Refined and mixed models for free vibration analysis of functionally graded material plates and shells

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SUMMARY. In this paper an exhaustive free vibration analysis of multilayered plates and shells embedding functionally graded layers, is considered. Both refined models, based on the principle of virtual displacements, and advanced mixed models, based on Reissner's mixed variational theorem, are employed to investigate the vibration problem of functionally graded structures. In order to obtain accurate values of frequency, the use of refined models is mandatory. Mixed models, which a priori model both displacement and transverse shear/normal stresses, permit also to correctly obtain the relative vibration modes in terms of displacements and stresses.

1 INTRODUCTION

The volume fraction of the constituents of a functionally graded material (FGM) changes gradually in a preferred direction (usually the thickness direction z) and consequently the elastic properties depend on the considered coordinate. FGMs have been presented as an alternative to laminated composite materials that show a mismatch in properties at the material interfaces. This material discontinuity in laminated composite materials leads to large interlaminar stresses and the possibility of the initiation and propagation of cracks.

FGMs were first proposed in Japan, by materials scientists in the Sendai area, in 1984 [1], as thermal barrier materials. Since then, high-performance heat resistant barriers in FGMs have been developed. The FGM concept has also been considered to improve energy conversion efficiency. For other application fields, readers can refer to the excellent review by Koizumi [2]. An other interesting review paper has been provided by Birman and Byrd [3].

The special feature of graded spatial compositions associated to FGMs provides freedom in the design and manufacturing of novel structures; on the other hand, it also poses great challenges in numerical modeling and simulation of the FGM structures. In this paper free vibration analysis of multilayered plates and shells, embedding FGM layers, is considered. Some interesting natural frequencies and free vibration analysis of FGM plates can be found in [4] and [5], where higher-order two dimensional models are proposed. The extension to shell geometry is made in [6] and [7], in this last paper a buckling analysis of cylindrical shells subjected to a temperature-specified boundary condition is also made.

The above quoted papers demonstrate the importance of higher-order models to investigate functionally graded structures. The models proposed in this work are refined and advanced/mixed two-dimensional (2D) models obtained in the framework of Carrera's Unified Formulation (CUF) [8], this permits to obtain in a unified manner several two-dimensional models. They can differ for the order of expansion in the thickness direction for primary variables ($N = 1, ..., 4$) and for the multilayer approach: Equivalent Single Layer (ESL) or Layer Wise (LW). CUF has been extended to FGM structures in [9], where refined displacements models have been considered. Mixed models, with "a priori" displacement and transverse/shear stresses, have been extended to FGMs in [10] and

[11]. In case of vibration analysis, in order to obtain the correct frequencies, the use of refined models is enough (if higher orders of expansion are employed). The use of mixed models is mandatory for a correct evaluation of vibration modes in terms of displacements and stresses.

The paper has been organized as follows: Carrera's Unified Formulation is described and extended to FGMs in Section 2; Section 3 gives the employed variational statements and the relative consistent constitutive and geometrical equations; governing equations for the vibration problem are described in Section 4; Section 5 discusses the main results for plate and shell geometries; conclusions are given in Section 6.

2 CARRERA'S UNIFIED FORMULATION

In case of bi-dimensional multi-layered structures (plates and shells), Carrera's Unified Formulation (CUF) [8] permits to obtain a large variety of 2D models that differ in the order of used expansion in thickness direction and in the manner the variables are modelled (Equivalent Single Layer (ESL) or Layer Wise (LW) approach). The salient feature of CUF is the unified manner in which all considered variables and fields (displacement, transverse shear/normal stresses, material) can be treated. As usual in plate/shell theories, the considered variables and their variation are split in a set of thickness functions depending on transverse coordinate z, and the relative terms depending on in-plane coordinates (α,β) only. According to this separation, a general variable $\mathbf{a}$ and its respective variation $\delta\mathbf{a}$ can be written as:

$$\mathbf{a}(\alpha,\beta,z)=F_\tau(z)\;\mathbf{a}_\tau(\alpha,\beta)\;,\quad \delta\mathbf{a}(\alpha,\beta,z)=F_s(z)\;\delta\mathbf{a}_s(\alpha,\beta)\;,\quad \text{with}\quad \tau,s=1,\ldots,N\;, \tag{1}$$

where N is the order of expansion in the thickness direction.

