Linear stability analysis of multiparameter dynamical systems via a numerical-perturbation approach

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SUMMARY. A numerical-perturbation method, based on eigenvalue sensitivity analysis, is developed, able to build-up linear stability diagrams of multiparameter dynamical systems. The algorithm furnishes the critical combinations of parameters causing multiple bifurcations, of static, dynamical or mixed type. Moreover, it determines the manifolds in the parameter space on which simple bifurcations take place. Although the method is general, it is illustrated here for codimension-2 bifurcations only, sufficient to highlight the underlying ideas.

1 INTRODUCTION

Dynamical systems usually depend on parameters. When some of them are varied in some region of the parameter-space $\{\boldsymbol{\mu}\} \subseteq \mathbb{R}^M$, bifurcations from an equilibrium position are likely to occur. The bifurcations can be both of static (divergence) or dynamic (Hopf) nature, according to the values assumed by the critical eigenvalue(s) of the Jacobian matrix of system, namely zero or purely imaginary, respectively. These basic bifurcations occur on smooth codimension-1 surfaces; if the surfaces intersect each other along lines or points, *multiple bifurcations* occur at these higher-codimension geometrical loci [1]. The critical surfaces divide the parameter-space in regions in which the system has qualitatively similar behaviors, being different in each region. The plot of the organized parameter-space is called the *linear stability diagram*, and gives a comprehensive scenario of the system behavior near the equilibrium. Aim of the linear stability analysis is to build-up such a diagram.

When the system to be analyzed has few degrees-of-freedom, it is not difficult to find the bifurcation loci by manipulating the closed-form characteristic polynomial of the Jacobian matrix. The task requires finding the combinations of parameters $\boldsymbol{\mu}_c$ such that a number of critical eigenvalues $\lambda_k = \lambda_k(\boldsymbol{\mu}_c)$ are zero, or have zero real part. In contrast, when the dimension of the system increases, pure numerically-based analyses would entail the quite expansive reiterate computation of the eigenvalues of a large matrix, in order to evaluate their parameter-derivatives $\partial^n \lambda_k / \partial \boldsymbol{\mu}^n$ (called *sensitivities*). Moreover, when two or more critical eigenvalues coalesce, such methods fail, since the eigenvalues are *not analytical* at the coalescence point, in the sense that their first derivatives with respect to the parameters are infinite.

As opposed to pure numerical methods, evaluating eigenvalue sensitivities by perturbation methods is cheap and of easy implementation, since only solution of a chain of linear algebraic equations is required. Therefore, it seems important, to formulate an algorithm that, exploiting the strong potentialities of perturbation methods, is able to find: (1) the multiple bifurcation loci and, (2) the branches of critical surfaces originating from it. That constitutes the main goal of this paper.

Reference is made here to a general system, undergoing codimension-2 bifurcations. Two bifurcation parameters are therefore sufficient to exhaustively describe the scenario. In Sect 2 the simplest case of non-coincident critical eigenvalues is studied. In Sect 3 the more complex case of coincident

critical eigenvalues is analyzed. In Sect 4 some numerical examples are illustrated. Finally, some conclusions are drawn in Sect 5.

2 NON-COINCIDENT CRITICAL EIGENVALUES

2.1 *Eigenvalue sensitivity*

Let us assume, first, that the Jacobian matrix $\mathbf{A} = \mathbf{A}(\boldsymbol{\mu})$ of the system admits two distinct critical eigenvalues at the (unknown) bifurcation point C. In Ref. [2] it was shown that, given a matrix $\mathbf{A} = \mathbf{A}(\boldsymbol{\mu})$, depending on two real parameters $\boldsymbol{\mu} := \{\mu, \nu\}^{\mathrm{T}}$, the following series expansion holds for its eigenvalues $\lambda_k(\boldsymbol{\mu})$:

$$\begin{pmatrix}\lambda_1(\boldsymbol{\mu}+\delta\boldsymbol{\mu})\\ \lambda_2(\boldsymbol{\mu}+\delta\boldsymbol{\mu})\end{pmatrix} = \begin{pmatrix}\lambda_1(\boldsymbol{\mu})\\ \lambda_2(\boldsymbol{\mu})\end{pmatrix} + \begin{pmatrix}S_{1\mu}(\boldsymbol{\mu}) & S_{1\nu}(\boldsymbol{\mu})\\ S_{2\mu}(\boldsymbol{\mu}) & S_{2\nu}(\boldsymbol{\mu})\end{pmatrix}\begin{pmatrix}\delta\mu\\ \delta\nu\end{pmatrix} + \mathrm{O}(|\delta\boldsymbol{\mu}|^2) \tag{1}$$

in which $S_{k\alpha}(\boldsymbol{\mu})$ ($k = 1, 2; \alpha = \mu, \nu$) are the *eigenvalue sensitivities* at $\boldsymbol{\mu}$, given by:

$$\left[S_{k\alpha}(\boldsymbol{\mu})\right] := \begin{bmatrix}\mathbf{y}_1^{\mathrm{H}}(\boldsymbol{\mu})\mathbf{A}_\mu(\boldsymbol{\mu})\mathbf{x}_1(\boldsymbol{\mu}) & \mathbf{y}_1^{\mathrm{H}}(\boldsymbol{\mu})\mathbf{A}_\nu(\boldsymbol{\mu})\mathbf{x}_1(\boldsymbol{\mu})\\ \mathbf{y}_2^{\mathrm{H}}(\boldsymbol{\mu})\mathbf{A}_\mu(\boldsymbol{\mu})\mathbf{x}_2(\boldsymbol{\mu}) & \mathbf{y}_2^{\mathrm{H}}(\boldsymbol{\mu})\mathbf{A}_\nu(\boldsymbol{\mu})\mathbf{x}_2(\boldsymbol{\mu})\end{bmatrix} \tag{2}$$

and where$\mathbf{A}_\mu := \partial\mathbf{A}/\partial\mu$, $\mathbf{A}_\nu := \partial\mathbf{A}/\partial\nu$ and, moreover, $\mathbf{x}_k(\boldsymbol{\mu})$ and $\mathbf{y}_k(\boldsymbol{\mu})$ are right and left eigenvectors of $\mathbf{A}(\boldsymbol{\mu})$, respectively, associated with $\lambda_k(\boldsymbol{\mu})$.

