Stochastic Stability of Elastic and Viscoelastic Columns

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SUMMARY. The problem of the stochastic stability of elastic and viscoelastic columns compressed by a Gaussian white noise axial force is addressed in the context of the dynamic stability. The former column has proportional damping, the other has both proportional damping and viscoelastic damping with memory expressed by the superposition integral. The stability is checked in statistical moments by examining the eigenvalues of the matrix of the coefficients of the ODE ruling the time evolution of them, which are obtained by means of Itô's differential rule.

1 INTRODUCTION

In classic Mechanics distinction is made between static and dynamic stability. However, for practical conservative engineering systems the static buckling load coincides with the dynamic buckling load (e.g. see Corradi dell'Acqua, Vol. 3, [14]). Static and dynamic buckling loads are equal even for an Euler's column compressed by an axial load.

This paper is concerned with the study of the dynamic stability of an Euler's column. In a stochastic setting the matter is different from a deterministic one even if the trivial solution is still $\boldsymbol{X} = \dot{\boldsymbol{X}} = 0$, being $\boldsymbol{X}$ the vector of system states, and the aim of a stability analysis is to discover whether beside the trivial solution there are non trivial solutions $\boldsymbol{X} \neq 0, \dot{\boldsymbol{X}} \neq 0$, and whether they are bounded or not. Differently from a deterministic analysis there are various definitions of stochastic stability that lead to different buckling loads.

There most common definitions of stochastic stability are (e.g. see Lin and Cai [26], Chap. 6): (1) almost sure stability also named Lyapunov stability with probability 1 (WP1); (2) stability in probability; (3) stability in the r-th moment. For linear systems the stability in the second moments is more stringent than the criterion (1). Different definitions of stochastic stability lead to different buckling loads. Moreover, for a given definition of stochastic stability there are more methods for finding the buckling load.

From now on attention will be restricted to the problem of the stochastic stability of an Euler's column compressed by a stochastic axial load $N(t) = \mu_P + \sqrt{\pi w_P} W(t)$, where μ_P is a constant, and $W(t)$ is a Gaussian white noise (the reason of this choice will be given later). This and similar problems have deserved attention since the sixties. Some authors studied the stability of a stochastic ODE, which is similar to the modal equations of the column: Caughey and Gray [10], Infante [21], Kozin and Prodromou [24], Kozin and Wu [25], Arnold [7], Ariaratnam and Xie [6], Cottone and Di Paola [15]. Arnold, Cottone and Di Paola excepted, previous studies look for analytic conditions of almost sure stability. Notwithstanding the mathematic refinement and complication, the results do not agree among them very well. Arnold [7] establishes a relationship between almost sure stability and moment stability, but the mathematics are cumbersome. Cottone and Di Paola propose a numerical method, the path integral solution, to study the moment stability.

Other authors make explicit reference to an elastic Euler-Bernoulli's bar starting the study from the stochastic PDE that governs the motion of the bar: Plaut and Infante [29], Ahmadi and Mostaghel [4], Ahmadi e Glockner [3], Pavlovic et al. [28]. Modal analysis is used in all the studies with the exception of that by Plaut and Infante, who consider a non separable system. Almost sure stability is considered.

Other authors overcome the Euler's model by including viscous effects in the analysis: here the term viscous means delayed effects with memory, that is depending on the past. Only two papers have been found that consider a damping with memory: Ariaratnam [5], Cai and Lin [9]. In both papers an ordinary integro-differential equation is analyzed: the partial integro-differential equation of motion can be reduced to this form by modal analysis. In the other papers found in literature viscous deformations affect the flexural curvature, but do not influence the damping. Since this case is not treated here, we recall Potapov [30, 31], Drozdov [17, 18], Huang and Xie [20], Xie and Huang [33].

As regards the methods of analysis, the literature examination reveals a strong preference for the Lyapunov exponent either considering the almost sure stability or the moment stability. Clearly, the tools that are used for computing it are very different. A different approach is outlined by Drozdov and Kolmanovskij [19], and by Potapov [30]. However, these authors do not pursue the approach further, and in more recent works return back to the Lyapunov exponent methods.

