

1 INTRODUCTION

The analytical models of several physical systems show that eigenvalues and eigenvectors strongly depend on a set of mechanical parameters. This dependence can be effectively studied through the analysis of the eigenvalue loci curves versus one or more significant parameters. Two curves approaching to each other in a certain region of the parameter space may exhibit different typical behaviours. The intersection of the curves for a critical parameter value, corresponding to a double frequency, is known as a *crossing* condition. Otherwise the sudden divergence of the curves causes an avoided crossing, also known as a *veering* phenomenon. The eigenvectors associated to a couple of veering eigenvalues completely interchange their shapes in a rapid but continuous way, giving birth to a pair of parameter-evolving *hybrid* forms. Hence veering regions are qualitatively characterized by high sensitivities of the eigensolution, that is large variation of the eigenvalues and eigenvectors resulting from small changes in the control parameters. Research attention towards veering phenomena is called by different motivations; among them, the presence of internally-resonant modes, potentially leading to strong energy exchange (with technical application in the field of passive control [1]), and the strict correlation with phenomena of vibration localization [2],[3].

The frequency veering has been often identified in assemblies of weakly-coupled equal subsystems (pendulum chains, multi-span beams, bladed disks), as a consequence of imperfections disturbing the nominal structural periodicity. Both the weak coupling (*mistuning*) and the imperfections (*disorder*) can be regarded as small perturbations of an ideal perfect (*tuned* and *ordered*) system, with known eigensolutions. Therefore, the eigensolutions of close systems, with small mistuning and disorder, can be approximated constructing asymptotical expansions based on perturbation methods. The traditional approach, commonly used for conservative systems [2],[4], consists in including the disorder into the unperturbed reference system, and then performing a single-parameter analysis considering only the coupling as perturbation. Nonetheless, such perturbation scheme presents some drawbacks, relying essentially in the small-range validity of the achievable solutions.

A refined approach consists instead in retaining both the disorder and the coupling as independent perturbations of the tuned and ordered system, and then performing a multi-parameter analysis [5]. Among other features, this refinement allows the extention of the solution validity over a wider parameter range, together with the treatment of non-conservative discrete systems governed by nearly defective matrices, which are the object of its original development.

The main line of the present study consists in applying the second approach to conservative discrete system exhibiting multiple frequency veering, such as the paradigmatic mechanical model presented in Figure 1a. The model is made of a principal sub-system, the rectangular body SP, and two secondary sub-systems, the point bodies SS_1 and SS_2. The system could be intended as a minimal sectional model of a cable-stayed or a suspended bridge, and was originally formulated to synthetically reproduce the strong dynamical interactions which may couple the vertical/torsional motion of the deck, represented by the principal system, and the transversal motion of many resonant stay cables, represented by the secondary systems [6]. In certain parameter regions, the system frequencies are found to exhibit important veering phenomena involving two, or even three modes (Figure 1b), with different amount of localization (Figure 1c).

Aiming to analyze the eigensolution sensitivity in the critical regions of the parameter space through the multi-parameter perturbation scheme, the *real* (or *experimental*) system is supposed to be originated by the perturbation of an *ideal*, though unknown, non-defective system with double or triple frequency. Differently speaking, the existence of a crossing close to the veering region is postulated. According to this perturbation strategy, the paper tackles first the *direct problem*, consisting in the description of the experimental system eigensolution as a generic multi-parameter perturbation of the ideal system eigenvalues and eigenvectors. Then, focus is made on the solution of an *inverse problem*, consisting in the identification of the unknown ideal system to be perturbed. As dealing with an inverse problem, questions regarding the existence and uniqueness of the solution are discussed, and the need of imposing compatibility conditions on the data, namely the eigenvalues and the eigenvectors of the experimental system, is evidenced. Finally, the complete procedure is effectively tested on the relevant case of a three dofs systems with two resonant, interacting modes and a third mode with passive role.

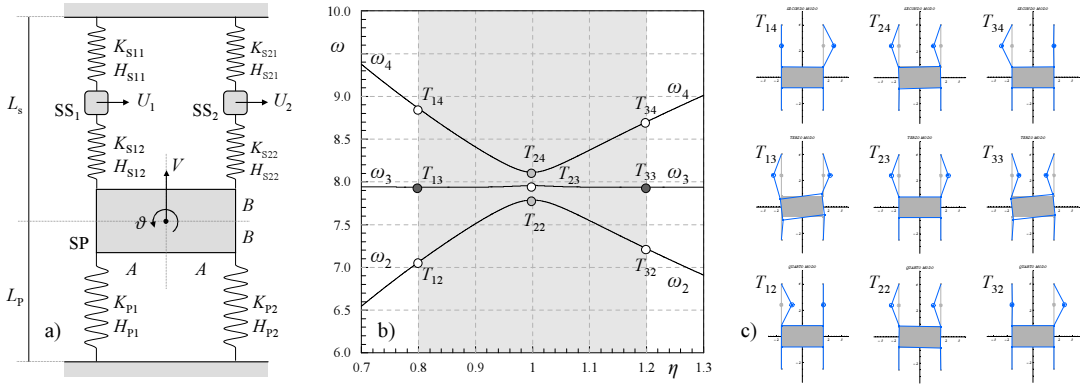

Figure 1: Paradigmatic mechanical system with multiple frequency veering: (a) sketch, (b) veering phenomenon involving three frequency loci, (c) localization and hybridization on the modes.