2.2 *Searching for the critical point*

The search for the (unknown) critical point C, at which $\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_c)] = 0$ ($k = 1, 2$) (dynamic bifurcation), is performed through an iterative scheme (Newton method), based on the linear extrapolation of the eigenvalues. If an approximation $\boldsymbol{\mu}_i = \{\mu_i, \nu_i\}^{\mathrm{T}}$ of $\boldsymbol{\mu}_c$ is known, we look for parameter-increments $\delta\boldsymbol{\mu} := \boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i$ such that $\mathrm{Re}[\lambda_k(\boldsymbol{\mu}+\delta\boldsymbol{\mu})] = 0$; from Eqs (1), by neglecting the reminder, it follows:

$$\begin{pmatrix}\mathrm{Re}[S_{1\mu}(\boldsymbol{\mu}_i)] & \mathrm{Re}[S_{1\nu}(\boldsymbol{\mu}_i)]\\ \mathrm{Re}[S_{2\mu}(\boldsymbol{\mu}_i)] & \mathrm{Re}[S_{2\nu}(\boldsymbol{\mu}_i)]\end{pmatrix}\begin{pmatrix}\mu_{i+1}-\mu_i\\ \nu_{i+1}-\nu_i\end{pmatrix} = -\begin{pmatrix}\mathrm{Re}[\lambda_1(\boldsymbol{\mu}_i)]\\ \mathrm{Re}[\lambda_2(\boldsymbol{\mu}_i)]\end{pmatrix} \tag{3}$$

If, in contrast, the bifurcation is of mixed static-dynamic type, the Re operator must be omitted in a row (the double-zero case will be analyzed ahead). It should be noted, that Eq (3) does not require numerical evaluations of the sensitivities via incremental ratios, as usually done in purely numerical methods, since these are furnished by the perturbation analysis (Eq (2)).

By resuming, the algorithm is the following:

1) evaluate, e.g. by the QR-method, the two eigenvalues of matrix $\mathbf{A} = \mathbf{A}(\boldsymbol{\mu}_i)$ with smallest real parts, candidate to become critical (e.g. one real and the other complex, or two complex *not* conjugate), with the associated right and left eigenvectors $\mathbf{x}_k(\boldsymbol{\mu}_i)$, $\mathbf{y}_k(\boldsymbol{\mu}_i)$;

2) compute, via Eq (2), the four sensitivities $S_{k\alpha}(\boldsymbol{\mu}_i)$;

3) solve the linear system (3) for the $i+1$ approximation of the parameter vector $\boldsymbol{\mu}_{i+1} = \{\mu_{i+1}, \nu_{i+1}\}^{\mathrm{T}}$;

4) if $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i\right| > toll_1$ *and* $\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_{i+1})] > toll_2$ ($k = 1, 2$), then execute a new iteration, with $i = i+1$; if, instead, $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i\right| \le toll_1$ *or* $\left|\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_{i+1})]\right| \le toll_2$ ($k = 1, 2$), then assume $\boldsymbol{\mu}_c = \boldsymbol{\mu}_{i+1}$.

2.3 *Building-up the bifurcation loci*

Once the critical point C has been determined in the parameter plane, the two curves originating from it, which are loci of simple bifurcations, are sought for. On each of them, *just one* eigenvalue (λ_1 or λ_2, respectively) is critical, the other having non-zero real part; consequently, the relevant equation is $\mathrm{Re}[\lambda_k(\mu,\nu)] = 0$ (or $\lambda_k(\mu,\nu) = 0$) for $k = 1$ *or* $k = 2$. This equation, implicitly defines a curve in the (μ,ν)-plane, passing through C. After linearization around a point $\boldsymbol{\mu}_i = \{\mu_i, \nu_i\}^{\mathrm{T}}$, and according to Eqs (1), it reads:

$$\mathrm{Re}[S_{k\mu}(\boldsymbol{\mu}_i)](\mu_{i+1} - \mu_i) + \mathrm{Re}[S_{k\nu}(\boldsymbol{\mu}_i)](\nu_{i+1} - \nu_i) = -\,\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_i)] \quad k = 1 \textit{ or } 2 \tag{4}$$

An iterative scheme could directly be applied to Eq (4), in order to obtain a Cartesian representation for the curve of the form $\mu = \mu(\nu)$ or $\nu = \nu(\mu)$; however, as well known, such a representation fails at turning points. Therefore, a parametric representation of the curve, namely $\mu = \mu(s)$, $\nu = \nu(s)$, with s a parameter, is preferable. To obtain it, a (constraint) scalar equation must be appended to Eq (4), in order to define the meaning of s; the more common choices for the constraint are referred in literature as the *arclength method*, or the *pseudo-arclength method* [3] (Fig. 1). When a point $\boldsymbol{\mu}_0$ is known on the curve, i.e. $\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_0)] = 0$, a close point is sought iteratively as $\boldsymbol{\mu}_i$, $\boldsymbol{\mu}_{i+1}, \cdots$. According to the arclength method, $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_0\right| = |\Delta s|$ is fixed for some small increment $|\Delta s|$ of the modulus of the parameter (with $\Delta s > 0$ or $\Delta s < 0$), and the following constraint equation appended to Eq (4):

