Analytical and numerical methods for the dynamic analysis of slender masonry structures

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SUMMARY. A comparison is presented between some explicit approximated solutions and the corresponding numerical results regarding the nonlinear oscillations of a slender masonry column, subjected to periodic excitations with variable amplitudes and frequencies.

1 INTRODUCTION

The analysis of masonry structures subjected to time–dependent loads is a matter of growing interest for researchers, not only for its technical applications in earthquake engineering but also for the great difficulties involved in their mathematical modelling. In fact, the dynamic behaviour of masonry buildings depends on many parameters, such as the mechanical characteristics of constituent materials and the soil, the construction type and geometry and the kind of acceleration applied to the structural supports. Moreover, masonry materials typically exhibit different behaviour under tensile and compressive stresses. A number of nonlinear mechanical models have been proposed during recent years; these are mainly based on the use of elastic–plastic constitutive equations or homogenization techniques for masonry [1, 2, 3] or the definition of macro–elements [1, 4] to reduce computational effort. However, the choice of constitutive models able to realistically simulate the dynamic and static behaviour of masonry structures is still an open problem.

Experimental tests [2, 3] show that the dynamic behaviour of simple masonry elements, such as panels or columns, is strongly influenced by their slenderness (i.e. the ratio between length and height). In fact, for growing slenderness, the influence of shear forces on dynamic equilibrium tends to decrease and the nonlinearities are essentially due to the opening of cracks. Moreover, the amount of the energy dissipated during the movement also decreases with slenderness. This particular behaviour can be modeled reasonably using the constitutive equation of masonry–like materials [5], by wich masonry is represented as a nonlinear elastic material with zero tensile strength and infinite compressive strength. This nonlinear elastic equation has been implemented in the finite element code NOSA and successfully applied to the static analysis of several historical masonry buildings [5] and the dynamic analysis of masonry pillars and beams [6].

Some approximate explicit solutions to the dynamic problem have recently been proposed [7, 8] for beam–columns, by using a masonry–like constitutive equation expressed in terms of generalized stresses and strains [9]. The equation of the motion has been solved using a variational approach based on the averaged Lagrangian of the system [10, 11].

In the present work a nonlinear dynamic analysis is performed of a rectangular cross – sectional masonry column, hinged at the supports and subjected to a constant axial load and periodic excitations of variable frequency and amplitude – see Figure 2). The influence of slenderness and damping on the beam's response is investigated as well. The numerical results obtained via the code NOSA [5] are compared with the corresponding explicit solutions [7, 8].

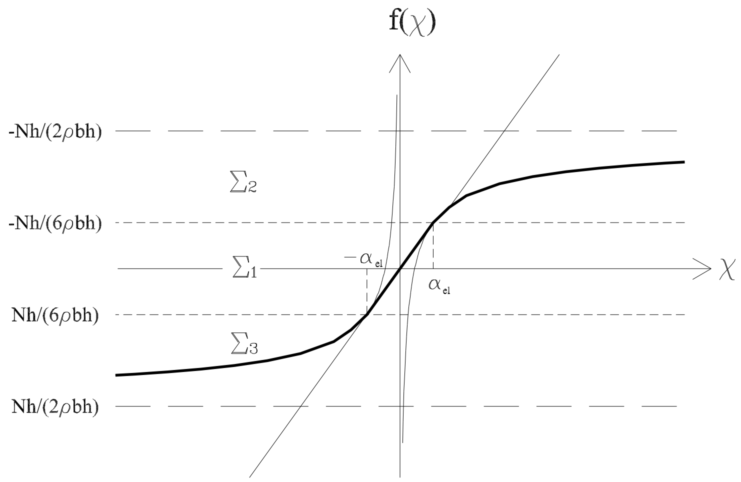


Figure 1: The constitutive equation for a rectangular section made of masonry–like material with infinite compressive strength.