In a multi-layered plate/shell the thickness functions of the considered variables can be assumed for the whole structure (ESL approach) or for each single layer (LW approach). In the former case Taylor polynomials are employed as thickness functions while in the latter, combinations of Legendre polynomials are used. For further details about CUF for multi-layered structures and the relative assembling procedure we refer the reader to [8]. Shells have a curvilinear reference system (α,β,z), when the radii of curvature are infinite these coordinates degenerate in the rectilinear ones for plates (x,y,z).

2.1 *CUF for displacement components*

Due to the unified treatment of all variables, the three displacement components u_α, u_β and u_z and their relative variations can be modelled via CUF, irrespective of whether FGM layers or constant property layers are considered. In case of the ESL model, the expansion of the displacement components is assumed for the whole multi-layer:

$$(u_\alpha,u_\beta,u_z)=F_\tau\;(u_{\alpha\tau},u_{\beta\tau},u_{z\tau})\;,\quad (\delta u_\alpha,\delta u_\beta,\delta u_z)=F_s\;(\delta u_{\alpha s},\delta u_{\beta s},\delta u_{zs})\;, \tag{2}$$

with Taylor expansions from first up to 4^{th} order: $F_0=z^0=1$, $F_1=z^1=z$, $F_2=z^2$, $F_3=z^3$, $F_4=z^4$.

The LW model is obtained if we consider separately each layer k of the given multi-layered structure:

$$(u^k_\alpha,u^k_\beta,u^k_z)=F^k_\tau\;(u^k_{\alpha\tau},u^k_{\beta\tau},u^k_{z\tau})\;,\quad (\delta u^k_\alpha,\delta u^k_\beta,\delta u^k_z)=F^k_s\;(\delta u^k_{\alpha s},\delta u^k_{\beta s},\delta u^k_{zs})\;. \tag{3}$$

In this case, a combination of Legendre polynomials is employed as thickness functions:

$$F_t=\frac{P_0+P_1}{2}\;,\quad F_b=\frac{P_0-P_1}{2}\;,\quad F_l=P_l-P_{l-2}\;,\quad \text{with}\;\;\tau,s=t,b,l\;\;\text{and}\;\;l=2,\ldots,4\;. \tag{4}$$

Here, t and b indicate the top and bottom values for each layer. The Legendre polynomials are $P_0 = 1$, $P_1 = \zeta_k$, $P_2 = \frac{(3\zeta_k^2-1)}{2}$ and so on, with $\zeta_k = \frac{2z^k}{h^k}$ as the non-dimensionalized thickness coordinate ranging from -1 to $+1$ in each layer k. z_k is the local coordinate and h_k the thickness of the k^{th} layer. Displacement components u_α, u_β and u_z for shells degenerate in u_x, u_y and u_z for plates, respectively.

2.2 *CUF for transverse shear/normal stresses*

Transverse shear/normal stresses σ_n in case of mixed models, are always modelled a priori in LW form. For a generic layer k:

$$(\sigma^k_{\alpha z}, \sigma^k_{\beta z}, \sigma^k_{zz}) = F^k_\tau \; (\sigma^k_{\alpha z\tau}, \sigma^k_{\beta z\tau}, \sigma^k_{zz\tau}), \quad (\delta\sigma^k_{\alpha z}, \delta\sigma^k_{\beta z}, \delta\sigma^k_{zz}) = F^k_s \; (\delta\sigma^k_{\alpha zs}, \delta\sigma^k_{\beta zs}, \delta\sigma^k_{zzs}). \quad (5)$$

The chosen thickness functions are the combinations of Legendre polynomials as seen for the displacement approximation. In case of plates, the transverse shear/normal stress vector is $\sigma^k_n = (\sigma^k_{xz}, \sigma^k_{yz}, \sigma^k_{zz})$.