$$\sqrt{(\mu_{i+1} - \mu_0)^2 + (\nu_{i+1} - \nu_0)^2} = |\Delta s| \tag{5}$$

According to the pseudo-arclength method, $(\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_0) \cdot \mathbf{t}_{k0} = \Delta s$, is instead prescribed, namely:

$$a_{k0}\,(\mu_{i+1} - \mu_0) + b_{k0}\,(\nu_{i+1} - \nu_0) = \Delta s \tag{6}$$

where $\mathbf{t}_{k0} = \{a_{k0}, b_{k0}\}^{\mathrm{T}}$ is the unit vector tangent to the curve (k=1 or 2) at $\boldsymbol{\mu}_0$ and, moreover:

$$\begin{aligned} a_{k0} &:= +\frac{\mathrm{Re}\left[S_{k\nu}(\boldsymbol{\mu}_0)\right]}{\sqrt{\left\{\mathrm{Re}\left[S_{k\mu}(\boldsymbol{\mu}_0)\right]\right\}^2 + \left\{\mathrm{Re}\left[S_{k\nu}(\boldsymbol{\mu}_0)\right]\right\}^2}},\ k = 1 \textit{ or } 2 \\ b_{k0} &:= -\frac{\mathrm{Re}\left[S_{k\mu}(\boldsymbol{\mu}_0)\right]}{\sqrt{\left\{\mathrm{Re}\left[S_{k\mu}(\boldsymbol{\mu}_0)\right]\right\}^2 + \left\{\mathrm{Re}\left[S_{k\nu}(\boldsymbol{\mu}_0)\right]\right\}^2}},\ k = 1 \textit{ or } 2 \end{aligned} \tag{7}$$

are its components. Thus, in the two approaches, $\boldsymbol{\mu}_{i+1}$ respectively moves on a circle of radius $|\Delta s|$ centered at $\boldsymbol{\mu}_0$, or along a line which is parallel to the normal $\mathbf{n}_{k0}$ to the curve at $\boldsymbol{\mu}_0$, at a distance $|\Delta s|$ from it. In both cases, the ambiguity of the sign of Δs refers to the two opposite directions in which the curve can be traveled.

The constraint equation (5) is nonlinear, whereas the constraint equation (6) is linear. In order to keep the whole problem linear, the pseudo-arclength method is adopted here. The relevant algorithm is detailed below.

1) take the critical point $\boldsymbol{\mu}_c$ as initial point $\boldsymbol{\mu}_0$, and evaluate the sensitivities $S_{k\mu}(\boldsymbol{\mu}_0), S_{k\nu}(\boldsymbol{\mu}_0)$ via Eq (2); then, compute the direction cosines a_{k0}, b_{k0} (Eqs (7));

2) select a new point $\boldsymbol{\mu}_1$(*predictor phase*) on the tangent $\mathbf{t}_{k0} = \{a_{k0}, b_{k0}\}^{\mathrm{T}}$ at $\boldsymbol{\mu}_0$, at a sufficiently small distance $|\Delta s|$ from $\boldsymbol{\mu}_0$, having coordinates:

$$\mu_1 = \mu_0 + a_k\,\Delta s, \quad \nu_1 = \nu_0 + b_k\,\Delta s, \quad k = 1 \textit{ or } 2 \tag{8}$$

3) solve iteratively (for $i = 1, 2, \cdots$) the following equations in the unknown μ_{i+1}, ν_{i+1} (*corrector* phase):

$$\begin{pmatrix} \mathrm{Re}[S_{k\mu}(\boldsymbol{\mu}_i)] & \mathrm{Re}[S_{k\nu}(\boldsymbol{\mu}_i)] \\ a_{k0} & b_{k0} \end{pmatrix} \begin{pmatrix} \mu_{i+1} - \mu_i \\ \nu_{i+1} - \nu_i \end{pmatrix} = \begin{pmatrix} -\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_i)] \\ \Delta s \end{pmatrix} \tag{9}$$

4) if $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i\right| > toll_1$ *and* $\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_{i+1})] > toll_2$ ($k = 1$ *or* 2), then execute a new iteration (9); if, instead, $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i\right| \leq toll_1$ *or* $\left|\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_{i+1})]\right| \leq toll_2$ ($k = 1$ *or* 2), then assume as a new point $\boldsymbol{\mu}_0 = \boldsymbol{\mu}_{i+1}$, and restart from step 2).

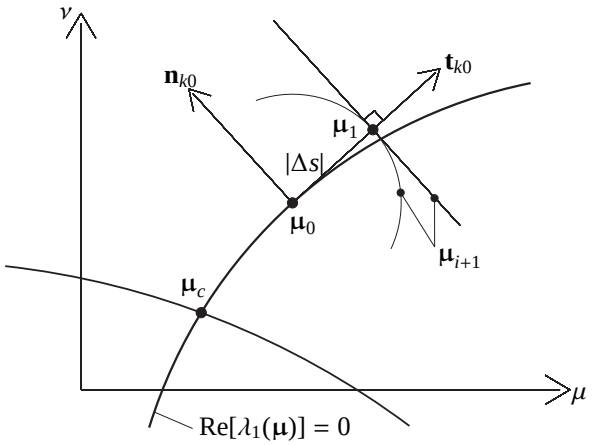

Figure 1: Arclength and pseudo-arclength iterative methods.

3 COINCIDENT CRITICAL EIGENVALUES

3.1 *Eigenvalue sensitivity*

The eigenvalue sensitivity analysis for a Jacobian matrix $\mathbf{A}(\boldsymbol{\mu})$ is more difficult when several eigenvalues coincide at the multiple bifurcation point C, being all zero (multiple zero-bifurcation) or equal to the same pair of complex conjugate purely imaginary numbers (multiple Hopf bifurcation). In these cases $\mathbf{A}(\boldsymbol{\mu}_c)$ is (generally) *defective* at the bifurcation, i.e. it does not posses a complete set of eigenvalues; consequently, it is *nearly-defective* close to the critical point, i.e. a complete set of eigenvectors does exist, but some of them are nearly-coincident. It was shown in Ref. [4], that sensitivities of nearly-defective eigenvalues cannot be evaluated independently, as in Eqs (1), but they are, in contrast, coupled. Moreover, the non-analytical nature of the eigenvalues requires using *fractional power expansions* in the parameters.