2 PERTURBATION ANALYSIS FOR THE EIGENSOLUTION SENSITIVITY

The mechanical model presented in the previous section can be considered a paradigmatic example of a generic class of N-dimensional linear Hamiltonian systems. These systems are governed only by the mass matrix $\mathbf{M} = [m_{ij}]$ and the stiffness matrix $\mathbf{K} = [k_{ij}]$ (where $i, j = 1, \ldots, N$), and the associated eigensolution consists of real (and generally distinct) eigenvalues, listed in ascending order in the diagonal matrix $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, and a complete set of eigenvectors, listed columnwise in the modal matrix $\mathbf{\Phi} = \{\boldsymbol{\phi}_1 | \ldots | \boldsymbol{\phi}_N\}$. The eigensolution depends on the vector $\mathbf{p}$, collecting all the M independent parameters $p_1, \ldots, p_M$ defining the mass and stiffness coefficients.

Multi-parameter perturbation (MPP) methods represent a valid alternative to the computationally-consuming numerical techniques of solution continuation (SC), applied in the previous section to characterize the parameter-dependence of the eigensolution. The algorithm requires to fix an initial point $\mathbf{p}_0$ in the M-dimensional parameter space Π, corresponding to a reference system $\mathcal{S}_0$, with unperturbed governing matrices $\mathbf{M}_0(\mathbf{p}_0)$, $\mathbf{K}_0(\mathbf{p}_0)$, and known eigensolution $\mathbf{\Lambda}_0(\mathbf{p}_0)$, $\mathbf{\Phi}_0(\mathbf{p}_0)$. A close point $\mathbf{p}$, corresponding to a new system $\mathcal{S}$, can be treated as a perturbation of the initial point, that is $\mathbf{p} = \mathbf{p}_0 + \varepsilon \mathbf{p}_1$. It is worth noting here that $\mathbf{p}_1$ is a generic multi-parameter modification of the initial point $\mathbf{p}_0$, whose (small) amplitude is regulated by the perturbation parameter $\varepsilon \ll 1$. The MPP method allows the construction of asymptotic expansions $\mathbf{\Lambda}_0(\mathbf{p}_0) + \varepsilon \mathbf{\Lambda}_1(\mathbf{p}_0, \mathbf{p}_1) + \mathcal{O}(\varepsilon^2)$ and $\mathbf{\Phi}_0(\mathbf{p}_0) + \varepsilon \mathbf{\Phi}_1(\mathbf{p}_0, \mathbf{p}_1) + \mathcal{O}(\varepsilon^2)$, well-approximating the exact eigensolution $\mathbf{\Lambda}(\mathbf{p})$, $\mathbf{\Phi}(\mathbf{p})$. The approximation is uniformly valid over a certain region around the initial point. The coefficients of the asymptotic expansion are called *eigensolution sensitivities*.

On the light of these considerations, the frequency veering and modal hybridization can be recognized as particular phenomena characterized by rapid variations of the eigenvalues and the eigenvectors over a small range of the system parameters, that is high eigensolution sensitivities in certain critical regions of the parameter space. The leading idea here consists in systematically treating the nearly-resonant systems $\mathcal{S}^*$, which are observed to exhibit these phenomena (therefore called *experimental*, or *real* systems) as the perturbation of a resonant, though unknown, non-defective system $\bar{\mathcal{S}}_0$, with repeated frequencies (therefore called *ideal* system). The task is twofold; given a resonant system $\mathcal{S}_0$, (i) the *direct problem* consists in determining the approximated eigensolution of

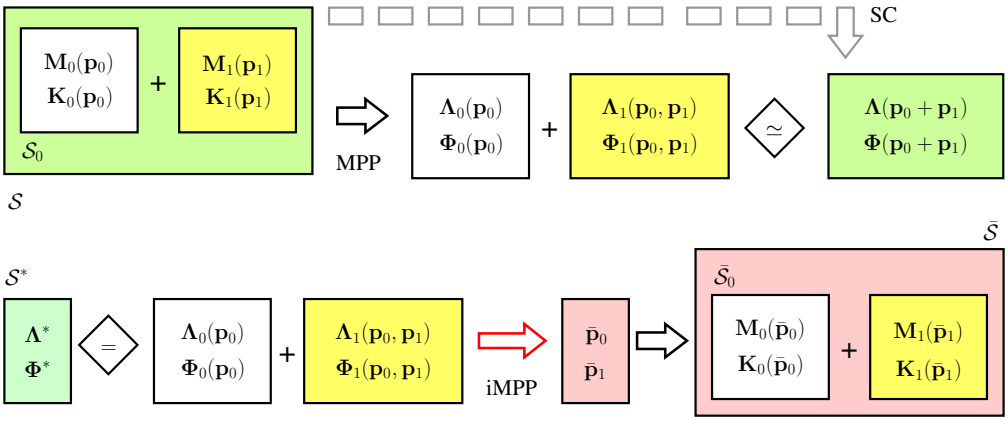

Figure 2: Multi-parameter perturbation scheme (first order approximation): (a) direct problem for the approximation of the real eigensolution, (b) inverse problem for the identification of the reference system.

a nearly-resonant system $\mathcal{S}$, by employing a generic multi-parameter perturbation of the $\mathcal{S}_0$ eigensolution (see Figure 2a); otherwise, given an experimental system $\mathcal{S}^*$, (ii) the *inverse problem* consists in identifying the unknown unperturbed resonant system $\bar{\mathcal{S}}_0$, by forcing the eigensolution of the perturbed system $\bar{\mathcal{S}}$ to match that of $\mathcal{S}^*$ (see Figure 2b). It is worth noting that the multi-parameter perturbation originating the nearly-resonant system is a priori unknown, and has to be identified too.

2.1 *Direct problem*

Since the starting point of the perturbation expansion can be built artfully, a set of independent linear oscillators can be suitably chosen. Therefore, the reference system $\mathcal{S}_0$ is governed by a diagonal stiffness matrix $\mathbf{K}_0 = \mathrm{diag}(k_{01}, ..., k_{0\mathrm{N}})$, and a unitary mass matrix $\mathbf{M}_0 = \mathbf{I}$ is assumed for sake of simplicity. Systems with non-unitary mass matrix do not require an independent problem statement, so that the effect of mass differences between the oscillators can be treated a posteriori.

