Fractional calculus for the solution of non-linear stochastic oscillators with viscous damper devices

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Fluid viscoelastic dampers are of great interest in different fields of engineering. Examples of their applications can be found in seismic mitigation design of structures or in vibration absorption in airplane suspension. Such devices introduce a non-linear dissipative term in the equation of motion and therefore, the solution of even a single degree of freedom system excited by a white noise process, becomes prohibitive. The solution is usually obtained by approximated methods, like the stochastic linearization technique.

In this paper it is shown that, by means of fractional operators, it is possible to find the solution of oscillators provided with fluid viscoelastic devices, approaching the problem in its originally non-linear form. Comparison with solutions obtained by Monte Carlo simulation are finally discussed.

1 Introduction

Viscous fluid dampers have found applications in passive control of civil structures for their ability in energy absorbing. Some examples are the damper devices in buildings for the mitigation of earthquake ground motion and airplane shock absorber. Applications to highway overcrossing equipped with such devices are given in [1], while their constitutive relations and dynamical properties are presented in [2], [3] and [4].

In the fluid viscous-elastic element, the damping force is generated by a moving piston into a chamber filled with a viscous fluid. Its constitutive equation is characterized by the shape of the orifices where the fluid itself flows. As shown in [4], the constitutive equation for such devices can be written as

$$F_d = c_d \left|\dot{x}\right|^{\gamma} \operatorname{sign}(\dot{x}) \tag{1}$$

where F_d is the piston force, x its relative displacement, γ is a real exponent, c_d is a damping parameter (in Ns^{γ}) and $\operatorname{sign}(\cdot)$ is the signum function defined as follow

$$\operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{2}$$

The motion equation of an oscillator equipped with a fluid viscous-elastic damper excited by a white noise process $W(t)$ is consequently written as

$$\ddot{X}(t) + 2\zeta\omega_0 \dot{X}(t) + \zeta_d \left|\dot{X}(t)\right|^{\gamma} \operatorname{sign}\left(\dot{X}(t)\right) + \omega_0^2 X(t) = \frac{W(t)}{m} \tag{3}$$

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having indicated with $X(t)$ the resulting displacement process, ζ the damping ratio, $\omega_0 = \sqrt{\frac{k}{m}}$ the natural radial frequency (being m the mass and k the stiffness respectively), $\zeta_d = c_d/m$ and $W(t)$ a white noise process. Due to the presence of the fluid viscoelastic device, eq.(3) is a non-linear stochastic differential equation.

Probabilistic characterization of the solution of eq.(3) is necessary both for design and reliability purposes, but it is very hard to be found because of the strong non-linearity, varying with the parameter γ. Monte Carlo simulation is very useful to provide approximated solutions but is highly time consuming. Recent analysis of multi-degree of freedom systems in the context of stochastic linearization is given in [5], where very good solutions are provided up to second order approximation.

In this paper we provide a solution of the non-linear oscillator provided with a viscoelastic fluid device by using stochastic calculus and fractional operators, achieving very good accuracy of the solution in probability. In order to give great generality to our results, we will consider the oscillator excited by a Lévy α-stable white noise indicated by $W_\alpha(t)$. The parameter α of the noise might assume values in the range $0 < \alpha \leq 2$, giving the classical Gaussian noise for $\alpha = 2$. In this way, the Gaussian case will be just a particular case derived from the general one. In the following we give a brief summary on the principal feature of α-stable variables and noise processes, but readers are referred to the book [6] for a deep treatment on both theory and application.

The marginal distribution of a Lévy α-stable white noise is the so-called α-stable distribution which has attracted physicians since the thirties, because it relies on a generalized central limit theorem. Indeed, as well as the sum of independent random variables with finite variance tends to a normal distribution, dropping off the hypothesis of finite variance, the sum converges to a stable distribution. The main peculiarity of such distributions is their typical inverse power law asymptotic behavior, i.e. the tails of the density function goes to zero as fast as $|x|^{-\alpha-1}$ for $x \to \infty$. From this trend, it can be noted that a stable variable X has only moments of order q such that $0 < q < \alpha$ and consequently, if $1 < \alpha \leq 2$, the mean and the variance does not exist and for, $0 < \alpha \leq 1$, even the mean diverges. Then, well-known relations of probability theory based on moments are useless for stable random variables. A new attempt for handling with such variables has been given in [7], leading more amenable their mathematical treatment.
The Lévy α-stable motion has become only more recently in the engineers background mainly because of its connection to fractal geometry and for this reason will be considered in this paper.