2.3 *CUF for elastic properties of FGMs*

In FGM layers the elastic properties change continuously in the thickness direction. The variation of the elastic characteristics is usually given in terms of exponential and/or polynomial functions applied directly to the engineering constants such as Young's Moduli E_i, Shear Moduli G_{ij}, Bulk Moduli B_i and/or Poisson ratio ν_{ij} or directly to the material elastic coefficients C_{ij}. Actually, since in each point of the plate a relation between the engineering constants and the material elastic coefficients holds, only the second case can be treated. Generally, the variation of the elastic coefficients matrix in the thickness direction can be described by multiplying a material constant by a function of z, i.e.:

$$\mathbf{C}(z) = \mathbf{C}_0 \cdot f(z) \; , \quad (6)$$

where $\mathbf{C}$ can be the matrix of elastic coefficients $\mathbf{Q}$ in case of refined models, or the matrix of modified coefficients $\hat{\mathbf{Q}}$ in case of mixed models.

The procedure does not depend on the thickness laws $f(z)$. Thus, any possible material gradient can be accounted for. Now, applying the ideas behind CUF, the following expansion is made [9]:

$$\mathbf{C}(z) = F_b(z)\,\mathbf{C}_b + F_\gamma(z)\,\mathbf{C}_\gamma + F_t(z)\,\mathbf{C}_t = F_r\,\mathbf{C}_r \; , \quad (7)$$

where the thickness functions F_r are taken in the same manner as in the LW expansion for displacements and transverse shear/normal stresses:

$$F_t = \frac{P_0 + P_1}{2} \; , \quad F_b = \frac{P_0 - P_1}{2} \; , \quad F_\gamma = P_\gamma - P_{\gamma-2} \; , \quad \text{with} \quad \gamma = 2, \ldots, N_r \; . \quad (8)$$

The actual values of $\mathbf{C}$ are then recovered as a weighted summation on the terms $\mathbf{C}_r$. The weights are given by the thickness functions F_r. The order of the expansion can be freely chosen as for the displacements. In this paper the maximum value of N_r is 10. It is mandatory to choose such a high order of expansion to ensure the necessary accuracies [9], [10].

The procedure to include the varying elastic coefficients in the model requires the computation of the $\mathbf{C}_r$ arrays. This task can be accomplished by solving for each component C_{ijr} a simple algebraic system of order N_r. The actual values are calculated at N_r different locations along the thickness

$(z_1, \ldots, z_{N_r})$:

$$\begin{bmatrix} (C_{ij})(z_1) \\ \vdots \\ (C_{ij})(z_{N_r}) \end{bmatrix} = \begin{bmatrix} F_b(z_1) & \cdots & F_\gamma(z_1) & \cdots & F_t(z_1) \\ \vdots & & \vdots & & \vdots \\ F_b(z_{N_r}) & \cdots & F_\gamma(z_{N_r}) & \cdots & F_t(z_{N_r}) \end{bmatrix} \begin{bmatrix} C_{ijb} \\ \vdots \\ C_{ijr} \\ \vdots \\ C_{ijt} \end{bmatrix} . \tag{9}$$

3 VARIATIONAL STATEMENTS

Two different variational statements can be applied. In case of refined models, only displacements are considered as variables of the problem, and the Principle of Virtual Displacements (PVD) is employed. For mixed models, Reissner's Mixed Variational Theorem (RMVT) is applied in order to a priori modelling both displacements $\mathbf{u}$ and transverse shear/normal stresses σ_n.