The nearly-resonant close system $\mathcal{S}$ is generated by a small symmetric perturbation of the only stiffness matrix $\mathbf{K}_1 = [\, k_{1,\mathrm{ij}}]$. In view of the problem inversion, it is a priori convenient to include all the stiffness coefficients in the vector $\mathbf{p}$ of the independent perturbation parameters. The mass matrix is supposed instead to remain unperturbed, since any perturbation of the mass matrix could be transformed in an equivalent perturbation of the stiffness matrix. The $\mathbf{p}$-dependent eigensolution of $\mathcal{S}$, expanding and truncating at the first-order approximation, can be expressed as

$$\mathbf{\Lambda}(\mathbf{p}_0, \mathbf{p}_1) = \mathbf{\Lambda}_0(\mathbf{p}_0) + \varepsilon \mathbf{\Lambda}_1(\mathbf{p}_0, \mathbf{p}_1), \qquad \mathbf{\Phi}(\mathbf{p}_0, \mathbf{p}_1) = \mathbf{\Phi}_0(\mathbf{p}_0) + \varepsilon \mathbf{\Phi}_1(\mathbf{p}_0, \mathbf{p}_1) \tag{1}$$

Applying a classical perturbation scheme to the eigenproblem of the system $\mathcal{S}$, that is expanding the modal equation and collecting the terms of the same ε-power, the following equations are obtained

$$(\mathbf{K}_0 - \lambda_0 \mathbf{I})\boldsymbol{\phi}_0 = \mathbf{0} \tag{2}$$

$$(\mathbf{K}_0 - \lambda_0 \mathbf{I})\boldsymbol{\phi}_1 = -(\mathbf{K}_1 - \lambda_1 \mathbf{I})\boldsymbol{\phi}_0 \tag{3}$$

where the zeroth-order Eq.(2) is the modal equation of the unperturbed system $\mathcal{S}_0$, and furnishes the trivial solution $\mathbf{\Lambda}_0 = \mathbf{K}_0$. It simply states that the requested resonance conditions in the unperturbed system $\mathcal{S}_0$ can be forced a priori imposing internal constraints among the diagonal terms of its stiffness matrix $\mathbf{K}_0$. Moreover, it suggests a different physical interpretation of the perturbation terms $k_{1,\mathrm{ij}}$, since the diagonal coefficients ($\mathrm{i} = \mathrm{j}$) can be recognized as *mistuning* term, whereas the non-diagonal coefficients ($\mathrm{i} \neq \mathrm{j}$) can be recognized as *coupling* terms.

The solution of the first-order Eq.(3) gives instead the first-order correction of the eigenvalues $\mathbf{\Lambda}_1 = \mathrm{diag}(\lambda_{11}, ..., \lambda_{1\mathrm{N}})$ and eigenvectors $\mathbf{\Phi}_1 = \{\boldsymbol{\phi}_1 | ... | \boldsymbol{\phi}_\mathrm{N}\}$, as a function of the perturbation parameters $\mathbf{p}$. Both $\mathbf{\Lambda}_1$ and $\mathbf{\Phi}_1$ are expected to depend non-linearly on the unperturbed $\mathbf{p}_0$ and, due to the coalescence of the unperturbed eigenvalues, also on the parameter perturbation $\mathbf{p}_1$.

The eigensolution of the nearly-resonant system $\mathcal{S}$ could be certainly better approximated if the expansion (1) is extended to higher orders. Moreover, it is expected that a better approximation in the direct problem could ensure a greater accuracy of the inverse problem solution. On the other hand, a higher-order solution of the direct problem may reduce the advantageous possibility to achieve a closed-form solution of the inverse problem, which is generally nonlinear. In practice, the first-order approximation is usually sufficient to balance these two competing requirements, and iterative procedures can be suitably employed to rapidly reduce the approximation error.

To understand the behaviour shown by mechanical systems, like that presented in Figure 1 as a paradigmatic case-study, a 3-dofs nearly-resonant system is investigated in the following. Focus is made on the particular case of two close frequencies, corresponding to a couple of *active* modes interacting to each other, and a third *passive* mode.

Resonant system with a double eigenvalue

The direct problem states as follows: given the reference system $\mathcal{S}_0$, made of three uncoupled linear oscillators, two of them being perfectly resonant, determine the first-order approximated eigensolution of the close nearly-resonant system $\mathcal{S}$, obtained from a generic perturbation.

According to the statement, the solution of the zeroth-order Eq.(2) is requested to have a double eigenvalue ($\lambda_{01}=\lambda_{02}\neq\lambda_{03}$), corresponding to a complete set of eigenvectors

$$\boldsymbol{\phi}_{0,1}=\phi_{0,11}\mathbf{i}+\phi_{0,12}\mathbf{j},\quad \boldsymbol{\phi}_{0,2}=\phi_{0,21}\mathbf{i}+\phi_{0,22}\mathbf{j},\quad \boldsymbol{\phi}_{0,3}=\phi_{0,33}\mathbf{k} \tag{4}$$

where the unit vectors $\mathbf{i}=\{1,0,0\}^\top$, $\mathbf{j}=\{0,1,0\}^\top$, $\mathbf{k}=\{0,0,1\}^\top$ have been introduced.

