On the stability of elastomeric bearings

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SUMMARY. The Timoshenko beam theory is considered according to the Engesser and the Haringx formulations showing that only the latter provides acceptable results when shear deformation is rather large as it occurs with elastomeric bearings. The physical parameters needed for the calculation of the critical load according to the Timoshenko-Haringx theory are briefly described. The formulation used in engineering practice for extending the results to large displacements is also summarized. Some numerical results available in the literature for strip bearings are briefly reviewed. A three-dimensional finite element model using a Neo-Hookean constitutive law for rubber is used to predict some buckling loads and modes of a square elastomeric bearing. A tensile fundamental buckling load confirms similar predictions from the Timoshenko – Haringx theory. A post-buckling analysis using Riks' method shows some aspects of the large displacement behavior of elastomeric bearings.

1 INTRODUCTION

The assessment of the stability of elastomeric isolation bearings proves to be a rather complex problem for several reasons. Firstly, simple formulations allowing for friendly use in engineering practice should be provided. Secondly, the stability of single bearings and/or of the whole isolation system and building should be ensured. At the moment the formulations available in the literature refer to the linear stability of single bearings in the undeformed configuration [1].

The Haringx theory is used for the evaluation of the critical load of a single isolation bearing with the base allowed to translate but with rotations prevented. The Haringx theory consists of the Timoshenko beam theory applied to compressed helical springs and/or rubber cylinders [2 - 4]. As it is well known, the stability of the Timoshenko beam may be based on two distinct assumptions, [5], the second one being more accurate than the first as already pointed out by Timoshenko. The first of such theories was developed by Engesser [6] while Haringx developed the second one with the objective of studying the stability of devices used in the isolation of machinery from vibrations. In the sixties an animated debate developed within the international scientific community on which of the two theories was the correct one with Timoshenko quite clearly leaning towards the second one [7].

Since the two theories provide results close to each other when the shear deformation is rather small, Timoshenko suggests the use of the second theory when the shear deformability is exceptionally large. A measure of such deformability is provided by the ratio between the Euler critical load and the critical load of a shear beam. Timoshenko limits his investigation to values of this ratio between zero and one, the lower bound pertaining to a rigid beam in shear and the upper bound obviously referring to an exceptionally high value of shear deformability.

Modern applications to the stability of elastomeric bearings imply values of the parameter above in the range of 10^3 to 10^4. In this range the difference between the two theories becomes extraordinary with the Engesser theory providing clearly unacceptable results. Just to give an idea of the actual behaviour, an elastomeric bearing of the type used in the Solarino buildings, with a diameter of 500 mm, a rubber height of 96 mm and a service load of 1200 kN, has a value of the shear deformability parameter of about 4000, an Engesser critical load of 92 kN and a Haringx critical load 5800 kN.

In a recent paper [8], the stiffness matrix for an elastomeric bearing has been evaluated for the Engesser and Haringx theories. This has been used for the calculation of the critical load of elastomeric bearings with the bottom base fixed and the top surface free to translate but prevented to rotate. It turned out that while the Engesser theory provided critical loads only in compression, the Haringx theory predicted critical loads also in tension. Actually the study was motivated because a critical load in tension was considered unacceptable. The fact shown above that the critical load provided by the Engesser theory was unreliably low, actually lower than the service load, together with the acknowledgement that the Engesser theory was based on a faulty assumption, led the authors to accept the Haringx theory and leave the check for the existence of critical loads in tension to more sophisticated analyses based on finite element models.

In the present work, after briefly describing the evaluation of the critical loads of base isolation bearings by using the Timoshenko beam theory and the formulation provided by Haringx, some results from two-dimensional and three-dimensional finite element models are used for comparison. The two-dimensional results are derived from the work by Takhirov and Kelly [9] while the three-dimensional results were obtained with a rather coarse mesh by the third author within a project for an Advanced Finite Element Analysis course at SUNY, University at Buffalo.

Subsequently, the current formulation adopted by the engineering practice for the evaluation of the load capacity of elastomeric bearings under large displacements is briefly reviewed and compared to the results of a three-dimensional finite element post-buckling analysis.