In this paper, firstly, by means of stochastic calculus, the equation ruling the characteristic function is found and then, a proper use of fractional operators will provide a solution scheme for the problem in hand. In the numerical section a comparison between the stable and the Gaussian white noise excitation will be provided, and Monte Carlo simulation will assess that the numerical solution proposed gives very good results. Moreover, the computational speed of the proposed method suggests that it might be a valid alternative for design and active control purposes.

2 Derivation of the spectral Fokker-Planck equation

Stochastic differential equations excited by white noise processes have been widely studied in literature by means of Itô calculus. In the framework of the generalized theory of random processes, the stationary white noise process $W_\alpha(t)$ is indeed defined as the formal derivative of a process with stationary orthogonal and independent increment $L_\alpha(t)$, that is

$$W_\alpha(t) \stackrel{def}{=} \frac{dL_\alpha(t)}{dt} \tag{4}$$

with $0 < \alpha \leq 2$. In the case of $\alpha = 2$, the process $L_\alpha(t)$ coincides with the Brownian motion $B(t)$, whose increments $dB(t)$ are zero mean standard distributed, with even moments of the form

$$E[dB(t)^{2k}] = (2k-1)!!\ q(t)^k dt^k,\ k = 1, 2, ... \tag{5}$$

where $q(t)$ is the noise strength. In this paper, we assume $q(t) = q$ and consequently a constant noise power spectral density $S_W(\omega) = q/2\pi$.
If α is strictly different from 2, the only property that can be claimed for the increments of the α-stable Lévy motion process $dL_\alpha(t)$ is related to the characteristic function $\phi(dL_\alpha)$

$$\phi(dL_\alpha) = E[\exp(i\,\theta\, dL_\alpha)] = \exp(-|\theta|^\alpha dt) \tag{6}$$

being $dL_\alpha(t)$ distributed as a standard α-stable distribution, that is with zero location and unitary scale parameters. It has been shown in [8] and [9] that the extended Itô's rule, proposed originally in [10] for Poisson excitation, can be applied to Lévy excitation in order to find the statistics of the response. Remanding to the references [8] and [9], we briefly report the results for a general non-linear oscillator. Given the equation of motion in the form

$$\ddot{X} + f(X, \dot{X}, t) = W_\alpha(t) \tag{7}$$

expressed in the state variable space as

$$\dot{\mathbf{Z}} = \mathbf{g}(\dot{\mathbf{Z}}, \mathbf{t}) + \boldsymbol{\ell}\, W_\alpha(t) \tag{8}$$

with $\mathbf{Z} = [X, \dot{X}]^T$; $\mathbf{g}(\mathbf{Z}, \mathbf{t}) = [\dot{\mathbf{X}},\ -\mathbf{f}(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{t})]^\mathbf{T}$ and $\boldsymbol{\ell} = [0, 1]^T$, bearing in mind eq.(4), it can be rewritten in the Itô form

$$d\mathbf{Z} = \mathbf{g}(\dot{\mathbf{Z}}, \mathbf{t})dt + \boldsymbol{\ell}\, dL_\alpha(t) \tag{9}$$

Applying the extended Itô's rule for a Lévy white noise process, it has been shown in [8] and [9] that the evolution of the characteristic function of the response, called $\phi_\mathbf{Z}(\vartheta_1, \vartheta_2) = E[\exp(i\,\theta_1 X + i\,\theta_2 \dot{X})]$, is ruled by the deterministic partial differential equation

$$\frac{\partial \phi_\mathbf{Z}(\boldsymbol{\theta}, t)}{\partial t} = i\,\boldsymbol{\theta}^T E\left[\mathbf{g}(\mathbf{Z}, t) \exp\left(i\,\boldsymbol{\theta}^T \mathbf{Z}\right)\right] - |\theta_2|^\alpha\, \phi_\mathbf{Z}(\boldsymbol{\theta}, t) \tag{10}$$