The geometrical relations for shells link the strains with the displacement components. For a generic layer k:

$$\begin{aligned} \epsilon_{pG}^k &= (\epsilon_{\alpha\alpha}^k, \epsilon_{\beta\beta}^k, \gamma_{\alpha\beta}^k) = (\mathbf{D}_p^k + \mathbf{A}_p^k)\mathbf{u}^k , \\ \epsilon_{nG}^k &= (\gamma_{\alpha z}^k, \gamma_{\beta z}^k, \epsilon_{zz}^k) = (\mathbf{D}_{np}^k + \mathbf{D}_{nz}^k - \mathbf{A}_n^k)\,\mathbf{u}^k . \end{aligned} \tag{10}$$

Strains ϵ are split in in-plane (p) and out-plane (n) components. The matrices $\mathbf{D}$ contain the differential operators. The matrices $\mathbf{A}$ include algebraic terms for the geometrical information of shells: they disappear in case of plates. The displacement vector $\mathbf{u}$ has components u_α, u_β and u_z in the three curvilinear directions α, β and z. In the case of plates, the curvilinear coordinates degenerate in the rectilinear ones x, y and z, and the displacement components are u_x, u_y and u_z. In-plane and out-plane strain components for plates are $\epsilon_{pG}^k = (\epsilon_{xx}^k, \epsilon_{yy}^k, \gamma_{xy}^k)$ and $\epsilon_{nG}^k = (\gamma_{xz}^k, \gamma_{yz}^k, \epsilon_{zz}^k)$, respectively.

3.1 Principle of Virtual Displacements

For a vibration problem and for multilayered plates and shells embedding FGM layers, PVD is written as [9]:

$$\int_V \Big(\delta\epsilon_{pG}^T \sigma_{pC} + \delta\epsilon_{nG}^T \sigma_{nC}\Big) dV = \delta L_e - \delta L_{in} , \tag{11}$$

where V stands for the volume of the considered multilayered structure and $\delta L_{in} = \int_V \rho \delta \mathbf{u}^T \ddot{\mathbf{u}}\; dV$ is the external virtual work made by the inertial forces. $\mathbf{u}$ is the displacement vector, $\ddot{\mathbf{u}}$ is the second temporal derivative, and ρ is the mass density. δL_e is the virtual work made by the external forces. The in-plane stresses are $\sigma_{pC}^k = (\sigma_{\alpha\alpha}^k, \sigma_{\beta\beta}^k, \sigma_{\alpha\beta}^k)$ for shells and $\sigma_{pC}^k = (\sigma_{xx}^k, \sigma_{yy}^k, \sigma_{xy}^k)$ for plates. The subscript G means geometrical relations, the subscript C indicates the substitution of constitutive equations.

The relative constitutive equations are the well-known Hooke's law, which for an FGM structure, use the approximation of elastic coefficients as indicated in Section 2.3:

$$\begin{aligned} \sigma_{pC}^k &= F_r \mathbf{Q}_{ppr}^k \epsilon_{pG}^k + F_r \mathbf{Q}_{pnr}^k \epsilon_{nG}^k , \\ \sigma_{nC}^k &= F_r \mathbf{Q}_{npr}^k \epsilon_{pG}^k + F_r \mathbf{Q}_{nnr}^k \epsilon_{nG}^k . \end{aligned} \tag{12}$$

Matrices $\mathbf{Q}_{ppr}^k$, $\mathbf{Q}_{pnr}^k$, $\mathbf{Q}_{npr}^k$ and $\mathbf{Q}_{nnr}^k$ contain the elastic coefficients for each layer k, and they have dimension $[3 \times 3]$.

3.2 Reissner's Mixed Variational Theorem

In case of mixed models the RMVT is employed, which for a multilayered FGM structure is written as [10]:

$$\int_V \Big(\delta\epsilon_{pG}^T\sigma_{pC} + \delta\epsilon_{nG}^T\sigma_{nM} + \delta\sigma_{nM}^T(\epsilon_{nG} - \epsilon_{nC})\Big)dV = \delta L_e - \delta L_{in}\ . \tag{13}$$