Finally it is shown that traditional beam theories may prove to be unrealistic when applied in the prediction of large displacement behaviour of elastomeric bearings because the underlying kinematical assumptions may no longer hold.

2 ENGESSER AND HARINGX CRITICAL LOADS

It has already been mentioned that this study was motivated by the unwillingness to accept critical loads in tension as predicted by the Haringx stability theory. However, in a previous report, [8], the first two of the present authors discussed the stability theory according to Engesser and Haringx showing that the two theories coincide in the absence of shear deformability and that the assumption behind the Haringx theory are the correct ones when shear deformation is included. The stability stiffness functions according to Haringx were also derived and applied for evaluating the critical loads of several structural systems with different end conditions. Here the critical loads calculated from the Engesser and the Haringx theory are shown in order to explain the paradoxical result described in the introduction. A structural element with end conditions similar to those of an elastomeric bearing in a base isolated building are considered. The typical parameters necessary for the calculation of the critical load are shown in table 1. The dimensionless translation stiffness $\gamma(\sigma, p)$ is a function of two other dimensionless parameters σ and p. The first is clearly related to the applied axial load P and may also be written in the form $\sigma = \pi\sqrt{P / P_E}$ which also shows its relation with the Euler critical load P_E.The second one is simply the ratio between the applied axial load and the critical load P_S of a shear beam. The translation stiffness has the same expression for the two theories but the key parameters α and D have different expressions obviously leading to different results. It is easy for the reader to realize that the two expressions become identical when $p \to 0$, that is when the shear deformability vanishes.

Table 1. Translation stiffness for structural element.

Engesser	Haringx
$\gamma = \sigma^3 s(\alpha)/D$	$\gamma = \sigma^3 s(\alpha)/D$
$\alpha = \sigma/\sqrt{1-p}$	$\alpha = \sigma\sqrt{1+p}$
$D = 2(1-c)/\sqrt{1-p} - s\sigma$	$D = 2\sqrt{1+p}(1-c(\alpha)) - s(\alpha)\sigma$
$\sigma^2 = Ph^2/EI$; $p = P/P_S$; $P_S = GA$	

The critical load P_C is the value of the axial load P for which the translation stiffness $\gamma(\sigma, p)$ vanishes. This leads to the following expression for the Engesser critical load:

$$\frac{P_C^E}{P_E} = \left(1 + \frac{P_E}{P_S}\right)^{-1} \tag{1}$$

It is clear from equation (1) that according to the Engesser theory there is not a critical load in tension and that in the absence of shear deformability ($P_S \to \infty$) the predicted critical load coincides with the Euler load P_E. However, the critical load tends asymptotically to zero as the shear deformability becomes larger and larger. The vanishing of the translation stiffness leads to the following expression for the critical load according to the Haringx theory:

$$\frac{P_C^H}{P_E} = \left(1 + \frac{P_C}{P_E}\frac{P_E}{P_S}\right)^{-1} \tag{2}$$

Equation (2), being quadratic, can admit and actually admits a negative solution thus predicting a critical load in tension; moreover the compression critical load coincides with the Euler critical load P_E when the shear deformability vanishes. The existence of a critical load in tension, always larger than the critical load in compression, is proved by the following expression for the Haringx critical loads:

$$P_C^H = \frac{P_S}{2}\left(-1 \pm \sqrt{1 + 4\frac{P_E}{P_S}}\right) \tag{3}$$

The following two inequalities derived from equations (1) and (2) show that the Engesser critical load is always smaller than the critical load of the corresponding shear beam and that the Haringx compression critical load is always larger than the corresponding one by Engesser.

$$P_C^E \leq P_S \tag{4}$$

$$P_C^H = \frac{P_E}{1+\frac{P_E}{P_S}\frac{P_C^H}{P_E}} \geq P_C^E \tag{5}$$

3 PHYSICAL PARAMETERS FOR ELASTOMERIC BEARINGS

The evaluation of the critical load of elastomeric bearings according to the Haringx theory requires the specification of two material parameters, E and G, and three geometrical parameters A, I and h. These must be evaluated from the properties of the constituent materials which are provided in the form of rubber layers and steel shims. Since herein only essential information can be provided, the interested reader is invited to consult the specialized literature for details [10,11]. Each rubber layer in a bearing is characterized by a thickness t, a confined area A_C and an unconfined area A_U. The shape factor S is defined as the ratio between the unconfined area A_U and the confined area A_C:

$$S = \frac{A_U}{A_C}, \tag{6}$$

which leads to the following expressions needed in ensuing parts of the present paper:

Circular bearings $$S = \frac{D}{4t} \tag{7}$$

Strip bearings $$S = \frac{B}{2t} \tag{8}$$

Square bearings $$S = \frac{B}{4t} \tag{9}$$

where D is the diameter of the confined rubber area for circular bearings and B is the width of the strip and the side of the square, for strip and square bearing respectively. The compression modulus

$$E_C = nGS^2 \tag{10}$$

is required for the definition of the bending stiffness of the bearing:

$$EI = \frac{1}{m} E_C \frac{h}{t_r} I \tag{11}$$

where

$$h = N_r t + \left(N_r - 1\right)\, t_s \tag{12}$$

is the total height of the bearing, N_r is the number of the rubber layers, t_r is the total rubber thickness, t_s is the thickness of steel shims and I is the second moment of area of the rubber confined area. Moreover, the area required for the definition of the shear rigidity takes the following expression:

$$A = A_C \frac{h}{t_r} \tag{13}$$

In expression (10), G is the shear modulus of the rubber while n is a parameter depending on the shape of the bearing and takes the value of 6 for circular bearings, 6.73 for square bearings and 4 for the infinite strip. Also, in expression (11) the parameter m takes the value of 3 for circular bearings, 5 for the infinite strip and 3.039 for square bearings. Considering that the Solarino bearings are composed of 12 rubber layers of 8 mm thickness and 11 steel shims of 3 mm thickness for a total height $h = 129\ mm$ and that the rubber shear modulus evaluated experimentally is $0.353\ MPa$, the critical load values provided in the introduction can easily be calculated using the information given above.

4 LOAD BEARING CAPACITY UNDER LARGE DISPLACEMENTS

In earthquake conditions elastomeric bearings must carry the weight of a building under large displacements which may sometimes be of the order of the height of the bearing. Therefore it is important to assess the bearing capacity in a deformed configuration. Since the stability theory described in Section 2 refers to small displacements, some approximate formulations have been devised to estimate the load capacity under large displacements in engineering practice [10,11]. The most popular of these formulations may be written as follows:

$$P_C = P_{C0} \frac{A_r}{A} \tag{14}$$

where P_C is an estimate of the critical load in the current configuration, P_{C0} is the critical load in the undeformed configuration, A is the area of the bottom base of the bearing and A_r is the intersection of the projection of the top base of the bearing with the bottom one, see Figure 1.

In the literature it has been shown that the formula (14) above may overestimate the critical load under small displacement and underestimate it under large displacements.

5 TWO-DIMENSIONAL FINITE ELEMENTS RESULTS

Takhirov and Kelly [9] have performed numerical studies on a set of five strip type elastomeric bearings with the intent of assessing the performance of the Timoshenko – Haringx theory in the prediction of the critical load under small displacements. Each bearing considered was characterized by a specific shape factor which varied from a minimum of 3 to a maximum of 14.