where $\boldsymbol{\theta} = [\theta_1, \theta_2]^T$. The latter rules the evolution in time of the characteristic function of the solution and is the *spectral* counterpart of the Fokker-Planck equation that, as it is well-known rules the evolution of the density. Eq.(10) is not a solvable partial differential equation of the characteristic function, unless the function $\mathbf{g}(\mathbf{Z}, t)$ representing the drift term of the equation of motion is specified. If the non-linearity is expressed in power series form, the solution has been presented in literature by wavelet in [9] or by modified orthogonal polynomial in [11]. When the non-linear drift is not expressible in power series form, as like as in the case under exam, the main difficulty is represented by finding the structure of the equation involving explicitly the characteristic function. Indeed, introducing in eq.(10) the non-linear term of the oscillator under study given in (3), that is $f(X, \dot{X}, t) = 2\zeta\omega_0\dot{X} + \zeta_d|\dot{X}|^\gamma \mathrm{sign}(\dot{X}) + \omega_0^2 X$ and making some algebra, eq.(10) is rewritten as

$$\begin{aligned} \frac{\partial \phi_\mathbf{Z}(\boldsymbol{\theta}, t)}{\partial t} \quad &= \quad (\theta_1 - \zeta\theta_2)E\left[i\,\dot{X}\exp\left(i\,\boldsymbol{\theta}^T\mathbf{Z}\right)\right] - \omega_0^2\theta_2 E\left[i\,X\exp\left(i\,\boldsymbol{\theta}^T\mathbf{Z}\right)\right] + \\ &\quad - \frac{|\theta_2|^\alpha}{m}\phi_\mathbf{Z}(\boldsymbol{\theta}, t) - \zeta_d\theta_2 E\left[i\left|\dot{X}\right|\mathrm{sign}(\dot{X})\exp\left(i\,\boldsymbol{\theta}^T\mathbf{Z}\right)\right] \end{aligned} \tag{11}$$

It is easy to prove from the definition of the characteristic function that the relations

$$E\left[i\,X\exp\left(i\,\boldsymbol{\theta}^T\mathbf{Z}\right)\right]=\frac{\partial\phi_{\mathbf{Z}}\left(\boldsymbol{\theta},t\right)}{\partial\theta_1} \tag{12}$$

$$E\left[i\,\dot{X}\exp\left(i\,\boldsymbol{\theta}^T\mathbf{Z}\right)\right]=\frac{\partial\phi_{\mathbf{Z}}\left(\boldsymbol{\theta},t\right)}{\partial\theta_2} \tag{13}$$

hold true.
Yet, the presence of the last term in eq.(11) with the average $E\left[i\left|\dot{X}\right|\mathrm{sign}(\dot{X})\exp\left(i\,\boldsymbol{\theta}^T\mathbf{Z}\right)\right]$ is not expressed in terms of the characteristic function and makes the solution of eq.(11) impossible to be found.
In the following it will be shown that, by using fractional calculus, eq.(11) will be transformed in a fractional differential equation amenable to be numerically solved.

3 Fractional operators for the solution of the non linear oscillator

Fractional calculus deals with derivatives or integrals of real or even complex order. Many books and papers are available in literature, showing an always increasing interest on this advanced mathematical tool. In this section we provide few definitions and properties necessary to solve the equation (11), referring the readers to the exceptional books [12] and [13] for deeper insight and applications.
The Riesz fractional integral of order γ is defined as

$$\left(I^{\gamma}f\right)(x)\overset{def}{=}\frac{1}{2\,\Gamma\left(\gamma\right)\,\cos\left(\gamma\,\pi/2\right)}\int\limits_{-\infty}^{+\infty}\frac{f\left(\xi\right)}{\left|\xi-x\right|^{1-\gamma}}d\xi \qquad Re\gamma>0 \quad \gamma\neq1,3,5,... \tag{14}$$

jointly to the Riesz fractional derivative

$$\left(D^{\gamma}f\right)(x)\overset{def}{=}\frac{1}{2\,\Gamma\left(\gamma\right)\,\cos\left(\gamma\,\pi/2\right)}\int\limits_{-\infty}^{+\infty}\frac{f\left(x-\xi\right)-f\left(x\right)}{\left|\xi\right|^{1+\gamma}}d\xi \tag{15}$$