In this case, the subscript M indicates "a priori" modelled variables. The consistent constitutive equations are obtained from the Hooke's law in Equation (12):

$$\begin{aligned} \sigma_{pC}^k &= F_r\hat{\mathbf{Q}}_{ppr}^k\epsilon_{pG}^k + F_r\hat{\mathbf{Q}}_{pnr}^k\sigma_{nM}^k\ ,\\ \epsilon_{nC}^k &= F_r\hat{\mathbf{Q}}_{npr}^k\epsilon_{pG}^k + F_r\hat{\mathbf{C}}_{nnr}^k\sigma_{nM}^k\ . \end{aligned} \tag{14}$$

$\hat{\mathbf{Q}}_{ppr}^k$, $\hat{\mathbf{Q}}_{pnr}^k$, $\hat{\mathbf{Q}}_{npr}^k$ and $\hat{\mathbf{Q}}_{nnr}^k$ are the modified matrices of dimension $[3\times3]$. The approximation of Section 2.3 is applied in case of FGM layers.

4 GOVERNING EQUATIONS

For both models, the governing equations are obtained by substituting the geometrical relations (G) and the constitutive equations (C) in the variational statements. Then, Carrera's Unified Formulation is employed for the two-dimensional approximation. Algebraic closed form solutions are obtained after integration by parts and supposing simply supported boundary conditions and harmonic forms for both displacements and stresses. Governing equations are written in terms of few fundamental nuclei which do not formally depend on the order of expansion N used in the z direction and on the description of variables (LW or ESL). By assembling such nuclei via indexes τ and s for the order of expansion in z, k for the multilayer procedure and r for FGM properties, the global matrices of governing equations are obtained.

4.1 Refined governing equation

In the case of refined models, the governing equation is:

$$\mathbf{K}_{uu}^{k\tau sr}\mathbf{u}_\tau^k = -\mathbf{M}_{uu}^{k\tau sr}\ddot{\mathbf{u}}_\tau^k\ . \tag{15}$$

For a free vibration problem, the external forces are discarded and only the inertial ones are included.

4.2 Mixed governing equation

In the case of mixed models, both displacements and transverse shear/normal stresses are a priori modelled, so governing equations are:

$$\begin{aligned} \mathbf{K}_{uu}^{k\tau sr}\mathbf{u}_\tau^k + \mathbf{K}_{u\sigma}^{k\tau sr}\sigma_{n\tau}^k &= -\mathbf{M}_{uu}^{k\tau sr}\ddot{\mathbf{u}}_\tau^k\\ \mathbf{K}_{\sigma u}^{k\tau sr}\mathbf{u}_\tau^k + \mathbf{K}_{\sigma\sigma}^{k\tau sr}\sigma_{n\tau}^k &= 0\ . \end{aligned} \tag{16}$$

A static condensation is applied for the free vibration analysis.

4.3 Free vibration analysis

The free vibration analysis leads to an eigenvalue problem. Upon substitution of harmonic expressions, the governing equations assume the form of a linear system of algebraic equations in the Ω_k domain:

$$\mathbf{K}^*\hat{\mathbf{U}} = \omega_{\mathbf{mn}}^{\mathbf{2}}\mathbf{M}\hat{\mathbf{U}}\ , \tag{17}$$

where $\mathbf{K}^*$ is the equivalent stiffness matrix obtained by means of static condensation in case of mixed model. $\mathbf{M}$ is the inertial matrix and $\hat{\mathbf{U}}$ is the vector of unknown variables. Only the free vibration analysis is investigated in this article, and the external loadings are therefore set to zero and the relative boundary conditions are exactly fulfilled. By defining $\lambda_{mn} = \omega_{mn}^2$, the solution of the associated eigenvalue problem becomes:

$$||\mathbf{K}^* - \lambda_{mn}\hat{\mathbf{M}}|| = 0 \ . \tag{18}$$

The eigenvectors $\hat{\mathbf{U}}$ associated to the eigenvalues λ_{mn} (or to circular frequencies ω_{mn}) define the vibration modes of the structure in terms of primary variables. Once the waves number (m,n) has been defined in the in-plane directions, the number of obtained frequencies is equal to the degrees of freedom of the employed two-dimensional model. The relative eigenvector can be obtained in terms of primary variables, for each value of frequency, in order to have the modes plotted in the thickness direction.