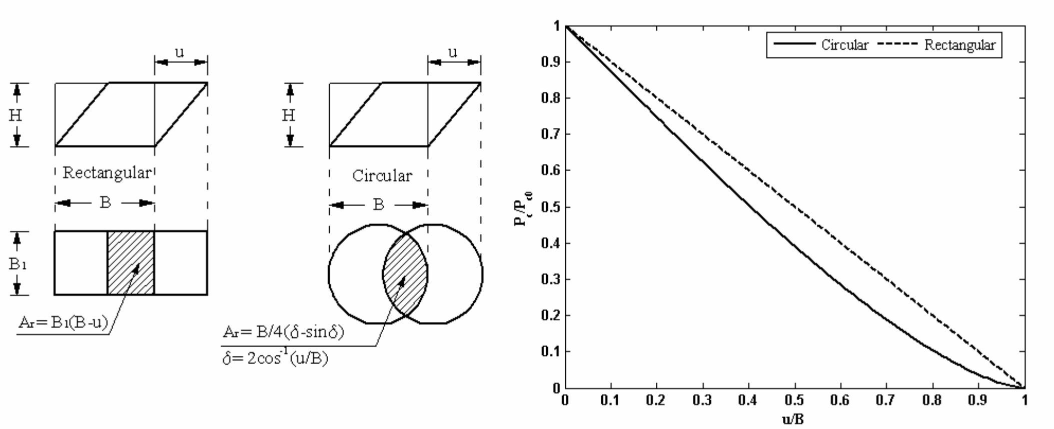

Figure 1: Estimate of critical load under large displacements

Table 2. Geometrical characteristics of elastomeric strip bearing.

Dimensions	in	mm
Steel shim thickness	0.1024	2.60
Total rubber thickness	3.15	80
Width	6.30	160
Rubber layer thickness	0.63	16

The authors considered two constitutive models for the rubber, one characterized by a 2nd order Polynomial form for strain energy and the other by a Neo-Hookean form with constants shown in Table 3. The mechanical properties for the steel shims were $E_S = 200000\ MPa$ and $\nu_S = 0.3$.

Table 3. Material parameters for 2nd order Polynomial form and Neo-Hookean form from experimental test data by Treloar (1940).

Rubber model	C_{10}, *psi*	C_{01}, *psi*	C_{20}, *psi*	C_{11}, *psi*	C_{02}, *psi*	D_1, *psi*	G, *psi*
Polynomial	28.03	-0.021	-0.1170	0.026	-0.0005	0	56.018
Neo-Hookean	50	0	0	0	0	0.0000067	100

For the evaluation of the critical load according to the Timoshenko – Haringx beam theory it is required to know the shear modulus *G* which is provided by expressions (15) and (16) for the Polynomial and Neo-Hookean models respectively.

$$G = 2\left(C_{10} + C_{01}\right) \tag{15}$$

$$G = 2C_{10} \tag{16}$$

Table 4. Critical loads (kN/mm) for strip bearing.

Analysis	Timoshenko – Haringx BT		Finite Element Analysis	
Material	Polynomial	Neo-Hookean	Polynomial	Neo-Hookean
Compression	0.47	0.83	0.67	1.20
Tension	0.54	0.96	0.39	0.70

The most important result that emerges from this analysis is that the Finite Element Analysis provides critical loads in compression that are larger than the ones given by the Beam Theory while the critical loads in tension from the Beam Theory are larger than those obtained from the Finite Element Analysis. Since the numerical analysis adopted a rather fine mesh the results obtained must be considered quite accurate. Therefore it appears that the Beam Theory provides safe estimates for the critical load in compression and unsafe estimates for the critical load in tension.

6 THREE-DIMENSIONAL FINITE ELEMENTS RESULTS

A three dimensional numerical model was developed for a square bearing having the same number of rubber layers and steel shims as the two dimensional strip bearing described in Section 5. The side of the square has the same length as the width of the strip.

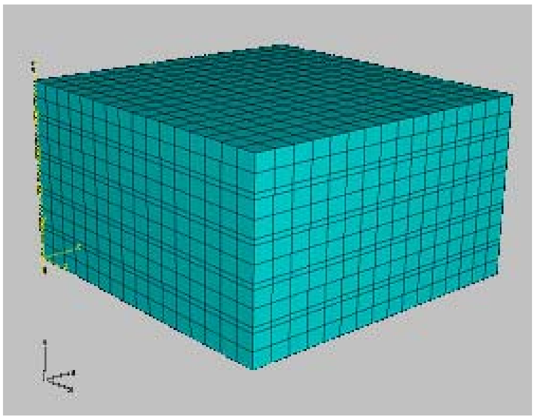

Figure 2: Finite element mesh of 3D model developed in ABAQUS [12].

The mesh for the finite element model is shown in Figure 2 where it can be seen that each steel shim is discretized by a single layer of C3D8 simple 8-noded linear brick elements while each rubber layer is discretized by two layers of C3D8H brick elements. The bottom base of the bearing is fixed while to the top one is associated a rigid surface to which a reference node is attached. This node is constrained in such a way that only one horizontal and the vertical translations are allowed. The material properties for the rubber layers were taken as specified in Table 3 for the Neo – Hookean material and the same properties for the steel shims as used for the strip model.