2 THE ANALYTICAL METHOD

Let us consider a rectilinear beam made of masonry–like material with zero tensile strength, infinite compressive strength, Young's modulus E and density ρ and having a rectangular cross section of height h and width b. Then, under the classical Euler–Bernoulli hypothesis, the strain is described by the extensional strain ε and the change in curvature χ of the beam's axis. Moreover, by considering solely the longitudinal component of the stresses, the tensional field of the beam can be described by two generalized forces, the axial force N and the bending moment B. We now consider the beam to be loaded by a constant axial force N; under such hypotheses, the dependence of the bending moment on the curvature is given by the constitutive equation [7, 9]

$$f(\chi) = c^2\chi, \quad |\chi| \leq \alpha_{el}, \tag{1}$$

$$f(\chi) = c^2\alpha_{el}\,\text{Sign}(\chi)\,(3 - 2\sqrt{\frac{\alpha_{el}}{|\chi|}}) \quad |\chi| > \alpha_{el}, \tag{2}$$

with

$$f(\chi) = \frac{B}{\rho bh}, \qquad \alpha_{el} = -\frac{2N}{Ebh^2}, \qquad c = \sqrt{\frac{EJ}{\rho bh}}, \tag{3}$$

where α_{el} represents the curvature corresponding to the elastic limit and EJ is the bending stiffness. Equations (1) – (3) are represented in Figure 1), where, denoted by Σ the set of all generalized admissible stresses for the section defined by $|B| \leq -\frac{1}{2}Nh$, Σ_1 is the subset of Σ with $|B| \leq -\frac{1}{6}Nh$, in which the section is not cracked and $\Sigma_2 \cup \Sigma_3$ is the complement of Σ_1 (relative to Σ).

Let $v(x,t)$ be the transverse displacement at time t of the point having abscissa x along the beam's axis (Figure 2). We assume that v and its derivative v_x with respect to x are small, so that the curvature is given by

$$\chi + v_{xx} = 0. \tag{4}$$

Let $p(x,t)$ be the transverse load per unit length acting along the beam and C the viscous damping coefficient. Thus, the motion equation is

$$\ddot{v} - f_{xx} + \frac{C\dot{v} - p}{\rho b h} = 0, \tag{5}$$

where the dot represents the derivative with respect to time. The Lagrangian $\mathcal{L}$ associated to the system can be expressed by

$$\mathcal{L} = \left[\frac{1}{2}\,\dot{v}^2 - f'\chi_x v_x + \chi f - F\right]\rho b h + p\,v, \tag{6}$$

where $F(\chi)$ is a primitive function of $f(\chi)$ with $F(0) = 0$ [7, 10].

With the aim of finding some approximate solutions to (5), we can describe the beam's behaviour by means of some parameters, whose variation in space and time can be considered "slow". Let us limit the problem to studying excitations in the primary resonance of the first mode (i.e. excitation frequency near the beam's fundamental elastic one), which take the form

$$p(x,t) = k\sin\left[2\pi(\nu_e + \lambda)t\right], \tag{7}$$

where ν_e represents the fundamental elastic frequency of the beam, λ is a variable parameter expressing the nearness of the exciting frequency to the fundamental elastic one and k is the excitation amplitude. Provided that no internal resonance phenomena are allowed, the solution can be expressed through the simple unimodal expression

$$v(x,t) = A(t)\,\phi(x)\,\sin\left[2\pi(\nu_e + \lambda)t + \beta(t)\right], \tag{8}$$

where $\phi(x)$ is the first elastic vibration mode of the beam and amplitude A and phase displacement β are the slowly varying parameters of the system, describing the nonlinear response of the beam. Most classical perturbation methods, such as the multiple–scale one, use series developments and periodicity conditions to solve the problem. The following approach, instead, is based on an averaging technique leading to the equations [10, 11, 12]

$$\left(\frac{\partial\bar{\mathcal{L}}}{\partial\dot{A}}\right)^{\cdot} - \frac{\partial\bar{\mathcal{L}}}{\partial A} = \bar{Q}_A, \tag{9}$$

$$\left(\frac{\partial\bar{\mathcal{L}}}{\partial\dot{\beta}}\right)^{\cdot} - \frac{\partial\bar{\mathcal{L}}}{\partial\beta} = \bar{Q}_\beta, \tag{10}$$

with

$$\bar{\mathcal{L}} = \int_0^{T_e}\int_0^L \mathcal{L}\,dx\,dt, \quad \bar{Q}_A = \int_0^{T_e}\int_0^L -C\dot{v}\frac{\partial v}{\partial A}\,dx\,dt, \quad \bar{Q}_\beta = \int_0^{T_e}\int_0^L -C\dot{v}\frac{\partial v}{\partial\beta}\,dx\,dt, \tag{11}$$

where T_e is the beam's fundamental elastic period, $\mathcal{L}$ is given in (6), v in (8) and the integrals on the elastic period are performed by taking the parameters A and β to be constant. $\bar{Q}_A$ and $\bar{Q}_\beta$ in (9) and (10) are small with respect to the conservative terms and are calculated by using the hypothesis of modal damping.