4.4 Acronyms

Equivalent single layer theories based on PVD are named as ED1-ED4 where the last digit indicates the order of expansion in the thickness direction. The relative layer wise models are named as LD1-LD4 where the letter E (equivalent single layer) is replaced with the letter L (layer wise). In the case of mixed models, based on RMVT, the letter D, for PVD, is replaced with the letter M; so the acronyms are EM1-EM4 and LM1-LM4 for ESL and LW theories, respectively. Classical Lamination Theory (CLT) and First order Shear Deformation Theory (FSDT) are obtained from ED1 model via a typical penalty technique.

5 NUMERICAL RESULTS

In order to validate the proposed models, two different assessments are here proposed: - a one-layered simply supported FGM plate with polynomial material law as proposed in [5]; - a one-layered simply supported FGM cylindrical shell with material polynomial law as given in [6]. Further results will be given at the conference, in particular for the multilayered configurations.

First, the plate geometry is investigated. The plate is in aluminium at the top, and the properties gradually change from metallic to ceramic (at the top). Further details about the material properties, the geometry and the reference solution can be found in [5]. The solution by Matsunaga [5] is a higher-order two-dimensional theory, so a quasi-3D solution is provided as suggested in [9] via a discrete layer method. In Table 1, only PVD models are given because the use of mixed theories does not improve the results in terms of circular frequencies; however mixed models are mandatory in order to obtain the correct mode in terms of displacements and stresses, as indicated in Figure 1. The LM4 model permits to obtain correct evaluations of vibration modes in terms of transverse shear/normal stresses. The plate is one-layered, so if higher orders of expansion are employed in the thickness direction ($N = 4$), the quasi-3D results are obtained for both ESL and LW models. Classical theories such as CLT and FSDT result inappropriate for FGM plates analysis. Imposed the waves number ($m = n = 1$ in this case), only refined models give the higher frequencies related to the further introduced degrees of freedom. CLT gives 3 frequencies because it has only 3 degrees of freedom (dof) in the thickness direction; FSDT has 5 dof and it gives 5 frequencies, and so on for the other refined models.

The second case is a cylindrical shell with the same material properties and thickness law of the plate case, it has been proposed by Matsunaga in [6]. In [6], a higher-order two-dimensional model has been proposed, these results can be improved by the use of refined and mixed models

	[5]	$LD4$	$ED4$	$ED1$	$FSDT$	CLT	$3D$
$\bar{\omega}_1$	6.1932	6.1932	6.1932	6.2112	6.2112	6.3405	6.1932
$\bar{\omega}_2$	30.685	30.685	30.685	30.686	30.686	30.687	30.685
$\bar{\omega}_3$	51.795	51.795	51.795	49.455	51.867	51.873	51.795
$\bar{\omega}_4$	222.43	222.43	222.43	246.37	246.37	-	222.43
$\bar{\omega}_5$	227.29	227.30	227.30	251.59	251.59	-	227.30
$\bar{\omega}_6$	403.06	403.73	403.73	414.70	-	-	403.74
$\bar{\omega}_7$	433.49	436.90	436.90	-	-	-	436.90
$\bar{\omega}_8$	444.99	447.93	447.93	-	-	-	447.93
$\bar{\omega}_9$	645.98	808.78	808.78	-	-	-	808.78

Table 1: FGM isotropic plate with polynomial material law [5] and exponent $K = 1$. Free vibration problem, non-dimensional circular frequencies $\bar{\omega} = \omega h \sqrt{\frac{\rho_m}{E_m}}$ for $m = n = 1$. Thickness ratio $a/h = 10$.

with higher-order of expansions in the thickness direction, as clearly reported in Table 2. Here, for several combinations of wavelengths m and n, the fundamental frequency is given. The Matsunaga