Here we limit ourselves to the simplest case of two *real* critical eigenvalues (double-zero, or Takens-Bogdanov, bifurcation), which is generic in two-parameter families of systems. As shown in Ref. [4], the problem of two nearly-coincident eigenvalues $\lambda_1(\boldsymbol{\mu}) \simeq \lambda_2(\boldsymbol{\mu})$ of a nearly-defective matrix $\mathbf{A}(\boldsymbol{\mu})$ is overcome by starting the expansion *not* from the actual system, but rather by an *ideal*

system $\mathbf{A}_0(\boldsymbol{\mu};\xi(\boldsymbol{\mu}))$, belonging to an enlarged parameter space $\{\boldsymbol{\mu},\xi\}$, in which the two eigenvalues coalesce at $\lambda_0(\boldsymbol{\mu}) := (\lambda_1(\boldsymbol{\mu}) + \lambda_2(\boldsymbol{\mu}))/2$. To achieve this goal, an *inverse problem* must be solved, in which the *small* additional parameter $\xi \in \mathbb{R}$ must determined in order to render $\mathbf{A}_0(\boldsymbol{\mu};\xi(\boldsymbol{\mu}))$ defective. After that, the sensitivities of $\lambda_0(\boldsymbol{\mu})$ must be evaluated.

According to Ref. [4], the ideal defective matrix is:

$$\mathbf{A}_0(\boldsymbol{\mu};\xi(\boldsymbol{\mu})) = \mathbf{A}(\boldsymbol{\mu}) - \xi(\boldsymbol{\mu})\,\mathbf{x}_{20}(\boldsymbol{\mu})\,\mathbf{y}_{10}^{\mathrm{H}}(\boldsymbol{\mu}) \tag{10}$$

where:

$$\begin{gathered} \xi(\boldsymbol{\mu}) = \tfrac{1}{4}\left[\lambda_1(\boldsymbol{\mu}) - \lambda_2(\boldsymbol{\mu})\right]^2, \\ \mathbf{x}_{20}(\boldsymbol{\mu}) = \tfrac{1}{2}\xi^{-1/2}(\boldsymbol{\mu})\left[\mathbf{x}_1(\boldsymbol{\mu}) - \mathbf{x}_2(\boldsymbol{\mu})\right], \quad \mathbf{y}_{10}(\boldsymbol{\mu}) = \mathbf{y}_1(\boldsymbol{\mu}) + \mathbf{y}_2(\boldsymbol{\mu}), \\ \mathbf{x}_{10}(\boldsymbol{\mu}) = \tfrac{1}{2}\left[\mathbf{x}_1(\boldsymbol{\mu}) + \mathbf{x}_2(\boldsymbol{\mu})\right], \quad \mathbf{y}_{20}(\boldsymbol{\mu}) = \xi^{1/2}\left[\mathbf{y}_1(\boldsymbol{\mu}) - \mathbf{y}_2(\boldsymbol{\mu})\right] \end{gathered} \tag{11}$$

are the additional perturbation parameter ξ and the generalized right and left eigenvectors $(\mathbf{x}_{20}, \mathbf{y}_{10})$ of $\mathbf{A}_0(\boldsymbol{\mu};\xi(\boldsymbol{\mu}))$, respectively; the proper right and left eigenvectors $(\mathbf{x}_{10}, \mathbf{y}_{20})$ of the same matrix will be used later. All these quantities are evaluated from the (nearly coincident) eigenvalues, λ_k, and associated right and left eigenvectors, $\mathbf{x}_k$ and $\mathbf{y}_k$, of the given matrix $\mathbf{A}(\boldsymbol{\mu})$. It should be noticed that, if the eigenvalues λ_k are complex conjugate, than $\mathbf{x}_{10}$, $\mathbf{y}_{10}$ are real, while $\mathbf{x}_{20}$, $\mathbf{y}_{20}$ are purely imaginary; based on this, it is easy to check that all the quantities involved in the following analysis are real.

Second-order sensitivity analysis of $\mathbf{A}_0(\boldsymbol{\mu};\xi(\boldsymbol{\mu}))$, carried out along the lines of Ref. [4], leads, after some manipulations, to the following second-degree *sensitivity equation*, in the increment $\Delta\lambda_{1,2} := \lambda_{1,2}(\boldsymbol{\mu} + \delta\boldsymbol{\mu}) - \lambda_0(\boldsymbol{\mu})$:

$$\Delta\lambda^2 - [S_{1\mu}(\boldsymbol{\mu})\delta\mu + S_{1\nu}(\boldsymbol{\mu})\delta\nu]\Delta\lambda - [\xi(\boldsymbol{\mu}) + S_{2\mu}(\boldsymbol{\mu})\delta\mu + S_{2\nu}(\boldsymbol{\mu})\delta\nu] + \mathrm{O}(|\delta\boldsymbol{\mu}|^{3/2}) = 0 \tag{12}$$

where:

$$\begin{aligned} S_{1\alpha}(\boldsymbol{\mu}) &:= \mathbf{y}_{20}^{\mathrm{H}}(\boldsymbol{\mu})\mathbf{A}_\alpha(\boldsymbol{\mu})\mathbf{x}_{20}(\boldsymbol{\mu}) - \mathbf{y}_{20}^{\mathrm{H}}(\boldsymbol{\mu})\mathbf{u}_\alpha^*(\boldsymbol{\mu}), \\ S_{2\alpha}(\boldsymbol{\mu}) &:= \mathbf{y}_{20}^{\mathrm{H}}(\boldsymbol{\mu})\mathbf{A}_\alpha(\boldsymbol{\mu})\mathbf{x}_{10}(\boldsymbol{\mu}), \quad \alpha = \mu, \nu \end{aligned} \tag{13}$$