Equations (9), (10) become a system of two nonlinear differential equations in the variables A and β, having the form

$$C_1\frac{d\Theta}{dA} + A\dot{\beta} + C_2 k\cos(\lambda t - \beta) = 0, \tag{12}$$

$$\dot{A} + C_3\zeta A - C_4 k\sin(\lambda t - \beta) = 0, \tag{13}$$

where ζ is the modal damping coefficient, Θ is a function of A, L and α_{el}, and C_1, C_2, C_3, C_4 are constant coefficients. Since A and β are slowly oscillating functions, the system (12) – (13) can be easily solved numerically. If interested only in the stationary part of the motion, we can put $\dot{A} = 0$ and $\dot{\beta} = 0$ and transform (12) – (13) into a system of transcendent equations. Moreover, by putting $k = 0$, from (12) we can obtain the function $\dot{\beta}(A)$ that describes the variation of the beam's fundamental frequency with the amplitude A of the motion. All results depend on the dimensionless parameter $\alpha = \alpha_{el} \cdot L$, which comprises all the data regarding the beam's geometry and the axial force acting.

Further details can be found in [7, 8], while [10, 11] present a theoretical justification of the method.

3 THE NUMERICAL METHOD

This section briefly recalls the constitutive equation of masonry–like materials and a numerical procedure for dynamic analysis of masonry constructions. Greater details can be found in [5] and [6].

Masonry is assumed to be a nonlinear elastic material characterized by Young's modulus $E > 0$, Poisson ratio ν with $0 \leq \nu < \frac{1}{2}$, infinite compressive strength and zero tensile strength. We denote by Sym the vector space of symmetric tensors with inner product $\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{AB})$, where $\mathbf{A}, \mathbf{B} \in Sym$ and tr is the trace. The subsets of Sym constituted by the negative and positive semidefinite tensors are called Sym^- and Sym^+, respectively. We assume that the infinitesimal strain $\mathbf{E} \in Sym$ is the sum of an elastic part $\mathbf{E}^e \in Sym$ and an orthogonal inelastic part $\mathbf{E}^f \in Sym^+$, called fracture strain

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^f, \tag{14}$$

and that the Cauchy stress $\mathbf{T}$ depends linearly and isotropically on $\mathbf{E}^e$,

$$\mathbf{T} = \frac{\mathbf{E}}{1+\nu}[\mathbf{E}^e + \frac{\nu}{1-2\nu} tr(\mathbf{E}^e)\mathbf{I}], \tag{15}$$

with $\mathbf{I}$ the identity tensor. Lastly, we assume that $\mathbf{T}$ and $\mathbf{E}^f$ satisfy the conditions

$$\mathbf{T} \in Sym^-, \tag{16}$$

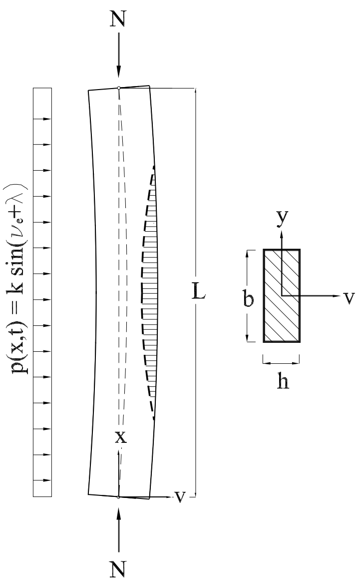
$$\mathbf{E}^f \cdot \mathbf{T} = 0. \tag{17}$$

The constitutive equation (14)–(17) has been implemented in the finite – element code NOSA to model the structural behaviour of masonry structures [5, 6].

In [6] a numerical procedure is proposed for solving the dynamic problem of masonry structures via the finite-element method. The equation of motion is integrated directly. More precisely, time integration is performed of the system of ordinary differential equations obtained by dividing the structure into finite elements. In particular, at each time step a system of the type

$$K[u(t)]\Delta u + C\Delta\dot{u} + M\Delta\ddot{u} = \Delta f, \tag{18}$$

is solved via the Newmark method. In (18), $u(t)$ is the vector of nodal displacements at time t, Δf is the load increment and Δu, $\Delta\dot{u}$ and $\Delta\ddot{u}$ are respectively the incremental nodal displacements, velocities and accelerations. K, C and M are the stiffness, damping and mass matrices of the structure, respectively. In conformity with Rayleigh's assumption, we have $C = \gamma M + \delta K$, where γ and δ are constants to be determined from the linear elastic vibration frequencies of the structure and



Fixed parameters:

$L = 6$ m; $b = 1$ m;
$\rho = 1800$ kg/m^3; $E = 3 \cdot 10^9$ Pa;
$N = 100$ KN.