Herein, the stochastic stability is checked in moments by developing the approach envisaged by the above cited authors. The method operates according the following steps: (1) basing on the assumption that the dynamic system is separable, the modal analysis is applied; (2) as in general only the first mode is relevant to stability, the second order ODE for the first mode is converted into two first order Itô's differential equations; (3) by applying Itô's differential rule, the differential equations ruling the response moment evolution are written; (4) the stability of these equations is studied through an eigenvalue analysis.

Steps (2) and (3) can be runned along only if the excitation is a Gaussian white noise (for the stochastic differential calculus [16, 26, 22, 23]). This is the reason why the random part of the axial load has been chosen to be such a stochastic process. This choice is a limitation only partially as a colored process can be approximated by the output of linear filters having a Gaussian white noise as a primary excitation. For clarity's sake, the case of a merely elastic column and that of a viscoelastic column will be analyzed separately.

2 FORMULATION OF THE GOVERNING EQUATIONS

Consider an Euler's column subjected to the axial load $N(t) = \mu_P + \sqrt{\pi w_P} W(t)$ applied in the centroid of an end cross section, being $W(t)$ a unit strength Gaussian white noise stochastic process. The column has the banal rectilinear shape. We make the assumption that μ_P is smaller than the Euler's buckling load. However, this is not sufficient in order that the column is stable in stochastic sense as the stochastic perturbation caused by $\sqrt{\pi w_P} W(t)$ may cause the loss of stability. In other words, there exist values of the intensity w_P which cause the loss of stability in a stochastic sense. With reference to the dynamic criterion of stability, the column is stochastically stable when it returns to the banal straight undeformed shape after an external agency perturbates this equilibrium configuration. The column is unstable when the external agency causes the divergence of the column's deformed configuration.

As advanced in the Introduction, the stochastic stability analysis is performed by studying the equations that govern the evolution of the response moments. Two cases are considered: (1) elastic column with viscous damping without memory; (2) viscoelastic column with both damping without memory and damping with memory, that is described by the classic hereditary model of viscosity (the linear theory of the viscosity is exposed in [8, 12]).

2.1 *Elastic column*

The motion of an elastic Bernoulli-Navier column is governed by the equation

$$EI\frac{\partial^4 w}{\partial x^4}+\left[\,\mu_P+\sqrt{\pi w_P}\,\right]\frac{\partial^2 w}{\partial x^2}+c\frac{\partial w}{\partial t}+\rho\frac{\partial^2 w}{\partial t^2}=0 \qquad (1),$$

with initial and boundary conditions to be specified, where E is the Young's modulus, I the second moment of the cross section area, c the viscous damping coefficient, and ρ the column's density for unit length. It can be demonstrated that the system (1) is separable. Thus, a solution is sought in the form [13]

$$w(x,t)=\sum_{1}^{\infty}{}_j\,\phi_j(x)V_j(t) \qquad (2).$$

The eigenfunctions $\phi_j(x)$ are a complete set in $\mathscr{L}^2$, and enjoy the following properties: $\int_0^L \phi_j\phi_k \mathrm{d}x=\int_0^L \phi'_j\phi'_k \mathrm{d}x=\delta_{jk}$, $\int_0^L \phi''_j\phi''_k \mathrm{d}x=\lambda_k\,\delta_{jk}$, $\int_0^L \phi_j^2 \mathrm{d}x=M_j/\rho$, $EI\int_0^L \phi_k\phi_k^{IV}\mathrm{d}x=\omega_k^2 M_k$; they satisfy the boundary conditions. In previous relationships the eigenvalues λ_k and the pulsations ω_k are related by $\omega_k^2=\left(\lambda_k^4 EI\right)/\left(L^2\rho\right)$, being L the column length.

By inserting Eq. (2) into (1), multiplying this by $\phi_k(x)$, integrating from zero to L, and profiting from the above mentioned properties, the modal equations are obtained as

$$\frac{\mathrm{d}^2V_k}{\mathrm{d}t^2}+2\zeta_k\omega_k\frac{\mathrm{d}V_k}{\mathrm{d}t}+\omega_k^2\left(1-\frac{\mu_P}{P_{Ek}}\right)V_k+\frac{I_k}{M_k}\sqrt{\pi w_P}W\,V_k=0 \qquad (3),$$

where the ratio of critical damping is given by $\zeta_k=c/2\rho\omega_k$, $I_k=\int_0^L\sum_j\left(\phi_j\phi''_k\right)\mathrm{d}x$, and P_{Ek} is the k-th Euler's buckling load. Without loss of generality from now on reference is made to a hinged-hinged column, for which the boundary conditions, the eigenvalues and the eigenfunctions are, respectively: $w(0,t)=w(L,t)=w''(0,t)=w''(L,t)=0$, $\lambda_k=k\pi$ (k = 1,2, …), $\phi_k(x)=\sin(k\pi x/L)$; moreover, $I_k/M_k=-k^2\pi^2/\rho L^2=-\omega_k^2/P_{Ek}$.