are called *sensitivities of order-1 and order-1/2*, respectively, and moreover $\mathbf{u}_\mu^*(\boldsymbol{\mu})$, $\mathbf{u}_\nu^*(\boldsymbol{\mu})$ are solutions for the following linear problems:

$$\begin{cases} [\mathbf{A}_0(\boldsymbol{\mu}) - \lambda_0(\boldsymbol{\mu})\mathrm{I}]\,\mathbf{u}_\alpha^*(\boldsymbol{\mu}) = S_{2\alpha}\mathbf{x}_{20}(\boldsymbol{\mu}) - \mathbf{A}_\alpha(\boldsymbol{\mu})\mathbf{x}_{10}(\boldsymbol{\mu}) \\ \mathbf{e}_h^{\mathrm{T}}\mathbf{u}_\alpha^*(\boldsymbol{\mu}) = 0, \quad \alpha = \mu, \nu \end{cases} \tag{14}$$

made unique by a normalization condition (here $\mathbf{e}_h$ is the hth N-dimensional canonical vector). Note that when $\delta\mu = \delta\nu = 0$, Eq (12) correctly leads to λ_1, λ_2. Therefore, $\xi(\boldsymbol{\mu})$ brings back from the ideal $\mathbf{A}_0(\boldsymbol{\mu};\xi(\boldsymbol{\mu}))$ to the actual system $\mathbf{A}(\boldsymbol{\mu})$, while $\delta\mu, \delta\nu$ account for the true perturbation; the two effects, however, cannot be separated.

Equation (12) shows that $\Delta\lambda = \mathrm{O}\left((\xi + |\Delta\boldsymbol{\mu}|)^{1/2}\right)$. If $\xi = 0$ (i.e. $\boldsymbol{\mu} = \boldsymbol{\mu}_c$), then $\Delta\lambda/\Delta\boldsymbol{\mu} \to \infty$ when $\Delta\boldsymbol{\mu} \to 0$, this denoting that $\lambda(\boldsymbol{\mu}_c)$ is not analytical at the coalescence point. The increment $\Delta\lambda$ is therefore mainly governed by sensitivities of order-1/2, $S_{2\alpha}$; however, there always exist a special combination of the increments of the parameters $\delta\mu$ (i.e. a singular direction in the parameter space) for which $S_{2\alpha} = 0$, this entailing that $\Delta\lambda = \mathrm{O}(|\Delta\boldsymbol{\mu}|)$ in a narrow angular sector containing this direction.

3.2 *Searching for the critical point*

Let us assume to know a trial set of parameters $\boldsymbol{\mu}_i$, close to $\boldsymbol{\mu}_c$, for which the two critical conditions $\lambda_{1,2}(\boldsymbol{\mu}_c) = 0$ are approximately satisfied. In order to refine the approximation, we can use the sensitivity equation (12) (with the remainder neglected), which furnishes, with $\boldsymbol{\mu} = \boldsymbol{\mu}_i$ and $\delta\boldsymbol{\mu} = \boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i$, the eigenvalues $\lambda_{1,2}(\boldsymbol{\mu}_{i+1}) = \lambda_0(\boldsymbol{\mu}_i) + \Delta\lambda_{1,2}(\boldsymbol{\mu}_i, \delta\boldsymbol{\mu})$; this guides us in choosing the increment $\delta\boldsymbol{\mu}$ making $\lambda_{1,2}(\boldsymbol{\mu}_{i+1}) = 0$. The operation is easily carried out if we rewrite the sensitivity equation (12) in the form of a *reduced characteristic equation*:

$$\lambda^2 - I_1(\boldsymbol{\mu}, \delta\boldsymbol{\mu})\lambda - I_2(\boldsymbol{\mu}, \delta\boldsymbol{\mu}) + \mathrm{O}(|\delta\boldsymbol{\mu}|^{3/2}) = 0 \tag{15}$$

where:

$$\begin{aligned} I_1(\boldsymbol{\mu}, \delta\boldsymbol{\mu}) &:= S_{1\mu}(\boldsymbol{\mu})\delta\mu + S_{1\nu}(\boldsymbol{\mu})\delta\nu + 2\lambda_0(\boldsymbol{\mu}) \\ I_2(\boldsymbol{\mu}, \delta\boldsymbol{\mu}) &:= S_{2\mu}(\boldsymbol{\mu})\delta\mu + S_{2\nu}(\boldsymbol{\mu})\delta\nu - \lambda_0(\boldsymbol{\mu})I_1(\boldsymbol{\mu}, \delta\boldsymbol{\mu}) + \xi(\boldsymbol{\mu}) + \lambda_0^2(\boldsymbol{\mu}) \end{aligned} \tag{16}$$

and we require the invariants vanish simultaneously, namely $I_1(\boldsymbol{\mu}_i, \delta\boldsymbol{\mu}) = 0$, $I_2(\boldsymbol{\mu}_i, \delta\boldsymbol{\mu}) = 0$, i. e.:

$$\begin{pmatrix} S_{1\mu}(\boldsymbol{\mu}_i) & S_{1\nu}(\boldsymbol{\mu}_i) \\ S_{2\mu}(\boldsymbol{\mu}_i) & S_{2\nu}(\boldsymbol{\mu}_i) \end{pmatrix} \begin{pmatrix} \mu_{i+1} - \mu_i \\ \nu_{i+1} - \nu_i \end{pmatrix} = -\begin{pmatrix} \lambda_1(\boldsymbol{\mu}) + \lambda_2(\boldsymbol{\mu}) \\ \left[\lambda_1^2(\boldsymbol{\mu}) + \lambda_2^2(\boldsymbol{\mu})\right]/2 \end{pmatrix} \tag{17}$$

From these equations an enhanced approximation for the critical parameters, $\boldsymbol{\mu}_{i+1} = \{\mu_{i+1}, \nu_{i+1}\}^{\mathrm{T}}$, is drawn, and the procedure can be reiterated up to the desired tolerance. It is worth stressing that, while the invariants are nonlinear in $\boldsymbol{\mu}$, *they are linear in the increments* $\delta\boldsymbol{\mu}$, so that, in the iterative approach, Eqs (17) still appear in the linear form, as in the non-defective case (Eqs (3)).