The complementary Riesz fractional integral is instead defined as

$$\left(H^{\gamma}f\right)(x)\overset{def}{=}\frac{1}{2\,\Gamma\left(\gamma\right)\,\sin\left(\gamma\,\pi/2\right)}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{sign}\left(x-\xi\right)}{\left|x-\xi\right|^{1-\gamma}}f\left(\xi\right)d\xi \qquad Re\gamma>0 \quad \gamma\neq2,4,6,... \tag{16}$$

jointly to the complementary Riesz fractional derivative

$$\left(H^{-\gamma}f\right)(x)\overset{def}{=}\frac{1}{2\,\Gamma\left(-\gamma\right)\,\sin\left(\gamma\,\pi/2\right)}\int\limits_{-\infty}^{+\infty}\frac{f\left(x-\xi\right)-f\left(x\right)}{\left|\xi\right|^{1+\gamma}}\mathrm{sign}\left(\xi\right)d\xi \tag{17}$$

The operators in (14)-(17) are very useful once their Fourier transform is given. Indeed, it has been proved that the following Fourier relations

$$\left(\mathcal{F}\,I^{\gamma}f\left(x\right)\right)\left(\theta\right)=\left|\theta\right|^{-\gamma}\left(\mathcal{F}\,f\left(x\right)\right)\left(\theta\right) \tag{18}$$

$$\left(\mathcal{F}\, H^{\gamma} f\,(x)\right)(\theta) = i\,\mathrm{sign}\,(\theta)\,|\theta|^{-\gamma}\left(\mathcal{F}\, f\,(x)\right)(\theta) \tag{19}$$

hold true for the fractional integrals, having indicated by $\mathcal{F}$ the Fourier transform defined as

$$\left(\mathcal{F} f(x)\right)(\theta) = \int_{-\infty}^{+\infty} f\,(x) \exp\left(i\, x\, \theta\right)\, dx \tag{20}$$

Similarly, for the fractional derivatives, the properties

$$\left(\mathcal{F}\, D^{\gamma} f\,(x)\right)(\theta) = |\theta|^{\gamma}\left(\mathcal{F}\, f\,(x)\right)(\theta) \tag{21}$$

$$\left(\mathcal{F}\, H^{-\gamma} f\,(x)\right)(\theta) = i\,\mathrm{sign}\,(\theta)\,|\theta|^{\gamma}\left(\mathcal{F}\, f\,(x)\right)(\theta) \tag{22}$$

hold true.
By means of relation (22) we will show that the term in eq.(11), accounting the non-linear viscoelastic damping, can be expressed in terms of the complementary Riesz fractional derivative of the characteristic function.
Indeed eq.(22) can be extended to $\phi_{\mathbf{Z}}\,(\theta_1, \theta_2)$ as follow

$$\begin{aligned}\left(\mathcal{F}^{-1}\left({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}\right)(\boldsymbol{\theta})\right)\left(X, \dot{X}\right) &= -i\,\mathrm{sign}\left(\dot{X}\right)\left|\dot{X}\right|^{\gamma}\left(\mathcal{F}^{-1}\,\phi_{\mathbf{Z}}\,(\boldsymbol{\theta})\right)(x, \dot{x}) \\ &= -i\,\mathrm{sign}\left(\dot{X}\right)\left|\dot{X}\right|^{\gamma} p_{\mathbf{Z}}\,(x, \dot{x})\end{aligned} \tag{23}$$

where $\left({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}\right)(\boldsymbol{\theta})$ is the partial fractional complementary Riesz derivative, that reads

$$\left({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}\right)(\boldsymbol{\theta}) = \frac{1}{2\,\Gamma\,(-\gamma)\,\sin\left(\gamma\,\pi/2\right)} \int_{-\infty}^{+\infty} \frac{\phi\,(\theta_1, \theta_2 - u) - \phi\,(\theta_1, \theta_2)}{|u|^{1+\gamma}} \mathrm{sign}\,(u)\, du \tag{24}$$

Fourier transform of both sides of eq.(23) latter produce the result searched

$$\left({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}\right)(\theta_1, \theta_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} -i\,\mathrm{sign}\left(\dot{X}\right)\left|\dot{X}\right|^{\gamma}\, p_{\mathbf{Z}}\left(X, \dot{X}\right)\, \exp\left(i\,\theta^{T}\,\mathbf{z}\right) dx\, d\dot{x} \tag{25}$$