For $\zeta = 0.02$ and $\lambda = 0$:

h [m]	k [N/m]					
0.30	100	200	300	400	600	800
0.40	100	200	300	400	600	800
0.50	100	200	300	400	600	800

For $\zeta = 0.05$ and $\lambda = 0$:

h [m]	k [N/m]					
0.40	100	200	300	400	600	800

For $\zeta = 0.02$ and $k = 400$ N/m:

h [m]	λ [Hz]							
0.40	-1.5	-1.25	-1	-0.75	-0.5	0	0.25	0.5

Figure 2: Geometry of the beam and data used for numerical tests.

the corresponding damping ratios. The Newton-Raphson scheme is applied to solve the nonlinear algebraic system obtained at each time step, and the tangent stiffness matrix is calculated by using the explicit expression of the derivative of the stress with respect to the strain.

4 COMPARING ANALYTICAL AND NUMERICAL RESULTS

The numerical tests have been performed using the scheme shown in Figure 2). Three values were chosen for slenderness, with the corresponding section height h equal to 0.30 m, 0.40 m and 0.50 m; two values of damping ratio ζ were considered, 2% and 5%. The beam is subjected to a sinusoidal load of variable amplitude k and frequency $(\nu_e + \lambda)$; for the three slenderness values chosen, the numerical values of the linear fundamental frequency are $\nu_e(0.3 \text{ m}) = 4.9$ Hz, $\nu_e(0.4 \text{ m}) = 6.5$ Hz and $\nu_e(0.5 \text{ m}) = 8.1$ Hz. For all tests, null initial displacements and velocities have been imposed to the beam. With the aim of optimizing the comparisons between the analytical and numerical results, different kinds of elements were tested, while varying the number of elements as well. Lastly, the eight–node isoparametric thin shell element described in [5] was chosen, and the beam divided into 120 finite–elements.

Figures 3) and 4) show the displacements of the mid–point of the beam for $h = 0.40$ m, $k = 400$ N/m and different values of the damping ratio ζ. In both cases, the analytical and numerical results are quite consistent. Note that after a brief transient the oscillations tend toward a stationary behaviour. Figures 5), 6) instead show the stress σ_x at the extrados of the mid–section vs. t for $h = 0.40$ m, $k = 400$ N/m and different values of the damping ratio ζ. Figures 7) and 8)

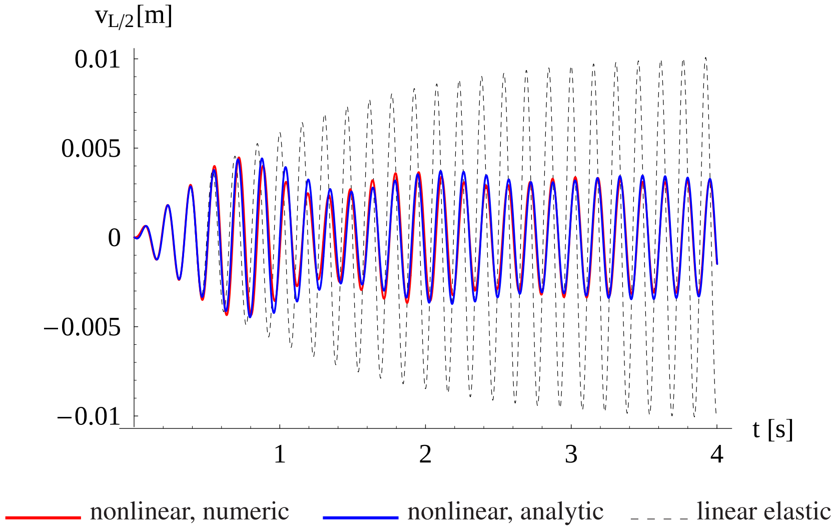


Figure 3: Displacements of the mid–section of the beam vs. time t for $\zeta = 0.02$, $k = 400$ N/m, $h = 0.4$ m.