A generic perturbation is introduced including six independent coefficient of the stiffness matrix $k_{1,\mathrm{ij}}$ in the parameter vector $\mathbf{p}$. Substituting the zeroth-order solution into the first-order Eq.(3), the left-hand operator is found to be singular. Therefore, solvability conditions have to be imposed, separately for the double ($\lambda_{01}=\lambda_{02}$) and the single eigenvalue (λ_{03}). Once the first-order correction of the eigenvalues $\boldsymbol{\Lambda}_1$ are determined in order to satisfy the solvability conditions, the equation can be solved in the first-order correction of the eigenvectors $\boldsymbol{\Phi}_1$. After reconstruction, re-absorbing the perturbation parameter ε, the first-order approximation of the eigenvalues reads

$$\lambda_{1,2}=k_{0,11}+\tfrac{1}{2}(k_{1,11}+k_{1,22}\mp\Delta_{12}),\qquad \lambda_3=k_{0,33}+k_{1,33} \tag{5}$$

whereas the corresponding first-order approximation of the eigenvectors read

$$\begin{aligned}\boldsymbol{\phi}_{1,2}&=\left\{\frac{k_{1,11}-k_{1,22}\mp\Delta_{12}}{2k_{1,12}},\,1\,,-\frac{[\,2k_{1,23}k_{1,12}+k_{1,13}(k_{1,11}-k_{1,22}\mp\Delta_{12})]}{2k_{1,12}(k_{0,33}-k_{0,11})}\right\}^\top\\ \boldsymbol{\phi}_3&=\left\{\frac{k_{1,13}}{k_{0,33}-k_{0,11}},\,\frac{k_{1,23}}{k_{0,33}-k_{0,11}},1\right\}^\top\end{aligned} \tag{6}$$

where the auxiliary parameter $\Delta_{12}=\sqrt{(k_{1,11}-k_{1,22})^2+4k_{1,12}^2}$ has been introduced.

A few advances can be pointed out from the direct problem solution towards understanding the observed behaviour of mechanical systems:

i. The veering phenomenon, due to the splitting of the double eigenvalue, is originated by both the *mistuning* and the *coupling* between the resonant oscillators. They equally affect the *veering amplitude*, that is the frequency difference $\Delta\lambda=\lambda_2-\lambda_1$, which is exactly equal to Δ_{12}.

ii. The modal hybridization phenomenon, due to the interaction between the eigenvectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$, depends only on the *coupling* between the resonant oscillators. In fact, if the coupling term vanishes ($k_{1,12}=0$), two possibilities arise: if a non-null *mistuning* term exists ($k_{1,11}\neq k_{1,22}$), the resonance is lost ($\lambda_1\neq\lambda_2$), but the two oscillators remain uncoupled; otherwise ($k_{1,11}=k_{1,22}$), the initial condition of two resonant and uncoupled oscillators is recovered, except for the shift of the double eigenvalue.

iii. The frequency veering phenomenon, due to the interaction between the eigenvalues of the two resonant oscillators, does not depend on the *coupling* with the third non-resonant oscillator. If a non-null coupling exists ($k_{1,12}\neq 0$, or $k_{1,13}\neq 0$), a small effect interests just the third component of the resonant eigenvectors. This effect apparently grows as the third eigenvalue approaches the resonant ones ($k_{0,33}\simeq k_{0,11}$). Nonetheless, in this case the initial hypotheses, on which the perturbation scheme is based, are essentially violated, and the achieved solution is expected to rapidly lose validity.

2.2 *Inverse problem*

Assume to experimentally, or pseudo-experimentally, know the eigenvalues $\mathbf{\Lambda}^* = \mathrm{diag}\,(\lambda_1^*, ..., \lambda_{\mathrm{N}}^*)$ and eigenvectors $\mathbf{\Phi}^* = \{\boldsymbol{\phi}_1^* | ... | \boldsymbol{\phi}_{\mathrm{N}}^*\}$ of a nearly-resonant system $\mathcal{S}^*$. According to the leading idea of the present work, the existence of a close, though unknown, resonant system $\mathcal{S}_0$ is postulated, and the experimental system $\mathcal{S}^*$ is assumed to born from a certain unknown perturbation $\mathbf{p}_1$ of its parameters $\mathbf{p}_0$. Once determined, $\mathcal{S}_0$ can be assumed as initial point for a generic multi-parameter perturbation analysis, which is expected to well-approximate the eigensolution of each perturbed system $\mathcal{S}$ close to $\mathcal{S}_0$, including the experimental system $\mathcal{S}^*$ as particular case.

Determining the unknown resonant system $\mathcal{S}_0$, through the assessment of the unperturbed parameters $\mathbf{p}_0$, as well as determining the same unknown parameter perturbation $\mathbf{p}_1$, represents an *identification* problem, which is governed by the nonlinear equations

$$\mathbf{\Lambda}_0(\mathbf{p}_0) + \mathbf{\Lambda}_1(\mathbf{p}_0, \mathbf{p}_1) = \mathbf{\Lambda}^*, \qquad \mathbf{\Phi}_0(\mathbf{p}_0) + \mathbf{\Phi}_1(\mathbf{p}_0, \mathbf{p}_1) = \mathbf{\Phi}^* \tag{7}$$

in which the unknowns $\mathbf{p}_0$ and $\mathbf{p}_1$ are requested to satisfy the coincidence between the approximated and the experimental eigensolution, as far as the approximation given by the maximum order of the multi-parameter expansion can be considered acceptable.

The closed-form solution of the equations (7) expresses the unperturbed parameters and their perturbation as a function of the experimental eigenvalues and eigenvectors

$$\bar{\mathbf{p}}_0(\mathbf{\Lambda}^*, \mathbf{\Phi}^*), \qquad \bar{\mathbf{p}}_1(\mathbf{\Lambda}^*, \mathbf{\Phi}^*) \tag{8}$$

where the bar denotes *identified* values. Therefore, the stiffness matrix $\bar{\mathbf{K}}_0$ (known $\bar{\mathbf{p}}_0$) of the resonant system $\mathcal{S}_0$, as well as its perturbation $\bar{\mathbf{K}}_1$ (known $\bar{\mathbf{p}}_0$, and $\bar{\mathbf{p}}_1$) result completely assessed.