By summarising, the *i*-th iteration of the algorithm is the following:

1) evaluate, e.g. by the QR-method, the two eigenvalues of matrix $\mathbf{A} = \mathbf{A}(\boldsymbol{\mu}_i)$ having the smallest real part (both real or complex conjugate) and the associated right and left eigenvectors;

2) compute the quantities in Eqs (10) and (11);

3) calculate $\mathbf{u}_\mu^*(\boldsymbol{\mu})$, $\mathbf{u}_\nu^*(\boldsymbol{\mu})$ by using Eqs (14) and the sensitivity coefficients in Eqs (13);

4) solve Eqs (17) for the new parameter set $\boldsymbol{\mu}_{i+1} = \{\mu_{i+1}, \nu_{i+1}\}^{\mathrm{T}}$;

5) if $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i\right| > toll_1$ *and* $\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_{i+1})] > toll_2$ $(k = 1, 2)$, then execute a new iteration, with $i = i+1$; if, instead, $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i\right| \le toll_1$ *or* $\left|\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_{i+1})]\right| \le toll_2$ $(k = 1, 2)$, then assume $\boldsymbol{\mu}_c = \boldsymbol{\mu}_{i+1}$.

3.3 *Building-up the bifurcation loci*

After having determined the critical point *C*, the construction of the critical manifolds must be tackled. The two invariants (16), evaluated at $\boldsymbol{\mu} = \boldsymbol{\mu}_c$, and equated to zero, provide the equations of the tangents to the two loci at the critical point. Namely, $I_2(\boldsymbol{\mu}_c, \delta\boldsymbol{\mu}) = 0$ is the (straight line) tangent to the divergence locus, while $I_1(\boldsymbol{\mu}_c, \delta\boldsymbol{\mu}) = 0$, $I_2(\boldsymbol{\mu}_c, \delta\boldsymbol{\mu}) < 0$ is the (straight semi-line) tangent to the Hopf locus. An iterative predictor-corrector scheme, based on the pseudo-arclength method, is used again. A point $\boldsymbol{\mu}_1$ close to $\boldsymbol{\mu}_c$ is taken on one of these two lines (predictor phase); then (corrector phase) the associated invariant is zeroed (i.e. $I_k(\boldsymbol{\mu}_1, \delta\boldsymbol{\mu}) = 0$, $k = 1$ *or* 2) together with a linear constraint equation; a new approximation $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_1 + \delta\boldsymbol{\mu}$ is obtained, and the procedure reiterated. When convergence has been reached, a new point is predicted on the tangent, to follow the curve in the whole region of interest.

By summarizing, the algorithm is the following:

1) take the critical point $\boldsymbol{\mu}_c$ as initial point $\boldsymbol{\mu}_0$, and evaluate the sensitivities $S_{k\mu}(\boldsymbol{\mu}_0)$, $S_{k\nu}(\boldsymbol{\mu}_0)$ (k = 1 *or* 2) and the vectors $\mathbf{u}^*_\mu(\boldsymbol{\mu})$, $\mathbf{u}^*_\nu(\boldsymbol{\mu})$ via Eqs (13), (14);

2) select a new point $\boldsymbol{\mu}_1$ (*predictor phase*) on the tangent $\mathbf{t}_{k0} = \{a_{k0}, b_{k0}\}^{\mathrm{T}}$ at $\boldsymbol{\mu}_0$, at a sufficiently small distance $|\Delta s|$ from $\boldsymbol{\mu}_0$, having coordinates:

$$\mu_1 = \mu_0 + a_{k0}\,\Delta s, \quad \nu_1 = \nu_0 + b_{k0}\,\Delta s, \quad k = 1 \text{ or } 2 \tag{18}$$

where:

$$a_{k0} := +\frac{S_{1\nu}(\boldsymbol{\mu}_0)}{\sqrt{S^2_{1\mu}(\boldsymbol{\mu}_0) + S^2_{1\nu}(\boldsymbol{\mu}_0)}}, \quad b_{k0} := -\frac{S_{1\mu}(\boldsymbol{\mu}_0)}{\sqrt{S^2_{1\mu}(\boldsymbol{\mu}_0) + S^2_{1\nu}(\boldsymbol{\mu}_0)}}, \quad \text{if } k = 1 \tag{19}$$

or:

$$\begin{aligned} a_{k0} &:= +\frac{S_{2\nu}(\boldsymbol{\mu}_0) - \lambda_0(\boldsymbol{\mu}_0) S_{1\nu}(\boldsymbol{\mu}_0)}{\sqrt{\left[S_{2\mu}(\boldsymbol{\mu}_0) - \lambda_0(\boldsymbol{\mu}_0) S_{1\mu}(\boldsymbol{\mu}_0)\right]^2 + \left[S_{2\nu}(\boldsymbol{\mu}_0) - \lambda_0(\boldsymbol{\mu}_0) S_{1\nu}(\boldsymbol{\mu}_0)\right]^2}}, \\ b_{k0} &:= -\frac{S_{2\mu}(\boldsymbol{\mu}_0) - \lambda_0(\boldsymbol{\mu}_0) S_{1\mu}(\boldsymbol{\mu}_0)}{\sqrt{\left[S_{2\mu}(\boldsymbol{\mu}_0) - \lambda_0(\boldsymbol{\mu}_0) S_{1\mu}(\boldsymbol{\mu}_0)\right]^2 + \left[S_{2\nu}(\boldsymbol{\mu}_0) - \lambda_0(\boldsymbol{\mu}_0) S_{1\nu}(\boldsymbol{\mu}_0)\right]^2}}, \quad \text{if } k = 2 \end{aligned} \tag{20}$$