6.1 *Buckling analysis*

A buckling analysis was performed on the model described in the introductory part of the present section and the first five critical loads and buckling modes where evaluated. Four of the first five critical loads were in tension, the first mode being in tension and the second one in compression, as shown in Table 5 and Figure 3.

Table 5. First five critical loads for model in Figure 2.

Mode	1	2	3	4	5
Pc (kN)	-135	151	-186	-194	-194

The existence of critical loads in tension both in the two-dimensional model and in the three-dimensional one ensures that the critical load in tension predicted by the Timoshenko-Haringx theory is not an inessential by-product of the analysis but is an actual instability mode predicted also by more complex and reliable models.

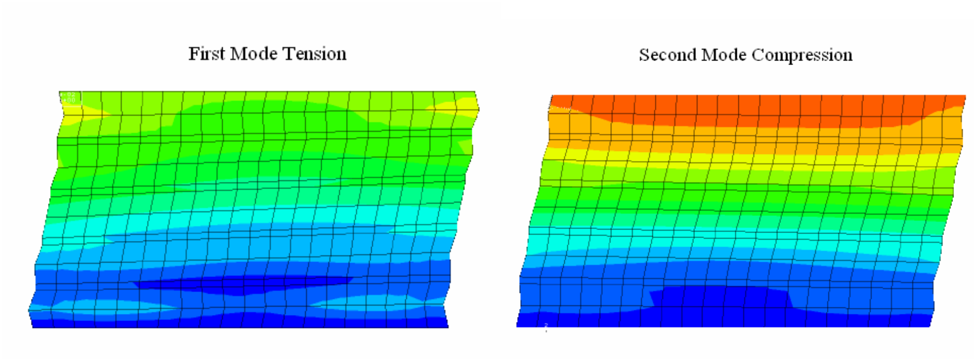

Figure 3: First two buckling modes for model in Figure 2.

6.2 *Comparison with Timoshenko – Haringx beam theory*

A comparison in terms of critical loads can be made with the results of the Timoshenko – Haringx beam theory and with an extension of the two-dimensional analysis of Takhirov and Kelly [9]. If the critical load of the strip bearing is multiplied by the width, an upper bound to the critical load of the corresponding square bearing should be found.

Table 6. Comparison from different models

Timoshenko – Haringx beam theory			FE Model
Strip bearing		Square bearing	Square bearing
kN/mm	kN	kN	kN
0.83	133	110	151
-0.96	-154	-130	-135

The comparison with the strip model shows that the mesh of the three-dimensional model is still rather coarse and a refinement accompanied by a convergence analysis is required. However, the general trend showing a critical load smaller in tension than in compression is confirmed, pointing out that the elementary beam theory can be unsafe when estimating the critical load in tension.

6.3 *Post-buckling analysis*

A post-buckling Riks Analysis has been performed by applying to the model an initial imperfection proportional to the first buckling mode and by gradually increasing the vertical load and/or the horizontal displacement. The assumed initial configuration is shown by the picture on the left of Figure 4 while the picture on the right shows the configuration assumed by the bearing at the maximum considered displacement.

The imposed displacement of the top base of the bearing was 10 mm while the maximum considered displacement was equal to the width of the bearing, that is 160 mm.

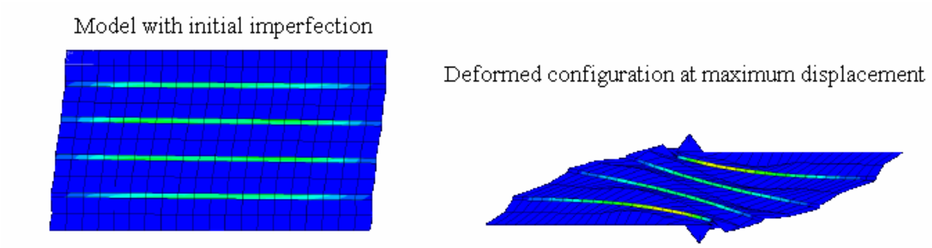

Figure 4: Initial configuration and configuration at maximum considered displacement.