It is worth noting here that the inverse problem may suffer for non-existence and/or non-uniqueness of the solution. If the existence of the solution is subordinated to solvability conditions to be satisfied by the data, it means that not all the experimental nearly-resonant systems may be obtained as perturbation of the resonant ideal system, at least in the class in which it is defined. Differently, if the solution is non-unique, for instance when part of the parameters remain indeterminate, it means that the same experimental nearly-resonant system may be obtained indifferently perturbing many resonant ideal systems. The two conditions may also co-exist.

Resonant system with a double eigenvalue

The inverse problem states as follows: given the experimental system $\mathcal{S}^*$ with two nearly-coincident eigenvalues, identify (if it exists), a close system $\mathcal{S}_0$, made of three uncoupled linear oscillators, two of them being perfectly resonant, whose first-order approximated eigensolution, due to an unknown multi-parameter perturbation, exactly matches the experimental one.

Imposing the coincidence between the approximated eigensolution, as obtained from Eq.(5), and the experimental eigensolution $\mathbf{\Lambda}^* = \mathrm{diag}\,(\lambda_1^*, \lambda_2^*, \lambda_3^*)$ and $\mathbf{\Phi}^* = \{\boldsymbol{\phi}_1^* | \boldsymbol{\phi}_2^* | \boldsymbol{\phi}_3^*\}$, the set of nonlinear equations governing the identification problem can be expressed in the form

$$\lambda_1(k_{0,11}, k_{1,11}, k_{1,12}, k_{1,22}) = \lambda_1^*, \quad \boldsymbol{\phi}_1(k_{0,11}, k_{0,33}, k_{1,11}, k_{1,12}, k_{1,22}, k_{1,13}, k_{1,23}) = \boldsymbol{\phi}_1^* \tag{9}$$

$$\lambda_2(k_{0,22}, k_{1,11}, k_{1,12}, k_{1,22}) = \lambda_2^*, \quad \boldsymbol{\phi}_2(k_{0,22}, k_{0,33}, k_{1,11}, k_{1,12}, k_{1,22}, k_{1,13}, k_{1,23}) = \boldsymbol{\phi}_2^* \tag{10}$$

$$\lambda_3(k_{0,33}, k_{1,33}) = \lambda_3^*, \qquad \boldsymbol{\phi}_3(k_{0,11}, k_{0,33}, k_{1,13}, k_{1,23}) = \boldsymbol{\phi}_3^* \tag{11}$$

in which, once the resonance condition ($k_{0,11} = k_{0,22}$) is forced, the 8 remaining independent unknowns (2 coefficients of the diagonal matrix $\mathbf{K}_0$, and 6 coefficients of the symmetric matrix $\mathbf{K}_1$) have to be determined from 9 independent experimental data (3 eigenvalues and 6 independent eigenvector coefficients). The solution has to be sought by proper handling different equation subsets.

The Eqs. (9)a, (10)a, (11)a, concerning the eigenvalues, are tackled first. For convenience, the nearly coincident eigenvalues λ_1^* and λ_2^* are replaced, as equivalent experimental data, by their half-sum $\lambda_{\rm hs}^* = \frac{1}{2}(\lambda_1^* + \lambda_2^*)$ and half-difference $\lambda_{\rm hd}^* = \frac{1}{2}(\lambda_2^* - \lambda_1^*)$. After simple algebra, these equations give a partial solution for the independent coefficients of the diagonal matrix $\mathbf{K}_0$

$$k_{0,11}+k_{0,22}=2\lambda_{\rm hs}^*-(k_{1,11}+k_{1,22}), \quad k_{0,22}-k_{0,11}=2\lambda_{\rm hd}^*+\Delta_{12}, \quad k_{0,33}=\lambda_3^*-k_{1,33} \tag{12}$$

Recalling that the initial system $\mathcal{S}_0$ possesses a double eigenvalue ($k_{0,11}=k_{0,22}$), Eq.(12a) gives

$$k_{0,11} = k_{0,22} = \lambda_{\rm hs}^* - \tfrac{1}{2}(k_{1,11} + k_{1,22}) \tag{13}$$

whereas Eq.(12b) furnishes a *resonance condition* involving the only coefficients of the matrix $\mathbf{K}_1$

$$(k_{1,11}-k_{1,22})^2 + 4k_{1,12}^2 = (2\lambda_{\rm hd}^*)^2 \tag{14}$$

In the spirit of the inverse problem, this relation substantially states how the assigned veering amplitude of the experimental eigenvalues $2\lambda_{\rm hd}^*$ can be obtained perturbing the uncoupled resonant system $\mathcal{S}_0$ with a mistuning ($k_{1,11}$, or $k_{1,22}$), or a coupling term ($k_{1,12}$).