As advanced, in general the study can be limited to the first mode as it gives raise to the lowest limit of instability. Hence, for k = 1, which is omitted for simplicity's sake, Eq. (3) is converted into two first order Itô's differential equations by putting $z_1=V$, $z_2=\mathrm{d}V/\mathrm{d}t$:

$$\mathrm{d}z_1 = z_2\,\mathrm{d}t$$
$$\mathrm{d}z_2 = -\left(\beta z_2 + \Omega^2 z_1\right)\mathrm{d}t + c\sqrt{\pi w_P}\, z_1\,\mathrm{d}B \qquad (4\text{ a,b}),$$

where $\beta = 2\zeta\omega$, $\Omega^2 = \omega^2\left(1-\mu_P/P_E\right)$, $c = |I/M|$, and dB is the increment of a standard Brownian motion $B(t)$, for which the formal relationship $\mathrm{d}B/\mathrm{d}t = W(t)$ holds. Eq. (3) is a parametric one, but the Wong-Zakai-Stratonovich corrective term in Eq. (4 b) results to be zero [32, 34].

By applying Itô's differential rule to the non anticipating function $\psi(z_1, z_2) = z_1^{p_1} z_2^{p_2}$ [16, 22, 23], one writes the differential equations ruling the evolution of response moments of order $p_1 + p_2 = r$. In symbolic notation, they are expressed as

$$\dot{\boldsymbol{m}}_r(t) = \boldsymbol{A}\,\boldsymbol{m}_r(t) \qquad (5),$$

where $\boldsymbol{m}_r$(t) is the vector with all the moments of order r of system states. Eq. (5) has the solution

$$\boldsymbol{m}_r(t) = \boldsymbol{m}_0\,\exp(\boldsymbol{A}t) \qquad (6),$$

where $\boldsymbol{m}_0$ is the vector with the initial conditions for the moments, which constitute the perturbation to the column banal equilibrium configuration.

Whenever the matrix $\boldsymbol{A}$ has negative real eigenvalues and complex eigenvalues with negative real part, the system is stable, and the moments tend to zero as t grows. Since the matrix $\boldsymbol{A}$ depends on the stochastic axial load intensity w_P, there exists a critical value of this for which the condition above is no longer satisfied, and the moments grow without limits. Thus, the problem of the stochastic stability of the column is solved by studying the eigenvalues of the matrix of the coefficients of the moment equations.

Stability of the statistical averages requires $\zeta > 0$ and that μ_P is smaller than the Euler's buckling load. The three second order moments are governed by the equations:

$$\begin{Bmatrix}\dot{\mu}_{11}\\ \dot{\mu}_{12}\\ \dot{\mu}_{22}\end{Bmatrix} = \begin{bmatrix} 0 & 2 & 0\\ -\Omega^2 & -\beta & 1\\ \pi c^2 w_P & -2\Omega^2 & -2\beta\end{bmatrix}\begin{Bmatrix}\mu_{11}\\ \mu_{12}\\ \mu_{22}\end{Bmatrix} \qquad (7),$$

where $\mu_{pq} = E\left[z_1^p\, z_2^q\right]$ $(0 \le p+q \le 2)$ and the dot means derivative with respect to the time t. The characteristic equation for finding the eigenvalues of $\boldsymbol{A}$ is:

$$\det\left[\lambda\boldsymbol{I} - \boldsymbol{A}\right] = \lambda^3 + 3\beta\lambda^2 + \left(2\beta^2 + 4\Omega^2\right)\lambda + 4\Omega^2\beta - 2\pi c^2 w_P = 0 \qquad (8).$$

Eq. (8) is an algebraic third degree equation: thus, its roots have analytical expressions (it is recalled that one root is real and the other two are complex conjugate). From a theoretical point of view, the critical value of w_P could be obtained by studying these roots: for $w_P < w_{Pcr}$ the roots have negative real parts, while for $w_P > w_{Pcr}$ a real part at least becomes positive. On the other hand, there are several parameters that influence the results, and not only the ratio of critical damping ζ as claimed by some authors, so that the study has notable complexity even because of the cumbersome expressions of the roots. It seems to be preferable to use a mathematical software, to write a specific program, by which the critical value w_{Pcr} is found by increasing w_P till a positive real part is encountered.