3) Solve iteratively (for $i = 1, 2, \cdots$) the following equations in the unknowns μ_{i+1}, ν_{i+1} (*corrector* phase):

$$\begin{pmatrix} S_{1\mu}(\boldsymbol{\mu}_i) & S_{1\nu}(\boldsymbol{\mu}_i) \\ a_{k0} & b_{k0} \end{pmatrix} \begin{pmatrix} \mu_{i+1} - \mu_i \\ \nu_{i+1} - \nu_i \end{pmatrix} = \begin{pmatrix} -2\lambda_0(\boldsymbol{\mu}_i) \\ \Delta s \end{pmatrix}, \quad \text{if } k = 1 \tag{21}$$

or:

$$\begin{pmatrix} S_{2\mu}(\boldsymbol{\mu}_i) - \lambda_0(\boldsymbol{\mu}_i) S_{1\mu}(\boldsymbol{\mu}_i) & S_{2\nu}(\boldsymbol{\mu}_i) - \lambda_0(\boldsymbol{\mu}_i) S_{1\nu}(\boldsymbol{\mu}_i) \\ a_{k0} & b_{k0} \end{pmatrix} \begin{pmatrix} \mu_{i+1} - \mu_i \\ \nu_{i+1} - \nu_i \end{pmatrix} = \begin{pmatrix} \lambda_0^2(\boldsymbol{\mu}) - \xi(\boldsymbol{\mu}) \\ \Delta s \end{pmatrix}, \quad \text{if } k = 1 \tag{22}$$

4) if $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i\right| > toll_1$ *and* $\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_{i+1})] > toll_2$ (k = 1 *or* 2), then execute a new iteration (21) or (22); if, instead, $\left|\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i\right| \le toll_1$ *or* $\left|\mathrm{Re}[\lambda_k(\boldsymbol{\mu}_{i+1})]\right| \le toll_2$ (k = 1 *or* 2), then assume as a new point $\boldsymbol{\mu}_0 = \boldsymbol{\mu}_{i+1}$, and restart from step 2).

4 NUMERICAL EXAMPLES

The classical mechanical systems illustrated in Fig. 2 are analyzed. They are double-pendula with lumped inertia (m), elastic (k_i) and damping (c_i) properties, under forces applied at the free end. The structure of Fig. 2 (a) is loaded by a dead weight P and a follower force F; the stucture of Fig. 2 (b) is loaded by a follower force F, only. The rods are rigid and massless; the elastic springs and the viscous devices are linear. The follower force F is taken as μ-parameter in both systems, while the dead load P, or the k_2 stiffness, are taken as ν-parameter in the two systems, respectively.

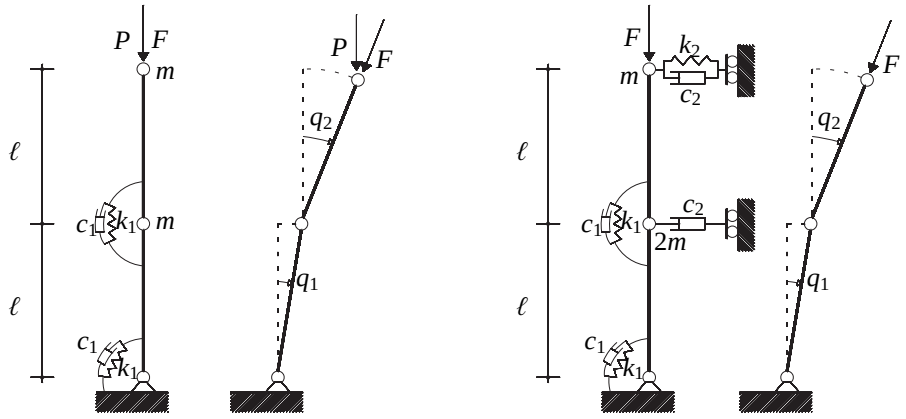

Figure 2: Double pendulum: (a) loaded by a follower force and a dead load; (b) loaded by a follower force and visco-elastically braced.

By assuming the rotations q_i $(i = 1, 2)$ as Lagrangian parameter, and $\mathbf{x} = (q_1, \dot{q}_1, q_2, \dot{q}_2)^{\mathrm{T}}$ as state-vector, the following Jacobian matrices are obtained for systems of Fig. 2, respectively:

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3\kappa+\mu+\nu & -3\eta & 2\kappa-\mu-\nu & 2\eta \\ 0 & 0 & 0 & 1 \\ 4\kappa-\mu-\nu & 4\eta & -3\kappa+\mu+2\nu & -3\eta \end{bmatrix}, \\
\mathbf{A} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-3\kappa+\mu}{2} & \frac{\eta(-3-\zeta)}{2} & \frac{2\kappa-\mu}{2} & \eta \\ 0 & 0 & 0 & 1 \\ \frac{5\kappa-2\nu-\mu 1}{2} & \frac{\eta(5-\zeta)}{2} & \frac{\mu-4\kappa-2\nu}{2} & \eta(-2-\zeta) \end{bmatrix}
\end{aligned} \tag{23}$$

where the following nondimensional parameters appear: $\kappa := k_1/(m\omega^2\ell^2)$, $\eta := c_1/(m\omega\ell^2)$, $\mu := F/(m\omega^2\ell)$, for both systems; $\nu := P/(m\omega^2\ell)$ for system 1 and $\zeta = c_2\ell^2/c_1$, $\nu := k_2/(m\omega^2)$ for system 2.