Each of the Eqs. (9)b, (10)b, (11)b, concerning the eigenvectors, includes three scalar equations. Only two of them can be used to solve the inverse problem, since the third is automatically satisfied once the experimental and the analytical eigenvectors are scaled according to the same normalization rule. Here unitary-valued second (third) component is the criterion used to normalize the first and second (third) eigenvector, as in Eqs.(6). The six remaining equations are separated, since three pairs of coupled equations can be recognized and solved independently from each other:

1. the first scalar equations from the Eqs. (9)b, (10)b give the partial solution

$$k_{1,11}-k_{1,22} = 2\lambda_{\rm hd}^* \frac{\phi_{21}^* + \phi_{11}^*}{\phi_{21}^* - \phi_{11}^*}, \qquad k_{1,12} = \frac{2\lambda_{\rm hd}^*}{\phi_{21}^* - \phi_{11}^*}, \tag{15}$$

2. the third scalar equations from the Eqs. (9)b, (10)b give the solution

$$k_{1,13} = (k_{0,33} - k_{0,11})\, \frac{\phi_{23}^* - \phi_{13}^*}{\phi_{21}^* - \phi_{11}^*}, \qquad k_{1,23} = (k_{0,33} - k_{0,11})\, \frac{\phi_{11}^*\phi_{13}^* - \phi_{21}^*\phi_{23}^*}{\phi_{21}^* - \phi_{11}^*} \tag{16}$$

3. the first and the third scalar equations from the Eqs. (11)b give the solution

$$k_{1,13} = (k_{0,33} - k_{0,11})\, \phi_{31}^*, \qquad k_{1,23} = (k_{0,33} - k_{0,11})\, \phi_{32}^* \tag{17}$$

It is worth noting, from Eqs.(12), that part of the solution (the coefficient $k_{1,33}$, together with one of the coefficients $k_{1,11}$ and $k_{1,22}$) remains indeterminate. On the other hand, substituting the solution (15) into the Eq.(12), after some manipulation, the *resonance condition* becomes

$$\phi_{11}^*\phi_{21}^* = -1 \tag{18}$$

which has to be satisfied by the experimental eigenvectors. Furthermore, the *compatibility conditions*

$$\phi_{31}^*\,(\phi_{21}^*-\phi_{11}^*) = \phi_{13}^*-\phi_{23}^*, \qquad \phi_{32}^*\,(\phi_{21}^*-\phi_{11}^*) = \phi_{11}^*\phi_{23}^*-\phi_{13}^*\phi_{21}^* \tag{19}$$

have to be imposed on the experimental eigenvectors to force the coincidence of the different solutions (16) and (17) obtained for the coefficients $k_{1,13}$ and $k_{1,23}$.

A complementary remark is that both the resonance (18) and the compatibility conditions (19) have to be satisfied by the experimental eigenvectors in an approximate manner, as far as they have been derived by the first-order approximated solution of the direct problem.

Assuming the resonance and the compatibility conditions satisfied, a simple manipulation of the Eqs.(15)-(17) leads to the general form of the inverse problem solution. Setting the indeterminate coefficient $k_{1,33}=0$ for sake of simplicity, the diagonal terms of the stiffness matrix $\bar{\mathbf{K}}_0$ read

$$\bar{k}_{0,11}=\lambda^*_{\rm hs}-k_{1,22}-\lambda^*_{\rm hd}\frac{\phi^*_{11}+\phi^*_{21}}{\phi^*_{11}-\phi^*_{21}},\quad \bar{k}_{0,22}=\lambda^*_{\rm hs}-k_{1,22}-\lambda^*_{\rm hd}\frac{\phi^*_{11}+\phi^*_{21}}{\phi^*_{11}-\phi^*_{21}},\quad \bar{k}_{0,33}=\lambda^*_3 \tag{20}$$

and the terms of the symmetric stiffness matrix perturbation $\mathbf{K}_1$ are

$$\bar{k}_{1,11}=k_{1,22}-2\lambda^*_{\rm hd}\frac{\phi^*_{11}+\phi^*_{21}}{\phi^*_{11}-\phi^*_{21}},\qquad \bar{k}_{1,13}=\left(\lambda^*_3-\lambda^*_{\rm hs}+k_{1,22}-\lambda^*_{\rm hd}\frac{\phi^*_{11}+\phi^*_{21}}{\phi^*_{11}-\phi^*_{21}}\right)\frac{\phi^*_{23}-\phi^*_{13}}{\phi^*_{11}-\phi^*_{21}} \tag{21}$$

$$\bar{k}_{1,21}=-\frac{2\lambda^*_{\rm hd}}{\phi^*_{11}-\phi^*_{21}},\qquad \bar{k}_{1,23}=\left(\lambda^*_3-\lambda^*_{\rm hs}+k_{1,22}-\lambda^*_{\rm hd}\frac{\phi^*_{11}+\phi^*_{21}}{\phi^*_{11}-\phi^*_{21}}\right)\frac{\phi^*_{13}\phi^*_{21}-\phi^*_{11}\phi^*_{23}}{\phi^*_{11}-\phi^*_{21}} \tag{22}$$

where the second indeterminate coefficient $k_{1,22}$ remains to be fixed arbitrarily. Among different possible choices, two relevant assessments can be recognized

$$k_{1,22}=\pm\lambda^*_{\rm hs}\frac{\phi^*_{11}+\phi^*_{21}}{\phi^*_{11}-\phi^*_{21}},\qquad k_{1,22}=0 \tag{23}$$

corresponding respectively to $\bar{k}_{0,11}\simeq\lambda^*_{1,2}$ (the double eigenvalue of the identified resonant system approximately equal to one or the other experimental eigenvalue of the nearly-resonant system), and $\bar{k}_{0,11}\simeq\lambda^*_{\rm hd}$ (the double eigenvalue of the identified resonant system equal to the half-sum of the experimental eigenvalues of the nearly-resonant system).