K	h/R_β	m	[6]	$LM4$	$ED4$	$LD3$	$ED2$	CLT	$FSDT$
1	0.500	2	0.2720	0.2768	0.2765	0.2768	0.2771	0.2811	0.2778
	0.200	4	0.7218^{-1}	0.7113^{-1}	0.7117^{-1}	0.7113^{-1}	0.7119^{-1}	0.7206^{-1}	0.7137^{-1}
	0.100	6	0.2821^{-1}	0.2766^{-1}	0.2771^{-1}	0.2766^{-1}	0.2768^{-1}	0.2792^{-1}	0.2770^{-1}
	0.050	6	0.1020^{-1}	0.1013^{-1}	0.1013^{-1}	0.1013^{-1}	0.1013^{-1}	0.1015^{-1}	0.1014^{-1}
	0.010	10	0.9454^{-3}	0.9434^{-3}	0.9446^{-3}	0.9434^{-3}	0.9434^{-3}	0.9438^{-3}	0.9436^{-3}
	0.001	18	0.3068^{-4}	0.3090^{-4}	0.3090^{-4}	0.3090^{-4}	0.3090^{-4}	0.3090^{-4}	0.3090^{-4}
4	0.500	2	0.2209	0.2261	0.2258	0.2262	0.2267	0.2313	0.2280
	0.200	4	0.5995^{-1}	0.5879^{-1}	0.5884^{-1}	0.5880^{-1}	0.5893^{-1}	0.5980^{-1}	0.5912^{-1}
	0.100	6	0.2391^{-1}	0.2330^{-1}	0.2334^{-1}	0.2330^{-1}	0.2335^{-1}	0.2359^{-1}	0.2336^{-1}
	0.050	6	0.8449^{-2}	0.8372^{-2}	0.8381^{-2}	0.8372^{-2}	0.8375^{-2}	0.8395^{-2}	0.8384^{-2}
	0.010	10	0.7879^{-3}	0.7856^{-3}	0.7856^{-3}	0.7856^{-3}	0.7856^{-3}	0.7860^{-3}	0.7858^{-3}
	0.001	18	0.2571^{-4}	0.2569^{-4}	0.2569^{-4}	0.2569^{-4}	0.2569^{-4}	0.2569^{-4}	0.2569^{-4}
10	0.500	2	0.1972	0.2021	0.2018	0.2021	0.2030	0.2000	0.2000
	0.200	4	0.5438^{-1}	0.5344^{-1}	0.5348^{-1}	0.5345^{-1}	0.5367^{-1}	0.5469^{-1}	0.5384^{-1}
	0.100	6	0.2224^{-1}	0.2174^{-1}	0.2178^{-1}	0.2174^{-1}	0.2181^{-1}	0.2210^{-1}	0.2182^{-1}
	0.050	6	0.7667^{-2}	0.7601^{-2}	0.7609^{-2}	0.7601^{-2}	0.7604^{-2}	0.7628^{-2}	0.7615^{-2}
	0.010	10	0.7219^{-3}	0.7198^{-3}	0.7198^{-3}	0.7198^{-3}	0.7199^{-3}	0.7203^{-3}	0.7201^{-3}
	0.001	18	0.2351^{-4}	0.2349^{-4}	0.2349^{-4}	0.2349^{-4}	0.2349^{-4}	0.2349^{-4}	0.2349^{-4}

Table 2: FGM isotropic cylindrical shell with polynomial material law [6] and several exponents K. Free vibration problem, non-dimensional circular frequencies $\bar{\omega} = \omega h \sqrt{\frac{\rho_c}{E_c}}$. For several thickness ratios h/R_β, the fundamental frequencies are calculated for $n = 1$ and m from 2 to 18.

solution [6] gives good results for low values of thickness ratios (thin shells) or for low values of