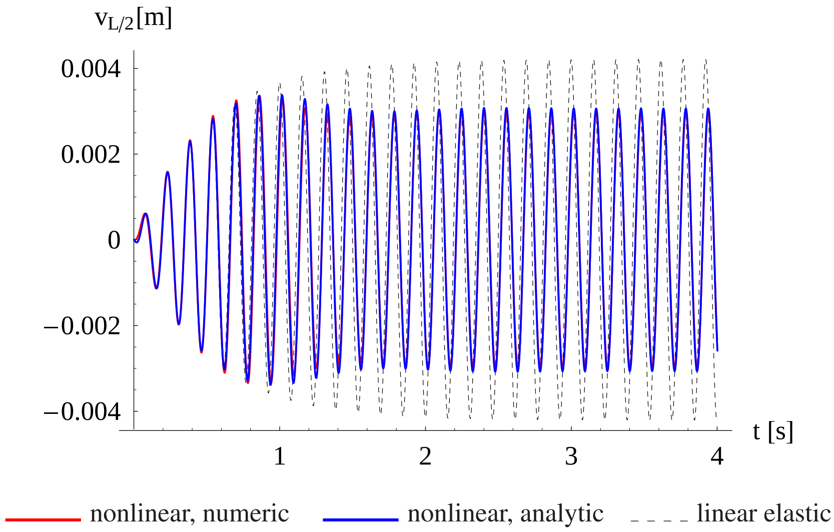


Figure 4: Displacements of the mid–section of the beam vs. time t for $\zeta = 0.05$, $k = 400$ N/m, $h = 0.4$ m.

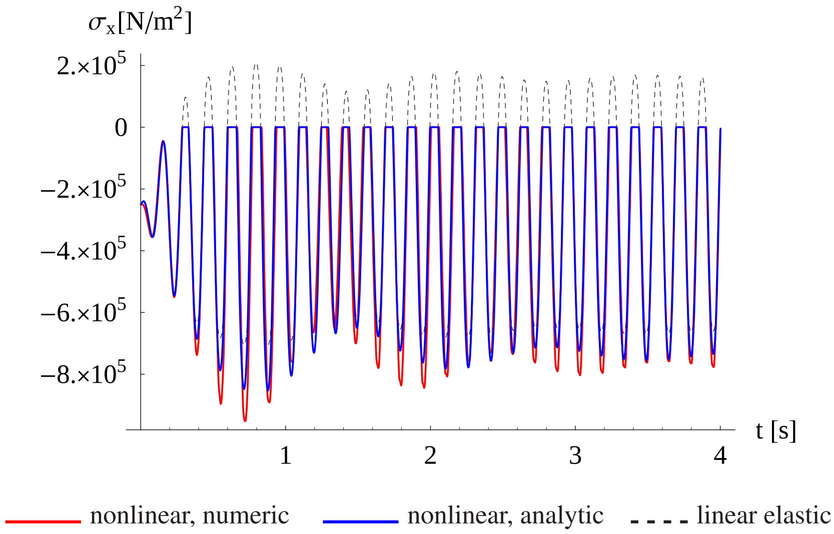


Figure 5: Stress σ_x at the extrados of the mid–section of the beam vs. time t for $\zeta = 0.02$, $k = 400$ N/m, $h = 0.4$ m.