2.2 *Viscoelastic column*

The motion equation of a viscoelastic Bernoulli-Navier column is

$$EI\frac{\partial^4 w}{\partial x^4}+\left[\mu_P+\sqrt{\pi w_P}\right]\frac{\partial^2 w}{\partial x^2}+c\frac{\partial w}{\partial t}+\rho\frac{\partial^2 w}{\partial t^2}+\int_0^t \Gamma(t-t')\frac{\partial w(t')}{\partial t'}\mathrm{d}t'=0 \qquad (9),$$

where the fifth term accounts for the damping with memory, being $\Gamma(t-t')$ some relaxation kernel of hereditary type (among others, the dynamic of the oscillator with hereditary damping is studied in [1, 2, 27]). Even if the motion equation is a partial integro-differential one, modal analysis is still applicable [2], and the solution is still expressed by Eq. (2). For a hinged-hinged column, the modal equations are

$$\frac{\mathrm{d}^2 V_k}{\mathrm{d}t^2}+2\zeta_k\omega_k\frac{\mathrm{d}V_k}{\mathrm{d}t}+\omega_k^2\left(1-\frac{\mu_P}{P_{Ek}}\right)V_k-\frac{\omega_k^2}{P_{Ek}}\sqrt{\pi w_P}W\,V_k+\frac{1}{\rho}\int_0^t \Gamma(t-t')\frac{\mathrm{d}V_k(t')}{\mathrm{d}t'}\mathrm{d}t'=0 \qquad (10).$$

In order to transform Eq. (10) into a set of first order Itô's differential equations, the method of the additional state variables is used [27]. The most general hereditary kernel is expressed by a Dirichlet-Prony series [8]:

$$\Gamma(t)=\sum_{1}^{m}{}_i\,\varphi_i\,\alpha_i\exp(-\alpha_i t) \qquad (11).$$

The number of additional variables equates the number m of exponential functions retained in the series. Herein, for simplicity's the classic Kelvin-Voigt model is adopted:

$$\Gamma(t)=G\varphi_\infty\,\alpha\exp\left[-\alpha(1+\varphi_\infty)t\right] \qquad (12).$$

In this way, the system has three states, the third of which is given by

$$Y_k(t)=\overline{\varphi}_\infty\int_0^t \exp\left[-\overline{\alpha}(t-t')\right]\dot{V}_k(t')\mathrm{d}t' \qquad (13),$$

where $\overline{\varphi}_\infty=G\varphi_\infty\alpha/\rho,\ \overline{\alpha}=\alpha(1+\varphi_\infty)$. With reference to the first mode (k = 1 is omitted), by putting $z_1=V,\ z_2=\mathrm{d}V/\mathrm{d}t,\ z_3=Y$, and taking the derivative of Eq. (13) we have:

$$\begin{aligned}\mathrm{d}z_1&=z_2\,\mathrm{d}t\\ \mathrm{d}z_2&=-\left(\beta z_2+\Omega^2 z_1-z_3\right)\mathrm{d}t+c\sqrt{\pi w_P}\,z_1\,\mathrm{d}B\\ \mathrm{d}z_3&=\left(-\overline{\alpha}z_3+\overline{\varphi}_\infty\,z_2\right)\mathrm{d}t\end{aligned} \qquad (14\text{ a - c}).$$