Introducing eq.(24) in eq.(11), the characteristic function's equation of the non-linear oscillator is achieved, and reads

$$\frac{\partial \phi_{\mathbf{Z}}}{\partial t} = (\theta_1 - \zeta\theta_2)\frac{\partial \phi_{\mathbf{Z}}}{\partial \theta_2} - \omega_0^2\theta_2\frac{\partial \phi_{\mathbf{Z}}}{\partial \theta_1} + \zeta_d\theta_2({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}) - \frac{|\theta_2|^{\alpha}}{m}\phi_{\mathbf{Z}} \tag{26}$$

with arguments omitted. It is worth to stress that the fluid viscoelastic damping is represented in this equation by the contribute of the fractional derivative.

4 Stationary solution

Aim of this section is to provide a method of solution of eq.(25). The hypothesis of stationarity is taken for simplicity's sake, because we want to stress how to tackle the fractional and integral order derivatives in the space of variable θ_1 and θ_2. Extensions to tackle also transitory states would

involve a step-by-step procedure and follows plainly.
To this aim, let us consider a bounded domain $-\overline{\theta}_1 \leq \theta_1 \leq \overline{\theta}_1$, $-\overline{\theta}_2 \leq \theta_2 \leq \overline{\theta}_2$, where each axes is portioned in *m* even intervals of size $\Delta\theta_1$ and $\Delta\theta_2$, respectively. This imply that the domain o be discretized by a grid of $(m+1)^2$ nodes. We will indicate the unknown values of the function $\phi(\theta_1, \theta_2)$ in the grid as $\varphi_{i,j} = \phi(-\overline{\theta}_1 + (j-1)\Delta\theta_1, -\overline{\theta}_2 + (i-1)\Delta\theta_2)$ for $i, j = 1, 2, ..., m+1$. In this way, the unknown variables are represented by the vector

$$\boldsymbol{\varphi} = \left[\varphi_{1,1}, ..., \varphi_{1,m+1}, \varphi_{2,1}, ..., \varphi_{2,m+1}, ..., \varphi_{m+1,1}, \varphi_{m+1,2}, ..., \varphi_{m+1,m+1}\right]^T$$

The fractional operator $\left({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}\right)(\theta_1, \theta_2)$ can be rewritten in terms of a kind of finite difference expression, called *Grünwald-Letnikov fractional operator,* that reads

$$\begin{aligned}\left({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}\right)(\theta_1, \theta_2) = \lim_{\Delta\theta_2 \to 0} -\tfrac{\Delta\theta_2^{-\gamma}}{2\sin(\gamma\pi/2)} & \left(\sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \gamma \\ k \end{pmatrix} \phi_{\mathbf{Z}}(\theta_1, \theta_2 - k\Delta\theta_2) + \right. \\ & \left. - \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \gamma \\ k \end{pmatrix} \phi_{\mathbf{Z}}(\theta_1, \theta_2 + k\Delta\theta_2)\right)\end{aligned} \tag{27}$$

in the exact form for unbounded domain, while as

$$\begin{aligned}\left({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}\right)(\theta_1, \theta_2) \cong -\tfrac{\Delta\theta_2^{-\gamma}}{2\sin(\gamma\pi/2)} & \left(\sum_{k=0}^{\frac{\theta_2 - \bar{\theta}_2}{\Delta\theta_2}} (-1)^k \begin{pmatrix} \gamma \\ k \end{pmatrix} \phi_{\mathbf{Z}}(\theta_1, \theta_2 - k\Delta\theta_2) + \right. \\ & \left. - \sum_{k=0}^{\frac{\bar{\theta}_2 - \theta_2}{\Delta\theta_2}} (-1)^k \begin{pmatrix} \gamma \\ k \end{pmatrix} \phi_{\mathbf{Z}}(\theta_1, \theta_2 + k\Delta\theta_2)\right)\end{aligned} \tag{28}$$