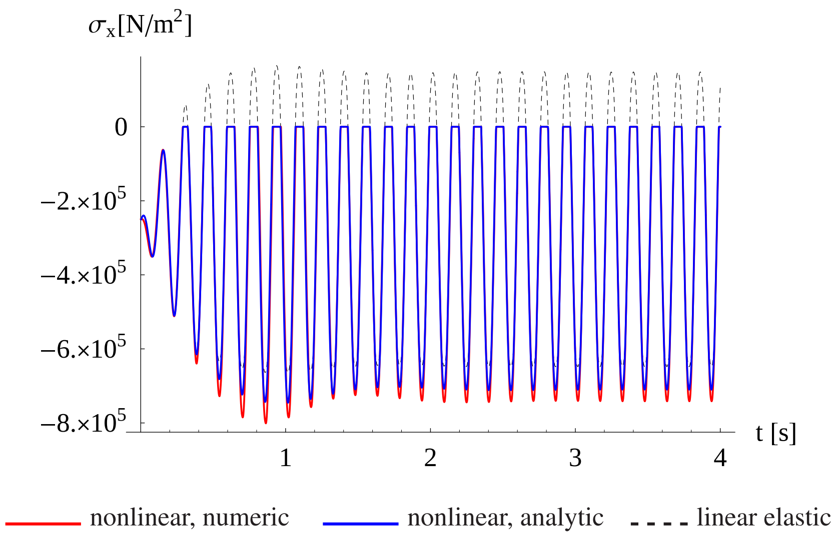


Figure 6: Stress σ_x at the extrados of the mid–section of the beam vs. time t for $\zeta = 0.05$, $k = 400$ N/m, $h = 0.4$ m.

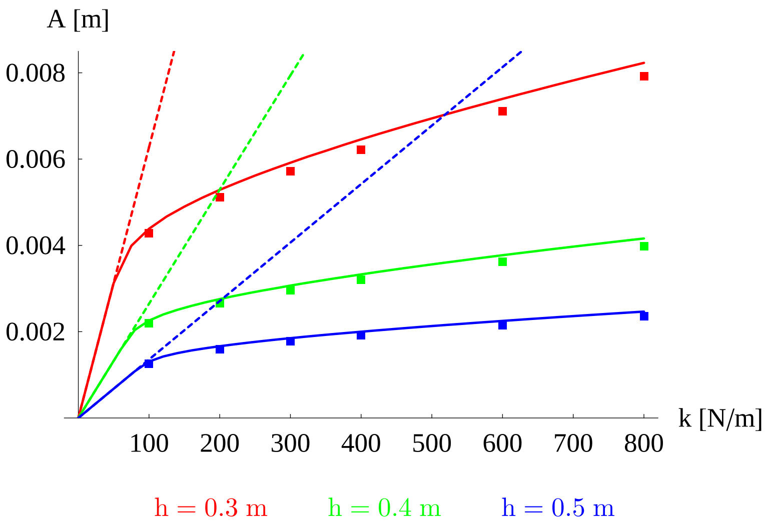


Figure 7: Maximum stationary displacement of the beam vs. k for $\lambda = 0$ and different values of h.
■ nonlinear, numeric ——— nonlinear, analytic - - - - linear elastic.

show the stationary amplitude A vs. k for different values of h and ζ. The nonlinear values are quite different from the corresponding linear ones and all curves tend to exhibit very marked softening behaviour. It is worth noting that the curves related to different damping values in Figure 8) tend to become coincident for large values of k. The numerical values are a bit lower than the corresponding analytical ones: numerical tests tend to exhibit a more softening behaviour. Figure 9) shows the phase displacements β vs. k; numerical values are obtained by using the Fast Fourier transform. The analytical solution shows two branches; however, for all numerical solutions, phase displacements are on the lower branch. The horizontal line for $\beta = \frac{\pi}{2}$ represents the linear elastic solution. Lastly, Figure 10) shows a comparison among the linear elastic frequency response function (dashed curve), the corresponding analytical nonlinear function (continuous curve) and the results of the numerical tests with variable frequency excitations (red curve). The differences between the linear and nonlinear responses are considerable, particularly in the range centred on the linear fundamental frequency. The nonlinear analytical curve presents the typical shift towards low frequencies characteristic of all systems exhibiting softening behaviour. Moreover, for excitation of given frequency and amplitude, the curve presents more than one solution, depending on the initial conditions. In our tests, for the chosen initial conditions, the numerical solution presents a jump to about 5.6 Hz, between the upper to the lower branch of the analytical curve.

5 CONCLUSIONS

We have presented explicit and numerical solutions to the dynamic problem of a masonry–like column, subjected to a constant axial force and a sinusoidal transverse excitation. We have investigated the dependence of the solution on various parameters such as the slenderness and damping ratio of the structure, and the amplitude and frequency of the excitation. The analytical and numerical results have proved to be consistently in good agreement. The numerical methods enable solving