The effectiveness and robustness of the identification procedure is illustrated by a numerical example. A resonant system $\mathcal{S}^*_0$, governed by the stiffness matrix $\mathbf{K}^*_0$, and affected by the artificial perturbation $\mathbf{K}^*_1$ (originating a nearly-resonant system $\mathcal{S}^*$), is used to generate the pseudo-experimental eigenvalues $\mathbf{\Lambda}^*$ and eigenvectors $\mathbf{\Phi}^*$. The inverse problem is solved using the Eqs.(20)-(22), where the indeterminate coefficients are fixed according to Eq.(23)a (case A) or Eq.(23)b (case B). Finally, the resonant system $\bar{\mathcal{S}}_0$, governed by the stiffness matrix $\bar{\mathbf{K}}_0$, as well as the perturbation $\bar{\mathbf{K}}_1$, requested to originate the nearly-resonant system $\bar{\mathcal{S}}$, which should approximately match the pseudo-experimental $\mathcal{S}^*$, are identified. The pseudo-experimental data and the identification results are reported in Tables 1 and 2-3, respectively. The two cases give obviously different solutions in terms of the resonant system (compare the $\bar{\mathbf{K}}_0$-columns in Tables 2-3) and its perturbation (compare the $\bar{\mathbf{K}}_1$-columns). Moreover, due to the approximation inherent to the perturbation analysis,

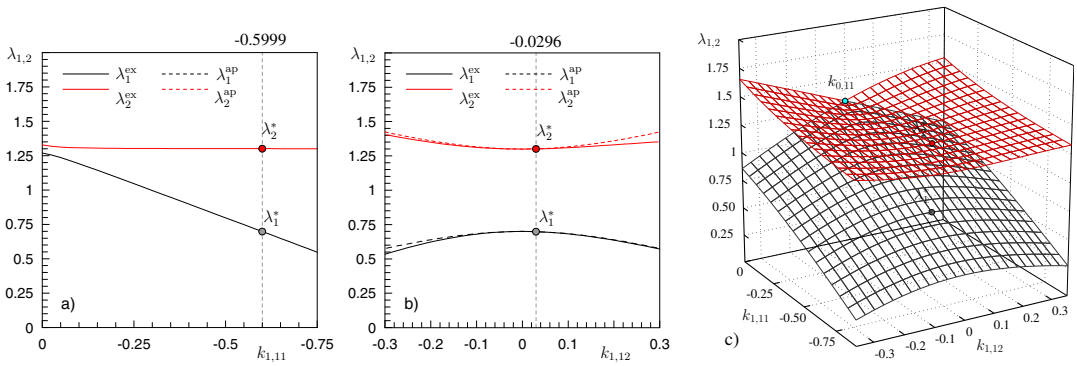

Figure 3: Eigenvalues loci versus (a) the mistuning parameter $k_{1,11}$ (fixing $k_{1,12}=-0.0296$), (b) the coupling parameter $k_{1,12}$ (fixing $k_{1,11}=-0.5999$), (c) in the $k_{1,11}$-$k_{1,12}$ parameter space.

Table 1: Pseudo-experimental nearly resonant system $\mathcal{S}^*$, obtained perturbing the resonant system $\mathcal{S}_0^*$.

$\mathrm{diag}(\mathbf{K}_0^*)$	$\mathbf{K}_1^*$			$\mathrm{diag}(\mathbf{\Lambda}_{\mathrm{ex}}^*)$	$\mathrm{diag}(\mathbf{\Lambda}_{\mathrm{ap}}^*)$	$\mathbf{\Phi}_{\mathrm{ex}}^*$			$\mathbf{\Phi}_{\mathrm{ap}}^*$		
1.0000	-0.3000	0.0300	0.0300	0.6982	0.6985	1.0000	0.0491	0.0117	1.0000	0.0499	0.0150
1.0000	...	0.3000	0.0300	1.3010	1.3015	-0.0493	1.0000	0.0152	-0.0499	1.0000	0.0150
3.0000	sym	...	0.3000	3.3008	3.3000	-0.0110	-0.0157	1.0000	-0.0146	-0.0082	1.0000

Table 2: Inverse problem solution (case A): identified resonant system $\bar{\mathcal{S}}_0$ and nearly resonant system $\bar{\mathcal{S}}$.

$\mathrm{diag}(\bar{\mathbf{K}}_0)$	$\bar{\mathbf{K}}_1$			$\mathrm{diag}(\mathbf{\Lambda}_{\mathrm{ex}})$	$\mathrm{diag}(\mathbf{\Lambda}_{\mathrm{ap}})$	$\mathbf{\Phi}_{\mathrm{ex}}$			$\mathbf{\Phi}_{\mathrm{ap}}$		
1.2995	-0.5999	0.0296	0.0234	0.6980	0.6982	1.0000	0.0487	0.0092	1.0000	0.0493	0.0117
1.2995	...	-	0.0304	1.3005	1.3010	-0.0489	1.0000	0.0153	-0.0493	1.0000	0.0152
3.3008	sym	...	-	3.3015	3.3008	-0.0084	-0.0157	1.0000	-0.0110	-0.0157	1.0000

Table 3: Inverse problem solution (case B): identified resonant system $\bar{\mathcal{S}}_0$ and nearly resonant system $\bar{\mathcal{S}}$.

$\mathrm{diag}(\bar{\mathbf{K}}_0)$	$\bar{\mathbf{K}}_1$			$\mathrm{diag}(\mathbf{\Lambda}_{\mathrm{ex}})$	$\mathrm{diag}(\mathbf{\Lambda}_{\mathrm{ap}})$	$\mathbf{\Phi}_{\mathrm{ex}}$			$\mathbf{\Phi}_{\mathrm{ap}}$		
0.9996	-0.2999	0.0296	0.0269	0.6979	0.6982	1.0000	0.0485	0.0106	1.0000	0.0493	0.0117
0.9996	...	0.2999	0.0349	1.3004	1.3010	-0.0487	1.0000	0.0176	-0.0493	1.0000	0.0152
3.3008	sym	...	-	3.3017	3.3008	-0.0097	-0.0181	1.0000	-0.0110	-0.0157	1.0000

neither of the two identified resonant systems $\bar{\mathcal{S}}_0$ exactly correspond to the unperturbed pseudo-experimental system $\mathcal{S}_0^*$. To check the solutions, the respective identified perturbations have been re-applied, and the perturbed eigensolution exactly ($\mathbf{\Lambda}_{\mathrm{ex}}$,$\mathbf{\Phi}_{\mathrm{ex}}$) or approximately ($\mathbf{\Lambda}_{\mathrm{ap}}$,$\mathbf{\Phi}_{\mathrm{ap}}$) calculated. As expected, the exact (approximated) eigensolution approximately (exactly) matches the exact pseudo-experimental eigensolution ($\mathbf{\Lambda}_{\mathrm{ex}}^*$,$\mathbf{\Phi}_{\mathrm{ex}}^*$) of the perturbed system $\mathcal{S}^*$.