in the approximated form for bounded domain. The structure of eq.(26) reveals that the evolution of the fractional derivative $\left({}_{\theta_2}H^{-\gamma}\phi_{\mathbf{Z}}\right)$ in a point θ_2 is performed summing all the weighted values of the function preceding θ_2 (first sum in (28)) and all the weighted values of the function following θ_2. Moreover, the partial derivative in eq.(26) are expressed in finite difference form by the Fornberg's algorithm modified. Here we point out that, by means of the Fornberg's algorithm one can approximate a derivative by a finite difference calculated on an arbitrary chosen number of grid points. The numerical solution of the partial integro-differential equation can therefore be performed on the bounded domain, just enforcing the condition that $\phi(0,0) = 1$. Finally, the eq.(26) is rewritten in the form

$$\mathbf{A}\boldsymbol{\varphi} = \mathbf{b} \tag{29}$$

where $\mathbf{A}$ is the coefficient matrix, $\boldsymbol{\varphi}$ is the unknown vector and $\mathbf{b}$ is a vector composed by 1 in the $((m+1)^2/2)$ term and zero elsewhere resulting from the position of the unique boundary condition $\phi(0,0) = 1$.

Summing up, by proper application of fractional calculus we succeeded firstly to write an integro-differential equation ruling the evolution of the characteristic function of the response. Then by using the approximation scheme for fractional operators we provide a numerical scheme that leads to a linear algebraic system of equation. Solution of such system gives the characteristic function sampled in a set of points. In the following, some numerical examples are reported to highlight the effectiveness of the method

5 Numerical Application

In this section, the efficacy of the method proposed is highlighted comparing the solution obtained for a particular oscillator, with one coming from a Monte Carlo simulation. To this aim we consider an oscillator having unitary mass and characterized by damping ratios $\zeta = 0.008$ and $\zeta_d = 0.35$, natural radial frequency $\omega = 10 rad/sec$, and $\gamma = 0.5$.

This problem has been solved considering both a Gaussian and an α-stable white noise excitation. In both cases, the solution is reported in terms of characteristic function, probability density function and, together with the ones obtained by Monte Carlo simulation, in term of its probability marginal distributions.

In particular, Fig.(1) reports the Gaussian case, where panel (a) shows the stationary characteristic function that has been interpolated on the sampled values gathered in the vector $\boldsymbol{\varphi}$ of eq.(29). Panel (b) on the same figure shows the stationary density obtained by FFT and in panel (c) and (d) the comparison between the marginal densities of the displacement X and the velocity $\dot{X}$ obtained and the Monte Carlo results are plotted. The agreement of the results and the efficiency of the method are very satisfactory.

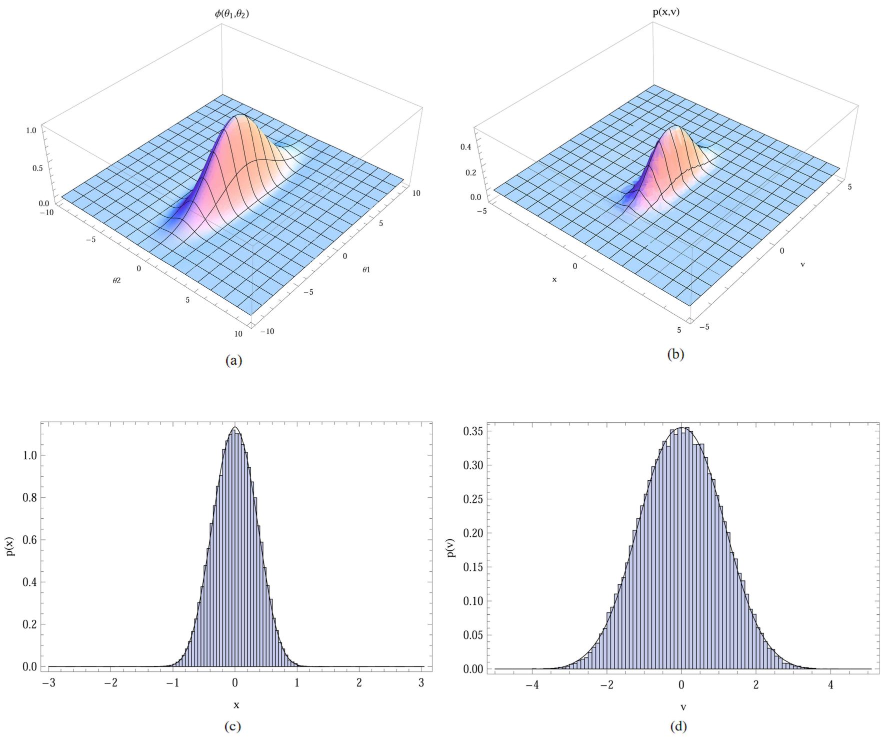

Figure 1: Oscillator excited by Gaussian white noise: a) characteristic function; b) density; c) marginal distribution of the displacement; d) marginal distribution of the veloocity