Applying Itô's differential rule, three equations for the statistical averages and six for the second order moments are obtained. Again, $\mu_P = P_{Ek}$ causes the system to be unstable, while not necessarily ζ must be larger than zero, which agrees with the finding of Drozdov and Kolmanovskij [19]. The second moment equations are:

$$\begin{Bmatrix} \dot{\mu}_{200} \\ \dot{\mu}_{110} \\ \dot{\mu}_{101} \\ \dot{\mu}_{020} \\ \dot{\mu}_{011} \\ \dot{\mu}_{002} \end{Bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ -\Omega^2 & -\beta & -1 & 1 & 0 & 0 \\ 0 & \overline{\phi}_\infty & -\overline{\alpha} & 0 & 1 & 0 \\ \pi c^2 w_P & -2\Omega^2 & 0 & -2\beta & -2 & 0 \\ 0 & 0 & -\Omega^2 & \overline{\phi}_\infty & -\overline{\alpha}-\beta & -1 \\ 0 & 0 & 0 & 0 & 2\overline{\phi}_\infty & -2\overline{\alpha} \end{bmatrix} \begin{Bmatrix} \mu_{200} \\ \mu_{110} \\ \mu_{101} \\ \mu_{020} \\ \mu_{011} \\ \mu_{002} \end{Bmatrix} \qquad (15).$$

where $\mu_{pqr} = E\left[z_1^p z_2^q z_3^r\right]$ $(0 \le p+q+r \le 2)$ The characteristic equation for the matrix $\boldsymbol{A}$ in Eq. (15) is of sixth degree so that an analytical expression for the critical intensity cannot be found. Thus, it is compulsory to use the numerical procedure outlined in 2.1 .

3 APPLICATIONS

The stochastic stability limit in mean square is looked for by considering a hinged-hinged column having the following values of the parameters: cross section area A = 0.0216 m^2, second moment of the area $I = 4.32 \cdot 10^{-4}$ m^4, length L = 15 m, mass density for unit length ρ = 169.56 kg/m, c = 215.53 $N{\cdot}s/m$. From these values we obtain: ω_1 = 71.50 rad/s, first Euler's buckling load P_{1E} = $3.90363 \cdot 10^6$ *newton*, ζ = 0.002 in the first mode. It is chosen $\mu_P = 0.4 P_{1E}$. The parameter w_P governs the strength of the white noise axial force, and it is varied to find its critical value w_{Pcr}. The autocorrelation function and the one-sided power spectral density of $W(t)$ are, respectively: $R_{WW}(\tau) = \delta(t-\tau)$, $S_{WW}(\omega) = 1/\pi$ $(\omega > 0)$. Thus, if the term $\sqrt{\pi w_P}$ is kept into account, the strength of the white noise is just w_P.

The column with the memory damping has the same parameters. With reference to Eq. (12), this is characterized by G = 1000 $N{\cdot}s/m$, φ_∞ = 3, α = 1 s^{-1}. With this value of the relaxation modulus G, roughly the memory damping is five times stronger than the classical damping: the elastoviscous devices can be even 10 times stronger.

With reference to the column with classical damping only, the characteristic equation of the second moment equations [Eqs. (8) and (7), respectively] has one real root and two complex conjugate roots: by increasing w_P the real part of the latter becomes positive when $w_P > (85594.85)^2$ $newton^2$; a plot of Re(λ) against $\sqrt{w_P}$ is in Fig. 1. Thus $w_{Pcr} = (85594.85)^2$. It is noted that in the book by Lin and Cai, [26], in the Chap. 6 a stability limit is derived by Routh-Hurwitz criteria [11]. If this limit is adapted to the present case, it reads as

$$w_{Pcr} = \frac{4\zeta_1\omega_1}{\pi}\left(1 - \frac{\mu_P}{P_E}\right) EI\rho \qquad (16).$$

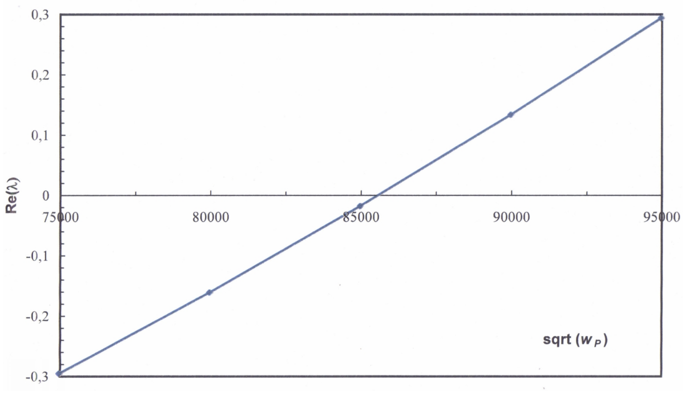

Figure 1: elastic column, real part of the eigenvalue λ against $\sqrt{w_P}$.