System 1 undergoes a divergence – Hopf bifurcation at $\boldsymbol{\mu}_c = (5.77, -2.35)$. Starting from the guess point $\boldsymbol{\mu}_0 = (5.2, -1.6)$, and applying the procedure of Sect 2.2, the iterations displayed in Fig. 3 (a) are performed, and convergence reached in few steps. Then, starting from $\boldsymbol{\mu}_c$, and applying the procedure of Sect 2.3, the bifurcation loci depicted in Fig. 3 (b) are obtained.

System 2 experiences a double-zero bifurcation at $\boldsymbol{\mu}_c = (5.83, 0.15)$. A tentative point $\boldsymbol{\mu}_0 = (8, 0.08)$ was chosen, and both the iterative procedure for non-defective and defective systems were applied. As it appears in Fig. 4 (a), the algorithm based on sensitivity of distinct eigenvalues diverges, while the method grounded on nearly coincident eigenvalues (Sect 3.2) converges fast. The bifurcation loci originating from $\boldsymbol{\mu}_c$ are shown in Fig. 4 (b).

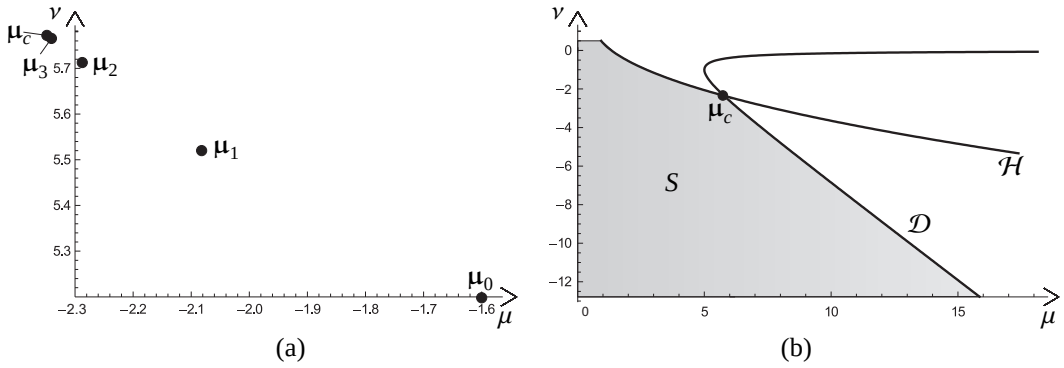

Figure 3: System 1: (a) iterations to the divergence-Hopf bifurcation point; (b) bifurcation loci; $\mathcal{D}$ Divergence locus, $\mathcal{H}$ Hopf locus, S Stable region; $\kappa = 1$, $\eta = 1$.

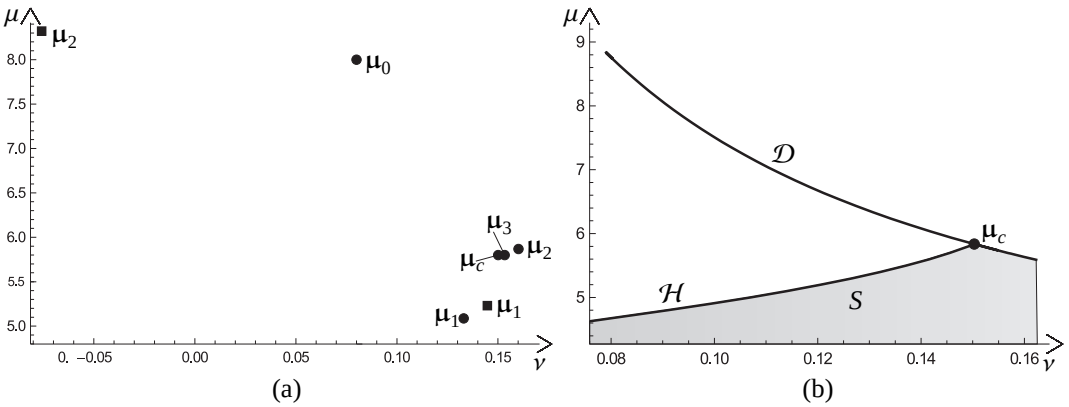

Figure 4: System 2: (a) iterations to the double-zero bifurcation point: ● sensitivity for nearly coincident eigenvalues, ■ sensitivity for distinct eigenvalues; (b) bifurcation loci; $\mathcal{D}$ Divergence locus, $\mathcal{H}$ Hopf locus, S Stable region; $\kappa = 1$, $\eta = 1.5$, $\zeta = 0.5$.

5 CONCLUSIONS

By exploiting the potentiality of eigenvalue sensitivity analysis, an iterative numerical- perturbation method was implemented, to build-up linear stability diagrams of two-parameter dynamical systems, undergoing codimension-2 bifurcations. Both non-defective and defective bifurcations were studied, and specific algorithms illustrated.

In particular, it was stressed, that sensitivity analisys avoids numerical evaluation of the derivatives of the eigenvalues with respect the parameters, which are necessary both in approaching the critical point and in building-up the bifurcation loci crossing there. Such a computational advantage is even more evident when defective bifurcations must be analyzed, for which the eigenvalue derivatives tend to infinity at the coalescence point. In these cases, sensitivity analysis furnishes uniformly

valid *reduced characteristic equations*, of degree equal to the number of interacting eigenvalues, which govern the eigenvalue behavior around the critical point, capturing their unavoidable singularities. The analysis of the invariants of such reduced equations guides the search for the multiple bifurcation point and bifurcation loci.

Few numerical examples were presented, relevant to mechanical systems exhibiting Hopf- divergence or double-zero bifurcations.

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