The case of stable diffusion is reported in Fig.(2) that reports the same scheme of results previously outlined. For this example we have chosen a value of $\alpha = 1.5$. Apart the very good agreement of the results also in this case, it is interesting to note that the effect of the decreasing of the stability index from $\alpha = 2$ (previous Gaussian) to $\alpha = 1.5$ is evident from the marginal densities. Indeed comparing Fig.(1c) with Fig.(2c) and Fig.(1d) with Fig.(2d) the slowest vanishing to zero behavior of the tails in the stable case is evident.

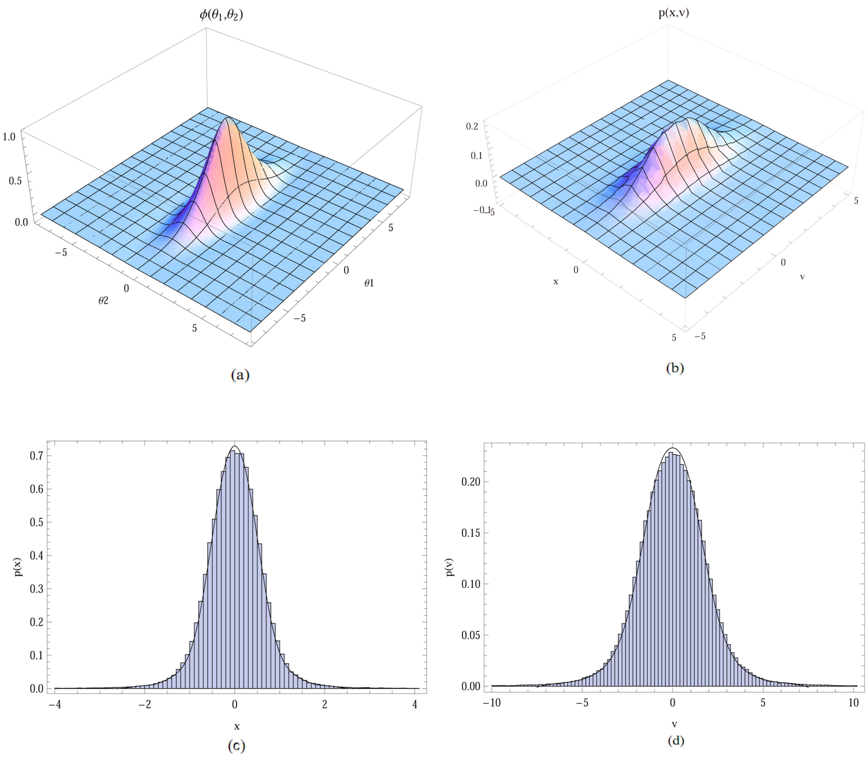

Figure 2: Oscillator excited by 1.5-stable white noise: a) characteristic function; b) density; c) marginal distribution of the displacement; d) marginal distribution of the velocity

6 Conclusions

In this paper, the stationary solution of a non-linear oscillator with non-linear viscoelastic damper excited by the general Lévy white noise process has been studied. By proper application of stochastic calculus the equation ruling the time evolution of the characteristic function has been derived.

It has been shown that such equation can be manipulated in order to involve Riesz fractional operators. In such a way the structure of the equation assumes the form of a partial integro-differential equation. The stationary solution has been found in terms of characteristic function solving such governing equation by a proper numerical scheme relying on the Grünwald-Letnikov numerical approximation of the fractional operators. The method can be plainly extended to the treatment of transitory states, by time step integration, and this is also feasible because of the computational efficiency. Comparisons with Monte Carlo simulation have been reported, showing that very accurate solution can be achieved by the proposed approach.

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