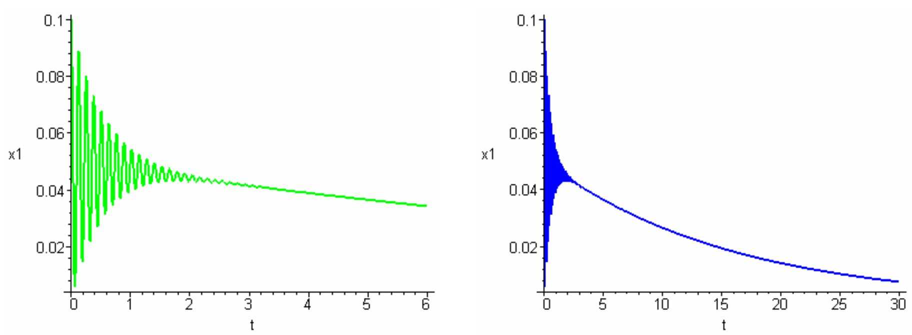

Figure 2: elastic column, time evolution of $E\lfloor z_1^2 \rfloor = E\lfloor V^2 \rfloor$ for $w_P = 0.95w_{Pcr}$.

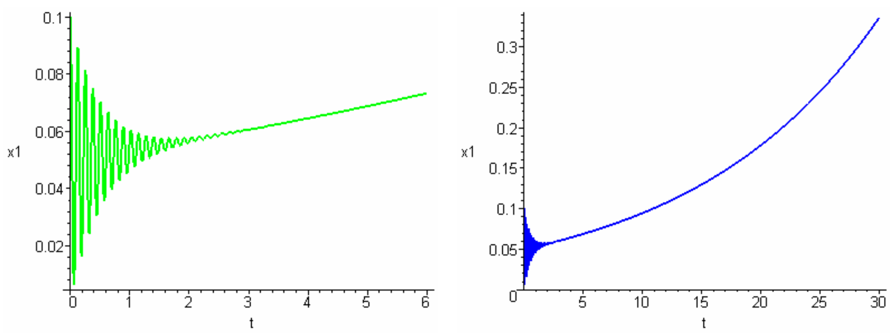

Figure 3: elastic column, time evolution of $E\lfloor z_1^2 \rfloor = E\lfloor V^2 \rfloor$ for $w_P = 1.05w_{Pcr}$.

In the present case Eq. (16) yields $w_{Pcr} = 0.924704 \cdot 10^{10}$ $newton^2$, which is larger than the finding of this paper by a 26.2 %. The exactness of the value determined by studying the eigenvalues of Eq. (8) has been controlled by solving the second moment equations [Eq. (8)] for $w_P = 0.95w_{Pcr}$, and $w_P = 1.05w_{Pcr}$, taking w_{Pcr} the value determined here. The plots are in Figs. 2,3, re-

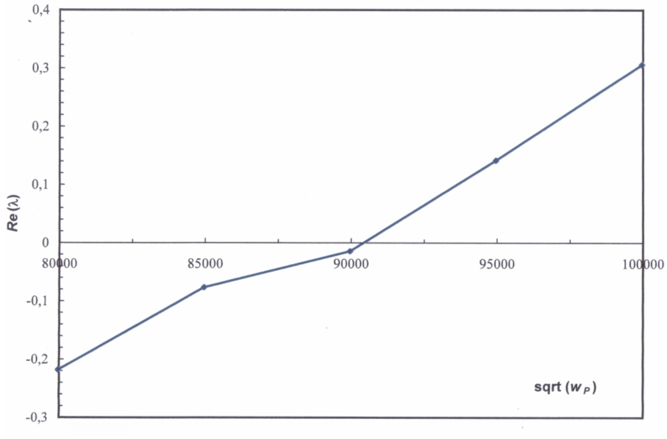

Figure 4: viscoelastic column, real eigenvalue λ against $\sqrt{w_P}$.