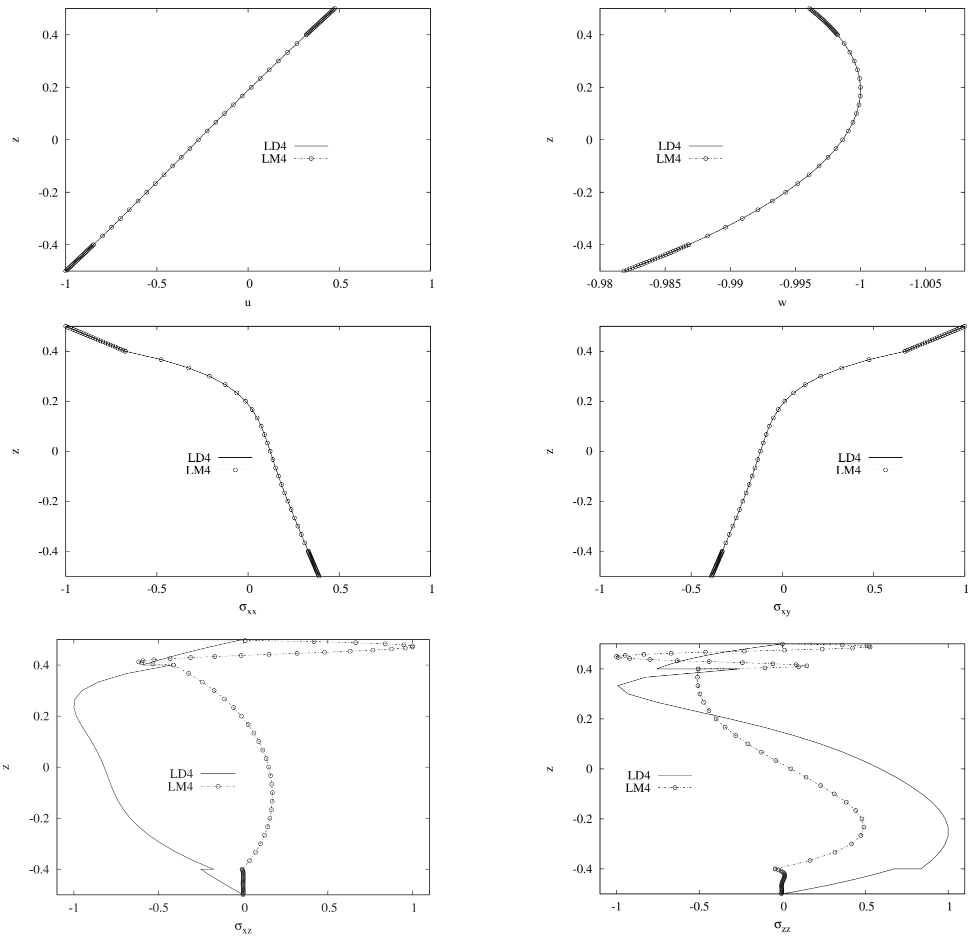

Figure 1: FGM isotropic plate with polynomial material law [5] and exponent $K = 5$, thickness ratio $a/h = 10$. Mode in terms of displacements and stresses for the fundamental frequency for $m = n = 1$. LD4 vs. LM4.

m (low modes). For thick shells and/or higher modes (higher values of the wavelength m), the use of refined and mixed models, with higher orders of expansion in the thickness direction, results mandatory. The shell is one-layered, so no differences are exhibited for ESL and LW models. The curvature does not introduce further considerations: so, as in the case of plate, the use of mixed models is mandatory for a correct evaluation of vibration modes in terms of displacements and stresses. At the conference, the Table 2 will be given in a complete way, by separately studying the effect of the thickness ratio and the effect of the wavelengths: refined models are mandatory to study thick shells and plates, and higher values of wavelengths m and n.

6 CONCLUSIONS

A free vibration analysis of multilayered plates and shells, embedding FGM layers, has been presented in this paper. Refined and mixed models have been employed, and their validity, in the case of functionally graded layers, has been demonstrated. Refined models permit to obtain a quasi-3D evaluation of frequencies in case of vibration problem, but the use of mixed models is mandatory in the case of vibration modes given in terms of displacements and stresses. Further results will be presented at the conference, in particular the importance of refined and advanced models will be remarked in the case of higher order modes, and/or thick and moderately thick plates and shells.

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