Once identified, the resonant system $\bar{\mathcal{S}}_0$ must be considered the initial point of a generic multi-parameter perturbation, which obviously has to include the experimental nearly-resonant system $\mathcal{S}^*$. The Figure 3c shows the exact surfaces representing the eigenvalues loci around the resonant system $\bar{\mathcal{S}}_0$ (case A), in the two-dimensional parameter space defined by the mistuning $k_{1,11}$ and the coupling parameter $k_{1,12}$. Making focus on the iso-$\bar{k}_{1,11}$ (Figure 3a) and the iso-$\bar{k}_{1,12}$ section (Figure 3b), the correctness of the inverse problem solution can be appreciated looking at the experimental data (circles), really lying on the eigenvalue loci, whereas the effectiveness of the perturbation scheme is confirmed by the good agreement between the eigenvalue exact (continuous lines) and approximated loci (dashed lines), over a wide parameter range.

3 CONCLUSIONS AND FUTURE DEVELOPMENTS

Perturbation methods are efficient tools to track the eigensolution sensitivity of internally-resonant Hamiltonian discrete systems. Given an experimental nearly-resonant system, the paper tackles the key point of identifying a resonant system, a priori unknown (*inverse problem*), suited to serve as initial reference system for a multi-parameter perturbation analysis (*direct problem*). The effectiveness of the designed procedure is successfully tested on the relevant case of a three dofs systems with two resonant, interacting modes and a third mode with passive role. The compatibility conditions for the existence and uniqueness of the inverse problem solution are discussed, and the validity of the direct problem formulation is confirmed by numerical examples. The ongoing research is regarding the generalization of the procedure for N-dofs systems, with multiple internal resonances and several passive modes, and therefore, the straightforward extension to naturally continuous systems.

To formulate such a general procedure, and in particular to avoid any undesired increase in the dimension of the multi-parameter perturbation-based inverse problem, a simplified approach consisting in neglecting the role of all the passive (non-resonant) degrees of freedom is under current investigation. Further future developments will consider also that additional experimental data, as for instance the modal participation factors, may reveal that the mass ratio between the resonant modes has to assume asymptotic (small or large) values.

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APPENDIX

The direct problem presented in paragraph 2.1 deals with the approximated solution of the eigenproblem $(\mathbf{K}-\lambda\mathbf{I})\boldsymbol{\phi}=\mathbf{0}$. Such formulation is said to be the *standard form* of the eigenproblem, since the governing equation depends on a real symmetric matrix only. The problem statement is consistent with the hypothesis to assume the initial resonant system made of uncoupled oscillators with unitary mass ($\mathbf{M}=\mathbf{I}$), so that the stiffness matrix $\mathbf{K}$ is sufficient to govern the problem. Nonetheless, the illustrated multi-parameter perturbation technique can be easily generalized to describe the eigensolution sensitivity when the oscillators possess different masses, and the resulting eigenproblem is formulated in the *non-standard form* $(\mathbf{K}-\lambda\mathbf{M})\boldsymbol{\psi}=\mathbf{0}$. In fact, since the corresponding mass matrix $\mathbf{M}$ is real and positive definite, it can be decomposed as $\mathbf{M}=\mathbf{Q}^{\top}\mathbf{Q}$, where $\mathbf{Q}$ is a real non-singular matrix. If the mass matrix is diagonal $\mathbf{M}=\mathrm{diag}(m_1,...,m_{\mathrm{N}})$, the decomposition is unique, and $\mathbf{Q}=\mathrm{diag}(\sqrt{m_1},...,\sqrt{m_{\mathrm{N}}})$. Therefore, the non-standard eigenproblem can be reduced to an equivalent standard form $(\hat{\mathbf{K}}-\lambda\mathbf{I})\hat{\boldsymbol{\phi}}=\mathbf{0}$, where $\hat{\mathbf{K}}=\mathbf{Q}^{-\top}\mathbf{K}\mathbf{Q}^{-1}$, and $\hat{\boldsymbol{\phi}}=\mathbf{Q}\boldsymbol{\psi}$. It is worth noting that the equivalent eigenproblems yield the same eigenvalues, but different eigenvectors. In conclusion, the eigensolution based on the perturbation technique can be applied to resonant systems with generic mass matrix with the steps: (i) decomposition of the mass matrix $\mathbf{M}=\mathbf{Q}^{\top}\mathbf{Q}$ and transformation of the stiffness matrix $\hat{\mathbf{K}}=\mathbf{Q}^{-\top}\mathbf{K}\mathbf{Q}^{-1}$, (ii) perturbation-based solution of the equivalent standard form eigenproblem $(\hat{\mathbf{K}}-\lambda\mathbf{I})\hat{\boldsymbol{\phi}}=\mathbf{0}$, (iii) linear transformation of the eigenvectors $\boldsymbol{\psi}=\mathbf{Q}^{-1}\hat{\boldsymbol{\phi}}$.