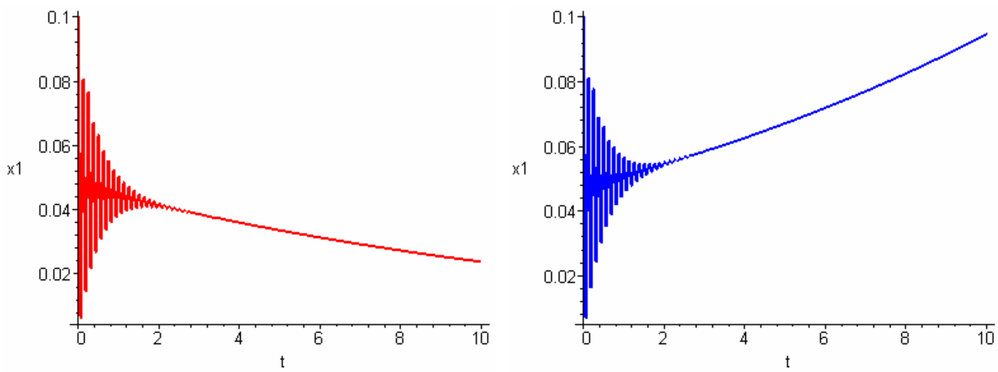

Figure 5: viscoelastic column, time evolution of $E\left[V^2\right]$: $w_P = 0.95w_{Pcr}$ (left), $w_P = 1.05w_{Pcr}$ (right).

spectively. It is clear that the value of w_{Pcr} obtained in this paper is the true critical value: in fact, when w_P is $0.95w_{Pcr}$, if an initial perturbation $E[\,z_1^2(0)\,] = 0.1$ is applied, after some cycles of oscillations the second moments decay to zero as does $E[\,z_1^2\,]$ shown in Fig. 2. When w_P is $1.05w_{Pcr}$, the second moments grow without limits if they are subjected to an initial perturbation (Fig. 3).

The characteristic equation of the second system that is examined, the viscoelastic column, is obtained from det $[\lambda\boldsymbol{I} - \boldsymbol{A}] = 0$, being the matrix $\boldsymbol{A}$ in Eq. (15). It has two real roots and two couples of complex conjugate roots. In this case, it is one of the real roots that becomes positive as w_P increases. The critical value is found to be $w_{Pcr} = 90464.5^2$ newton; a plot of λ against $\sqrt{\ w_{Pcr}}$ is in Fig. 4. The behavior of the second moments for $w_P = 0.95w_{Pcr}$ and $w_P = 1.05w_{Pcr}$ is analogous to that of the previous case: the plots of $E\left[z_1^2\right] = E\left[V^2\right]$ are shown in Fig. 5 for the two cases. It is worth noting that the supplemental damping of the viscoelastic system has little effect on the critical value, which is increased by a 5.69 % only with respect to the first case having classical damping only. The stochastic perturbation that causes instability is small if compared to the first Euler's buckling load: the ratio $\sqrt{w_{Pcr}}\,/P_E$ is only 0.022 and 0.023 in the two cases respectively.

4 CONCLUSIONS

This paper examines the stochastic stability of Euler's columns compressed by a stochastic axial force. This has a non zero mean, and the variable part is a Gaussian white noise. As is frequently done in stochastic mechanics, the dynamic stability is considered. Thus, the damping mechanism must be exactly defined. Herein, two damping types are assumed: (1) classic damping, that is the dissipative force is linearly related to the velocity; (2) classic damping and damping with memory, being the latter expressed by the convolution integral of the linear viscosity, in which the relaxation kernel is the Kelvin-Voigt's hereditary one.

If the banal rectilinear shape of the column is perturbed, after some cycles of motion a stable column tends to it; viceversa, from a stochastic point of view instability means divergent sample paths or divergent response statistical moments. The second aspect is studied here. In particular, the equations that govern the time evolution of the second moments are written by means of Itô's differential rule. Differently from other authors that look for the Lyapunov exponents of the moments or construct a Lyapunov functional, herein the eigenvalues of the coefficient matrix of the moment equations are analyzed: the stability limit is reached when the real part of an eigenvalue becomes positive or a real eigenvalue is positive. The availabilty of computer algebra programs makes it easier to perform the computations.

In the applications, it is found that the damping with memory has little effect on the stability limit even if it is generally believed to be very efficacious in suppressing the vibrations. In any case, the strength of the white noise that causes the loss of stability is small, which must be taken into account in engineering design.

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