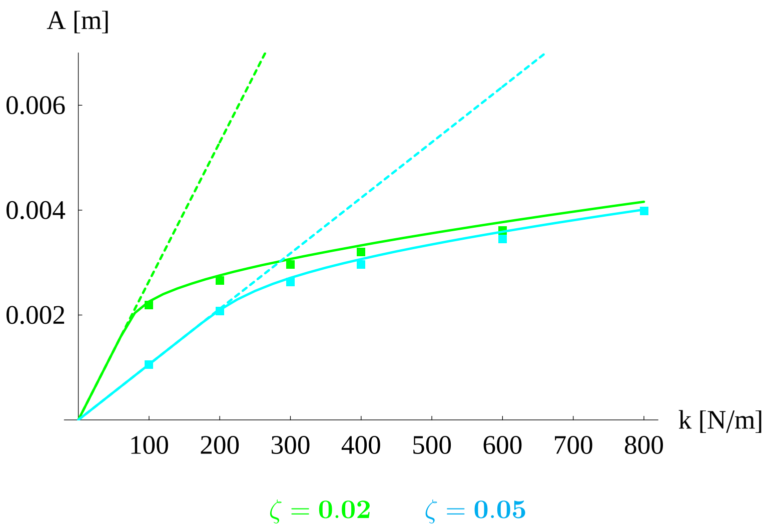


Figure 8: Maximum stationary displacement of the beam vs. k for $\lambda = 0$, $h = 0.4$ m and different values of ζ. ■ nonlinear, numeric ——— nonlinear, analytic - - - - linear elastic.

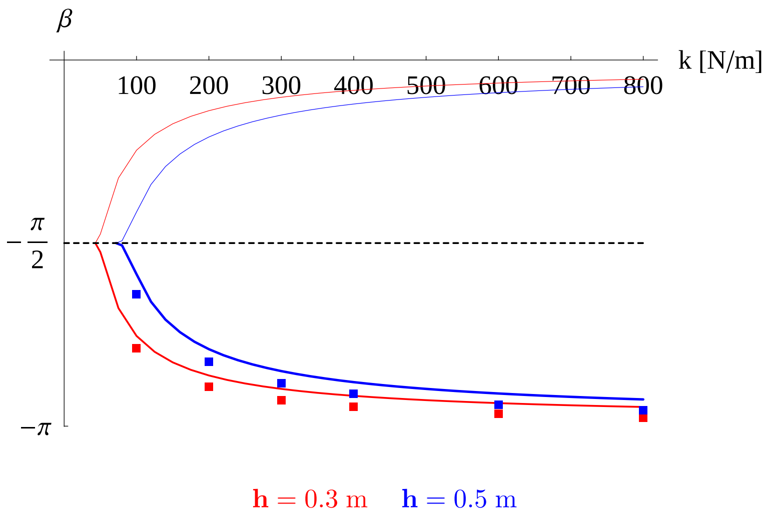


Figure 9: Stationary values of β vs. k for $\lambda = 0$, $\zeta = 0.02$ and different values of h. ■ nonlinear, numeric ——— nonlinear, analytic - - - - linear elastic.

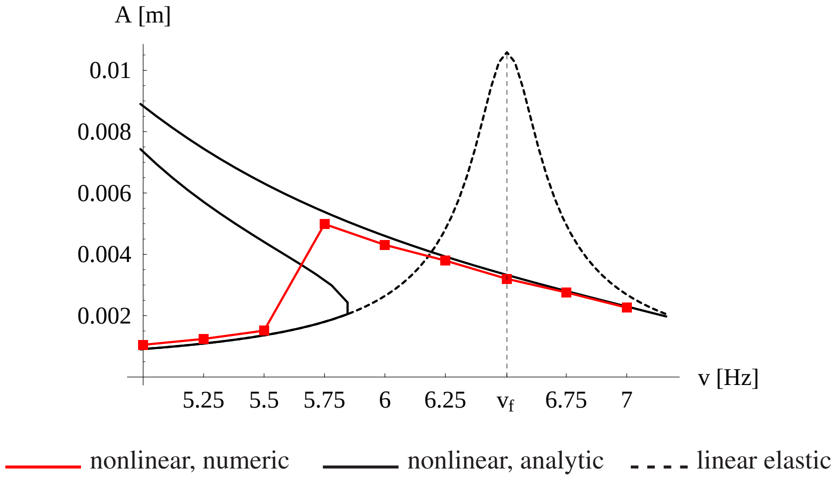


Figure 10: Maximum stationary displacement of the beam vs. the excitation frequency ν for $\zeta = 0.02$ and $k = 400$ N/m, $h = 0.4$ m.

problems for very general conditions of geometry and loading. However, the analytical solutions, albeit limited to some particular cases, provide synthetical descriptions of nonlinear phenomena and contribute to better understanding the overall behaviour of masonry structures.

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