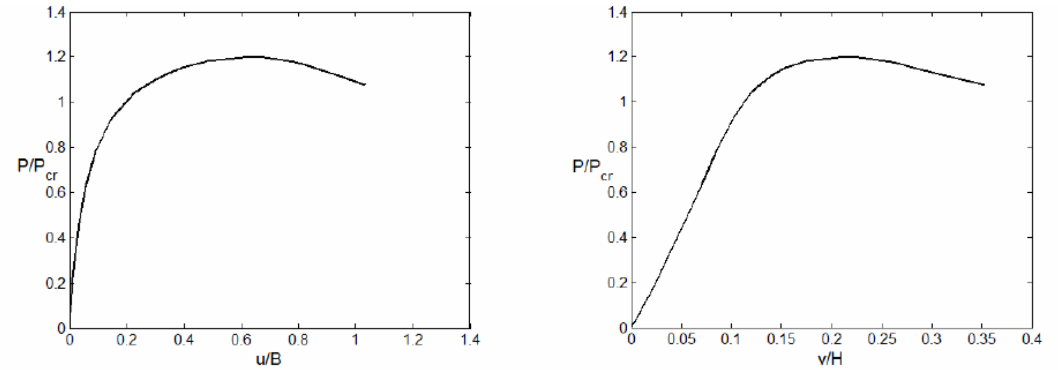

Figure 5: Vertical load versus horizontal displacement (left) and vertical displacement (right).

The vertical load – horizontal displacement curve shows that the post-buckling behavior is initially stable but becomes unstable at displacements equal or larger than about half the width of the bearing. Also, the vertical load – vertical displacement curve allows for the evaluation of the vertical stiffness of the bearing as a function of the horizontal displacement. An actual calculation shows that the vertical stiffness initially increases up to a maximum value under a vertical load of about 80 % of the critical load in the initial configuration and then decreases at a roughly constant rate. The vertical stiffness at the maximum considered displacement is about 35% of the maximum.

7 CONCLUSIONS

The Timoshenko-Haringx beam theory has been compared to the Engesser-Timoshenko beam theory in the stability theory of elastomeric bearings showing that the latter provides unacceptable results. The Timoshenko-Haringx beam theory provides a critical load in tension as well as one in compression, the former one being larger than the latter. Two-dimensional numerical analyses conducted on 5 strip bearings with shape factors equal to 3,5,7,10 and 14 have shown that the critical load in tension is always much smaller than the one in compression, thus contradicting the results from the beam theory. The critical load in compression derived from the numerical model is larger than the one predicted by the beam theory, ensuring that the latter provides results on the safe side. However, for the tensile critical load, the beam theory overestimates by a large amount the critical value predicted by the numerical method. The buckling analyses conducted on the strip model can be considered very accurate because a very refined mesh was used. One of the strip models was assumed as the basis for the development of a three-dimensional finite element model of a square elastomeric bearing. The choice fell on the model with shape factor equal to 5 which, because of the change of shape, became 2.5 in the 3D model. The analyses conducted on this model were only preliminary and for opportunistic reasons a rather coarse mesh was used. However, the trend predicted by the more accurate 2D models of a tensile critical load much smaller than the compression one was confirmed. The compression critical load resulted larger than the corresponding one predicted by the strip model, but this may be explained by the coarseness of the mesh used. A smaller critical load can be expected when the 3D mesh has been sufficiently refined.

The post-buckling analyses conducted on the strip models showed a rather stable behaviour in compression up to a shear deformation of about 0.10 and then, up to a shear deformation of nearly 0.35, the load carrying capacity either remained constant or increased at a small rate.

The post-buckling analysis conducted on the 3D model showed an initially stable post-buckling behaviour and an unstable behaviour for a lateral displacement larger than about 60% of the width of the bearing. Another important aspect shown by the post-buckling analysis is that cross-sections do not remain plane at large shear deformations and that beam theories based on such assumptions perhaps cannot be expected to provide reliable results on the load capacity of elastomeric bearings under large displacements.

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